mathematics

Article

A Second-Order Time Discretization for Second Kind Volterra Integral Equations with Non-Smooth Solutions

Boya Zhou [1] and Xiujun Cheng [2,*]

[1] School of Mathematics and Big Data, Foshan University, Foshan 528225, China; byzhou@fosu.edu.cn
[2] College of Science, Zhejiang Sci-Tech University, Hangzhou 310018, China
* Correspondence: xiujuncheng@zstu.edu.cn

Abstract: In this paper, a novel second-order method based on a change of variable and the symmetrical and repeated quadrature formula is presented for numerical solving second kind Volterra integral equations with non-smooth solutions. Applying the discrete Grönwall inequality with weak singularity, the convergence order $\mathcal{O}(N^{-2})$ in L^{∞} norm is proved, where N refers to the number of time steps. Numerical results are conducted to verify the efficiency and accuracy of the method.

Keywords: volterra equations; weakly singular kernels; a discrete Grönwall inequality; change of variable; error analysis

MSC: 34A12; 34A34; 34A45

check for updates

Citation: Zhou, B.; Cheng, X. A Second-Order Time Discretization for Second Kind Volterra Integral Equations with Non-Smooth Solutions. *Mathematics* **2023**, *11*, 2594. https://doi.org/10.3390/math11122594

Academic Editor: Xiaobing Feng

Received: 21 April 2023
Revised: 25 May 2023
Accepted: 26 May 2023
Published: 6 June 2023

1. Introduction

In this paper, we consider numerical solutions of the following second kind of Volterra integral equations (VIEs) [1],

$$y(t) = g(t) + \int_0^t \frac{\kappa(t,s)y(s)}{(t-s)^{\beta}}ds,\ t \in [0,T],\ 0 < \beta < 1, \tag{1}$$

where $\kappa \in C^m(\Omega)$, $\Omega = (t,s)\colon 0 \le s \le t \le T$ and $g(t)$ is a given function and we assume that $g(t^{\frac{1}{1-\beta}})$ satisfies sufficiently smooth. The second kind VIEs have been widely used in many areas, such as science, mathematical physics, and engineering [2–4]. However, in most cases, the integration part can not be solved analytically by taking a weakly singular kernel into account.

In the past several years, there appears to be an increasing interest in finding numerical methods for solving the general weak singular VIEs of the second kind, such as product integration methods [5], fractional multistep methods [6], collocation methods [7–12] and so on [13–19]. For the sake of the comprehensive analysis and the difficulty caused by the weak regularity of the kernel, more and more investigators pay attention to the collocation methods. As pointed out in [2,20–22], the usual collection methods enjoy high accuracy for problems with high regularity restriction on the solutions. We shall know the convergence order will be $\mathcal{O}(h^{1-\beta})$ for non-smooth solutions with uniform meshes whatever the degree of polynomials one chooses. In order to fill this gap, one of the most popular methods used by researchers is applying graded meshes [23–28]. In fact, as mentioned by Tang [26], graded meshes did not well in computing because enough small interval in initial time is needed. They introduced a new method with suitable transportation to solve this problem which can obtain high orders and avoid the difficulty of computation. For more detailed results, we refer readers to [29–33]. To our best knowledge, only fixed parameter β is considered when they numerically solved Equation (1) using variable transformations in previous works. Then, giving a simple and straightaway approach to solve the second kind VIEs is the main purpose to do this research.

In this paper, we investigate the analysis and computation of the trapezoidal rule for the second kind of Volterra integral equations. Firstly, a common change of variable $s = t^{1-\beta}$ is used to gain a new equation whose exact solution is smooth even at the initial time. Then, the trapezoidal rule could be applied to estimate the integral part. The convergence results are conducted and proved with weakly singular discrete Grönwall inequality which was first proposed in [34] and has been widely used in the analysis of numerical schemes [9,10,26,35]. Dixon [35] applied the discrete Grönwall inequality and collocation methods to solve VIEs. The error estimate $\mathcal{O}(N^{2-\alpha})$ is established at $t = T$. However, the previous results are obtained by applying collocation methods. The proof of present results is much more technical making use of variable changes and the non-locality of the problem.

The rest of this article is organized as follows. Numerical schemes using the product trapezoidal rule are structured under variables change in Section 2. We give a rigorous convergence analysis of the proposed method in Section 3. In Section 4, numerical tests are conducted to justify our theoretical results. Finally, we conclude this literature in Section 5.

2. The Product Trapezoidal Rule and Main Results

In this section, we present the trapezoidal scheme for solving problems (1). In order to guarantee the solution gained by the variable change is smooth even at the initial time, we assume that $g(t^{\frac{1}{1-\beta}})$ satisfies sufficiently smooth. The solution of Equation (1) can be expressed as follows, which is the same as [35],

$$y(t) = g(t) + \sum_{n=1}^{\infty} \phi_n(t;\beta) t^{n(1-\beta)}, t \in [0, T], \tag{2}$$

where $\phi_n \in C^m[0, T]$ for any n. Based on the expansion (2), we introduce the change of variable

$$t^{1-\beta} = s.$$

Let $\alpha = 1 - \beta$, $h = \frac{T^\alpha}{N}$ and denote $s_i = ih$, $i = 0, 1, 2, \ldots, N$ as a uniform mesh on $[0, T^\alpha]$, where N is a positive integer. For ease of exposition, we suppose that $u(s) := y(s^{\frac{1}{\alpha}})$, $f(s) := g(s^{\frac{1}{\alpha}})$. Then, we can rewrite Equation (1) as follows

$$\begin{aligned} u(s) &= \int_0^{s^{\frac{1}{\alpha}}} \frac{\kappa(s^{\frac{1}{\alpha}}, z) y(z)}{(s^{\frac{1}{\alpha}} - z)^{1-\alpha}} dz + f(s) \\ &= \frac{1}{\alpha} \int_0^{s} \frac{\kappa(s^{\frac{1}{\alpha}}, r^{\frac{1}{\alpha}}) y(r^{\frac{1}{\alpha}})}{(s^{\frac{1}{\alpha}} - r^{\frac{1}{\alpha}})^{1-\alpha}} r^{\frac{1}{\alpha}-1} dr + f(s) \\ &= \frac{1}{\alpha} \int_0^{s} \frac{\kappa(s^{\frac{1}{\alpha}}, r^{\frac{1}{\alpha}}) u(r)}{(s^{\frac{1}{\alpha}} - r^{\frac{1}{\alpha}})^{1-\alpha}} r^{\frac{1}{\alpha}-1} dr + f(s), s \in [0, T^\alpha]. \end{aligned} \tag{3}$$

Taking $s = s_i$, $u_i = u(s_i)$, $i = 0, 1, 2, \ldots, N$ and applying interpolation approximation at each interval $[s_j, s_{j+1}]$, $j = 0, 1, 2, \ldots, i-1$, we arrive at

$$\begin{aligned} u_i &= \frac{1}{\alpha} \int_0^{s_i} \frac{\kappa(s_i^{\frac{1}{\alpha}}, r^{\frac{1}{\alpha}}) u(r)}{(s_i^{\frac{1}{\alpha}} - r^{\frac{1}{\alpha}})^{1-\alpha}} r^{\frac{1}{\alpha}-1} dr + f(s_i) \\ &= \frac{1}{\alpha} \sum_{j=0}^{i-1} \int_{s_j}^{s_{j+1}} \frac{\kappa(s_i^{\frac{1}{\alpha}}, r^{\frac{1}{\alpha}}) u(r)}{(s_i^{\frac{1}{\alpha}} - r^{\frac{1}{\alpha}})^{1-\alpha}} r^{\frac{1}{\alpha}-1} dr + f(s_i) \\ &\approx \frac{1}{\alpha} \sum_{j=0}^{i-1} \int_{s_j}^{s_{j+1}} \frac{r^{\frac{1}{\alpha}-1}}{(s_i^{\frac{1}{\alpha}} - r^{\frac{1}{\alpha}})^{1-\alpha}} \left[\frac{r - s_{j+1}}{-h} u_j \kappa(s_i^{\frac{1}{\alpha}}, s_j^{\frac{1}{\alpha}}) + \frac{r - s_j}{h} u_{j+1} \kappa(s_i^{\frac{1}{\alpha}}, s_{j+1}^{\frac{1}{\alpha}}) \right] dr \\ &\quad + f(s_i). \end{aligned} \tag{4}$$

Omitting the truncation error and denoting U_i as the approximation to $u(s_i)(0 \le i \le N)$, we can obtain the product trapezoidal rule for Equation (3),

$$\begin{aligned} U_0 &= f(s_0), \\ U_i &= \textstyle\sum_{j=0}^{i} w_{i,j}\kappa(s_i^{\frac{1}{\alpha}}, s_j^{\frac{1}{\alpha}})U_j + f_i,\ 1 \le i \le N, \end{aligned} \tag{5}$$

where

$$\begin{aligned} w_{i,0} &= \tfrac{1}{\alpha h}\int_{s_0}^{s_1} \tfrac{r^{\frac{1}{\alpha}-1}(s_1-r)}{(s_i^{\frac{1}{\alpha}}-r^{\frac{1}{\alpha}})^{1-\alpha}}dr,\ 1 \le i \le N, \\ w_{i,j} &= \tfrac{1}{\alpha h}\int_{s_j}^{s_{j+1}} \tfrac{r^{\frac{1}{\alpha}-1}(s_{j+1}-r)}{(s_i^{\frac{1}{\alpha}}-r^{\frac{1}{\alpha}})^{1-\alpha}}dr + \tfrac{1}{\alpha h}\int_{s_{j-1}}^{s_j} \tfrac{r^{\frac{1}{\alpha}-1}(r-s_{j-1})}{(s_i^{\frac{1}{\alpha}}-r^{\frac{1}{\alpha}})^{1-\alpha}}dr, \\ &\quad 1 \le j \le i-1,\ 2 \le i \le N, \\ w_{i,i} &= \tfrac{1}{\alpha h}\int_{s_{i-1}}^{s_i} \tfrac{r^{\frac{1}{\alpha}-1}(r-s_{i-1})}{(s_i^{\frac{1}{\alpha}}-r^{\frac{1}{\alpha}})^{1-\alpha}}dr,\ 1 \le i \le N. \end{aligned}$$

Considering the formate of coefficient $w_{i,j}$ is more complex, we give the following proposition for exact expansion.

Lemma 1. *There exists a positive real number M_1 such that*

$$0 \le w_{i,j} \le \frac{M_1}{\alpha} j^{\frac{1}{\alpha}-1} h (i^{\frac{1}{\alpha}} - j^{\frac{1}{\alpha}})^{\alpha-1},\ 1 \le j < i \le N. \tag{6}$$

Moreover

$$0 \le w_{i,i} \le \frac{[i^{\frac{1}{\alpha}} - (i-1)^{\frac{1}{\alpha}}]^{\alpha} h}{\alpha},\ 1 \le i \le N. \tag{7}$$

Proof. Here, we give the rigorous calculated progress of $w_{i,j}$ and $w_{i,i}$. Recall the mean-value theorem, there exist $\xi \in (s_j, s_{j+1})$ and $\eta \in (s_{j-1}, s_j)$,

$$\begin{aligned} w_{i,j} &= \frac{\xi^{\frac{1}{\alpha}-1}(s_{j+1}-\xi)}{\alpha(s_i^{\frac{1}{\alpha}} - \xi^{\frac{1}{\alpha}})^{1-\alpha}} + \frac{\eta^{\frac{1}{\alpha}-1}(\eta - s_{j-1})}{\alpha(s_i^{\frac{1}{\alpha}} - \eta^{\frac{1}{\alpha}})^{1-\alpha}} \\ &\le \frac{h}{\alpha}\frac{s_{j+1}^{\frac{1}{\alpha}-1}}{(s_i^{\frac{1}{\alpha}} - s_j^{\frac{1}{\alpha}})^{1-\alpha}} + \frac{h}{\alpha}\frac{s_j^{\frac{1}{\alpha}-1}}{(s_i^{\frac{1}{\alpha}} - s_j^{\frac{1}{\alpha}})^{1-\alpha}} \\ &= \frac{h}{\alpha}(j+1)^{\frac{1}{\alpha}-1}(i^{\frac{1}{\alpha}} - j^{\frac{1}{\alpha}})^{\alpha-1} + \frac{h}{\alpha} j^{\frac{1}{\alpha}-1}(i^{\frac{1}{\alpha}} - j^{\frac{1}{\alpha}})^{\alpha-1} \\ &= \frac{h}{\alpha} j^{\frac{1}{\alpha}-1}(i^{\frac{1}{\alpha}} - j^{\frac{1}{\alpha}})^{\alpha-1}[1 + (\frac{j+1}{j})^{\frac{1}{\alpha}-1}] \\ &\le M_1 h j^{\frac{1}{\alpha}-1}(i^{\frac{1}{\alpha}} - j^{\frac{1}{\alpha}})^{\alpha-1}. \end{aligned}$$

Similarly, the estimate of $w_{i,i}$ gives that

$$\begin{aligned} w_{i,i} &= \frac{1}{\alpha h}\int_{s_{i-1}}^{s_i} \frac{r^{\frac{1}{\alpha}}}{(s_i^{\frac{1}{\alpha}} - r^{\frac{1}{\alpha}})^{1-\alpha}}dr - \frac{1}{\alpha h}\int_{s_{i-1}}^{s_i} \frac{r^{\frac{1}{\alpha}-1}s_{i-1}}{(s_i^{\frac{1}{\alpha}} - r^{\frac{1}{\alpha}})^{1-\alpha}}dr \\ &= \frac{1}{h}\int_{s_{i-1}^{\frac{1}{\alpha}}}^{s_i^{\frac{1}{\alpha}}} z^{\alpha}(s_i^{\frac{1}{\alpha}} - z)^{\alpha-1}dz - \frac{1}{h}\int_{s_{i-1}^{\frac{1}{\alpha}}}^{s_i^{\frac{1}{\alpha}}} s_{i-1}(s_i^{\frac{1}{\alpha}} - z)^{\alpha-1}dz \\ &= \frac{s_i^2}{h}\int_{s_{i-1}^{\frac{1}{\alpha}}/s_i^{\frac{1}{\alpha}}}^{1} v^{\alpha}(1-v)^{\alpha-1}dv - \frac{s_{i-1}s_i}{h}\int_{s_{i-1}^{\frac{1}{\alpha}}/s_i^{\frac{1}{\alpha}}}^{1}(1-v)^{\alpha-1}dv \\ &\leq \Big(\frac{s_i^2 - s_{i-1}s_i}{h}\Big)\int_{s_{i-1}^{\frac{1}{\alpha}}/s_i^{\frac{1}{\alpha}}}^{1}(1-v)^{\alpha-1}dv \\ &= \frac{[i^{\frac{1}{\alpha}} - (i-1)^{\frac{1}{\alpha}}]^{\alpha}h}{\alpha}, \end{aligned}$$

which completes the proof. □

The weak singularity discrete Grönwall inequality is very important in the classical result; thus, we display it as follows.

Lemma 2 ([34]). *Suppose $\{x_i\}$ and $\{\psi_i\}$ are two sequences of non-negative real numbers and $\{\psi_i\}$ increases monotonously, where $0 \leq i \leq N$. Given the parameters $\sigma, M \geq 0$, $0 < \mu < 1$, and $\lambda = \sigma + 1$, if*

$$x_i \leq \psi_i + Mh^{\sigma+1-\lambda\mu}\sum_{j=0}^{i-1}\frac{j^{\sigma}x_j}{(i^{\lambda} - j^{\lambda})^{\mu}},\ 0 \leq i \leq N,$$

then

$$x_i \leq \psi_i E_{1-\mu}\left(\frac{M\Gamma(1-\mu)}{\lambda}(ih)^{\lambda(1-\mu)}\right),\ 0 \leq i \leq N,$$

where $E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\alpha)}$ is the Mittag-Leffler function.

We present our main results in the following theorem and leave its proof in the next section.

Theorem 1. *Suppose that the exact solution of Equation (3) $u \in C^2[0,T]$, $\kappa \in C^2[0,T]$ and $f \in C^2[0,T]$, then the scheme defined in (5) has a unique solution U_i satisfying*

$$\|u_i - U_i\|_{L^{\infty}} \leq C^* h^2, i = 1, 2, \ldots, N, \tag{8}$$

where C^* is a positive constant independent of h.

Remark 1. *Since that $u(s_n) = y(s_n^{\frac{1}{\alpha}})$, the convergence results also can be imposed as*

$$\|y_i - y(s_i^{\frac{1}{\alpha}})\|_{L^{\infty}} \leq C^* h^2,\ i = 1, 2, \ldots, N,$$

where y_i is the numerical approximation to $y(s_i^{\frac{1}{\alpha}})$.

Remark 2. *The conclusion in the theorem can be extended by using collocation methods to approximate $u(r)$. In order to satisfy the effectiveness of the change thoughts, we only consider the case that the product trapezoidal rule.*

3. Convergence of the Product Trapezoidal Rule

In this section, we will focus on the proof of Theorem 1. Considering $s = s_i$ in Equation (3) yields

$$u_i = \sum_{j=0}^{i} w_{i,j}\kappa(s_i^{\frac{1}{\alpha}}, s_j^{\frac{1}{\alpha}})u_j + T_i,\ 0 \le i \le N, \tag{9}$$

where $T_0 = 0$ and for $1 \le i \le N$,

$$T_i = \sum_{j=0}^{i-1} \int_{s_j}^{s_{j+1}} \left[u(r)\kappa(s_i^{\frac{1}{\alpha}}, r^{\frac{1}{\alpha}}) - \left(\frac{r - s_{j+1}}{-h} u_j \kappa(s_i^{\frac{1}{\alpha}}, s_j^{\frac{1}{\alpha}}) + \frac{r - s_j}{h} u_{j+1}\kappa(s_i^{\frac{1}{\alpha}}, s_{j+1}^{\frac{1}{\alpha}}) \right) \right] \frac{r^{\frac{1}{\alpha}-1}}{\alpha(s_i^{\frac{1}{\alpha}} - r^{\frac{1}{\alpha}})^{1-\alpha}} dr.$$

Substituting (5) into (9) and taking $e_i := u_i - U_i$, we can obtain

$$e_i = \sum_{j=0}^{i} w_{i,j}\kappa(s_i^{\frac{1}{\alpha}}, s_j^{\frac{1}{\alpha}})e_j + T_i,\ 1 \le i \le N, \tag{10}$$

which further implies that for any $1 \le i \le N$,

$$\begin{aligned} |e_i| &\le \sum_{j=0}^{i} w_{i,j}|e_j||\kappa(s_i^{\frac{1}{\alpha}}, s_j^{\frac{1}{\alpha}})| + |T_i| \\ &\le \sum_{j=0}^{i-1} w_{i,j}|e_j||\kappa(s_i^{\frac{1}{\alpha}}, s_j^{\frac{1}{\alpha}})| + w_{i,i}|e_i||\kappa(s_i^{\frac{1}{\alpha}}, s_i^{\frac{1}{\alpha}})| + |T_i|. \end{aligned}$$

Combining inequality (6), (7) with the assumption $M_1 \max_{0\le s\le t\le T} |\kappa(t,s)|h < 1$, it yields

$$|e_i| \le C_2|T_i| + C_2 \sum_{j=1}^{i-1} \frac{M_1}{\alpha} j^{\frac{1}{\alpha}-1} h (i^{\frac{1}{\alpha}} - j^{\frac{1}{\alpha}})^{\alpha-1}|e_j|, \tag{11}$$

where C_2 is a positive constant independent on h and i.

Applying the error in Lagrange interpolation and the assumptions of the exact solution and $\kappa(t,s)$, there exists a positive constant M_2, for any $s \in [s_j, s_{j+1}]$,

$$u(s)\kappa(s_i^{\frac{1}{\alpha}}, s^{\frac{1}{\alpha}}) - \left[\frac{s_{j+1} - s}{h} u_j\kappa(s_i^{\frac{1}{\alpha}}, s_j^{\frac{1}{\alpha}}) + \frac{s - s_j}{h} u_{j+1}\kappa(s_i^{\frac{1}{\alpha}}, s_{j+1}^{\frac{1}{\alpha}}) \right] \le M_2 h^2. \tag{12}$$

Together with Equation (10), we can further obtain

$$\begin{aligned} |T_i| &\le \frac{M_1M_2h^2}{\alpha} \sum_{j=0}^{i-1} \int_{s_j}^{s_{j+1}} \frac{r^{\frac{1}{\alpha}-1}}{(s_i^{\frac{1}{\alpha}} - r^{\frac{1}{\alpha}})^{1-\alpha}} dr = \frac{M_1M_2h^2}{\alpha} \int_0^{s_i} \frac{r^{\frac{1}{\alpha}-1}}{(s_i^{\frac{1}{\alpha}} - r^{\frac{1}{\alpha}})^{1-\alpha}} dr \\ &= M_1M_2h^2 \int_0^{s_i^{\frac{1}{\alpha}}} \frac{1}{(s_i^{\frac{1}{\alpha}} - t)^{1-\alpha}} dt \\ &= M_1M_2h^2 \int_0^{s_i^{\frac{1}{\alpha}}} [s_i^{\frac{1}{\alpha}}(1 - \frac{t}{s_i^{1/\alpha}})]^{\alpha-1} dt \\ &= \frac{M_1M_2h^2 s_i^{\frac{1}{\alpha}-1}}{\alpha}. \end{aligned} \tag{13}$$

Then substituting above inequality (13) into (11), we can obtain

$$|e_i| \le C_3h^2 + C_3 \sum_{j=1}^{i-1} j^{\frac{1}{\alpha}-1} h (i^{\frac{1}{\alpha}} - j^{\frac{1}{\alpha}})^{\alpha-1}|e_j|, \tag{14}$$

where $C_3 = \max\{\frac{C_2 M_1 M_2 s_i^{\frac{1}{\alpha}-1}}{\alpha}, \frac{C_2 M_1}{\alpha}\}$.

Applying Lemma 2, Equation (14) gives that

$$|e_i| \le C^* h^2, \tag{15}$$

where $C^* = C_3 E_{1-\alpha}(C_3 \alpha \Gamma(1-\alpha)(ih))$, which completes the proof of Theorem 1.

4. Numerical Example

In this section, some examples are given to verify our theoretical results. Here we take $\kappa = -1$ in Equation (1). Introduce the following notation

$$errors(\tau) = \max_{1 \le n \le N} |ye(t_n) - y^n|, \quad err(t_N) = ye(t_N) - y^N$$

and the convergence order

$$orders := \frac{log(errors(\tau_1))/log(errors(\tau_2))}{log(\tau_1/\tau_2)},$$

where τ_1 and τ_2 mean the time steps.

Example 1. *Consider the following Volterra integral equation of the second kind with a weakly singular kernel*

$$y(t) = -\int_0^t (t-s)^{-\beta} y(s) ds + g(t), 0 \le t \le 1,$$

and $g(t)$ satisfies that the exact solution is $y = t + t^{1-\beta}, 0 < \beta < 1$.

To examine the effectiveness of our numerical methods, we take $N = 100, 200, 300, 400$ for different $\beta = 0.2,\ 0.4,\ 0.6$ and $\tau = 1/N$. Tables 1 and 2 show that the maximum errors and orders present that the convergence rate of our scheme is 2. Moreover, we consider the classical product trapezoidal rule without variable exchange to solve the second VIE with nonsmooth solution and the results found in Tables 1 and 2 give that the order is $2(1-\beta)$ which is less than 2. This finding coincides with our theoretical results. In order to make our method more useful, we also test the errors and orders at $t = 1$. The results found in Tables 2 and 3 state that the scheme without variable change observes $2-\beta$ convergence orders.

Table 1. The errors and orders with L^∞-norm.

		β = 0.2		**β = 0.4**		**β = 0.6**	
	N	**Errors**	**Orders**	**Errors**	**Orders**	**Errors**	**Orders**
Our Scheme	100	3.91×10^{-6}	$*$	6.95×10^{-6}	$*$	2.11×10^{-5}	$*$
	200	9.85×10^{-7}	1.99	1.74×10^{-6}	2.00	5.34×10^{-6}	1.97
	300	4.39×10^{-7}	1.99	7.75×10^{-7}	2.00	2.39×10^{-6}	1.98
	400	2.47×10^{-7}	1.99	4.36×10^{-7}	2.00	1.35×10^{-6}	1.98
Original Scheme	100	3.97×10^{-5}	$*$	6.20×10^{-4}	$*$	6.41×10^{-3}	$*$
	200	1.32×10^{-5}	1.59	2.76×10^{-4}	1.17	3.89×10^{-3}	0.72
	300	6.92×10^{-6}	1.59	1.71×10^{-4}	1.18	2.89×10^{-3}	0.73
	400	4.37×10^{-6}	1.59	1.22×10^{-4}	1.18	2.33×10^{-3}	0.74

* means here is no order.

Table 2. The errors and orders for $\beta = 0.8$.

	N	**Errors**		$err(t_N)$	
Our Scheme	1000	1.07×10^{-6}	$*$	1.07×10^{-6}	$*$
	2000	2.78×10^{-7}	1.95	2.78×10^{-7}	1.95
	3000	1.26×10^{-7}	1.96	1.26×10^{-7}	1.96
	4000	7.14×10^{-8}	1.96	7.14×10^{-8}	1.96
Original Scheme	1000	1.80×10^{-2}	$*$	4.40×10^{-6}	$*$
	2000	1.46×10^{-2}	0.30	1.80×10^{-6}	1.29
	3000	1.29×10^{-2}	0.31	1.07×10^{-7}	1.28
	4000	1.18×10^{-2}	0.31	7.40×10^{-6}	1.28

* means here is no order.

Table 3. The errors and orders at $t = 1$.

		β = 0.2		β = 0.4		β = 0.6	
	N	$err(t_N)$	**Orders**	$err(t_N)$	**Orders**	$err(t_N)$	**Orders**
Our Scheme	100	2.94×10^{-6}	$*$	6.95×10^{-6}	$*$	2.11×10^{-5}	$*$
	200	7.36×10^{-7}	2.00	1.74×10^{-6}	2.00	5.34×10^{-6}	1.97
	300	3.28×10^{-7}	2.00	7.75×10^{-7}	2.00	2.39×10^{-6}	1.98
	400	1.84×10^{-7}	2.00	4.36×10^{-7}	2.00	1.35×10^{-6}	1.98
Original Scheme	100	6.96×10^{-6}	$*$	1.82×10^{-5}	$*$	3.85×10^{-5}	$*$
	200	2.02×10^{-6}	1.78	5.88×10^{-6}	1.63	1.39×10^{-5}	1.47
	300	9.80×10^{-7}	1.79	3.04×10^{-6}	1.62	7.70×10^{-6}	1.46
	400	5.86×10^{-7}	1.79	1.91×10^{-6}	1.62	5.08×10^{-6}	1.45

* means here is no order.

Example 2. *We consider the problem* (1) *with the following right hand function*

$$g(t, u(t)) = u(t) + t^{3+2\alpha} B(\alpha, 4 + \alpha)$$

where $B(p, q)$ means the Beta function. The corresponding exact function is $u(t) = t^{3+\alpha}$.

Similarly, we choose $N = 100, 200, 300, 400$ for different $\alpha = 0.2, 0.4, 0.6, 0.8$. Tables 4 and 5 give the errors with maximum norm and from which we can obtain the order is of 2.

Table 4. The errors and orders of $\alpha = 0.2, 0.4$.

	α = 0.2		α = 0.4	
N	**Errors**	**Orders**	**Errors**	**Orders**
100	9.04×10^{-4}	*	2.74×10^{-4}	*
200	2.48×10^{-4}	1.86	7.07×10^{-5}	1.95
300	1.15×10^{-4}	1.89	3.19×10^{-5}	1.96
400	6.68×10^{-5}	1.90	1.81×10^{-5}	1.97

* means here is no order.

Table 5. The errors and orders of $\alpha = 0.6, 0.8$.

	α = 0.6		α = 0.8	
N	**Errors**	**Orders**	**Errors**	**Orders**
100	1.06×10^{-4}	*	4.93×10^{-5}	*
200	2.68×10^{-5}	1.99	1.24×10^{-5}	2.00
300	1.20×10^{-5}	1.99	5.50×10^{-6}	2.00
400	6.75×10^{-6}	1.99	3.92×10^{-6}	2.00

* means here is no order.

Example 3. *We consider the problem*

$$y(t) = -\int_0^t (t-s)^{-\beta} y(s)ds + g(t), 0 \leq t \leq 1,$$

with $y(0) = 1$ *and right hand function* $g(t) = 1$ *without knowing the analytical solution.*

We give the following notation for more readability.
Define:

$$Err := \max_{1 \leq n_2 \leq N} \left| y^{n_2} - y_{ref}^{n_1} \right|$$

where y_{ref} means the reference solution and $y_{ref}^{n_1}$ approximates to $ye(t_{n1})$, $n_1 = \frac{N_0}{N} n_2$, $n_2 = 1, 2, \ldots, N$.

We use proposed methods to solve the initial problem 3 and the reference solutions are computed with $N_0 = 2000$. The optimal error estimate is observed by taking $N = 50, 100, 200, 400$. Tables 6 and 7 declare that the convergence order is 2 for different $\beta = 0.2, 0.4, 0.6, 0.8$, which verifies the efficiency of our scheme.

Table 6. The errors and orders of $\beta = 0.2, 0.4$.

	β = 0.2		β = 0.4	
N	**Err**	**Orders**	**Err**	**Orders**
50	1.68×10^{-5}	*	2.56×10^{-5}	*
100	4.20×10^{-6}	2.00	6.42×10^{-6}	1.99
200	1.04×10^{-6}	2.01	1.60×10^{-6}	2.00
400	2.53×10^{-7}	2.04	3.90×10^{-7}	2.04

* means here is no order.

Table 7. The errors and orders of $\beta = 0.6, 0.8$.

	β = 0.6		β = 0.8	
N	**Err**	**Orders**	**Err**	**Orders**
50	4.42×10^{-5}	*	8.67×10^{-5}	*
100	1.14×10^{-5}	1.95	2.44×10^{-5}	1.83
200	2.92×10^{-6}	1.97	6.69×10^{-6}	1.88
400	7.22×10^{-7}	2.02	1.76×10^{-6}	1.93

* means here is no order.

5. Conclusions

This paper mainly presents the scheme produced by applying the trapezoidal rule to the changed equation for any $0 < \beta < 1$, which can be used in more general occasions. More rigorous analysis can be obtained with the help of the discrete Grönwall equality and a typical example is tested to verify our theoretical result.

Author Contributions: Conceptualization, X.C.; methodology, B.Z.; software, B.Z.; validation, X.C.; formal analysis, B.Z.; investigation, X.C.; writing—original draft preparation, B.Z.; writing—review and editing, X.C. and B.Z.; funding acquisition, X.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Liang, H.; Brunner H. The fine error estimation of collocation methods on uniform meshes for weakly singular Volterra integral equations. *J. Sci. Comput.* **2020**, *84*, 12.
2. Brunner, H.; Nørsett, S. Superconvergence of collocation methods for Volterra and Abel integral equations of the second kind. *Numer. Math.* **1981**, *36*, 347–358. [CrossRef]
3. Gripenberg, G.; Londen, S.; Staffans, O. *Volterra Integral and Functional Equations*; Cambridge University Press: Cambridge, MA, USA, 2009. [CrossRef]
4. Chen, X.; Di, Y.; Duan, J.; Li, D. Linearized compact ADI schemes for nonlinear time-fractional Schrödinger equations. *Appl. Math. Lett.* **2018**, *84*, 160–167.
5. Lima, P.; Diogo, T. An extrapolation method for a Volterra integral equation with weakly singular kernel. *Appl. Numer. Math.* **1997**, *24*, 131–148. [CrossRef]
6. Lubich, C. Fractional linear multistep methods for Abel-Volterra integral equations of the first kind. *Math. Comput.* **1987**, *45*, 463–469. [CrossRef]
7. Kocirik, M.; Tschirch, G.; Struve, P. Application of the volterra integral equation to the mathematical modelling of adsorption kinetics under constant-volume/variable-concentration conditions. *J. Chem. Soc. Faraday Trans.* **1988**, *84*, 2247–2257. [CrossRef]
8. Brunner, H. High-order collocation methods for singular Volterra functional equations of neutral type. *Appl. Numer. Math.* **2007**, *57*, 533–548. [CrossRef]
9. Zaky, M.A.; Ameen, I.G. A novel Jacob spectral method for multi-dimensional weakly singular nonlinear Volterra integral equations with nonsmooth solutions. *Eng. Comput.* **2021**, *37*, 2623–2631. [CrossRef]
10. Huang, C.; Tang, T.; Zhang, Z. Supergeometric convergence of spectral collocation methods for weakly singular Volterra and fredholm integral equations with smooth solutions. *J. Comput. Math.* **2011**, *29*, 698–719. [CrossRef]
11. Diogo, T.; Lima, P. Superconvergence of collocation methods for a class of weakly singular Volterra integral equations. *J. Comput. Appl. Math.* **2008**, *218*, 307–316. [CrossRef]
12. Mahmound, A.Z. An accurate spectral collocation method for nonlinear systems of fractional differential equations and related integral equations with nonsmooth solutions. *Appl. Numer. Math.* **2020**, *154*, 205–222. [CrossRef]
13. Hoog, F.R.; Anderssen, R.S. Kernel perturbations for a class of second-kind convolution Volterra equations with non-negative kernels. *Appl. Math. Lett.* **2012**, *25*, 1222–1225.
14. Ma, J.; Brunner, H. A posteriori error estimates of discontinuous Galerkin methods for non-standard Volterra integro-differential equations. *IMA J. Numer. Anal.* **2006**, *26*, 78–95. [CrossRef]
15. Chen, Y.; Tang, T. Spectral methods for weakly singular Volterra integral equations with smooth solutions. *J. Comput. Appl. Math.* **2009**, *233*, 938–950. [CrossRef]
16. Li, D.; Sun, W.; Wu, C. A novel numerical approach to time-fractional parabolic equations with nonsmooth solutions. *Numer. Math. Theor. Meth. Appl.* **2021**, *14*, 355–376. [CrossRef]
17. Li, L.; Li, D. Exact solutions and numerical study of time fractional Burgers' equations. *Appl. Math. Lett.* **2020**, *100*, 106011.
18. She, M.; Li, D.; Sun, H. A transformed L1 method for solving the multi-term time-fractional diffusion problem. *Math. Comput. Simu.* **2022**, *193*, 584–606. [CrossRef]
19. Gu X.; Wu, S.-L. A parallel-in-time iterative algorithm for Volterra partial integro-differential problems with weakly singular kernel. *J. Comput. Phys.* **2020**, *417*, 109576. [CrossRef]
20. Brunner, H. The numerical solution of weakly singular Volterra integral equations by collocation on graded meshes. *Math. Comput.* **1985**, *45*, 417–437. [CrossRef]
21. Liang, H.; Brunner, H. On the convergence of collocation solutions in continuous piecewise polynomial spaces for Volterra integral equations. *BIT* **2016**, *56*, 1339–1367. [CrossRef]
22. Brunner, H. *Collocation Methods for Volterra Integral and Related Functional Differential Equations*; Cambridge University Press: Cambridge, MA, USA, 2004. [CrossRef]
23. Li, D.; Wu, C.; Zhang, Z. Linearized Galerkin FEMs for nonlinear time fractional parabolic problems with non-smooth solutions in time direction. *J. Sci. Comput.* **2019**, *80*, 403–419.
24. Li, D.; Liao, H.L.; Sun, W.; Wang, J.; Zhang, J. Analysis of L1-Galerkin FEMs for time-fractional nonlinear parabolic problems. *Commun. Comput. Phys.* **2018**, *24*, 86–103. [CrossRef]
25. Wu, Q. On graded meshes for weakly singular Volterra integral equations with oscillatory trigonometric kernels. *J. Comput. Appl. Math.* **2014**, *263*, 370–376. [CrossRef]
26. Diogo, T.; Mckee, S.; Tang, T. Collocation methods for second-kind Volterra integral equations with weakly singular kernels. *Proc. R. Soc. Edinb. Sect. Math.* **1994**, *124*, 199–210. [CrossRef]
27. Zhao, Y.L.; Gu, X.M.; Ostermann, A. A preconditioning technique for an all-at-once system from Volterra subdiffusion equations with graded time steps. *J. Sci. Comput.* **2021**, *88*, 11. [CrossRef]
28. Zhou, B.; Chen, X.; Li, D. Nonuniform Alikhanov Linearized Galerkin Finite Element Methods for Nonlinear Time-Fractional Parabolic Equations. *J. Sci. Comput.* **2020**, *85*, 39. [CrossRef]
29. Yang, C.; Hou, J. Numerical method for solving Volterra integral equations with a convolution kernel. *Int. J. Appl. Math.* **2013**, *43*, 185–189. [CrossRef]

30. Basseem, M.; Alnaja, K.M. A mixed numerical method for solving Volterra-Fredholm integral equation. *Int. J. Numer. Appl.* **2012**, *7*, 35–40.
31. Li, M.; Huang, C.; Ming, W. Barycentric rational collocation methods for Volterra integral equations with weakly singular kernels. *Comput. Appl. Math.* **2019**, *38*, 120.
32. Katani, R.; McKee, S. Numerical solution of two-dimensional weakly singular Volterra integral equations with non-smooth solutions. *J. Comput. Appl. Math.* **2022**, *402*, 113779. [CrossRef]
33. Yuan, W.; Li, D.; Zhang, C. Linearized transformed L1 Galerkin FEMs with unconditional convergence for nonlinear time fractional Schrödinger equations. *Numer. Math. Theory Methods Appl.* **2023**, *16*, 348–369. [CrossRef]
34. Dixon, J.; Mckee, S. Weakly singular discrete Grönwall inequalities. *Z. Angew. Mech.* **1986**, *66*, 535–544. [CrossRef]
35. Dixon, J.A. On the order of the error in discretization methods for weakly singular second kind Volterra equations with non-smooth solutions. *BIT* **1985**, *25*, 624–634. [CrossRef]