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On Kirchhoff-Type Equations with Hardy Potential and Berestycki–Lions Conditions

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Abstract: The purpose of this paper is to investigate the existence and asymptotic properties of solutions to a Kirchhoff-type equation with Hardy potential and Berestycki–Lions conditions. Firstly, we show that the equation has a positive radial ground-state solution u_λ by using the Pohozaev manifold. Secondly, we prove that the solution u_{λ_n}, up to a subsequence, converges to a radial ground-state solution of the corresponding limiting equations as $\lambda_n \to 0^-$. Finally, we provide a brief summary.

Keywords: Kirchhoff equation; Pohozaev manifold; radial ground-state solution

MSC: 35J20; 35B09; 35B40

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1. Introduction

In the paper, we investigate the following Kirchhoff-type equations:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u-\frac{\lambda}{|x|^2}u=f(u),\quad x\in\mathbb{R}^3, \tag{1}$$

where $a>0, b>0$, $\frac{-\lambda}{|x|^2}$ is called the Hardy potential and f satisfies the following Berestycki–Lions-type conditions:

(f_1) $f\in C(\mathbb{R},\mathbb{R})$ is odd;
(f_2) $-\infty<\liminf\limits_{s\to0^+}\frac{f(s)}{s}\le\limsup\limits_{s\to0^+}\frac{f(s)}{s}=-m<0$;
(f_3) $\lim\limits_{s\to+\infty}\frac{f(s)}{s^5}=0$;
(f_4) There exists $\zeta>0$ such that $F(\zeta):=\int_0^\zeta f(\tau)d\tau>0$.

Because the Kirchhoff-type equation has a wide range of applications in many fields, such as it models several physical and biological systems, it has been widely considered in the last two decades by using variational methods, see [1–11] and references therein. We just introduce several results closely related to Equation (1) here. Under (f_1)–(f_4), Azzollini [1,2] studied ground-state solutions for the following limiting equations of Equation (1):

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u=f(u),\ \ x\in\mathbb{R}^3. \tag{2}$$

Additionally, under (f_1)–(f_4), Liu et al. [7] considered the following Kirchhoff equations with abstract potential:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=f(u),\ \ x\in\mathbb{R}^3,$$

where V satisfies

(V_1) $V\in C(\mathbb{R}^3,(-\infty,0])$ and $\lim\limits_{|x|\to+\infty}V(x)=0$;

(V_2) $\inf\limits_{0\neq u\in H^1(\mathbb{R}^3)}\frac{\int_{\mathbb{R}^3}[a|\nabla u|^2+(V(x)+m)u^2]dx}{\int_{\mathbb{R}^3}u^2dx}>0$;

(V_3) V is weakly differentiable and

$$\int_{\mathbb{R}^3}(\nabla V,x)u^2dx\leq 2a\int_{\mathbb{R}^3}|\nabla u|^2dx \text{ for any } u\in H^1(\mathbb{R}^3).$$

By verifying, we know that for the Hardy potential $\frac{-\lambda}{|x|^2}$, except for the continuity at origin, it satisfies (V_1)–(V_3) if $0<\lambda<\frac{a}{4}$. However, the singularity does not affect the proof in [7]. In addition, if $\lambda<0$, then $\frac{-\lambda}{|x|^2}>0$ and it does not allow us to utilize the concentration-compactness lemma to overcome the difficulty of lacking compactness, as in [7]. Thus, a natural question is if $\lambda<0$, does the equation still have a nontrivial solution? On the other hand, Li et al. [12] recently researched Schrödinger equations with Hardy potential and Berestycki–Lions-type conditions. So our purpose is to generalize some of the results in [12] to the Kirchhoff equations.

The main result of the paper reads as follows:

Theorem 1. *Suppose that $a>0,b>0,\lambda<0$ and (f_1)–(f_4) hold. Then, Equation (1) has a positive solution u_λ.*

Remark 1. *Although we cannot use the concentration-compactness lemma to overcome the difficulty of lacking compactness, fortunately, due to the symmetry of $\frac{-\lambda}{|x|}$, the radial function space $H^1_r(\mathbb{R}^3)$ can restore the compactness of spatial embedding. In fact, the solution u_λ in Theorem 1 is a radial ground-state solution, namely, a solution minimizing the action among all the nontrivial radial solutions.*

Next, we consider the asymptotic behavior of u_λ as $\lambda\to0^-$. We set $(E,\|\cdot\|)$ as the usual Hilbert space, where $E=H^1_r(\mathbb{R}^3)$ and

$$\|\cdot\|=\left[\int_{\mathbb{R}^3}(|\nabla\cdot|^2+|\cdot|^2)dx\right]^{\frac{1}{2}}.$$

The relevant result is the following theorem.

Theorem 2. *Suppose that $a>0,b>0,\lambda<0$ and (f_1)–(f_4) hold. Assume that u_n is a positively radial ground-state solution of Equation (1) with $\lambda=\lambda_n$ and $\lambda_n\to0^-$. Then there exists a positive radial function $u\in E$ such that $u_n\to u$ in E and u satisfies Equation (2).*

The structure of this paper is as follows: In Section 2, we introduce some preliminary content. In Section 3, the proof of Theorem 1 is completed. Section 4 involves the proof of Theorem 2. Finally, a simple summary is provided in Section 5.

2. Preliminaries

First, we introduce some notations below:

- $(L^p(\mathbb{R}^3), |\cdot|_p)$ is the Lebesgue space, where $2 \le p < \infty$ and
$$|\cdot|_p = \left[\int_{\mathbb{R}^3} |\cdot|^p dx\right]^{\frac{1}{p}}.$$
- $S = \inf\limits_{0 \ne u \in D^{1,2}(\mathbb{R}^3)} \frac{|\nabla u|_2^2}{|u|_6^2}$.

Now, we set $I_\lambda : E \to \mathbb{R}$ as
$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 - \frac{\lambda u^2}{|x|^2}\right)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^3} F(u)dx.$$

According to (f_1)–(f_3) and the Hardy inequality,
$$\int_{\mathbb{R}^3}\frac{u^2}{|x|^2}dx \le 4\int_{\mathbb{R}^3}|\nabla u|^2 dx,$$
we know that I is of C^1 and
$$\langle I_\lambda'(u), v\rangle = \int_{\mathbb{R}^3}\left(a\nabla u\cdot\nabla v - \frac{\lambda uv}{|x|^2}\right)dx + b\int_{\mathbb{R}^3}|\nabla u|^2 dx\int_{\mathbb{R}^3}\nabla u\cdot\nabla v dx - \int_{\mathbb{R}^3} f(u)v dx$$
for any $u, v \in E$. If $u \in E$ is a solution of Equation (1), multiplying both sides of Equation (1) by $v \in C_0^2(\mathbb{R}^3)$, integrating over $\mathbb{R}^3$, and using Green's formula, it holds that
$$\int_{\mathbb{R}^3}\left(a\nabla u\cdot\nabla v - \frac{\lambda uv}{|x|^2}\right)dx + b\int_{\mathbb{R}^3}|\nabla u|^2 dx\int_{\mathbb{R}^3}\nabla u\cdot\nabla v dx = \int_{\mathbb{R}^3} f(u)v dx.$$
Therefore, the critical points of I_λ correspond to the weak solutions of Equation (1). It is easy to obtain that I_λ satifies the mountain pass geometry under our weak assumption of f, but it seems insufficient to indicate that the Palais–Smale sequence at the mountain pass level is bounded. To avoid this difficulty, we use the Pohozaev manifold. The method we adopt is that of C. Keller [13] and was used in [1]. Thus, we define the Pohozaev manifold
$$\mathcal{P}_\lambda = \{u \in E : P_\lambda(u) = 0 \text{ and } u \ne 0\},$$
where
$$P_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 - \frac{\lambda u^2}{|x|^2}\right)dx + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 - 3\int_{\mathbb{R}^3} F(u)dx$$
and $P_\lambda(u) = 0$ is called the Pohozaev identity. In fact, if $u \in E\backslash\{0\}$ is a solution of Equation (1), from Lemma 2.2 in [14] we see that $u \in \mathcal{P}_\lambda$. Considering constraint minimization,
$$c_\lambda = \inf\{I_\lambda(u) : u \in \mathcal{P}_\lambda\},$$
we will see that $\mathcal{P}_\lambda$ is a good constraint and c_λ is a critical level in the next section.

3. Proof of Theorem 1

In this section, we always assume that $a > 0, b > 0, \lambda \le 0$, and (f_1)–(f_4) hold and prove that Equation (1) has a positive radial solution. First, we prove some properties of $\mathcal{P}_\lambda$ and c_λ.

Lemma 1. *$\mathcal{P}_\lambda$ is a nonempty set.*

Proof. According to [15], there is a function $u \in E$ such that $\int_{\mathbb{R}^3} F(u)dx > 0$. For $t > 0$, we define $u_t = u(\cdot/t)$ and obtain

$$P_\lambda(u_t) = \frac{t}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 - \frac{\lambda u^2}{|x|^2}\right)dx + \frac{bt^2}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 - 3t^3\int_{\mathbb{R}^3}F(u)dx.$$

Thus, $P_\lambda(u_t) > 0$ for $0 < t << 1$ and $P_\lambda(u_t) < 0$ for $t >> 1$. So, there is a constant $t_0 > 0$ such that $P_\lambda(u_{t_0}) = 0$. That is, $u_{t_0} \in \mathcal{P}_\lambda$. □

Lemma 2. *c_λ has a positive lower bound.*

Proof. Because (f_1)–(f_3) hold, there is a constant $C > 0$ such that

$$F(s) \leq -\frac{ms^2}{4} + Cs^6,\ \forall s \in \mathbb{R}. \tag{3}$$

Note that $S = \inf\limits_{0\neq u\in D^{1,2}(\mathbb{R}^3)} \frac{|\nabla u|_2^2}{|u|_6^2} > 0$, see ([16], p. 26) Thus, $\forall u \in \mathcal{P}_\lambda$, we have

$$\begin{aligned}
\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2 dx &\leq \frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 - \frac{\lambda u^2}{|x|^2}\right)dx + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 \\
&= 3\int_{\mathbb{R}^3}F(u)dx \\
&\leq 3C\int_{\mathbb{R}^3}u^6 dx \\
&\leq \frac{3C}{S^3}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^3.
\end{aligned}$$

Then $\int_{\mathbb{R}^3}|\nabla u|^2 dx \geq \sqrt{\frac{aS^3}{6C}}$. So, for any $u \in \mathcal{P}_\lambda$,

$$I_\lambda(u) = I_\lambda(u) - \frac{1}{3}P_\lambda(u) = \frac{1}{3}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 - \frac{\lambda u^2}{|x|^2}\right)dx + \frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 \geq \frac{a}{3}\sqrt{\frac{aS^3}{6C}}.$$

Therefore, $c_\lambda \geq \frac{a}{3}\sqrt{\frac{aS^3}{6C}}$. □

Lemma 3. *$\mathcal{P}_\lambda$ is a C^1 manifold.*

Proof. Suppose that there is a function $u \in \mathcal{P}_\lambda$ such that $P'_\lambda(u) = 0$, then similarly to Lemma 2.2 in [14], u satisfies

$$\frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 - \frac{\lambda u^2}{|x|^2}\right)dx + b\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 - 9\int_{\mathbb{R}^3}F(u)dx = 0.$$

Note that

$$\frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 - \frac{\lambda u^2}{|x|^2}\right)dx + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 - 3\int_{\mathbb{R}^3}F(u)dx = 0.$$

Thus

$$\int_{\mathbb{R}^3}\left(a|\nabla u|^2 - \frac{\lambda u^2}{|x|^2}\right)dx + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 = 0$$

which implies $u = 0$. It is a contradiction. □

Lemma 4. *c_λ is achieved by $u \in E$, where $u \geq 0$ in $\mathbb{R}^3$.*

Proof. Note that I_λ and P_λ are even functionals. There is a non-negative sequence $\{u_n\}$ in E such that $I_\lambda(u_n) \to c_\lambda, P_\lambda(u_n) = 0$. Recall that

$$I_\lambda(u_n) = I_\lambda(u_n) - \frac{1}{3}P_\lambda(u_n) = \frac{1}{3}\int_{\mathbb{R}^3}\left(a|\nabla u_n|^2 - \frac{\lambda u_n^2}{|x|^2}\right)dx + \frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\right)^2. \quad (4)$$

Thus, $\{|\nabla u_n|\}$ is bounded in $L^2(\mathbb{R}^3)$. From (3), we have

$$\begin{aligned} \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2 dx &\leq \frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u_n|^2 - \frac{\lambda u^2}{|x|^2}\right)dx + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\right)^2 \\ &= 3\int_{\mathbb{R}^3}F(u_n)dx \\ &\leq -\frac{3m}{4}\int_{\mathbb{R}^3}u_n^2 dx + 3C\int_{\mathbb{R}^3}u_n^6 dx \\ &\leq -\frac{3m}{4}\int_{\mathbb{R}^3}u_n^2 dx + \frac{3C}{S^3}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\right)^3. \end{aligned} \quad (5)$$

Thus, $\{|u_n|_2\}$ is bounded, so $\{\|u_n\|\}$ is bounded. There is a function $u \in E$, $u \geq 0$, such that up to a subsequence, $u_n \to u$ weakly in E, $u_n \to u$ in $L^p(\mathbb{R}^3)$ with $2 < p < 6$, and $u_n \to u$ a.e. in $\mathbb{R}^3$. Borrowing the method in [15], we set $f_1(s) = (f(s) + ms)^+$ and $f_2(s) = f_1(s) - f(s)$ for $s \geq 0$, where $(f(s) + ms)^+ = \max\{f(s) + ms, 0\}$. Extend f_1 and f_2 as odd functions for $s \leq 0$. Then $f(s) = f_1(s) - f_2(s)$, $f_2(s) \geq ms$ for all $s \geq 0$ and

$$\lim_{s\to 0}\frac{f_1(s)}{s} = 0, \ \lim_{s\to\infty}\frac{f_1(s)}{s^5} = 0.$$

Let $F_i(s) = \int_0^s f_i(t)dt$, $i = 1, 2$. Then, by using Strauss's lemma (see Theorem A.I in [15]), we have

$$\int_{\mathbb{R}^3}F_1(u_n)dx \to \int_{\mathbb{R}^3}F_1(u)dx. \quad (6)$$

Combining with Fatou's lemma implies that

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 - \frac{\lambda u^2}{|x|^2}\right)dx + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 + 3\int_{\mathbb{R}^3}F_2(u)dx + o(1) \\ \leq\ &\frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u_n|^2 - \frac{\lambda u_n^2}{|x|^2}\right)dx + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\right)^2 + 3\int_{\mathbb{R}^3}F_2(u_n)dx \\ =\ &3\int_{\mathbb{R}^3}F_1(u_n)dx \\ =\ &3\int_{\mathbb{R}^3}F_1(u)dx + o(1). \end{aligned}$$

That is

$$\frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 - \frac{\lambda u^2}{|x|^2}\right)dx + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 - 3\int_{\mathbb{R}^3}F(u)dx \leq 0.$$

Note that

$$P_\lambda(u_t) = \frac{t}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 - \frac{\lambda u^2}{|x|^2}\right)dx + \frac{bt^2}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 - 3t^3\int_{\mathbb{R}^3}F(u)dx.$$

Thus, $P_\lambda(u_1) \leq 0$ and $P_\lambda(u_t) > 0$ for $0 < t << 1$. So, there exists $t_0 \in (0, 1]$ such that $P_\lambda(u_{t_0}) = 0$. Suppose that $t_0 < 1$, then

$$\begin{aligned} c_\lambda &\leq I_\lambda(u_{t_0}) \\ &= I_\lambda(u_{t_0}) - \frac{1}{3}P_\lambda(u_{t_0}) \end{aligned}$$

$$
\begin{aligned}
&= \frac{t_0}{3}\int_{\mathbb{R}^3}\left(a|\nabla u|^2-\frac{\lambda u^2}{|x|^2}\right)dx+\frac{bt_0^2}{12}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2\\
&< \frac{1}{3}\int_{\mathbb{R}^3}\left(a|\nabla u|^2-\frac{\lambda u^2}{|x|^2}\right)dx+\frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2\\
&\leq \frac{1}{3}\int_{\mathbb{R}^3}\left(a|\nabla u_n|^2-\frac{\lambda u_n^2}{|x|^2}\right)dx+\frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)^2+o(1)\\
&= I_\lambda(u_n)-\frac{1}{3}P_\lambda(u_n)+o(1)\\
&= c_\lambda
\end{aligned}
$$

which is in contradiction. Thus, $P_\lambda(u)=0$ and

$$
\begin{aligned}
c_\lambda &\leq I_\lambda(u)\\
&= I_\lambda(u)-\frac{1}{3}P_\lambda(u)\\
&= \frac{1}{3}\int_{\mathbb{R}^3}\left(a|\nabla u|^2-\frac{\lambda u^2}{|x|^2}\right)dx+\frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2\\
&\leq \frac{1}{3}\int_{\mathbb{R}^3}\left(a|\nabla u_n|^2-\frac{\lambda u_n^2}{|x|^2}\right)dx+\frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)^2+o(1)\\
&= I_\lambda(u_n)-\frac{1}{3}P_\lambda(u_n)+o(1)\\
&= c_\lambda.
\end{aligned}
$$

We complete the proof. □

Now we begin to prove Theorem 1.

Proof of Theorem 1. According to Lemma 4, there is a function $u\in E$, $u\geq 0$ such that $I_\lambda(u)=c_\lambda$ and $u\in\mathcal{P}_\lambda$. By using the Lagrange multiplier theorem, we find that there is a constant $\mu\in\mathbb{R}$ such that $I'_\lambda(u)=\mu P'_\lambda(u)$, where

$$
P'_\lambda(u)=-\left(a+2b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u-\frac{\lambda}{|x|^2}u-3f(u).
$$

Similarly to Lemma 2.2 in [14], one has

$$
P_\lambda(u)=\mu\left\{\frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2-\frac{\lambda u^2}{|x|^2}\right)dx+b\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2-9\int_{\mathbb{R}^3}F(u)dx\right\}.
$$

From $P_\lambda(u)=0$, we have

$$
\frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2-\frac{\lambda u^2}{|x|^2}\right)dx+\frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2=3\int_{\mathbb{R}^3}F(u)dx.
$$

Thus,

$$
0=P_\lambda(u)=\mu\left\{-\int_{\mathbb{R}^3}\left(a|\nabla u|^2-\frac{\lambda u^2}{|x|^2}\right)dx-\frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2\right\}.
$$

So $\mu=0$. Therefore, $I'_\lambda(u)=0$. The positivity is from the strong maximum principle. □

4. Proof of Theorem 2

In this section, we consider the asymptotic behavior of a positive radial solution u_λ as $\lambda\to 0^-$. The following lemma indicates that c_λ is monotonic in $(-\infty,0]$.

Lemma 5. *c_λ is a strictly monotonically decreasing in $(-\infty,0]$.*

Proof. Suppose that $-\infty < \lambda_1 < \lambda_2 \le 0$, $I_{\lambda_1}(u) = c_{\lambda_1}$ and $P_{\lambda_1}(u) = 0$. Then

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 - \frac{\lambda_2 u^2}{|x|^2}\right)dx + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 &< \frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 - \frac{\lambda_1 u^2}{|x|^2}\right)dx \\ &\quad + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 \\ &= 3\int_{\mathbb{R}^3} F(u)dx. \end{aligned}$$

That is, $P_{\lambda_2}(u) < 0$. Thus, there is a constant $t_u \in (0,1)$ such that $P_{\lambda_2}(u_{t_u}) = 0$ and then

$$\begin{aligned} c_{\lambda_2} &\le I_{\lambda_2}(u_{t_u}) \\ &= I_{\lambda_2}(u_{t_u}) - \frac{1}{3}P_{\lambda_2}(u_{t_u}) \\ &= \frac{t_u}{3}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 - \frac{\lambda_2 u^2}{|x|^2}\right)dx + \frac{bt_u^2}{12}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 \\ &< \frac{1}{3}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 - \frac{\lambda_1 u^2}{|x|^2}\right)dx + \frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 \\ &= I_{\lambda_1}(u) - \frac{1}{3}P_{\lambda_1}(u), \\ &= c_{\lambda_1} \end{aligned}$$

i.e., c_λ is a strictly monotonically decreasing in $(-\infty, 0]$. □

Now we begin to prove Theorem 2.

Proof of Theorem 2. Because u_n is a positive radial solution of Equation (1) with $\lambda = \lambda_n$ and $\lambda_n \to 0^-$, we have $I_{\lambda_n}(u_n) = c_{\lambda_n}$, $I'_{\lambda_n}(u_n) = 0$ and $P_{\lambda_n}(u_n) = 0$. We may assume $\lambda_n \in [-1, 0)$. Then, from Lemma 1, we have $c_{\lambda_n} \le c_{-1}$ and $c_{\lambda_n} \to c_0$. Replacing λ with λ_n in (4) and (5), we obtain that $\{\|u_n\|\}$ is bounded. There is a function $u \in E$, $u \ge 0$, such that up to a subsequence, $u_n \to u$ weakly in E, $u_n \to u$ in $L^p(\mathbb{R}^3)$, $2 < p < 6$, and $u_n \to u$ a.e. in $\mathbb{R}^3$. Using (6) and the Fatou lemma, one has

$$\begin{aligned} &\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 + 3\int_{\mathbb{R}^3} F_2(u)dx + o(1) \\ \le\ &\frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u_n|^2 - \frac{\lambda_n u_n^2}{|x|^2}\right)dx + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\right)^2 + 3\int_{\mathbb{R}^3} F_2(u_n)dx \\ =\ &3\int_{\mathbb{R}^3} F_1(u_n)dx \\ =\ &3\int_{\mathbb{R}^3} F_1(u)dx + o(1). \end{aligned}$$

That is

$$\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 - 3\int_{\mathbb{R}^3} F(u)dx \le 0.$$

Note that

$$P_0(u_t) = \frac{at}{2}\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{bt^2}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 - 3t^3\int_{\mathbb{R}^3} F(u)dx.$$

Thus, $P_0(u_1) \le 0$ and $P_0(u_t) > 0$ for $0 < t << 1$. So there exists $t_0 \in (0,1]$ such that $P_0(u_{t_0}) = 0$. Suppose that $t_0 < 1$, then

$$c_0 \le I_0(u_{t_0})$$

$$
\begin{aligned}
&= I_0(u_{t_0}) - \frac{1}{3}P_0(u_{t_0}) \\
&= \frac{at_0}{3}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{bt_0^2}{12}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 \\
&< \frac{a}{3}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 \\
&\leq \frac{a}{3}\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)^2 + o(1) \\
&= I_{\lambda_n}(u_n) - \frac{1}{3}P_{\lambda_n}(u_n) + o(1) \\
&= c_{\lambda_n} + o(1) \\
&= c_0,
\end{aligned}
$$

which is in contradiction. Thus, $P_0(u) = 0$ and

$$
\begin{aligned}
c_0 &\leq I_0(u) \\
&= I_0(u) - \frac{1}{3}P_0(u) \\
&= \frac{a}{3}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 \\
&\leq \frac{1}{3}\int_{\mathbb{R}^3}\left(a|\nabla u_n|^2 - \frac{\lambda_n u_n^2}{|x|^2}\right)dx + \frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)^2 + o(1) \\
&= I_\lambda(u_n) - \frac{1}{3}P_\lambda(u_n) + o(1) \\
&= I_{\lambda_n}(u_n) - \frac{1}{3}P_{\lambda_n}(u_n) + o(1) \\
&= c_{\lambda_n} + o(1) \\
&= c_0,
\end{aligned}
$$

which implies $c_0 = I_0(u)$,

$$
\int_{\mathbb{R}^3}|\nabla u_n|^2dx \to \int_{\mathbb{R}^3}|\nabla u|^2dx \text{ and } \int_{\mathbb{R}^3}F(u_n)dx \to \int_{\mathbb{R}^3}F(u)dx.
$$

Combining with (6), we have

$$
\int_{\mathbb{R}^3}F_2(u_n)dx \to \int_{\mathbb{R}^3}F_2(u)dx.
$$

Recall that $f_2(s) \geq ms$ for all $s \geq 0$. We set $F_2(s) = \frac{m}{2}s^2 + G(s)$ for all $s \geq 0$, where G is a non-negative continuous function in $[0, +\infty)$. The Fatou lemma implies

$$
\begin{aligned}
\int_{\mathbb{R}^3}F_2(u)dx &= \frac{m}{2}\int_{\mathbb{R}^3}u^2dx + \int_{\mathbb{R}^3}G(u)dx \\
&\leq \frac{m}{2}\liminf_{n\to\infty}\int_{\mathbb{R}^3}u_n^2dx + \liminf_{n\to\infty}\int_{\mathbb{R}^3}G(u_n)dx \\
&\leq \frac{m}{2}\limsup_{n\to\infty}\int_{\mathbb{R}^3}u_n^2dx + \liminf_{n\to\infty}\int_{\mathbb{R}^3}G(u_n)dx \\
&\leq \limsup_{n\to\infty}\int_{\mathbb{R}^3}\left[\frac{m}{2}u_n^2 + G(u_n)\right]dx \\
&= \limsup_{n\to\infty}\int_{\mathbb{R}^3}F_2(u_n)dx \\
&= \int_{\mathbb{R}^3}F_2(u)dx.
\end{aligned}
$$

Thus

$$\int_{\mathbb{R}^3} u_n^2 dx \to \int_{\mathbb{R}^3} u^2 dx.$$

Therefore, $u_n \to u$ in E and $\forall \varphi \in E$,

$$0 = \langle I'_{\lambda_n}(u_n), \varphi\rangle + o(1) = \langle I'_0(u), \varphi\rangle.$$

We complete the proof. □

5. Summary

In this paper, a positive solution is obtained with the help of the Pohozaev manifold, and the asymptotic behavior of the positive solution u_λ is considered as $\lambda \to 0^-$, which complements the previous results. Moreover, the Kirchhoff-type equation has a wide range of applications in many fields, such as it models several physical and biological systems. Thus, the results of this paper are beneficial for people to better understand the Kirchhoff equation.

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