



Article **Combined** *m*-Consecutive-*k*-Out-of-*n*: *F* and Consecutive k_c -Out-of-*n*: *F* Structures with Cold Standby Redundancy

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Abstract: In the present work, we study the combined *m*-consecutive-*k*-out-of-*n*: *F* and k_c -out-of-*n*: *F* reliability systems, which consist of independent and identically distributed components. Two different redundancy policies are considered, and their general frameworks are described and illustrated. The main objective of the paper refers to the investigation of the effect of adding cold standby redundancy to the system at the the system level and the component level. Exact formulae for determining the crucial characteristics of the enhanced structure, such as its survival function, the mean time to failure and the mean residual lifetime, are provided. All formulae proved in the present manuscript are explicit expressions which are based on the signature vector of the resulting reliability schemes. An extensive numerical investigation is carried out to shed light on the performance of the combined *m*-consecutive-*k*-out-of-*n*: *F* and consecutive k_c -out-of-*n*: *F* reliability systems with cold standby redundancy. Some concluding remarks and comments are provided upon the determination of the optimal design parameters.

Keywords: signature; survival function; reliability consecutive-type structures; mean time to failure; mean residual lifetime; cold standby redundancy; systems with two failure criteria

MSC: 62G30; 62N05; 90B25

1. Introduction

In the field of statistical reliability modeling, several structures with two common failure criteria have been introduced. Their applicability to complex telecommunication networks and industrial or assurance engineering have made them crucial statistical tools for adequately handling several different real-life problems.

To the best of our knowledge, most reliability systems with two common failure criteria are generalizations of the well-known consecutive-*k*-out-of-*n*: *F* system with *n* linearly (or circularly) ordered components. Note that the latter system fails if and only if at least *k* consecutive components fail. The most popular applications of these systems, which were first introduced by [1], pertain to telecommunication and pipeline network modeling as well as integrated circuitry design. The term consecutive k-out-of-n: F system was first coined by [2], and since then, it has attracted considerable research interest (see, e.g., [3,4] and the references therein).

Among others, the (n,f,k) and the $\langle n,f,k \rangle$ structures (see, e.g., [5–7]) contain a k-outof-n stopping criterion and a consecutive-type one. The same holds true for the so-called constrained-(k,d)-out-of-n structures, which were introduced by [8] (see also [9]). Additional members of the class of structures with two common failure criteria are the consecutive- k_1 and k_2 -out-of-n: F structures (see, e.g., [10]) or the combined m-consecutivek-out-of-n: F and consecutive k_c -out-of-n: F structures (see, e.g., [11–13]), which operate under a dual failure scheme.



Citation: Triantafyllou, I.S. Combined *m*-Consecutive-*k*-Out-of-*n*: *F* and Consecutive *k*_c-Out-of-*n*: *F* Structures with Cold Standby Redundancy. *Mathematics* **2023**, *11*, 2597. https://doi.org/10.3390/ math11122597

Academic Editors: Radim Bris and Elena Zaitseva

Received: 15 May 2023 Revised: 2 June 2023 Accepted: 5 June 2023 Published: 6 June 2023



Copyright: © 2023 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). On the other hand, the enhancement of the performance of the underlying structures can be achieved by adding the so-called cold standby redundancy therein. The aforementioned cold standby sparing has been adopted by several authors in order to handle optimization problems (see, e.g., [14–16]). In addition, the dilemma of whether the cold standby redundancy is more effective when it is applied at the the system level or at the component level has attracted the attention of some researchers (see, e.g., [17–19]).

In the present paper, the combination of the *m*-consecutive-*k*-out-of-*n*: *F* and consecutive k_c -out-of-*n*: *F* structures ($C(n, m, k, k_c)$ hereafter) is investigated for two different cold standby redundancy schemes. Note that the ordinary *m*-consecutive-*k*-out-of-*n*: *F* system is a natural generalization of the classical *m*-out-of-*n*: *F* system and the consecutive-*k*-out-of-*n*: *F* system; it consists of n linearly ordered components such that the system fails if and only if there are at least m non-overlapping runs of k consecutive failed components. It is obvious that in the special cases m = 1 and k = 1, the *m*-consecutive-*k*-out-of-*n*: *F* system reduces to consecutive-*k*-out-of-*n*: *F* and *m*-out-of-*n*: *F* systems, respectively. For some advances on the topic, the interested reader is referred to [20] or [21].

In Section 2, the proposed reliability structure is described in some detail, while an illustrative example is presented. Section 3 provides the main theoretical results of the present work. Among others, closed and signature-based formulae for determining the survival function, the mean time to failure (MTTF) and the mean residual lifetime (MRL) of the combined *m*-consecutive-*k*-out-of-*n*: *F* and consecutive k_c -out-of-*n*: *F* structure with cold standby redundancy are provided. In Section 4, an extensive numerical experimentation is carried out, and several numerical comparisons are made. Finally, Section 5 summarizes the main contribution of the present manuscript, and some thoughts for future potential work are also discussed.

2. The General Redundancy Framework

In this section, we present the general framework of the $C(n, m, k, k_c)$ structure with redundancy. Generally speaking, the $C(n, m, k, k_c)$ structure consists of n components ordered in a line. The system fails if and only if there exists at least k_c consecutive failed components or at least m non-overlapping runs of k consecutive failed components $(k_c < mk)$. Note that for $k_c > mk$, the system reduces to the ordinary consecutive k_c -out-of-n: F structure.

Let us first assume that the *n* components of the primary (initial) structure are independent and identically distributed (*i.i.d.* hereafter), with lifetimes $T_1, T_2, ..., T_n$, respectively. We next denote by $S_1, S_2, ..., S_n$ the corresponding lifetimes of the components of the standby (spare) system, which are also considered *i.i.d.* random variables. Throughout the lines of the present manuscript, we assume that both primary and standby components follow the same lifetime distribution, namely, the variables $T_1, T_2, ..., T_n$ and $S_1, S_2, ..., S_n$ share a common distribution law.

Two different redundancy scenarios are considered. According to the first scenario, the standby redundancy is added at the the system level (*Scenario S* hereafter). Practically, this means that the spare system is of the same structure as the primary one, and it is activated whenever the first (primary) system ceases its operation. Once the primary system fails, the standby one takes its place.

We next denote by $LF(T_1, T_2, ..., T_n)$ and $LF(Q_1, Q_2, ..., Q_n)$ the lifetime of a $C(n, m, k, k_c)$ structure with the lifetimes of its components of $T_1, T_2, ..., T_n$ and $Q_1, Q_2, ..., Q_n$ respectively. The lifetime LF_S of the $C(n, m, k, k_c)$ structure under *Scenario S* can be expressed as

$$LF_{S} = LF(T_{1}, T_{2}, \dots, T_{n}) + LF(Q_{1}, Q_{2}, \dots, Q_{n}).$$
(1)

The expression (1) simply reveals that the lifetime of the $C(n, m, k, k_c)$ structure after standby redundancy at the the system level coincides to the sum of the lifetimes of the primary and the spare systems.

For illustration purposes, let us next consider that m = 3, k = 2 and $k_c = 4$, n = 8. For these specific choices of the design parameters, the resulting reliability structure fails if

and only if there exist at least four consecutive failed components or at least three nonoverlapping runs of two consecutive failed components. Let us assume that *Scenario S* is considered, namely, a standby redundancy is applied at the system level. In words, the primary C(8, 3, 2, 4) structure (with the lifetimes of its components $T_1, T_2, ..., T_8$) is replaced by a spare C(8, 3, 2, 4) structure (with the lifetimes of its components $Q_1, Q_2, ..., Q_8$) as soon as the primary one ceases its operation. Of course, in such a case, the lifetime of the resulting structure, namely, the lifetime of the C(8, 3, 2, 4) structure, with cold standby redundancy at the system level, is given by (see (1)):

$$LF_S = LF(T_1, T_2, \dots, T_8) + LF(Q_1, Q_2, \dots, Q_8)$$

Figure 1 displays three snapshots of the above-mentioned reliability structure at different time points of its operation. The first snapshot (starting point) shows that two C(8,3,2,4) structures with different the lifetimes of its components formulate the resulting C(8,3,2,4) structure with cold standby redundancy at the system level.





(b) Time point t_1 (2nd snapshot).



(c) Time point t_2 (3rd snapshot).

Figure 1. Combined 3-consecutive-2-out-of-8: *F* and consecutive 4-out-of-8: *F* structures with cold standby redundancy at the system level. (**a**) The components of both primary and spare systems are at working state; (**b**) the primary system fails, while the spare system starts working; (**c**) the spare system fails, leading to overall failure ($\stackrel{<}{\leftarrow}$: working component; ***** : failed component; $\stackrel{>}{\rightarrow}$: inactive component).

As it is easily observed, the second snapshot illustrates a possible scheme (e.g., at time point t_1) which results in the overall failure of the primary C(8,3,2,4) structure. Until the time point t_1 arrives, the secondary structure remains inactive and therefore, all of its components remain in a working state.

Since the primary C(8, 3, 2, 4) structure fails, its components do not change their state, while at the same time, the secondary one is activated. Finally, the third snapshot of the underlying structure illustrates a possible scheme (e.g., at time point t_2) which leads to the failure of the spare system and consequently to the overall failure of the underlying C(8, 3, 2, 4) structure with cold standby redundancy at the system level. Note that the third snapshot corresponds to a single failure scheme among other scenarios that can result in the overall failure of the C(8, 3, 2, 4) structure with cold standby redundancy at the system level.

On the other hand, the standby redundancy can be added at the component level (*Scenario C*, hereafter). Practically, this means that each component of the system has its own standby redundancy. Whenever a primary component ceases its operation, it is

replaced by the corresponding spare component. We next denote by $T_1, T_2, ..., T_n$ and $Q_1, Q_2, ..., Q_n$ the lifetimes of the primary and spare components of the combined *m*-consecutive-*k*-out-of-*n*: *F* and consecutive k_c -out-of-*n*: *F* structure with redundancy at the component level, respectively. The lifetime LF_C of the $C(n, m, k, k_c)$ structure under *Scenario C* can be expressed as:

$$LF_{C} = LF(T_{1} + Q_{1}, T_{2} + Q_{2}, \dots, T_{n} + Q_{n}).$$
⁽²⁾

Expression (2) shows that the lifetime of the $C(n, m, k, k_c)$ structure after standby redundancy at the component level is determined as the lifetime of an ordinary $C(n, m, k, k_c)$ structure with the lifetimes of its components $T_1 + Q_1, T_2 + Q_2, ..., T_n + Q_n$.

Figure 2 displays three snapshots of the above-mentioned reliability structure at different time points of its operation. The first snapshot shows that two sets of independent components formulate the resulting combined 3-consecutive-2-out-of-8: *F* and consecutive 4-out-of-8: *F* structure with cold standby redundancy at the component level.



(a) Starting point (1st snapshot).

(**b**) Time point t_1 (2nd snapshot).



(c) Time point t_2 (3rd snapshot).

Figure 2. Combined 3-consecutive-2-out-of-8: *F* and consecutive 4-out-of-8: *F* structure with cold standby redundancy at the component level. (a) Both primary and spare components of the system are in a working state; (b) some primary components of the system fail, and the corresponding spare components replace them; (c) additional primary components and some spare components fail and lead to the system's overall failure ($\stackrel{\text{there}}{=}$: working component; * : failed component; $\stackrel{\text{there}}{=}$: inactive component).

As it readily deduced, the second snapshot illustrates a possible scheme (say at time point t_1) wherein some primary components (those at second, third, fourth and fifth places of the structure line) have already failed and have been replaced by the corresponding spare ones. One may easily observe that the resulting combined 3-consecutive-2-out-of-8: *F* and consecutive 4-out-of-8: *F* structure with cold standby redundancy at the component level still operates since the failure criteria have not yet been met. At the time point t_1 , some spare components remain inactive (those at first, sixth, seventh and eighth places of the structure line) since the corresponding primary components are still in a working state.

Finally, the third snapshot of the underlying structure illustrates a possible scheme (e.g., at time point t_2) which leads to the overall failure of the underlying combined 3-consecutive-2-out-of-8: *F* and consecutive 4-out-of-8: *F* structure with cold standby redundancy at the component level. Indeed, at time point t_2 , there exist three non-overlapping runs of two consecutive failed components; consequently, the structure ceases its operation.

It is noticeable that at the third snapshot, there still exists an inactive spare component. That is easily explained since the corresponding primary component, namely, the component in the sixth place on the structure line, is still in a working state. Note that the third snapshot corresponds to a single failure scheme among other scenarios that can also result in the overall failure of the combined 3-consecutive-2-out-of-8: *F* and consecutive 4-out-of-8: *F* structure with cold standby redundancy at the component level.

3. Main Theoretical Results

In this section, we provide a reliability study for the combined *m*-consecutive-*k*-out-of-*n*: *F* and consecutive k_c -out-of-*n*: *F* ($C(n, m, k, k_c)$) structure with redundancy. Several performance characteristics of the underlying structure, such as the survival function, the MTTF and the MRL, are investigated in some detail.

Let us first consider the $C(n, m, k, k_c)$ structure with redundancy at the system level. We assume that the primary system consists of *n* independent and identically distributed components with a common continuous distribution *F*. In addition, we assume that the components of the spare system are also independent and follow the same lifetime distribution *F*.

The next proposition offers explicit expressions for determining some crucial reliability characteristics of the $C(n, m, k, k_c)$ structure with redundancy at the system level. The expressions provided in the following proposition are signature-based formulae (see, e.g., [22] or [23]).

Proposition 1. Let us consider the $C(n, m, k, k_c)$ structure with redundancy at the system level, while $s = (s_1, s_2, ..., s_n)$ corresponds to its signature vector. If $T_1, T_2, ..., T_n$ and $Q_1, Q_2, ..., Q_n$ are the lifetimes of the components of the primary and secondary systems with common cumulative distribution function F and probability density function f, then the following ensue:

(i) The survival function of the $C(n, m, k, k_c)$ structure with redundancy at the system level is given by

$$P(LF_{S} > t) = \sum_{i=k_{c}}^{n} s_{i} \sum_{r=n-i+1}^{n} {n \choose r} R_{1}(F(t), r) + \sum_{i=k_{c}}^{n} \sum_{j=k_{c}}^{n} \sum_{r=n-i+1}^{n} \sum_{k_{i}=k_{j}}^{n} \sum_{r=n-i+1}^{n} \frac{s_{i}s_{j}}{B(j,n-j+1)} {n \choose r} \cdot \int_{0}^{t} R_{1}(F(t-x), r) \cdot R_{2}(F(x), j) f(x) dx$$
(3)

where B(a, b) is the complete Beta function, while

$$\binom{n}{r} = \frac{n!}{r!(n-r)!},$$

$$R_1(y,w) = (1-y)^w y^{n-w}, \ 0 \le y \le 1$$
(4)

and

$$R_2(y,w) = (1-y)^{n-w} y^{w-1}, \ 0 \le y \le 1.$$
(5)

(ii) The MTTF of the $C(n, m, k, k_c)$ structure with redundancy at the system level is given by

$$MTTF_{S}(m,k,k_{c},n) = 2 \cdot \sum_{i=k_{c}}^{n} s_{i} \sum_{r=n-i+1}^{n} \binom{n}{r} \cdot \int_{0}^{\infty} R_{1}(F(t),r) dt.$$
(6)

(iii) The MRL of the $C(n, m, k, k_c)$ structure with redundancy at the system level is given by

$$MRL_{S}(m,k,k_{c},n,t) = \frac{1}{P(LF_{S}>t)} \left\{ \sum_{i=k_{c}}^{n} s_{i} \sum_{r=n-i+1}^{n} {n \choose r} \int_{t}^{\infty} R_{1}(F(z),r) dz + \sum_{i=k_{c}}^{n} \sum_{j=k_{c}}^{n} \sum_{r=n-i+1}^{n} \sum_{B(j,n-j+1)}^{n} {s_{i}s_{j} \choose r} \int_{t}^{\infty} \int_{0}^{z} R_{1}(F(z-x),r) \cdot R_{2}(F(x),j) f(x) dx dz \right\}$$
(7)

Proof. (i) The event $\{LF_S > t\}$ for some t > 0, is brought about whenever one of the following takes place:

- The lifetime of the primary system itself exceeds *t*, namely $LF(T_1, T_2, ..., T_n) > t$;
- The lifetime of the secondary system exceeds t x under the assumption that the lifetime of the primary system is not greater than x, namely, $LF(Q_1, Q_2, ..., Q_n) > t x | LF(T_1, T_2, ..., T_n) \le x$ for some x < t

Therefore, the survival function of the $C(n, m, k, k_c)$ structure with redundancy at the system level can be expressed as:

$$P(LF_{S} > t) = P(LF(T_{1}, T_{2}, ..., T_{n}) > t) +P(LF(Q_{1}, Q_{2}, ..., Q_{n}) > t - x | LF(T_{1}, T_{2}, ..., T_{n}) \le x).$$
(8)

It is known that the survival function at time *t* of any coherent system consisting of *n* independent and identically distributed components can be expressed as

$$\sum_{i=1}^{n} s_i P(X_{i:n} > t),$$

where s_i , i = 1, 2, ..., n corresponds to the i - th coordinate of the signature vector of the system, and $X_{1:n} < X_{2:n} < ... < X_{n:n}$ are the ordered the lifetimes of its components. Therefore, the right-hand side of (8) can be rewritten as

$$P(LF_{S} > t) = \sum_{i=1}^{n} s_{i}P(T_{i:n} > t) + \int_{0}^{t} P(LF(Q_{1}, Q_{2}, ..., Q_{n}) > t - x)dP(LF(T_{1}, T_{2}, ..., T_{n}) \le x)$$

$$= \sum_{i=1}^{n} s_{i}P(T_{i:n} > t) + \sum_{i=1}^{n} s_{i}\int_{0}^{t} P(Q_{i:n} > t - x)dP(LF(T_{1}, T_{2}, ..., T_{n}) \le x)$$

$$= \sum_{i=1}^{n} s_{i}P(T_{i:n} > t) + \sum_{i=1}^{n} s_{i}\int_{0}^{t} P(Q_{i:n} > t - x)\sum_{j=1}^{n} s_{j}dP(T_{j:n} \le x)$$

$$= \sum_{i=1}^{n} s_{i}P(T_{i:n} > t) + \sum_{i=1}^{n} \sum_{j=1}^{n} s_{i}s_{j}\int_{0}^{t} P(Q_{i:n} > t - x)f_{j:n}(x)dx.$$
(9)

In addition, the following equalities hold true:

$$P(Q_{i:n} > t - x) = P(\text{at least}(n - i + 1)Q'_{i}s \text{ are greater than } t - x)$$

= $\sum_{r=n-i+1}^{n} {n \choose r} s_{i} (1 - F(t - x))^{r} (F(t - x))^{n-r}$ (10)

and

$$f_{j:n}(x) = \frac{1}{B(j, n-j+1)} (F(x))^{j-1} (1-F(x))^{n-j} f(x).$$
(11)

Given that the first $k_c - 1$ coordinates of the signature vector for the $C(n, m, k, k_c)$ structure are equal to zero (see, e.g., [13]), the desired result is readily deduced after replacing Formulas (10) and (11) in (9).

(ii) The MTTF of the $C(n, m, k, k_c)$ structure with redundancy at the system level can be determined as

$$E(LF_S) = E(LF(T_1, T_2, \dots, T_n) + LF(Q_1, Q_2, \dots, Q_n))$$

= $E(LF(T_1, T_2, \dots, T_n)) + E(LF(Q_1, Q_2, \dots, Q_n)).$ (12)

As already mentioned, the components of both the primary and spare systems follow the same lifetime distribution *F*. Therefore, the two summands of the last expression are equal, and we may write (see, also [19,24]):

$$E(LF_S) = 2 \cdot E(LF(T_1, T_2, \dots, T_n)) = 2 \cdot \sum_{i=1}^n s_i E(T_{i:n}).$$
(13)

Based on the well-known identity for positive random variables, the expected value of $T_{i:n}$ is given by

$$E(T_{i:n}) = \int_0^\infty P(T_{i:n} > y) dy = \sum_{r=n-i+1}^n \binom{n}{r} \int_0^\infty (1 - F(t))^r (F(t))^{n-r} dt$$
(14)

and the result we are chasing after is effortlessly derived by replacing the last expression in (13).

(iii) The MRL of the $C(n, m, k, k_c)$ structure with redundancy at the system level is given by

$$MRL_{S}(m,k,k_{c},n,t) = E(LF_{S}-t|LF_{S}>t) = \frac{1}{P(LF_{S}>t)} \int_{t}^{\infty} P(LF_{S}>x) dx.$$
(15)

We next combine (3) and (15), and the result is straightforward. \Box

Let us next consider the $C(n, m, k, k_c)$ structure with redundancy at the component level. We assume that both primary and spare (secondary) components of the resulting system are independent and identically distributed with a common continuous distribution *F*. If $T_1, T_2, ..., T_n$ and $Q_1, Q_2, ..., Q_n$ correspond to the lifetimes of the primary and secondary components of the system with the common cumulative distribution function *F* and probability density function *f*, then the survival function of the $C(n, m, k, k_c)$ structure with redundancy at the component level is given by

$$P(LF_{C} > t) = P(LF(T_{1} + Q_{1}, T_{2} + Q_{2}, \dots, T_{n} + Q_{n}) > t) = \sum_{i=1}^{n} s_{i}P(TQ_{i:n} > t), \quad (16)$$

where $TQ_{1:n} < TQ_{2:n} < \cdots < TQ_{n:n}$ are the ordered random variables corresponding to lifetimes

$$TQ_1 = T_1 + Q_1, TQ_2 = T_2 + Q_2, \dots, TQ_n = T_n + Q_n.$$
 (17)

Following a similar argumentation with the one provided for the case of redundancy

at the system level (see also [19]), the survival function H(t) of $TQ_i = T_i + Q_i$, i = 1, 2, ..., n can be determined as

$$H(t) = 1 - H(t) = P(T_i + Q_i > t) = P(T_i > t) + P(Q_i > t - x | T_i \le x)$$

= $P(T_i > t) + \int_0^t P(Q_i > t - x) dP(T_i \le x)$
= $1 - F(t) + \int_0^t (1 - F(t - x)) f(x) dx.$ (18)

Since the survival function of $TQ_{i:n}$, i = 1, 2, ..., n can be determined via

$$P(TQ_{i:n} > t) = P(\text{at least}(n - i + 1)Q_i's \text{ are greater than } t)$$
$$= \sum_{r=n-i+1}^n {\binom{n}{r}} s_i (1 - H(t))^r (H(t))^{n-r},$$

the survival function of the $C(n, m, k, k_c)$ structure with redundancy at the component level can be rewritten as

$$P(LF_C > t) = \sum_{i=k_c}^{n} s_i \sum_{r=n-i+1}^{n} \binom{n}{r} (1 - H(t))^r (H(t))^{n-r},$$
(19)

where H(t) is provided with the aid of (18).

In addition, the MTTF of the $C(n, m, k, k_c)$ structure with redundancy at the component level can be expressed as

$$MTTF_{C}(m,k,k_{c},n) = E(LF(T_{1}+Q_{1},T_{2}+Q_{2},...,T_{n}+Q_{n})) = E(LF(TQ_{1},TQ_{2},...,TQ_{n}))$$

$$= \sum_{i=k_{c}}^{n} s_{i}E(TQ_{i:n}),$$
(20)

where the expected value of $TQ_{i:n}$, i = 1, 2, ..., n is determined as

$$E(TQ_{i:n}) = \int_0^\infty P(TQ_{i:n} > y) dy = \sum_{r=n-i+1}^n \binom{n}{r} \int_0^\infty (1 - H(t))^r (H(t))^{n-r} dt.$$

Furthermore, the MRL of the $C(n, m, k, k_c)$ structure with redundancy at the component level is given by

$$MRL_{C}(m,k,k_{c},n,t) = E(LF_{C}-t|LF_{C}>t) = \frac{1}{P(LF_{C}>t)} \int_{t}^{\infty} P(LF_{C}>x) dx.$$
(21)

where the survival function $P(LF_C > t)$ can be calculated by the aid of (19).

4. Numerical Results

In the present section, we carry out an extensive numerical investigation of the proposed combined *m*-consecutive-*k*-out-of-*n*: *F* and consecutive k_c -out-of-*n*: *F* ($C(n, m, k, k_c)$) structure with redundancy. The numerical results and graphical illustrations provided throughout the next lines were all produced with the aid of the theoretical outcomes proved in the previous section of the present manuscript.

We first focus on the survival function of the $C(n, m, k, k_c)$ structure with redundancy. More specifically, the impact of the design parameters m, k, k_c and n on the survival function is investigated. Figure 3 displays the survival probabilities for different design of the underlying system with redundancy at the system level. We assume that the lifetimes of both the primary and spare systems follow an exponential distribution with a unit mean. The numerical results provided at Figure 3 were produced with the aid of part (i) of Proposition 1 (see Section 3 of the present manuscript).

As can be readily observed, the survival probability of the $C(n, m, k, k_c)$ structure with redundancy at the system level becomes larger when:

- The parameter *k*_c increases under the assumption that the rest design parameters *m*, *k*and*n* remain unchanged (see Figure 3a);
- The parameter *m* increases under the assumption that the rest design parameters *k*_c, *k*and*n* remain unchanged (see Figure 3b);
- The parameter *n* decreases under the assumption that the rest design parameters *m*, *k*and*k*_c remain unchanged (see Figure 3c);
- The parameter *k* increases eases under the assumption that the rest design parameters *k*_c, *m*and*n* remain unchanged (see Figure 3d).

We next investigate the performance of the $C(n, m, k, k_c)$ structure with redundancy with the aid of its MTTF. We take into consideration not only the case of adding redundancy at the system level but also of adding redundancy at the component level. The corresponding MTTFs of the resulting systems are determined via Formulas (6) and (18), respectively. Table 1 displays numerical results for several designs of the combined *m*-consecutive-*k*-outof-*n*: *F* and consecutive k_c -out-of-*n*: *F* structure with redundancy under the exponential distribution model with a unit mean.



Figure 3. Combined *m*-consecutive-*k*-out-of-*n*: *F* and consecutive *k*_c-out-of-*n*: *F* structure with cold standby redundancy at the system level for several designs.

(m,k,k_c,n)	$MTTF_{S}(m,k,k_{c},n)$	$MTTF_{C}(m,k,k_{c},n)$
(2, 2, 3, 6)	1.96807	2.07025
(2, 2, 4, 6)	2.49800	2.45425
(2, 2, 4, 7)	2.13898	2.20239
(2, 3, 4, 7)	2.63381	2.55018
(2, 2, 3, 8)	1.52338	1.74424
(2, 2, 4, 8)	1.85206	1.99730
(2, 3, 4, 8)	2.40243	2.39091
(2, 3, 5, 8)	2.92320	2.75489
(2, 3, 6, 8)	3.47001	3.11523
(2, 3, 4, 9)	2.19870	2.25013
(2, 3, 5, 9)	2.67369	2.58602
(2, 3, 6, 9)	3.05157	2.84444
(2, 3, 4, 10)	2.04342	2.14076
(2, 3, 5, 10)	2.43495	2.42374
(2, 3, 6, 10)	2.76974	2.65516

Table 1. MTTF of the combined *m*-consecutive-*k*-out-of-*n*: *F* and consecutive k_c -out-of-*n*: *F* structure with redundancy at system and the component level.

The lifetimes of both primary and spare components follow exponential distribution with unit means.

Based on the numerical results provided at Table 1, we may readily deduce that the MTTF of the $C(n, m, k, k_c)$ structure with redundancy increases

- For fixed *m*, *k*and*n* as the parameter *k*_c increases;
- For fixed *m*, *k*and*k*_c as the parameter *n* decreases;
- For fixed *k_c*, *m*and*n* as the parameter *k* increases;
- For fixed *k*, *k*_c, *m*and*n* as *t* decreases.

For instance, let us consider the special case n = 10. Practically, this means that the underlying system consists of 10 components. For fixed values m = 2 and k = 3, we investigate the performance of the resulting structures under different choices for the parameter k_c . Based on Table 1, the MTTF of the resulting structure with redundancy at the system level is:

- Equal to 2.04342 for $k_c = 4$;
- Equal to 2.43945 for $k_c = 5$;
- Equal to 2.76974 for $k_c = 6$.

It is straightforward that the larger the parameter k_c is, the larger the MTTF values become.

In addition, for the aforementioned designs, redundancy was also applied at the component level. The numerical results illustrated at Table 1 reveal that the MTTF of the resulting structure with redundancy at the component level is:

- Equal to 2.14076 for $k_c = 4$;
- Equal to 2.42374 for $k_c = 5$;
- Equal to 2.65516 for $k_c = 6$.

Based on the above results, we can readily draw the same conclusion for the MTTF behavior of the structure with redundancy at the component level.

We next carry out a numerical investigation of the performance of the $C(n, m, k, k_c)$ structure with redundancy based on its MRL. We take into consideration not only the case of adding redundancy at the system level but also of adding redundancy at the component level. The corresponding MRLs of the resulting systems are determined via Formulas (15) and (21), respectively. Table 2 displays some numerical results for several designs of the $C(n, m, k, k_c)$ structure with redundancy under the exponential distribution model with a unit mean.

Table 2. MRLs of the combined *m*-consecutive-*k*-out-of-*n*: *F* and consecutive k_c -out-of-*n*: *F* structure with redundancy at system and the component level.

(m,k,k_c,n)	t	$MRL_S(m,k,k_c,n,t)$	$MRL_{C}(m,k,k_{c},n,t)$
(2, 2, 3, 6)	0.2	1.64148	1.87029
	0.4	1.56954	1.67184
	0.2	2.41646	2.25425
(2, 2, 4, 6)	0.4	2.24108	2.05442
(2, 2, 4, 7)	0.2	1.88761	2.00239
(2, 2, 4, 7)	0.4	1.77662	1.80264
(2, 3, 4, 7)	0.2	2.90118	2.35018
(2, 3, 4, 7)	0.4	2.73641	2.15030
(2, 2, 2, 2)	0.2	1.10698	1.54447
(2, 2, 3, 8)	0.4	1.08888	1.34645
(2, 2, 4, 8)	0.2	1.44086	1.79711
(2, 2, 4, 8)	0.4	1.35740	1.59743
$(2 \ 2 \ 4 \ 8)$	0.2	2.56983	2.19091
(2, 3, 4, 8)	0.4	2.44647	1.99105
(2, 2, 5, 8)	0.2	4.05953	2.55516
(2, 3, 5, 8)	0.4	3.85405	2.35517
$(2,2,\epsilon,8)$	0.2	8.47663	2.91523
(2, 3, 6, 8)	0.4	8.17625	2.71523

(m,k,k_c,n)	t	$MRL_S(m,k,k_c,n,t)$	$MRL_C(m,k,k_c,n,t)$
(2, 3, 4, 9)	0.2	2.16090	2.04990
	0.4	2.05420	1.85006
(2, 3, 5, 9)	0.2	3.62745	2.38602
	0.4	3.46091	2.18603
(2, 3, 6, 9)	0.2	5.31222	2.64444
	0.4	5.10282	2.44444
(2, 3, 4, 10)	0.2	1.83466	1.94076
	0.4	1.73471	1.74093
(2, 3, 5, 10)	0.2	2.88385	2.22374
	0.4	2.76974	2.02375
(2, 3, 6, 10)	0.2	4.55449	2.45489
	0.4	4.38012	2.25489

Table 2. Cont.

The lifetimes of both primary and spare components follow an exponential distribution with a unit mean.

Based on the numerical results provided at Table 2, we may readily deduce that the MRL of the $C(n, m, k, k_c)$ structure with redundancy increases:

- For fixed *m*, *k*and*n* as the parameter *k*_c increases;
- For fixed *m*, *k*and*k*_c as the parameter *n* decreases;
- For fixed *k_c*, *m*and*n* as the parameter *k* increases;
- For fixed *k*, *k*_{*c*}, *m*and*n* as *t* decreases.

For instance, let us consider the special case n = 9. Practically, this means that the underlying system consists of nine components. For fixed values m = 2 and k = 3, we investigate the performances of the resulting structures under different choices for the parameter k_c . Based on Table 2, the MRL of the resulting structure with redundancy at the system level is:

- Equal to 2.16090 and 2.05420 for t = 0.2, 0.4, respectively, for the choice $k_c = 4$;
- Equal to 3.62745 and 3.46091 for t = 0.2, 0.4, respectively, for the choice $k_c = 5$;
- Equal to 5.31222 and 5.10282 for t = 0.2, 0.4, respectively, for the choice $k_c = 6$.

It is straightforward that the larger the parameter k_c is, the larger the MRL values become. In addition, for the aforementioned designs, the redundancy was also applied at the component level. The numerical results illustrated at Table 2 reveal that the MRL of the resulting structure with redundancy at the component level is:

- Equal to 2.04990 and 2.1.85006 for t = 0.2, 0.4, respectively, for the choice $k_c = 4$;
- Equal to 2.38602 and 2.18603 for t = 0.2, 0.4, respectively, for the choice $k_c = 5$;
- Equal to 2.64444 and 2.44444 for t = 0.2, 0.4, respectively, for the choice $k_c = 6$.

Based on the above results, we can readily draw the same conclusion for the MRL behavior of the structure with redundancy at the component level.

In order to determine the most appropriate design, the practitioner should take into account several thoughts and restrictions. First of all, the system's size *n* should be properly selected in order to fit in the general framework of the underlying real-life problem. Apart from that, it seems that larger values of the parameters $kandk_c$ result in more reliable structures. For some recent advances on the design optimization of systems, the interested reader is referred to [25,26].

Furthermore, it is of some interest to compare the performance of the $C(n, m, k, k_c)$ structure under redundancy at the system level versus the corresponding structure with redundancy at the component level. For this reason, we shall focus on a particular design for both systems and then compare their performances. The numerical comparisons provided in Tables 1 and 2 reveal that the choice of the most effective redundancy type depends strongly on the structure of the system. In other words, there is no numerical evidence to support that redundancy at the system level overperforms (or is inferior to) redundancy at the component level. It is true that in some cases, the MTTF (and the MRL as well) of the underlying $C(n, m, k, k_c)$ structure with redundancy at the system level offers a more

reliable solution (see, e.g., the design $(m, k, k_c, n) = (2, 3, 6, 10)$ in Tables 1 and 2). On the other hand, there exist cases in which the combined *m*-consecutive-*k*-out-of-*n*: *F* and consecutive k_c -out-of-*n*: *F* structure with redundancy at the component level seems to be superior (see, e.g., the design $(m, k, k_c, n) = (2, 2, 3, 6)$ in Tables 1 and 2).

5. Discussion

The present article focuses on the effect of adding two different redundancy policies which are applied at the system level or the component level, respectively. Both redundancy strategies have been proven to result in some improvements to the underlying structures. For the implementation of an appropriate redundancy scheme, the practitioner should properly determine the design parameters so that the chosen redundancy framework fits well into the underlying real-life application. Regarding future, potential work, it seems quite intriguing to apply a similar methodological framework with the one developed here to provide a reliability study of additional systems. In addition, it is of some interest to investigate whether the implementation of different redundancy policies may offer better solutions in comparison with those applied in the present paper.

6. Conclusions

In the present article, the combined *m*-consecutive-*k*-out-of-*n*: *F* and consecutive k_c -out-of-*n*: *F* structure with redundancy is presented and investigated in some detail. Two different redundancy policies are taken into consideration. According to the first one, the redundancy is added at the system level, while the second scenario calls for applying the redundancy at the component level. Explicit formulae for determining the survival function, the MTTF and the MRL of the combined *m*-consecutive-*k*-out-of-*n*: *F* and consecutive k_c -out-of-*n*: *F* structure with cold standby redundancy for both types are provided. Through the aid of these theoretical results, a numerical investigation was carried out and offers some evidence that larger values of parameters $kandk_c$ result in more reliable structures, while as the parameter *n* increases, the system's MTTF- and MRL- performance grows worse.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

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