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Core-EP Monotonicity Characterizations for Property-*n* Matrices

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Abstract: A square matrix is said to have property n if there exists a positive integer w such that A^w is nonnegative. In this paper, we study the core-EP monotonicity for property-n matrices. Some necessary and sufficient conditions for a property-n matrix to be core-EP monotone are given. Moreover, a necessary and sufficient condition for a real square matrix to have a nonnegative core-EP inverse is also presented.

Keywords: property-n matrix; core-EP inverse; core-EP monotonicity; core monotonicity

MSC: 15A09

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1. Introduction

Throughout this paper, for a matrix A, $A \geq 0$ means that A is a nonnegative matrix; that is, each entry of A is a nonnegative real number. We use $\mathbb{C}^{m\times n}$ and $\mathbb{R}^{m\times n}$ to denote the set of all $m \times n$ complex matrices and $m \times n$ real matrices, respectively. In particular, $\mathbb{C}^n = \mathbb{C}^{n\times 1}$, $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. The symbols A^*, A^T, and $\mathrm{rk}(A)$, respectively, denote the conjugate transpose, transpose, and rank of a matrix A. For a matrix A, $\mathcal{R}(A)$ is the range of A and $\mathcal{N}(A)$ is the null space of A. The index of a matrix $A \in \mathbb{C}^{n\times n}$ is defined as the smallest nonnegative integer such that $\mathrm{rk}(A^k) = \mathrm{rk}(A^{k+1})$, and is denoted by $\mathrm{ind}(A)$. Moreover, I_n will refer to the $n \times n$ identity matrix.

We recall definitions of some generalized inverses. The Moore–Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$ [1], is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following Penrose equations

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

The Drazin inverse of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $X = A^D \in \mathbb{C}^{n\times n}$ [1] satisfying the relations

$$A^k XA = A^k, \;\; XAX = X, \;\; AX = XA,$$

where $k = \mathrm{ind}(A)$.

Baksalary and Trenkler [2] introduced the core inverse. For $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = 1$, the core inverse of A is defined to be the unique matrix $X = A^{\#\!\!\!\bigcirc}$ such that

$$AX = P_A, \;\; \mathcal{R}(X) \subseteq \mathcal{R}(A),$$

where P_A is the orthogonal projection onto $\mathcal{R}(A)$, i.e., $P_A = AA^\dagger$.

The core-EP inverse was proposed by Manjunatha Prasad and Mohana [3] for a square matrix of an arbitrary index, as an extension of the core inverse restricted to a square matrix of an index one. A matrix $X \in \mathbb{C}^{n\times n}$, denoted as $A^{\dagger\!\!\!\bigcirc}$, is called the core-EP inverse of A if it satisfies

$$XAX = X, \;\; \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k),$$

where $k = \mathrm{ind}(A)$. For more details of the core-EP inverse, the reader is referred to [4–7] and references therein.

A real square matrix A is called *monotone* if $Ax \geq 0$ implies $x \geq 0$. Collatz [8] treated square matrices of a monotone kind and showed that A is monotone if and only if A is nonsingular and $A^{-1} \geq 0$. The notion of the monotone matrix has been generalized in many different ways. Mangasarian [9] generalized Collatz's results to rectangular matrices and proved that the monotonicity of a rectangular matrix A is equivalent to the existence of a nonnegative left inverse of A. More generalizations of the notion of monotonicity can be found in [10]. Motivated by Collatz's results, characterizing those matrices (especially nonnegative matrices) which have a nonnegative generalized inverse has been a topic of interest in the past few decades. For example, Plemmons and Cline [11] gave some necessary and sufficient conditions for a nonnegative matrix to have a nonnegative Moore–Penrose inverse. Berman and Plemmons [12] characterized nonnegative matrices with nonnegative group inverse in terms of their nonnegative rank factorizations. Nonnegative matrices which have a nonnegative Drazin inverse were characterized in [13,14]. The motivation of the study of such matrices has its origin in the question of finding nonnegative least-squares solutions of linear systems [15]. Furthermore, numerous examples of applications of nonnegative generalized inverses that include numerical analysis and linear economic models can be found in the book by Berman and Plemmons [16].

Werner [17,18] studied the Drazin mononontonicity for a class of special matrices, which are called property-n matrices. The definition of property-n matrices is as follows.

Definition 1 ([17])**.** *A square matrix A is said to have property n if for some positive integer w, the w-th power of A is nonnegative. Such matrices are also called property-n matrices. We write $n(w)$ whenever A^w is nonnegative for the positive integer w.*

It is clear that nilpotent matrices and nonnegative matrices are property-n matrices.

Definition 2. *A square matrix A is called core-EP monotone if $A^{\textcircled{\dagger}} \geq 0$. In particular, A is called core-monotone if $A^{\textcircled{\#}}$ exists and $A^{\textcircled{\#}} \geq 0$.*

Definition 3 ([14])**.** *A nonnegative matrix A is called monomial if A has exactly one positive entry in each row and each column.*

It is well known that a nonnegative matrix A is monomial if and only if A is nonsingular and $A^{-1} \geq 0$.

As we introduced above, the nonnegativity characterizations of some classical generalized inverses (such as the Moore–Penrose inverse, the group inverse, and the Drazin inverse) received extensive research, and full characterizations for the nonnegativity of these generalized inverses were provided. It is natural to study the characterizations of classes of matrices with nonnegative generalized inverses introduced in recent years, such as the core inverse and the core-EP inverse. In this paper, we study the core-EP monotonicity for property-n matrices. The main contribution of this paper is providing some necessary and sufficient conditions for a property-n matrix to have a nonnegative core-EP inverse. In particular, we find that a property-n matrix A is core-EP monotone if and only if A has a matrix form coinciding with a special case of the well-known "core-EP Decomposition".

Lemma 1 ([19])**.** *The core-EP inverse of $A \in \mathbb{C}^{n\times n}$ is the unique solution to the system*

$$XA^{k+1} = A^k, \;\; AX^2 = X, \;\; (AX)^* = AX.$$

Lemma 2 ((Core-EP decomposition) [6])**.** *Let $A \in \mathbb{C}^{n\times n}$ be such that $ind(A) = k$. Then A can be written as the sum of matrices A_1 and A_2, i.e., $A = A_1 + A_2$, where*

(i) $rk(A_1) = rk(A_1^2)$;

(ii) $A_2^k = 0$;
(iii) $A_1^* A_2 = A_2 A_1 = 0$.

Lemma 3 ([6]). *Let the core-EP decomposition of A be as in Lemma 2. Then there exists a unitary matrix U such that*

$$A = U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*, \tag{1}$$

where T is nonsingular and N is nilpotent. Moreover, the core-EP inverse of A can be represented by

$$A^{\textcircled{\dagger}} = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{2}$$

Lemma 4 ([11]). *If E is a symmetric nonnegative idempotent matrix, then there exists a permutation matrix P such that*

$$PEP^T = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}, \tag{3}$$

where

$$J = \begin{bmatrix} x_1 x_1^T & & 0 \\ & \ddots & \\ 0 & & x_r x_r^T \end{bmatrix},$$

each $x_i (1 \le i \le r)$ is a positive unit vector, and $r = rk(E)$.

Lemma 5 ([11]). *Let A be an $m \times n$ nonnegative matrix of rank r. Then the following statements are equivalent.*

(i) $A^\dagger$ *is nonnegative.*
(ii) *There exists a permutation matrix P such that PA has the form*

$$PA = \begin{bmatrix} B_1 \\ \vdots \\ B_r \\ 0 \end{bmatrix},$$

where each B_i has rank 1 and where the rows of B_i are orthogonal to the rows of B_j whenever $i \neq j$.
(iii) $A^\dagger = DA^T$ *for some diagonal matrix D with positive diagonal elements.*

2. Drazin Monotonicity versus Core-EP Monotonicity

In this section, we investigate the relationship between the Drazin monotonicity and the core-EP monotonicity for property-n matrices. We show that for a property-n matrix, the core-EP monotonicity implies the Drazin monotonicity. However, the class of property-n matrices with Drazin monotonicity is not the same as the class of property-n matrices with core-EP-monotonicity. Hence, it is also of interest to characterize those property n-matrices which are core-EP monotone.

Lemma 6. *Let A be core-EP monotone and $ind(A) = k$. Then A has property n if and only if A has property $n(w)$ for each integer $w \ge k$.*

Proof. The sufficiency is clear and we only prove the necessity. Since A has property n, there exists a positive integer m such that $A^m \ge 0$. By the Archimedean principle, let t be the minimal positive integer such that $mt \geq k$. Then $A^{mt} = (A^m)^t \geq 0$. We first show that A has property $n(w)$ for each positive integer w in the interval $[k, mt]$. If $mt = k$, then it is clear that $A^k \geq 0$. Otherwise, if $mt \geq k+1$, then it follows from Lemma 1 that $A^{\textcircled{\dagger}} A^{k+1} = A^k$, which gives $A^{mt-1} = A^{\textcircled{\dagger}} A^{mt} \geq 0$. If $mt - 1 = k$, then the result follows. If $mt - 1 \geq k+1$, then $A^{mt-2} = A^{\textcircled{\dagger}} A^{mt-1} \geq 0$. Continuing in this way, we conclude that A has property $n(w)$ for each integer $w \in [k, mt]$. Similarly, we can show that A has property $n(w)$ for each positive integer $w \in [lm, (l+1)m]$, where $l \geq t$ is an arbitrary positive integer. This completes the proof. □

Lemma 7. *Let A be a square matrix such that $ind(A) = k$. Then for any positive integer $s \geq k$,*

(i) $(A^{\textcircled{\dagger}})^s = (A^s)^{\#}$.
(ii) $A^{\textcircled{\dagger}} = A^s (A^{s+1})^{\#}$.
(iii) $A^D = (A^{\textcircled{\dagger}})^{k+1} A^k$.

Proof. (i): For any positive integer $s \geq k$, it can be seen from $\mathrm{rk}(A^k) = \mathrm{rk}(A^s) = \mathrm{rk}(A^{s+1}) = \cdots = \mathrm{rk}(A^{2s})$ that $(A^s)^{\#}$ exists. Notice that A has the form in (1), then for any positive integer $s \geq k$,

$$A^s = U \begin{bmatrix} T^s & X \\ 0 & 0 \end{bmatrix} U^*,$$

where X is a corresponding matrix.

Hence,

$$(A^s)^{\#} = U \begin{bmatrix} T^{-s} & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Now, it is clear from (2) in Lemma 3 that $(A^{\textcircled{\dagger}})^s = (A^s)^{\#}$.

(ii) For any positive integer $s \geq k$,

$$A^s (A^{s+1})^{\#} = U \begin{bmatrix} T^s & X \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} T^{-s-1} & 0 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* = A^{\textcircled{\dagger}}.$$

(iii) It follows directly from [19,20]. □

Theorem 1. *Let A be a square matrix such that $ind(A) = k$. Suppose that A has property n. Then A is core-EP monotone if and only if $A^D \geq 0$ and $A^k (A^k)^\dagger \geq 0$.*

Proof. It follows from [5,21,22] that $A^{\textcircled{\dagger}} = A^D A^k (A^k)^\dagger$. If $A^D \geq 0$ and $A^k (A^k)^\dagger \geq 0$, then it is obvious that $A^{\textcircled{\dagger}} \geq 0$. Conversely, if A is core-EP monotone and A has property n, then by Lemma 6 we have $A^k \geq 0$. Hence, it can be observed from [6,20] and Lemma 7 (i) that $A^k (A^k)^\dagger = A^k (A^k)^{\#} = A^k (A^{\textcircled{\dagger}})^k \geq 0$. Moreover, we can see from Lemma 7 (iii) that $A^D = (A^{\textcircled{\dagger}})^{k+1} A^k \geq 0$. □

It should be noticed that for a square matrix A which has property n, $A^D \geq 0$ does not imply $A^{\textcircled{\dagger}} \geq 0$, i.e., the class of property-n matrix A with $A^D \geq 0$ is not the same as the class of property-n matrix A with $A^{\textcircled{\dagger}} \geq 0$. We will give an example to show this.

Example 1. *Let*

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 4 & 0 & 0 \\ 3 & 4 & 9 & 0 \end{bmatrix}$$

be a nonnegative matrix. Then A has property n.

Since $rk(A) = rk(A^2) = 3$, then $ind(A) = 1$. In this case, the Drazin inverse of A coincides with the group inverse of A. A direct computation shows that

$$A^D = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{6} & 0 & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{3}{8} & 0 \end{bmatrix} \geq 0.$$

However,

$$A^{\textcircled{\dagger}} = \begin{bmatrix} \frac{17}{63} & -\frac{2}{21} & -\frac{4}{63} & \frac{4}{63} \\ -\frac{1}{21} & -\frac{1}{14} & \frac{17}{84} & \frac{1}{21} \\ -\frac{1}{21} & \frac{2}{21} & -\frac{1}{21} & \frac{1}{21} \\ \frac{19}{126} & -\frac{3}{28} & \frac{97}{504} & \frac{23}{126} \end{bmatrix}$$

is not nonnegative.

3. Core-EP Monotonicity Characterizations for Property-n Matrices

In this section, we study the class of property-n matrices which have a nonnegative core-EP inverse. Some necessary and sufficient conditions for such matrices to have a nonnegative core-EP inverse are presented.

We first give a core-monotonicity characterization for nonnegative square matrices in terms of nonnegative full-rank decomposition. Recall that a matrix $X \in \mathbb{C}^{n\times m}$ is said to be a $\{1\}$-inverse of $A \in \mathbb{C}^{m\times n}$ if $AXA = A$ (see [1]).

Lemma 8. *Let A be a nonnegative square matrix with $ind(A) = 1$. Then A is core-monotone if and only if A has a nonnegative full-rank factorization $A = MN$, where NM is monomial and $M^\dagger \geq 0$.*

Proof. It is known that a nonnegative matrix having a nonnegative $\{1\}$-inverse always possesses a nonnegative full-rank factorization, and in every such factorization $A = MN$, M has a nonnegative left inverse, and N has a nonnegative right inverse (see [12]). Since the core inverse of A is also a $\{1\}$-inverse of A, if $A^{\textcircled{\#}} \geq 0$, then A has a nonnegative full-rank factorization $A = MN$. In this case, NM and M^*M are nonsingular and $A^{\textcircled{\#}} = M(NM)^{-1}(M^*M)^{-1}M^*$ (see [23]). Since M has a nonnegative left inverse M_L^{-1}, then $(NM)^{-1} = M_L^{-1}[M(NM)^{-1}(M^*M)^{-1}M^*]M = M_L^{-1}A^{\textcircled{\#}}M \geq 0$, i.e., NM is monomial. Moreover, $M^\dagger = (M^*M)^{-1}M^* = NMM_L^{-1}A^{\textcircled{\#}} \geq 0$. On the other hand, if A has a nonnegative full-rank factorization $A = MN$, where NM is monomial and $M^\dagger \geq 0$, then it can be seen from $A^{\textcircled{\#}} = M(NM)^{-1}(M^*M)^{-1}M^*$ that A is core-monotone. □

Theorem 2. *Let A be an $n \times n$ matrix with $ind(A) = k$ and $rk(A^k) = r$, and suppose that A has property n. Then A is core-EP monotone if and only if there exists a unitary matrix $U = [U_1 \;\; U_2]$ of order n such that*

$$A = U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* = U_1TU_1^* + U_1SU_2^* + U_2NU_2^*, \tag{4}$$

where T is monomial of order r, N is nilpotent of order $n - r$, U_1 is a nonnegative $n \times r$ matrix.

Proof. Suppose that A has the form in (4), then

$$A^{\textcircled{\dagger}} = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* = U_1T^{-1}U_1^* \geq 0.$$

Conversely, suppose that A has property n, then there exists a positive integer w such that $A^w \geq 0$. By the Archimedean principle, there exists a natural number t such that $tw \geq k$, where $k = \mathrm{ind}(A)$. Denote $m = tw$, if A is core-EP-monotone, then by Lemma 7 (i), $(A^m)^{\textcircled{\dagger}} = (A^{\textcircled{\dagger}})^m \geq 0$. Since $A^m = (A^w)^t$ is also nonnegative, then by Lemma 8, A^m has a nonnegative full-rank factorization $A^m = FG$, where GF is monomial and $F^\dagger \geq 0$.

Since F and $F^\dagger$ are nonnegative matrices, then $FF^\dagger$ is a symmetric nonnegative idempotent matrix. Noting that $\mathrm{rk}(FF^\dagger) = \mathrm{rk}(F) = \mathrm{rk}(A^m) = \mathrm{rk}(A^k) = r$. Then by Lemma 4, $FF^\dagger$ has the matrix form in (3), i.e., there exists a permutation matrix P such that

$$PFF^\dagger P^T = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}, \tag{5}$$

where

$$J = \begin{bmatrix} x_1 x_1^T & & 0 \\ & \ddots & \\ 0 & & x_r x_r^T \end{bmatrix},$$

each $x_i (1 \leq i \leq r)$ is a positive unit vector.

Moreover, since F and $F^\dagger$ are nonnegative matrices, it follows from Lemma 5 that PF has the form

$$PF = \begin{bmatrix} B_1 \\ \vdots \\ B_r \\ 0 \end{bmatrix},$$

where each B_i has rank 1 and where the rows of B_i are orthogonal to the rows of B_j whenever $i \neq j$, and the permutation matrix P is the same as that in (5).

We can observe from $(PFF^\dagger P^T)(PF) = PF$ that

$$\begin{bmatrix} x_1 x_1^T & & 0 \\ & \ddots & \\ 0 & & x_r x_r^T \end{bmatrix} \begin{bmatrix} B_1 \\ \vdots \\ B_r \end{bmatrix} = \begin{bmatrix} B_1 \\ \vdots \\ B_r \end{bmatrix}.$$

Hence, $x_i x_i^T B_i = B_i$ for any $1 \leq i \leq r$. Since each B_i has rank 1, then $B_i = u_i v_i^T$ for some nonzero vectors u_i and v_i. Substituting $B_i = u_i v_i^T$ into $x_i x_i^T B_i = B_i$ we obtain that $B_i = x_i x_i^T u_i v_i^T = x_i y_i^T$, where $y_i^T = x_i^T u_i v_i^T \geq 0$.

Now, we can rewrite PF as

$$PF = \begin{bmatrix} B_1 \\ \vdots \\ B_r \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 y_1^T \\ \vdots \\ x_r y_r^T \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_r \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_r^T \end{bmatrix} = XY,$$

where

$$X = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_r \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_r^T \end{bmatrix}.$$

Then X and Y are $n \times r$ and $r \times r$ nonnegative matrices, respectively. It is clear that $\mathrm{rk}(X) = r$. Moreover, since $\mathrm{rk}(PF) = \mathrm{rk}(F) = r$, then $\mathrm{rk}(Y) \geq r$. On the other hand, Y is of order r, so the rank of Y is r, i.e., Y is nonsingular.

Furthermore, since each x_i is a unit vector, then

$$(P^T X)^T (P^T X) = X^T X = \begin{bmatrix} x_1^T x_1 & & 0 \\ & \ddots & \\ 0 & & x_r^T x_r \end{bmatrix} = I_r.$$

Hence, the columns of $P^T X$ are mutually orthogonal unit vectors.

Let $U_1 = P^T X$. Notice that $dim\mathcal{N}[(A^k)^*] = dim\mathcal{N}(A^k) = n - r$, let U_2 be any $n \times (n-r)$ matrix whose columns are an orthonormal basis of $\mathcal{N}[(A^k)^*]$. Since $A^m = FG = P^T XYG$, or equivalently, $P^T X = A^m G_R^{-1} Y^{-1}$, then $\mathcal{R}(A^k) = \mathcal{R}(A^m) = \mathcal{R}(P^T X) = \mathcal{R}(U_1)$. Note that $\mathcal{R}(A^k)$ is the orthogonal complement of $\mathcal{N}[(A^k)^*]$. Then $U = [U_1 \vdots\, U_2]$ is a unitary matrix. Also, as $A\mathcal{R}(A^k) = \mathcal{R}(A^k)$, there exist some matrices T, S and N such that

$$A[U_1 \;\; U_2] = [U_1 \;\; U_2] \begin{bmatrix} T & S \\ 0 & N \end{bmatrix}, \tag{6}$$

where T is of order r, N is of order $n - r$, and T, S, and N are uniquely determined by $AU_1 = U_1 T$ and $AU_2 = U_1 S + U_2 N$.

Next, we show that N is nilpotent and T is monomial. First, we rewrite the equality (6) as

$$\begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} A = \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix}. \tag{7}$$

We can see from (7) that $U_2^* A = N U_2^*$, which implies that $U_2^* A^k = N^k U_2^*$. Since the columns of U_2 are a basis of $\mathcal{N}[(A^k)^*]$, then $(A^k)^* U_2 = 0$. Therefore, $U_2^* A^k = N^k U_2^* = 0$, which yields $N^k = N^k U_2^* U_2 = 0$, i.e., N is nilpotent.

It remains to show that T is monomial. It can be observed from (6) or (7) that

$$\begin{bmatrix} T & S \\ 0 & N \end{bmatrix} = \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} A[U_1 \;\; U_2].$$

Hence,

$$\begin{bmatrix} T^m & \widetilde{T} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T & S \\ 0 & N \end{bmatrix}^m = \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} A^m [U_1 \;\; U_2] = \begin{bmatrix} U_1^* A^m U_1 & U_1^* A^m U_2 \\ U_2^* A^m U_1 & U_2^* A^m U_2 \end{bmatrix}, \tag{8}$$

where $\widetilde{T}$ is a corresponding matrix.

It follows from (8) that $T^m = U_1^* A^m U_1 = U_1^* FGU_1 = U_1^* U_1 YGU_1 = YGU_1$. For any $x \in \mathcal{N}(YGU_1)$, $YGU_1 x = 0$. Since Y is nonsingular, then $GU_1 x = 0$, i.e., $U_1 x \in \mathcal{N}(G)$. It can be seen from $A^m = FG$ that $\mathcal{N}(G) = \mathcal{N}(A^m) = \mathcal{N}(A^k)$. Thus, $U_1 x \in \mathcal{N}(A^k)$. On the other hand, it is clear that $U_1 x \in \mathcal{R}(U_1) = \mathcal{R}(A^k)$. Hence, $U_1 x \in \mathcal{R}(A^k) \bigcap \mathcal{N}(A^k) = \{0\}$, i.e., $U_1 x = 0$. Therefore, $x = U_1^* U_1 x = 0$, which implies that T^m is nonsingular, and then T is also nonsingular.

By checking the definition of the core-EP inverse it can be further seen that

$$A^{\oplus} = [U_1 \;\; U_2] \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix}. \tag{9}$$

Then

$$A^m (A^{\oplus})^{m-1} = [U_1 \;\; U_2] \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} = U_1 T U_1^*.$$

Hence, $T = U_1^* A^m (A^{\textcircled{\dagger}})^{m-1} U_1 \geq 0$.

Moreover, it can be deduced from (9) that $A^{\textcircled{\dagger}} = U_1 T^{-1} U_1^*$. Consequently, $T^{-1} = U_1^* A^{\textcircled{\dagger}} U_1 \geq 0$, which means that T is monomial. $\square$

We remark that the matrix form of A in (4) coincides with the matrix form of A in (1). However, for a matrix A having property n, we have shown in Theorem 2 that in order to make such matrix A to be core-EP monotone, the unitary matrix U and the nonsingular matrix T given in (1) should be more special, i.e., U_1 is nonnegative and T is monomial. We illustrate this by the following two examples.

Example 2. *Let*

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 3 & 0 & \frac{12}{5} & \frac{9}{5} \\ 0 & 0 & \frac{24}{25} & \frac{18}{25} \\ 0 & 0 & -\frac{32}{25} & -\frac{24}{25} \end{bmatrix}.$$

Then $rk(A) = 3$, $rk(A^2) = rk(A^3) = 2$, i.e., $ind(A) = 2$. It can be seen from

$$A^2 = \begin{bmatrix} 6 & 0 & \frac{24}{5} & \frac{18}{5} \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0$$

that A has property n.

Moreover, A has the following decomposition

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & 0 & -\frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & -3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & 0 & -\frac{4}{5} & -\frac{3}{5} \end{bmatrix}.$$

Then A satisfies the conditions in Theorem 2. Hence, A has a nonnegative core-EP inverse

$$A^{\textcircled{\dagger}} = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 3. *Let*

$$A = \begin{bmatrix} 0 & 3 & 0 & \frac{1}{3} & 1 & \frac{5}{3} \\ 2 & 0 & 0 & \frac{13}{9} & \frac{1}{3} & \frac{5}{9} \\ 0 & 0 & 1 & \frac{5}{3} & \frac{5}{3} & \frac{11}{3} \\ 0 & 0 & 0 & -\frac{2}{9} & -\frac{8}{9} & -\frac{20}{9} \\ 0 & 0 & 0 & -\frac{8}{9} & \frac{4}{9} & -\frac{8}{9} \\ 0 & 0 & 0 & \frac{7}{9} & -\frac{8}{9} & -\frac{2}{9} \end{bmatrix}.$$

Then it is easy to see that $rk(A^2) = 4$, $rk(A^3) = rk(A^4) = 3$, i.e., $ind(A) = 3$. Observe that

$$A^3 = \begin{bmatrix} 0 & 18 & 0 & 1 & 2 & 0 \\ 12 & 0 & 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & \frac{4}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

Hence, A has property n.
Since A has the decomposition

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 & 3 & 0 & \frac{1}{3} & 1 & -\frac{5}{3} \\ 2 & 0 & 0 & 1 & \frac{11}{9} & -\frac{1}{9} \\ 0 & 0 & 1 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \end{bmatrix},$$

then by Theorem 2, the core-EP inverse of A is nonnegative. Indeed,

$$A^{\textcircled{\dagger}} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \geq 0.$$

We give some characterizations for core-EP monotonicity in terms of core monotonicity in the following.

Theorem 3. *Let A be a square matrix with ind$(A) = k$. Suppose that A has property $n(w)$ for some integer $w \geq k$. Then A is core-EP monotone if and only if A^{w+1} is core monotone.*

Proof. If A is core-EP monotone, then by Lemma 7 (i), $(A^{w+1})^{\textcircled{\#}} = (A^{\textcircled{\dagger}})^{w+1} \geq 0$. Hence, A^{w+1} is core monotone. On the other hand, If A has property $n(w)$ for some integer $w \geq k$ and A^{w+1} is core monotone, then it follows directly from Lemma 7 (ii) that A is core-EP monotone. □

Corollary 1. *Let $A \in \mathbb{R}^{n \times n}$ be nonnegative and ind$(A) = k$. Then A is core-EP monotone if and only if A^{k+1} is core-monotone.*

Theorem 4. *Let A be a square matrix with ind$(A) = k$, and suppose that A has property n. Then A is core-EP monotone if and only if A has property $n(k)$ and A^{k+1} is core-monotone.*

Proof. If A has property $n(k)$ and A^{k+1} is core-monotone, then by Lemma 7 (ii), $A^{\textcircled{\dagger}} = A^k (A^{k+1})^{\textcircled{\#}} \geq 0$. Conversely, if $A^{\textcircled{\dagger}} \geq 0$, then by Lemma 7 (i), $(A^{k+1})^{\textcircled{\#}} = (A^{\textcircled{\dagger}})^{k+1} \geq 0$. Moreover, if A has property n and A is core-EP monotone, then it follows from Lemma 6 that $A^k \geq 0$. □

Finally, we give a necessary and sufficient condition for a real square matrix to have a nonnegative core-EP inverse, which is a generalization of the result of Collatz [8].

Theorem 5. *Let $A \in \mathbb{R}^{n \times n}$ be such that ind$(A) = k$. Then A is core-EP monotone if and only if*

$$Ax \in \mathbb{R}^n_+ + \mathcal{N}[(A^k)^T] \text{ and } x \in \mathcal{R}(A^k) \Rightarrow x \geq 0, \tag{10}$$

where $\mathbb{R}^n_+$ is the set of all n-dimensional nonnegative vectors.

Proof. Assume that $A^{\textcircled{\dagger}} \geq 0$, we will show that (10) holds. Let $Ax = u + v$, where $u \geq 0$, $v \in \mathcal{N}[(A^k)^T]$ and $x \in \mathcal{R}(A^k)$. Since $\mathcal{N}(A^{\textcircled{\dagger}}) = \mathcal{N}[(A^k)^T]$ (see [7], Theorem 4.3), then $A^{\textcircled{\dagger}} v = 0$. Moreover, it follows from $\mathcal{R}(A^{\textcircled{\dagger}} A) = \mathcal{R}(A^{\textcircled{\dagger}}) = \mathcal{R}(A^k)$ (see [4], Theorem 3.7) that $x = A^{\textcircled{\dagger}} Ax$. Hence, $x = A^{\textcircled{\dagger}} Ax = A^{\textcircled{\dagger}} u + A^{\textcircled{\dagger}} v = A^{\textcircled{\dagger}} u \geq 0$. Conversely, since $\mathcal{R}(A^k) = \mathcal{N}[(A^k)^T]^{\perp}$, then $\mathbb{R}^n = \mathcal{R}(A^k) \oplus \mathcal{N}[(A^k)^T]$. For any $z \in \mathbb{R}^n_+$, we decompose z as

$z = u + v$, where $u \in \mathcal{R}(A^k)$ and $(A^k)^T v = 0$. Since $AA^{\oplus} = A^k(A^k)^{\dagger}$ is the orthogonal projector onto $\mathcal{R}(A^k)$ and $u \in \mathcal{R}(A^k)$, then $AA^{\oplus}z = AA^{\oplus}(u+v) = AA^{\oplus}u = u = z + (-v) \in \mathbb{R}^n_+ + \mathcal{N}[(A^k)^T]$. Moreover, since $A^{\oplus}z \in \mathcal{R}(A^{\oplus}) = \mathcal{R}(A^k)$, then by (10), $A^{\oplus}z \geq 0$, which means that $A^{\oplus}z \geq 0$ for every $z \geq 0$. Therefore, $A^{\oplus} \geq 0$. $\square$

4. Conclusions

The main contribution of this paper is that we provide some necessary and sufficient conditions for a property-n matrix to have a nonnegative core-EP inverse. We have shown that a property-n matrix A is core-EP monotone if and only if A has the form in (4), noting that the matrix form of A coincides with the core-EP decomposition of A. It is known that every square matrix has the core-EP decomposition, but for a property-n matrix A, the matrix U and T in (4) should be more special so that A is core-EP monotone. Moreover, most of the previous papers were concentrated on characterizing nonnegative matrices with nonnegative generalized inverses, and the classs of property-n matrices is larger than the class of nonnegative matrices. Since the investigation of nonnegative generalized inverses has applications in many fields, such as numerical analysis and linear economic models, this study may have the potential to encourage the researchers to further explore the topic of classes of matrices which have some kinds of nonnegative generalized inverse, such as the W-weighted core-EP inverse and the DMP inverse.

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