mathematics

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On the Structure of the Mislin Genus of a Pullback

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Abstract: The notion of genus for finitely generated nilpotent groups was introduced by Mislin. Two finitely generated nilpotent groups Q and R belong to the same genus set $\mathscr{G}(Q)$ if and only if the two groups are nonisomorphic, but for each prime p, their p-localizations Q_p and R_p are isomorphic. Mislin and Hilton introduced the structure of a finite abelian group on the genus if the group Q has a finite commutator subgroup. In this study, we consider the class of finitely generated infinite nilpotent groups with a finite commutator subgroup. We construct a pullback H_t from the l-equivalences $H_i \to H$ and $H_j \to H$, $t \equiv (i+j) \, mod \, s$, where $s = |\mathscr{G}(H)|$, and compare its genus to that of H. Furthermore, we consider a pullback L of a direct product $G \times K$ of groups in this class. Here, we prove results on the group L and prove that its genus is nontrivial.

Keywords: mislin genus; noncancellation; short exact sequence; pullback diagram; localization

MSC: 20B07; 20J99; 20D15; 20E9

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Citation: Tonisi, T.; Kwashira, R.; Mba, J.C. On the Structure of the Mislin Genus of a Pullback. *Mathematics* **2023**, *11*, 2672. https://doi.org/10.3390/math11122672

Academic Editors: Yilun Shang and Hilal Ahmad

Received: 4 May 2023
Revised: 8 June 2023
Accepted: 10 June 2023
Published: 12 June 2023

1. Introduction

Let Q be a finitely generated nilpotent group. The Mislin genus, $\mathscr{G}(Q)$, is defined to be the set of isomorphism classes of finitely generated nilpotent groups R such that for every prime p, the p-localizations R_p and Q_p are isomorphic. Mislin, in [1], gave a description of the genus set $\mathscr{G}(Q)$ if Q is finitely generated with a finite commutator subgroup. In particular, he showed how to compute the order of the genus set $\mathscr{G}(Q)$. Furthermore, the authors in [2] showed that the genus set $\mathscr{G}(Q)$, which is finite, admits an abelian group structure with Q as its identity element.

Let G be any group. We define the noncancellation set, denoted by $\chi(G)$, to be the set of all isomorphism classes of groups K such that $K \times \mathbb{Z} \cong G \times \mathbb{Z}$. Suppose that G is a finitely generated infinite nilpotent group with a finite commutator subgroup. Warfield's result [3], asserts that $\chi(G) = \mathscr{G}(G)$.

In this study, all groups considered are supposed to be nilpotent. We therefore study the class $\mathscr{N}_0$ of all finitely generated infinite nilpotent groups with finite commutator subgroups. In particular, we are interested in the class $\mathscr{K}$ of semidirect products of the form $T \rtimes \mathbb{Z}^k$, where T is a finite abelian group and $k \in \mathbb{Z}^+$. Many computations of the genus of nilpotent groups which belong to the class $\mathscr{K}$ can be found in the literature [2] and [4]. We note that there is no general method for the computation of $\mathscr{G}(Q)$ when $Q \in \mathscr{N}_0$. Hence, we describe the following subclass of $\mathscr{N}_0$.

Let Q be a nilpotent group. Consider the short exact sequence

$$TQ \rightarrowtail Q \twoheadrightarrow FQ,$$

where TQ is the torsion subgroup of Q and FQ is the torsion-free quotient. We have that $Q \in \mathscr{N}_0$ if and only if TQ is finite and FQ is free abelian of finite rank. The authors in [5] defined a subclass $\mathscr{N}_1 \subset \mathscr{N}_0$ if the following additional conditions hold.

1. TQ and FQ are commutative;
2. The sequence $TQ \rightarrowtail Q \twoheadrightarrow FQ$ splits for the action $\omega : FQ \to Aut(TQ)$;
3. $\omega(FQ) \subseteq Z(Aut(TQ))$, where $Z(Aut(TQ))$ is the center of $Aut(TQ)$.

In the presence of (1), condition (2) is equivalent to requiring that for each $\zeta \in FQ$ there exists $u \in \mathbb{Z}$ such that $\zeta \cdot x = ux$ for all $x \in TQ$, as observed in [5].

For a group $Q \in \mathcal{N}_1$, the authors in [5] gave a general method for the calculation of the genus set $\mathcal{G}(Q)$. Let d be the height of $Ker\,\omega$ in FQ; that is, $d = max\{h \in \mathbb{N} \mid Ker\,\omega \subseteq hFQ\}$, and let $(\mathbb{Z}/d)^*$ be the multiplicative group of units of $\mathbb{Z}/d$. The authors showed that

$$\mathcal{G}(Q) \cong (\mathbb{Z}/d)^*/\{\pm 1\}. \tag{1}$$

In the case where $d = 1$ or 2, the genus of the group Q is trivial. If $Q \in \mathcal{N}_1$ has a nontrivial genus, then FQ is cyclic. The only calculations of the genus of a group in $\mathcal{N}_0$ are those groups in the class $\mathcal{N}_1$ or the direct products of groups in $\mathcal{N}_1$ [2,4,6–8].

For any m prime to d, there is a group Q_m in the genus of Q. The groups Q_m are all supposed to be finitely generated infinite nilpotent. The authors in [5] proved that $Q_m \in \mathcal{G}(Q)$, and that $\mathcal{G}(Q)$ consists of isomorphism classes of groups Q_m. That is, there is a correspondence to $[m] \in (\mathbb{Z}/d)^*/\{\pm 1\}$, and this correspondence provides an isomorphism by (1). The groups Q_m have the same properties as those of Q.

The calculation of the Mislin genus was extended from the class $\mathcal{N}_1$ to a subclass in $\mathcal{N}_1$ of direct products of k copies of Q, where $k \geq 2$. If the direct product Q^k of groups Q_i, where $1 \leq i \leq k$ involves a group $Q_i \in \mathcal{N}_1$ with a noncyclic torsion-free quotient FQ_i, then the genus, $\mathcal{G}(Q^k)$ is trivial. The direct product of groups $Q_i \in \mathcal{N}_1$, each with a cyclic torsion-free quotient FQ_i, is a group Q^k with a noncyclic torsion-free quotient FQ^k. The authors in [4] proved that $\mathcal{G}(Q^k)$ need not be trivial. Let $Q \in \mathcal{N}_1$, and Q^k be the k^{th} direct power of Q, where $k \geq 2$. The authors in [5] proved that there is a surjective homomorphism

$$\rho : \mathcal{G}(Q) \to \mathcal{G}(Q^k)$$

given by $\rho(R) = R \times Q^{k-1}$, where $R \in \mathcal{G}(Q)$. Suppose that p is a prime. Let the torsion subgroup TQ be a p-group, and FQ be cyclic. The authors further showed that ρ is an isomorphism.

If $Q \in \mathcal{N}_1$, then for $k \geq 2$ we have that $Q^k \in \mathcal{N}_1$ if and only if Q is itself commutative since, in general, Q^k does not inherit condition (3) from Q.

Mislin, in [1], defined the genus set for groups in the class $\mathcal{N}_0$, of all finitely generated infinite nilpotent groups with finite commutator subgroups. This paper is a further contribution to the notion of the Mislin genus of groups in the class $\mathcal{N}_0$. For further studies regarding the genus set of a group not in the class $\mathcal{N}_0$, see [9]. In Section 2, we discuss the group structure of the genus of a group $H \in \mathcal{N}_1$ computed by the author in [10]. We construct a pullback H_t of l-equivalences $H_i \to H$ and $H_j \to H$, $t \equiv i + j \bmod s$, where for a prime p and $n \geq 1$, we have $s = p^{n-1}(p-1)/2$. We compute the group structure of the genus of the group H_t by studying the relationship between $\mathcal{G}(H_t)$ and $\mathcal{G}(H)$.

Let (n, u) be a relatively prime pair of integers. Let $G = G(11, u)$ and $K = (G(11^2, u))^{20}$ be $\mathcal{K}$-groups in $\mathcal{N}_0$. In Section 3, we prove that the group of the form $L = \langle x, y, z \mid (x, y)^n = 1, xy = yx, zxz^{-1} = x^u, zyz^{-1} = y^u \rangle$ is a subgroup of the direct product $K \times G$. In this case, the group L is a pullback. Moreover, we compute the structure of the genus set $\mathcal{G}(L)$ by a slight generalization of the results from [10]. Further, we describe the group L_m such that $L \ncong L_m$ but $L \times \mathbb{Z} \cong L_m \times \mathbb{Z}$; and show that $\mathcal{G}(L)$ consists of groups L_m. Section 4 then closes with exact calculations for our general construction of $\mathcal{G}(L)$ from Section 3. We show that $G \in \mathcal{N}_1$ and that $\phi_* : \mathcal{G}(L) \to \mathcal{G}(G)$ is a surjective group homomorphism, which then allows us to determine the exact structure and order of the genus group of L since we can calculate $\mathcal{G}(G)$.

2. Relations between Genus Groups

We discuss the genus set $\mathscr{G}(H)$ for H, a finitely generated infinite nilpotent group with finite commutator subgroup. Let us first recall some notation from [2] and [1]. Denote by ZH the center of H and by TZH the torsion subgroup of ZH. The group $FZH = \{x \in ZH \mid x = y^n; y \in ZH, n = |TZH|\}$ is the free center of H. In particular, the quotient H / FZH denotes the group QH. Then QH_{ab} is the abelianization group of QH.

Now let p be a prime and let $n, k \in \mathbb{N}$. Suppose $u = 1 + cp^k$, where $p \nmid c$ and $u \in \mathbb{Z}^+$ is prime to p^{n+k}. Let the cyclic group $\mathbb{Z}$ generated by ζ act on $\mathbb{Z}/p^{n+k}$ by $\zeta \cdot a = ua$, where $a \in \mathbb{Z}/p^{n+k}$. Then, we consider

$$H = \langle x, y \mid x^{p^{n+k}} = 1, yxy^{-1} = x^u \rangle.$$

Let p^n be the order of u *modulo* p^{n+k}. Note that the case where $p = 2, k = 1$ is excluded, since p^n fails to be the order of u *modulo* p^{n+k}. It is clear to see that the torsion subgroup $\mathbb{Z}/p^{n+k}$ is finite and the torsion-free quotient $\mathbb{Z}$ is free abelian of finite rank. Therefore, $H \in \mathscr{N}_0$.

In view of Proposition 3.2 of [11], we have the following proposition.

Proposition 1. *Let $p \in TH$, then the p-part of the exponent of QH_{ab} is given by $e = p^{\nu_i + \lambda_i}$.*

Proof. Assume that L nilpotent, then we find the following:
$ZL = \langle x^{n'}, y^{n'}, z^{p^{\nu_i}} \rangle$, where $n' = \frac{p^{2\kappa_i}}{p^{\lambda_i}}$,
$FZL = \langle z^{p^{\nu_i + \lambda_i}} \rangle$,
$QL = L/FZL = \langle x, y, z \mid x^n = 1, y^n = 1, xy = yx, zxz^{-1} = x^u, zyz^{-1} = y^u, z^{p^{\nu_i + \lambda_i}} = 1 \rangle$ and
$QL_{ab} = \langle \bar{x}, \bar{y}, \bar{z} \mid \bar{x}^{p^{\lambda_i}} = \bar{y}^{p^{\lambda_i}} = 1, \bar{z}^{p^{\nu_i + \lambda_i}} = 1 \rangle$.
Thus, it follows that $e = p^{\nu_i + \lambda_i}$ for all i. □

Let P be the set of prime divisors of the exponent of the torsion subgroup of H. Let $P - Aut(H)$ denote the semigroup of P-automorphisms of H. Since $H \in \mathscr{N}_0$, the exact sequence [2]

$$P - Aut(H) \xrightarrow{\theta} (\mathbb{Z}/p^{n+k})^* / \{\pm 1\} \overset{\delta}{\twoheadrightarrow} \mathscr{G}(H). \tag{2}$$

is used in the computation of $\mathscr{G}(H)$.

Let $\bar{x} \in (\mathbb{Z}/p^{n+k})^* / \{\pm 1\}$ be represented by $x \in (\mathbb{Z}/p^{n+k})^*$. The author in [1] defined a surjective group homomorphism $\delta : (\mathbb{Z}/p^{n+k})^* / \{\pm 1\} \to \mathscr{G}(H)$ by $\delta(\bar{x}) = H_1$, where $H_1 \in \mathscr{G}(H)$. Now suppose that $\phi : H \to H$ is a P-equivalence; Theorem 1.4 of [2] shows that $\delta(x) = \delta(y)$ if and only if $\theta(\phi) = xy^{-1}$. This implies that $y^{-1}x \in Im\,\theta$. Using this fact, and given that θ is multiplicative, we have that $Im\,\theta$ is a subgroup of $(\mathbb{Z}/p^{n+k})^* / \{\pm 1\}$. We can also show that

$$\delta(y^{-1}x) = \delta(y^{-1})\delta(x) = \delta(y^{-1})\delta(y) = e_{\mathscr{G}(H)}.$$

Thus, $y^{-1}x \in Ker\,\delta$. It is then easy to prove that $Ker\,\delta = Im\,\theta$. Since δ is a surjective group homomorphism, by the first isomorphism theorem, we have that

$$(\mathbb{Z}/p^{n+k})^* / \{\pm 1\} \Big/ Ker\,\delta \cong \mathscr{G}(H).$$

It is clear that the genus $\mathscr{G}(H)$ is a finite group; although, to deduce more about its structure, one must examine the image of θ, which, by [10], has order p^k. Thus, the genus group of H is a cyclic group of order $s = p^{n-1}(p-1)/2$, as shown by the author in [10].

Now, we use the notion of pullback in the category of nilpotent groups to describe the structure of the group $\mathscr{G}(H_t)$. We start by constructing a pullback H_t from the l-

equivalences $H_i \mapsto H$ and $H_j \mapsto H$, where $t \equiv (i+j)\, mod\, s$. Let H_i, H_j, and H be nilpotent groups. Consider the following commutative diagram [2] of group homomorphisms.

We prove the following proposition.

Proposition 2. *Let H, H_i, and H_j be nilpotent groups. Then H_t with $t \equiv (i+j)$ modulo s is a nilpotent group and the diagram in Figure 1 is a pullback.*

$$\begin{tikzcd} H_t \arrow[r, "\psi"] \arrow[d, "\phi"] & H_j \arrow[d, "\rho"] \\ H_i \arrow[r, "\sigma"] & H \end{tikzcd}$$

Figure 1. A commutative diagram of nilpotent groups and morphisms.

Proof. Let $H'_t = \{(x,y) \in H_i \times H_j \mid \sigma(x) = \rho(y)\}$. Then H'_t is a subgroup of $H_i \times H_j$. Figure 2 is a pullback diagram of the diagram in Figure 1. We show here that δ is an isomorphism. [12]. The two morphisms ϕ and ψ define a unique homomorphism $\delta : H_t \to H'_t$ such that $\phi'(\delta) = \phi$ and $\psi'(\delta) = \psi$. We must prove that δ is an isomorphism. Let e, e', e_i, and e_j be the identity elements of H_t, H'_t, H_i, and H_j, respectively. Let $a \in Ker\,\delta$. Then, we have that

$$e_i = \phi(a) = \phi'(\delta(a)) = \phi'(e'), \text{ and also } e_j = \psi(a) = \psi'(\delta(a)) = \psi'(e').$$

Suppose that $\alpha : H_t \to H$ is injective. Then, since the diagram in Figure 1 is commutative, we have that $\rho(\psi) = \alpha = \sigma(\phi)$. Thus, by Lemma 1.4 (iii) of [13] we have that ψ and ϕ are injective. Therefore, $Ker\,\psi \cap Ker\,\phi = \{e\}$ and so $a = e$. Hence, $Ker\,\delta = \{e\}$, and δ is injective.

Now let $(x,y) \in H'_t$. Since $\sigma(x) = \rho(y)$, we have that $y \in \rho^{-1}(\sigma(x))$. By assumption, we can write $y = \psi(a)$ for some a in H_t. Now, suppose that $x - \phi(a) = \phi(b) \in Ker\,\sigma$ for some $b \in H_t$. Then $\delta(a+b) = (\phi(a+b), \psi(a)) = (x,y)$. Thus, δ is surjective and is consequently an isomorphism. □

$$\begin{tikzcd} H_t \arrow[dr, dotted, "\delta"] \arrow[drr, bend left, "\psi"] \arrow[ddr, bend right, "\phi"'] & & \\ & H'_t \arrow[r, "\psi'"] \arrow[d, "\phi'"] & H_j \arrow[d, "\rho"] \\ & H_i \arrow[r, "\sigma"] & H \end{tikzcd}$$

Figure 2. Pullback diagram of nilpotent groups.

Theorem 1. *Let $H \in \mathcal{N}_1$ and let $H_t \in \mathcal{G}(H)$. Then $\beta : \mathcal{G}(H) \to \mathcal{G}(H_t)$ is an isomorphism.*

Proof. We have that $t \equiv (i+j)$ *modulo* s. This implies that the order of $\mathcal{G}(H_t)$ is s. Recall that the genus group of H is a cyclic group of finite order s. Thus, $\beta : \mathcal{G}(H) \to \mathcal{G}(H_t)$ is a bijection since $|\mathcal{G}(H)| = |\mathcal{G}(H_t)|$. Now, we have that $\mathcal{G}(H)$ and $\mathcal{G}(H_t)$ are cyclic groups; hence, we may find generators H_1 and X, respectively. Define the bijection $\beta : \mathcal{G}(H) \to$

$\mathscr{G}(H_t)$ by $\beta(H_1^r) = X^r$, for some $r \in \mathbb{Z}^+$. Let $H_2, H_3 \in \mathscr{G}(H)$. Since H_1 generates $\mathscr{G}(H)$, there exist $k, l \in \mathbb{Z}^+$ such that $H_2 = H_1^k, H_3 = H_1^l$. Thus,

$$\beta(H_2H_3) = \beta(H_1^{k+l}) = X^{k+l} = X^kX^l = \beta(H_1^k)\beta(H_1^l) = \beta(H_2)\beta(H_3).$$

Thus, β is a bijective homomorphism and is therefore an isomorphism. □

3. Group Structure on the Genus of A $\mathscr{K}-$Group

Let $\mathbb{Z}$ act on $\mathbb{Z}_n \times \mathbb{Z}_n$, that is, we have a nontrivial group homomorphism $\omega : \mathbb{Z} \to Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$. Suppose that L is given by $\mathbb{Z}_n \times \mathbb{Z}_n \rtimes_\omega \mathbb{Z}$, where the action ω is given by $\omega(1) : (x, y) \to (ux, uy)$. Then the group L is expressible as a short exact sequence $TL \rightarrowtail L \twoheadrightarrow FL$, where TL is the torsion subgroup of L and FL is the torsion-free quotient. Since $TL = \mathbb{Z}_n \times \mathbb{Z}_n$ is finite and $FL = \mathbb{Z}$ is free abelian of finite rank, we have that $L \in \mathscr{N}_0$. We want to interpret the structure of the group $\mathscr{G}(L)$. We will begin by proving some results on the group L.

Proposition 3. *Let $\mu : \mathbb{Z} \to Aut(\mathbb{Z}_n)$ be a nontrivial group homomorphism given by $\mu(1) : \mathbb{Z}_n \to \mathbb{Z}_n$, $t \mapsto ut$. Let K, G, and M be groups such that $G = K = \mathbb{Z}_n \rtimes_\mu \mathbb{Z}$ and $M = \mathbb{Z}$. Let Figure 3 be a commutative diagram of group homomorphisms.*

$$\begin{tikzcd} L \arrow[r, "\psi"] \arrow[d, "\phi"] & G \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"] & M \end{tikzcd}$$

Figure 3. A commutative diagram of group homomorphisms.

Define a subset L of $K \times G$ by

$$L = \{((x, y), z) \in K \times G \mid \beta(x, z) = \alpha(y, z)\}.$$

Then L is a subgroup of $K \times G$.

Proof. Let $\omega : \mathbb{Z} \to Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$ be a nontrivial group homomorphism given by $\omega(1) : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n \times \mathbb{Z}_n$, $(x, y) \mapsto (ux, uy)$. Let $((x, y), z), ((k, g), r) \in L$. Then, by definition, $\beta(x, z) = \alpha(y, z)$ and $\beta(k, r) = \alpha(g, r)$.
The multiplication on the set L is given by $((x, y), z)((k, g), r) = ((x, y)\omega_z(k, g), zr)$. Then we have

$$\begin{aligned} \beta((x, z)(k, r)) =& \beta(x, z)\beta(k, r) \\ =& \alpha(y, z)\alpha(g, r) \\ =& \alpha((k, z)(g, r)). \end{aligned}$$

Thus, $\beta((x, z)(k, r)) = \beta(x\mu_z(k), zr) = \alpha(y\mu_z(g), zr) = \alpha((y, z)(g, r))$. Therefore, $((x, y), z)$
$((k, g), r) = ((x, y)\omega_z(k, g), zr) \in L$.

Now consider

$$\begin{aligned}\beta((x,z)^{-1}) =&(\beta(x,z))^{-1}\\ =&(\alpha(y,z))^{-1}\\ =&\alpha((y,z)^{-1}).\end{aligned}$$

Therefore, $((x,y),z)^{-1} \in L$. Note that L is closed under the operation of $K \times G$. Thus, the result follows. $\square$

Proposition 4. *Let (n,u) be a relatively prime pair of integers. Let $\mu : \mathbb{Z} \to Aut(\mathbb{Z}_n)$ be a group action given by $\mu(1) : \mathbb{Z}_n \to \mathbb{Z}_n$, $t \mapsto ut$. Let $G = K = \mathbb{Z}_n \rtimes_\mu \mathbb{Z}$, and $M = \mathbb{Z}$. Suppose that α is injective. Then, the commutative diagram Figure 4 is a pullback if and only if the following conditions hold:*

$$\begin{tikzcd} L = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes_\omega \mathbb{Z} \arrow[r, "\psi"] \arrow[d, "\phi"'] & \mathbb{Z}_n \rtimes_\mu \mathbb{Z} = G \arrow[d, "\alpha"] \\ K = \mathbb{Z}_n \rtimes_\mu \mathbb{Z} \arrow[r, "\beta"'] & \mathbb{Z} = M \end{tikzcd}$$

Figure 4. A pullback diagram of nilpotent groups.

(1) $Ker\,\psi \cap Ker\,\phi = \{e\}$;
(2) $\alpha^{-1}(\beta(K)) = \psi(L)$;
(3) $\phi(Ker\,\psi) = Ker\,\beta$.

Proof. The two morphisms ϕ and ψ define a unique homomorphism $\gamma : L \to K \times G$. If the diagram is a pullback, then γ is an isomorphism and condition (1) and (2) are satisfied. Now we prove condition (3). Let $a \in Ker\,\beta$. By assumption, $\phi(x) = a$ for some $x \in L$. Then $\beta(a) = \beta(\phi(x)) = \alpha(\psi(x)) = 0$. Since α is injective, $\psi(x) = 0$, and so $x \in Ker\,\psi$. Thus $a = \phi(x) \in \phi(Ker\,\psi)$.

Conversely, let $a \in \phi(Ker\,\psi)$. Then there is $x \in Ker\,\psi$ such that $a = \phi(x)$. Since α is injective, we have that $\beta(a) = \beta(\phi(x)) = \alpha(\psi(x)) = 0$. Thus, $a \in Ker\,\beta$.

If the three conditions hold, then γ is injective by condition (1). Now let us take $(a,b) \in K \times G$. Then, since $\beta(a) = \alpha(b)$, we have that $b \in \alpha^{-1}(\beta(K))$, so $b = \psi(x)$ for some $x \in L$ by condition (2). Now, suppose that $a - \phi(x) = \phi(y) \in Ker\,\beta$, for some $y \in L$. Then $\gamma(x+y) = (\phi(x+y), \psi(x)) = (a,b)$. Thus γ is surjective. Therefore, the diagram is a pullback. $\square$

We fix some notation.

Let $\mathbb{Z}$ be generated by $\langle z\rangle$ and suppose that $\mathbb{Z}_n \times \mathbb{Z}_n = \langle x,y\rangle$. Then L is given by $\langle x,y,z\rangle$. Note that the subgroup $\langle x,y\rangle$ is normal in L, and so $zxz^{-1} \in \langle x\rangle$, and similarly $zyz^{-1} \in \langle y\rangle$. It is given that the group L is non-abelian, hence $zxz^{-1} = x^u$ and $zyz^{-1} = y^u$. We define two groups by

$$L = \langle x,y,z \mid x^n = 1, y^n = 1, xy = yx, zxz^{-1} = x^u, zyz^{-1} = y^u\rangle$$

and

$$L_m = \langle x,y,z \mid (x,y)^n = 1, xy = yx, zxz^{-1} = x^{u^m}, zyz^{-1} = y^{u^m}\rangle, \tag{3}$$

where $u \equiv 1\, modulo\, n^2$. We also have that $L_1 = L$. The actions of these groups are given by $z \cdot (x,y) = u(x,y)$ and $z \cdot (x,y) = u^m(x,y)$, where $(x,y) \in \mathbb{Z}_n \times \mathbb{Z}_n$, $u \in \mathbb{Z}$ and u is relatively prime to n of order d, respectively. We note that $m \not\equiv \pm 1\, modulo\, d$.
The properties of the group L_m were discussed in the papers [5,14–16]. Here, our focus is to study the relationship between the groups L_m and L. The author in [14] proved the following theorems.

Theorem 2. *Let $m \not\equiv \pm 1$ modulo d. Then $L \ncong L_m$.*

Theorem 3. *Let $L, L_m \in \mathcal{N}_0$ and let $\mathbb{Z}$ be a cyclic group. Then $L \times \mathbb{Z} \cong L_m \times \mathbb{Z}$.*

Let $|(x,y)|$ denote the order of $\mathbb{Z}_n \times \mathbb{Z}_n$. Set $u = 1 + cp_i^{\lambda_i}$, $p_i \nmid c$, $\lambda_i \geq 2$ and let $d = p_i^{\nu_i}$ be the order of u *modulo* $|(x,y)|$, where $n = p_i^{\kappa_i}$ for all i. We prove the following theorem.

Theorem 4. *Let $\mathbb{Z}$ and $\mathbb{Z}_n \times \mathbb{Z}_n$ be abelian groups. The group $L = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes_\omega \mathbb{Z}$ is nilpotent .*

Proof. Let $u = 1 + cp_i^{\lambda_i}$, with $p_i \nmid c$ and n^2 be the order of $\mathbb{Z}_n \times \mathbb{Z}_n$ where $n = p_i^{\kappa_i}$ for all i. Let us write $\mathbb{Z} = \langle \eta \rangle$ and $\mathbb{Z}_n \times \mathbb{Z}_n = \langle a, b \rangle$. Let $\Gamma^i_{\mathbb{Z}} \mathbb{Z}_n \times \mathbb{Z}_n$, $i = 1, 2, \cdots, r$ be the terms of the lower central series of the group L, where $\Gamma^0_{\mathbb{Z}} \mathbb{Z}_n \times \mathbb{Z}_n = L$. Then $\Gamma^1_{\mathbb{Z}} \mathbb{Z}_n \times \mathbb{Z}_n = \langle (x,y)^{(u-1)} \mid (x,y) \in \mathbb{Z}_n \times \mathbb{Z}_n \rangle, \Gamma^2_{\mathbb{Z}} \mathbb{Z}_n \times \mathbb{Z}_n = \langle (x,y)^{(u-1)(u-1)} \mid (x,y) \in \mathbb{Z}_n \times \mathbb{Z}_n \rangle, \cdots, \Gamma^i_{\mathbb{Z}} \mathbb{Z}_n \times \mathbb{Z}_n = \langle (x,y)^{(u-1)^i} \mid (x,y) \in \mathbb{Z}_n \times \mathbb{Z}_n \rangle$.

Therefore, for sufficiently large r, we have that $\Gamma^r_{\mathbb{Z}} \mathbb{Z}_n \times \mathbb{Z}_n = \{e\}$ if and only if $n^2 \mid (u-1)^i$ for some i. Therefore, the group L is nilpotent. □

Remark 1. Since $\Gamma^1_{\mathbb{Z}} \mathbb{Z}_n \times \mathbb{Z}_n = \langle (x,y)^{(cp_i^{\lambda_i})} \mid (x,y) \in \mathbb{Z}_n \times \mathbb{Z}_n \rangle$, it is clear to see that the commutators are powers of $(x,y)^{p_i^{\lambda_i}}$. Thus, $[L, L] = \langle x^{p_i^{\lambda_i}}, y^{p_i^{\lambda_i}} \rangle$, and is obviously finite.

We note that the torsion subgroup of L is noncyclic. Therefore, condition (3) of the definition of $\mathcal{N}_1$ is not automatically satisfied. Hence, we prove the following proposition.

Proposition 5. *Let $L \in \mathcal{N}_0$ with TL and FL commutative. Then L is contained in $\mathcal{N}_1$.*

Proof. We only need to prove condition (3) of the definition of $\mathcal{N}_1$. Let $\omega : \mathbb{Z} \to Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$ be a group homomorphism given by $\omega(z) = \phi_z$ with $\phi_z((a,b)) = z((a,b))z^{-1}$, for all $z \in \mathbb{Z}$ and $(a,b) \in \mathbb{Z}_n \times \mathbb{Z}_n$. The image $\omega(\mathbb{Z})$ is the inner automorphism group since it is a subgroup of $Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$. Since $\mathbb{Z}_n \times \mathbb{Z}_n$ is abelian, then $Z(\mathbb{Z}_n \times \mathbb{Z}_n) = \mathbb{Z}_n \times \mathbb{Z}_n$. Therefore, $Inn(\mathbb{Z}_n \times \mathbb{Z}_n) \cong \mathbb{Z}_n \times \mathbb{Z}_n / Z(\mathbb{Z}_n \times \mathbb{Z}_n) = \{e\}$. This implies that $\omega(\mathbb{Z}) = \{e\}$. The identity element commutes with every element of a group. Hence, $\omega(\mathbb{Z})$ must commute with every element of $Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$. Therefore, $\omega(\mathbb{Z})$ is contained in the center of $Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$. Henceforth, $L \in \mathcal{N}_1$. □

By Proposition 1, we note that the exponent of the group QL_{ab} is given by $p^{\nu_i + \lambda_i}$. We now proceed with the analysis of the structure of the genus $\mathcal{G}(L)$. Recall the exact sequence (2)

$$P - Aut(L) \xrightarrow{\theta} (\mathbb{Z}/p_i^{\nu_i + \lambda_i})^* / \{\pm 1\} \overset{\delta}{\twoheadrightarrow} \mathcal{G}(L). \tag{4}$$

Theorem 5. *Let $L \in \mathcal{N}_1$ with a cyclic torsion-free quotient, $FL = \mathbb{Z}$. Then $\mathcal{G}(L)$ is a finitely generated cyclic group.*

Proof. Suppose that the sequence (4) is exact, which implies that $\delta : (\mathbb{Z}/p_i^{\nu_i + \lambda_i})^* / \{\pm 1\} \to \mathcal{G}(L)$ is a surjective group homomorphism. In addition, we have that $Ker\, \delta \cong Im\, \theta$. Indeed, since δ is a surjective group homomorphism, we have that $Ker\, \delta$ is a subgroup of $(\mathbb{Z}/p_i^{\nu_i + \lambda_i})^* / \{\pm 1\}$. Further, we have that θ is multiplicative, so that $Im\, \theta$ is a subgroup of $(\mathbb{Z}/p_i^{\nu_i + \lambda_i})^* / \{\pm 1\}$, consisting of integers m prime to $p_i^{\nu_i + \lambda_i}$ such that $m \equiv 1$ *modulo* $p_i^{\nu_i}$ for all i, where $p_i^{\nu_i}$ is the order of u *modulo* $|(x,y)|$. Therefore, $|Im\, \theta| = p_i^{\lambda_i}$, and since we can conclude that $Ker\, \delta \cong Im\, \theta$, then by the first isomorphism theorem,

$$\mathcal{G}(L) \cong (\mathbb{Z}/p_i^{\nu_i})^* / \{\pm 1\}. \tag{5}$$

□

Since $\mathscr{G}(L)$ is a cyclic group, we can find a generator $h \in \mathbb{Z}^+$. Let c be the least exponent of h such that $h^c \equiv \pm 1 \, mod \, p_i^{v_i}$, where $c = p^{v_i - 1}(p-1)\,/\,2$.
This gives us the description of the genus group of L, $\mathscr{G}(L)$. However, we still do not know the exact order of this genus group. In the next section, we compute this order.

In the following theorem we want to prove that for each prime p, the two groups L and L_m such that $L \ncong L_m$ are $p-$equivalent. We will denote by P' the set of primes not in P.

Definition 1. Let L be a nilpotent group. We call the group L *P-local* if and only if for any $x \in L$, $x \mapsto x^n$ is a bijection for all $n \in P'$.

Theorem 6. *Let $L \in \mathscr{N}_1$. Then $L_m \in \mathscr{G}(L)$.*

Proof. Let $p \in P$. Suppose that there exists a positive integer m such that $p \nmid m$ and $hm \equiv 1 \, modulo \, d$. Let $\tau : TL \to TL$ and let $\gamma : \mathbb{Z} \to \mathbb{Z}$ be defined by $\gamma(\zeta_1) = \zeta_1^m$, $\gamma(\zeta_i) = \zeta_i$ for all $i \geq 2$. We claim that $\phi : L \to L_m$ is a group homomorphism. Indeed, in L_m we have that $yxy^{-1} = x^{u^m}$, so that $y^h x y^{-h} = x^{u^{mh}}$, but since $hm \equiv 1 \, modulo \, d$, then $y^h x y^{-h} = x^u$. Thus, ϕ is a group homomorphism and yields a commutative diagram [5] Figure 5.

$$\begin{tikzcd} TL \arrow[r, tail] \arrow[d, "="] & L \arrow[r, two heads] \arrow[d, "\phi"] & \mathbb{Z} \arrow[d, "\gamma"] \\ TL \arrow[r, tail] & L_m \arrow[r, two heads] & \mathbb{Z} \end{tikzcd}$$

Figure 5. A commutative diagram of short exact sequences of groups.

Since $m \in P'$, then $\mathbb{Z}$ is P-local, and hence $\gamma_p : \mathbb{Z}_p \cong \mathbb{Z}_p$. If we localize at p, then ϕ is a $p-$isomorphism. Thus, ϕ is a $p-$equivalence, and therefore L is P-equivalent to L_m. $\square$

The authors in [5] used Theorem 6 and Theorem 5 to prove the following corollary.

Corollary 1 (Corollary 2.3 [5])**.** *The group $\mathscr{G}(L)$ consists of isomorphism classes of groups L_m.*

Remark 2. *If L and L_m are nilpotent groups, then they are in the same Mislin genus.*

4. A Cyclic Genus of Finite Order

Consider the groups $G = \mathbb{Z}_{11} \rtimes_\rho \mathbb{Z}$, $K = (\mathbb{Z}_{11^2} \rtimes_\mu \mathbb{Z})^{20}$, and $M = \mathbb{Z}$, where $\rho : \mathbb{Z} \to Aut(\mathbb{Z}_{11})$ is a nontrivial homomorphism given by $\rho(1) : x \to ux$, and $\mu : \mathbb{Z}^{20} \to Aut((\mathbb{Z}_{11^2})^{20})$ is a nontrivial homomorphism given by $\mu(1) : x \to ux$, respectively. Consider the pullback diagram of group homomorphisms Figure 6.

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[d, "\phi"'] & G \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"'] & M \end{tikzcd}$$

Figure 6. Pullback diagram of group homomorphisms.

Proposition 6. *Let $G \in \mathscr{N}_1$. Let $\mathbb{Z}_{11}$ and $\mathbb{Z}$ be generated by $\langle x \rangle$ and $\langle y \rangle$, respectively. Suppose that $u \equiv 1$ modulo 11. Then the generators of G are subject to the relations $x^{11} = 1$ and $yxy^{-1} = x^u$.*

Proof. Let $\mathbb{Z}_{11} = \langle x \rangle$ and $\mathbb{Z} = \langle y \rangle$. Let $\rho : \mathbb{Z} \to Aut(\mathbb{Z}_{11})$ be a nontrivial group homomorphism. Then G is generated by $\langle x, y \rangle$. We have that the subgroup $\langle x \rangle$ is normal in G; therefore, $yxy^{-1} \in \langle x \rangle$. Note that since G is a non-abelian group, we have that $yxy^{-1} = x^u$. Thus, we have that

$$G = \langle x, y \mid x^{11} = 1, yxy^{-1} = x^u \rangle,$$

where $u \equiv 1$ *modulo* 11. □

Proposition 7. *Let $\bar{K}_i = \mathbb{Z}_{11^2} \rtimes_\mu \mathbb{Z}$, for all $i \in \{1, \cdots, 20\}$. Let $\bar{K_1}, \cdots, \bar{K_{20}} \in \mathscr{N}_1$ and suppose that $K = \bar{K_1} \times \cdots \times \bar{K_{20}}$. Then K is generated by the same generators as that of the groups $\bar{K}_i$.*

Proof. Let $\mathbb{Z}_{11^2} = \langle x \rangle$ and $\mathbb{Z} = \langle y \rangle$. Suppose that $\mu : \mathbb{Z} \to Aut(\mathbb{Z}_{11^2})$ is a nontrivial group homomorphism. Similarly to proposition 6, the group $\bar{K}_i$ has generators and relations x and y satisfying $x^{11^2} = 1$ and $yxy^{-1} = x^u$. Furthermore, since K is the direct product of each $\bar{K}_i$, for all $i \in \{1, \cdots, 20\}$, we have that K is generated by the same generators. □

The torsion subgroup of K is finite and the torsion-free quotient is free abelian of finite rank, therefore we have that $K \in \mathscr{N}_0$. Let $FK = \langle \zeta_1, \zeta_2, \cdots, \zeta_{20} \rangle$, with FK acting on TK by

$$\zeta_i a_i = u a_i, \quad a_i \in \mathbb{Z}_{11^2}^{20},$$

where u is relatively prime to 11^2.

Proposition 8. *Let $K \in \mathscr{N}_0$. Then the genus, $\mathscr{G}(K)$, is trivial.*

Proof. Let $K = (\mathbb{Z}_{11^2} \rtimes_\mu \mathbb{Z})_1 \times (\mathbb{Z}_{11^2} \rtimes_\mu \mathbb{Z})_2 \times \cdots \times (\mathbb{Z}_{11^2} \rtimes_\mu \mathbb{Z})_{20}$, where each $\mathbb{Z}_{11^2} \rtimes_\mu \mathbb{Z} \in \mathscr{N}_1$. Since the direct product of two or more infinite cyclic groups is not cyclic, then the torsion-free quotient FK is not cyclic. This implies that the genus of K is trivial. Furthermore, the torsion-free quotient of each $\mathbb{Z}_{11^2} \rtimes_\mu \mathbb{Z}$ is cyclic. Therefore, $\mathscr{G}(\mathbb{Z}_{11^2} \rtimes_\mu \mathbb{Z})$ need not be trivial. However, since the torsion subgroup $\mathbb{Z}_{11^2}$ is a cyclic p-group, where p is a prime, and the torsion-free quotient $\mathbb{Z}$ is cyclic, then $\mathscr{G}(\mathbb{Z}_{11^2} \rtimes_\mu \mathbb{Z})$ must be isomorphic to $\mathscr{G}(K)$. Thus, $\mathscr{G}(\mathbb{Z}_{11^2} \rtimes_\mu \mathbb{Z})$ is also trivial. □

We observe that if we take the direct product $K \times G$, then $K \times G \in \mathscr{N}_0$. From here, we can discuss the genus of this direct product. In the direct product of K and G, the torsion-free quotient is noncyclic; thus, the genus of this direct product, $\mathscr{G}(K \times G)$, is trivial. From this, we can deduce that $\varphi : \mathscr{G}(K) \to \mathscr{G}(K \times G)$ is an isomorphism. In addition, we have that $\mathscr{G}(K) \times \mathscr{G}(G) \cong \mathscr{G}(K \times G)$.

Proposition 9. *Let $G \in \mathscr{N}_1$ with a cyclic torsion-free quotient FG. Then the genus of G is trivial.*

Proof. It is sufficient for us to show that $d = 1$ or 2. Let $n = 11$ and let $u = 1 + 11^\lambda c$, $11 \nmid c, \lambda \geq 1$. We have that $\nu = 1 - \lambda$ if $1 > \lambda$ or $\nu = 0$ if $\lambda \geq 1$. In the second case, since $1 \leq \lambda < 1$, we have that $\lambda = 1$, and so $\nu = 0$. Therefore, $d = 11^0 = 1$; thus, the genus $\mathscr{G}(G)$ is trivial. □

In the next theorem, we determine the order of the genus of the group L. In particular, we prove that $\mathscr{G}(L)$ is nontrivial. For the proof of the theorem we need the following proposition, but let us first fix some notation.

Notation. Let a be an integer. We denote by $[a]_c$ the class which contains all elements a *modulo c*, where c is the order of the genus group, $\mathscr{G}(G)$.

Proposition 10. *Let $K \times G \in \mathcal{N}_0$. Suppose that L is a subgroup of $K \times G$. Then $\phi_* : \mathcal{G}(L) \to \mathcal{G}(G)$ is a surjective group homomorphism.*

Proof. Let $e_G \in \mathcal{G}(G)$ and let $\phi_* : \mathcal{G}(L) \to \mathcal{G}(G)$ be defined by $\phi_*([a]_c) = e_G$, for all $[a]_c \in \mathcal{G}(L)$, where c is the order of $\mathcal{G}(L)$. Let $[a]_c, [b]_c \in \mathcal{G}(L)$, then

$$\phi_*([a]_c + [b]_c) = e_G = e_G + e_G = \phi_*([a]_c) + \phi_*([b]_c).$$

Hence, ϕ_* is a homomorphism and is obviously surjective since every element in $\mathcal{G}(L)$ is mapped to the identity element. □

Theorem 7. *Let $L \in \mathcal{N}_1$ with a cyclic torsion-free quotient group. Then the group $\mathcal{G}(L)$ is a cyclic group of order* 2*.*

Proof. Since $\mathcal{G}(L)$ is mapped to the identity element of $\mathcal{G}(G)$, the kernel, $Ker\, \phi_*$ must be the group itself. This implies that $|Ker\, \phi_*| = |\mathcal{G}(L)|$. In Theorem 5, we showed that $\mathcal{G}(L)$ is a cyclic group. Therefore, the group $\mathcal{G}(L)$ can only have one subgroup of order $|\mathcal{G}(L)|$ in $\mathcal{G}(L)$, which must be the group itself. Then $\mathcal{G}(L)$ must have only two subgroups: the trivial subgroup and the group itself. Therefore, we can conclude that the genus group of L is cyclic of order 2. □

Example 1. *Let $Q \in \mathbb{N}_1$ with $TQ = \mathbb{Z}/27 \times \mathbb{Z}/27 = \langle x_1 \rangle \times \langle x_2 \rangle$. Let the action of $FQ = \langle \zeta \rangle$ on TQ be given by $\zeta \cdot x_1 = 4x_1$ and $\zeta \cdot x_2 = 7x_2$, for all $(x_1, x_2) \in TQ_1 \times TQ_2$. Thus, $u_1 = 1 + 3$ and $u_2 = 1 + 2 \cdot 3$. It is clear that $d = 3^2 = 9$. We claim that $\mathcal{G}(Q)$ is nontrivial. Indeed, $|(\mathbb{Z}/9)^*/\{\pm 1\}| = 3^{2-1}(3-1)/2 = 3$. Thus, $\mathcal{G}(Q) \cong \mathbb{Z}/3$.*

In Proposition 4, we proved that the group $L = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes_\omega \mathbb{Z}$ is a pullback, where $\omega : \mathbb{Z} \to Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$ is a nontrivial homomorphism given by $\omega(1) : (x, y) \mapsto (ux, uy)$. We then proceeded to show that the pullback L is nilpotent with a finite commutator subgroup. In particular, we proved in Proposition 5 that L is contained in the class $\mathcal{N}_1$.

We then considered the groups K and G, where $K = \bar{K}^{20}$. We proved that $\bar{K}$ is contained in $\mathcal{N}_1$. Since $\bar{K}$ is not abelian, K is not contained in $\mathcal{N}_1$. In this case, the condition that $\mu(FK) \subseteq Z(Aut(TK))$, where $Z(Aut(TK))$ is the center of $Aut(TK)$ and μ is the action of FK on TK, fails to hold; in general, direct products do not inherit this condition. Therefore, we see the importance of this condition in obtaining the genus of a group in $\mathcal{N}_1$. Propositions 8 and 9 show that the groups $\mathcal{G}(K)$ and $\mathcal{G}(G)$ are both trivial, respectively, and therefore are isomorphic to each other.

Finally, in Proposition 10, we showed that there is a surjective homomorphism between the groups $\mathcal{G}(G)$ and $\mathcal{G}(L)$. This surjective homomorphism enabled us to prove that the genus of L, $\mathcal{G}(L)$ is nontrivial. In particular, we showed in Theorem 7 that $\mathcal{G}(L)$ is a cyclic group of order 2.

It is clear that the group $\mathcal{G}(L)$ is an abelian group of finite order. We, therefore, realize an abelian group as a Mislin genus. Now that we have proven that the genus of a pullback L is abelian, we want to further study its subgroups in more detail.

Author Contributions: Conceptualization, T.T., R.K. and J.C.M.; Formal analysis, T.T., R.K. and J.C.M.; Writing—original draft, T.T.; Supervision, R.K. and J.C.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: We thank the University of Johannesburg for supporting the APC for the publication of the article.

Conflicts of Interest: The authors declare no conflict of interest.

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