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Harnack Estimation for Nonlinear, Weighted, Heat-Type Equation along Geometric Flow and Applications

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Abstract: The method of gradient estimation for the heat-type equation using the Harnack quantity is a classical approach used for understanding the nature of the solution of these heat-type equations. Most of the studies in this field involve the Laplace–Beltrami operator, but in our case, we studied the weighted heat equation that involves weighted Laplacian. This produces a number of terms involving the weight function. Thus, in this article, we derive the Harnack estimate for a positive solution of a weighted nonlinear parabolic heat equation on a weighted Riemannian manifold evolving under a geometric flow. Applying this estimation, we derive the Li–Yau-type gradient estimation and Harnack-type inequality for the positive solution. A monotonicity formula for the entropy functional regarding the estimation is derived. We specify our results for various different flows. Our results generalize some works.

Keywords: Harnack inequality; Harnack estimate; weighted Laplacian; parabolic equation; gradient estimate

MSC: 53C21; 58J35; 35B45

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1. Introduction

In 1887, Harnack [1] introduced an inequality while studying the regularity for weak solutions of certain types of partial differential equations, which is now known as the Harnack's inequality. It compares the values of a positive solution of PDEs at two distinct points. This inequality is used to prove Harnack's theorem as well as to show the Hölder regularity of weak solutions of harmonic functions. Serrin [2] and Moser [3,4] considered generalized Harnack inequalities for both elliptic and parabolic PDEs. In modern geometric analysis, the study of differential Harnack estimation is an effective area of research, and it was popularized after Li and Yau's [5] work on the differential Harnack inequality to the positive solution of a heat equation on a Riemannian manifold with positive scalar curvature. In [6], Hamilton found a version of the Harnack inequality, which was used to solve the Poincaré conjecture by Perelman [7]. An attractive fact about the Harnack estimation is that one can derive some properties of the solutions without solving the equation. In this method, one uses the maximum principle for the heat equation on the Harnack quantity and derives the result accordingly. This is by far the easiest technique that can be applied to study such solutions. The importance of the Harnack estimation is that it can produce a Li–Yau-type estimation without the help of a cut-off function. One can see the work of Li [8,9] regarding the Liouville-type theorem for a symmetric diffusion operator and can study Perelman's entropy formula for weighted Laplacian. Li and Zhu [10] studied

the Harnack estimation for a nonlinear parabolic equation on a Riemannian manifold evolving along a Ricci flow. There are other types of estimations such as the Hamilton-type and Souplet–Zhang-type estimations, which are equally important in the case of studying the properties of the solutions of PDE. In this context, Hui et al.'s [11,12] work on the Li–Yau-type gradient estimation, Hamilton and Souplet–Zhang-type estimation, along the general geometric flow, is worth studying. Hui et al. [13] also studied the later estimation in the case of a system of parabolic equations. We now take an n-dimensional closed Riemannian manifold $(M^n, g, e^{-\phi}d\mu)$, while $e^{-\phi}d\mu$ is the weighted volume measure with respect to the Riemannian metric g , and ϕ being a twice differentiable function on M. This ϕ depends on time t for a time-dependent metric g. We consider the following abstract geometric flow given by

$$\frac{\partial}{\partial t}g_{ij} = -2S_{ij}, \tag{1}$$

where $S_{ij}(t) := \mathcal{S}(e_i, e_j)(t)$ is the smooth symmetric two-tensor on $(M, g(t))$, and $\{e_i : i = 1, 2, \cdots n\}$ is an orthonormal frame on M. Let us take $S(t) = tr(\mathcal{S})(t) = g^{ij}(t)S_{ij}(t)$. The weighted Laplacian operator is defined by

$$\Delta_\phi = \Delta - \langle \nabla\phi, \nabla \rangle,$$

where Δ is the Laplace operator, and $\langle \cdot, \cdot \rangle$ is the induced inner product by g. Ma [14] proved a differential Harnack inequality for the following heat-type equation

$$\frac{\partial f}{\partial t} = \Delta f - f \log f + Sf \tag{2}$$

along the list's flow. In [15], Guo and Ishida derived several differential Harnack estimates for positive solutions to the nonlinear backward heat-type equation

$$\frac{\partial f}{\partial t} = -\Delta f + \gamma f \log f + aSf \tag{3}$$

with constants γ and a on the evolving Riemannian manifold. In [16], Abolarinwa formulated some differential Harnack quantities for positive solutions to a heat equation on the evolving Riemannian manifold. Along different geometric flows, he also obtained Li–Yau-type estimates and Perelman-type differential Harnack inequalities. Differential Harnack estimates and gradient estimates were studied in [6,17–25]. In recent studies, a group of researchers have made advancements in the field of special submanifolds within various spaces [26–38]. Further inspiration for our paper can be drawn from multiple articles as referenced in [39–57].

Here, we confine a generalized nonlinear parabolic heat equation involving weighted Laplacian and derive differential Harnack-type inequalities on M along the flow (1). For this, we consider a generalized non-linear parabolic equation by

$$\frac{\partial f}{\partial t} = \Delta_\phi f + \gamma(t) f \log f + \delta(t) Sf, \tag{4}$$

where $\gamma(t)$ and $\delta(t)$ are continuous functions of t. The weighted volume form, say $d\eta = e^{-\phi}d\mu$, evolves by

$$\frac{\partial}{\partial t}d\eta = -(S + \phi_t)d\eta, \tag{5}$$

along the flow (1). We also assume that $\mathcal{S}(X, X) \geq 0$, $\forall X \in \chi(M)$. Throughout this paper, we consider $f = e^{-u}$ as a positive solution of (4); hence, Equation (4) reduces to

$$\frac{\partial}{\partial t}u = \Delta_\phi u - |\nabla u|^2 + \gamma(t)u - \delta(t)S. \tag{6}$$

2. Preliminaries

This section is devoted to some basic lemmas and definitions.

Lemma 1 ([19]). *Under the geometric flow of Equation (1), we have*

1. $\frac{\partial}{\partial t}|\nabla u|^2 = 2\mathcal{S}(\nabla u, \nabla u) + 2\langle \nabla u, \nabla u_t\rangle$,
2. $\frac{\partial}{\partial t}(\Delta_\phi u) = \Delta_\phi u_t + 2S^{ij}\nabla_i\nabla_j u + \langle 2\,div\,\mathcal{S} - \nabla S, \nabla u\rangle - 2\mathcal{S}(\nabla\phi, \nabla u) - \langle \nabla u, \nabla\phi_t\rangle$, *where* $div\,\mathcal{S} :=$ *divergence of* S_{ij}.

This lemma gives us the evolution equations for $|\nabla u|^2$ and $\Delta_\phi u$ along the flow (1). It will play an important role for deriving the Harnack estimation.

Lemma 2 ([19]). *(Weighted Bochner formula) For any function* $u \in C^\infty(M)$, *we have*

$$\frac{1}{2}\Delta_\phi|\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla\Delta_\phi u, \nabla u\rangle + Ric_\phi(\nabla u, \nabla u),$$

where $Ric_\phi := Ric + \nabla^2\phi$, *is called the Bakry–Émery tensor, and* ∇^2 *is the Hessian operator. The* $(m-n)$*-Bakry–Émery tensor is defined by*

$$Ric_\phi^{m-n} := Ric + \nabla^2\phi - \frac{\nabla\phi\otimes\nabla\phi}{m-n}, \forall m > n,$$

where m, n *are integers.*

Definition 1 ([24]). *Let* $g(t)$ *be a solution to the geometric flow (1), and let* $X = X^i\frac{\partial}{\partial x^i} \in \chi(M)$. *Then, the quantities* $\mathcal{H}(S_{ij}, X), \mathcal{D}(S_{ij}, X)$ *are defined by*

$$\mathcal{H}(S_{ij}, X) = \frac{\partial S}{\partial t} + \frac{S}{t} - 2\nabla_i S X^i + 2S^{ij}X_iX_j, \tag{7}$$

$$\mathcal{D}(S_{ij}, X) = \frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2 - (4\nabla^i S_{il} - 2\nabla_l S)X^l + 2(Ric^{ij} - S^{ij})X_iX_j, \tag{8}$$

where $S^{ij} = g^{ik}g^{jl}S_{kl}$.

In this paper, we generalize the definition of $\mathcal{D}$ for weighted Laplacian, keeping the notation unchanged, by

$$\mathcal{D}(S_{ij}, X) = \frac{\partial S}{\partial t} - \Delta_\phi S - 2|S_{ij}|^2 - (4\nabla^i S_{il} - 2\nabla_l S)X^l + 2(Ric_\phi^{ij} - S^{ij})X_iX_j, \tag{9}$$

Definition 2. *For any* $X \in \chi(M)$, *we define*

$$\begin{aligned}\mathcal{P}_{a,\sigma}(S_{ij}, X) &= -(2a-3\sigma)\Delta_\phi S - 2(a+\sigma-1)(Ric_\phi^{ij} + S^{ij})X_iX_j \\ &- (2a+3\sigma)\frac{\partial S}{\partial t} + 2(a+\sigma-1)(2\nabla^i S_{il} - \nabla_l S)X^l,\end{aligned} \tag{10}$$

where a, σ *are any real numbers. For any real numbers* a, c, α, β, *we also define*

$$\begin{aligned}W_{c,\alpha,\beta} &= \frac{c}{t}\Delta_\phi\phi - 2\alpha\mathcal{S}(\nabla\phi, \nabla u) - \alpha\langle\nabla u, \nabla\phi_t\rangle - \frac{c}{t}\phi_t \\ &- (\frac{2c}{t} + 2(\alpha-\beta)\frac{\lambda}{t})\nabla\phi\nabla u,\end{aligned} \tag{11}$$

$$\begin{aligned}\mathcal{D}_{a,\alpha,\beta}(S_{ij}, X) &= a(\frac{\partial S}{\partial t} - \Delta_\phi S - 2|S_{ij}|^2) \\ &- \alpha(2\nabla^i S_{il} - \nabla_l S)X^l + 2\beta(Ric_\phi^{ij} - S^{ij})X_iX_j,\end{aligned} \tag{12}$$

for any $X \in \chi(M)$.

3. Harnack Estimation and Applications

We consider a generalized Harnack quantity involving the weighted Laplacian as follows

$$Q = \alpha\Delta_\phi u - \beta|\nabla u|^2 + aS - b\frac{u}{t} - d\frac{n}{t} - c\frac{\phi}{t}, \tag{13}$$

$a, b, c, d, \alpha, \beta$ are real constants. Next, we derive the evolution equation for the Harnack quantity Q along the flow (1) in its most generalized form and in its alternate form, which is required for applying the maximum principle.

Theorem 1. *The Harnack quantity (13) subject to the parabolic Equation (6) under the geometric flow (1) evolves by*

$$\begin{aligned}\frac{\partial}{\partial t}Q &= \Delta_\phi Q - 2\nabla Q\nabla u + 2(a+\beta\delta(t))\nabla S\nabla u - 2(\alpha-\beta)|\nabla^2 u|^2 - 2\alpha Ric_\phi(\nabla u, \nabla u)\\ &+ 2\alpha S^{ij}\nabla_i\nabla_j u - \alpha\delta(t)\Delta_\phi S - \frac{b}{t}|\nabla u|^2 + \frac{b}{t}\delta(t)S + \frac{bu}{t^2} + \frac{dn}{t^2} + 2a|S_{ij}|^2\\ &+ \mathcal{D}_{a,\alpha,\beta}(S_{ij}, -\nabla u) + \alpha\gamma(t)\Delta_\phi u - 2\beta\gamma(t)|\nabla u|^2 - \frac{b}{t}\gamma(t)u\\ &+ \left(\frac{c}{t}\Delta_\phi\phi - \frac{2c}{t}\nabla\phi\nabla u - 2\alpha\mathcal{S}(\nabla\phi, \nabla u) - \alpha\langle\nabla\phi_t, \nabla u\rangle - \frac{c}{t}\phi_t + \frac{c\phi}{t^2}\right).\end{aligned} \tag{14}$$

Proof. From (13), we have

$$\nabla Q = \alpha\nabla(\Delta_\phi u) - \beta\nabla|\nabla u|^2 + a\nabla S - b\frac{1}{t}\nabla u - c\frac{1}{t}\nabla\phi, \tag{15}$$

$$\Delta_\phi Q = \alpha\Delta_\phi(\Delta_\phi u) - \beta\Delta_\phi|\nabla u|^2 + a\Delta_\phi S - b\frac{1}{t}\Delta_\phi u - c\frac{1}{t}\Delta_\phi\phi. \tag{16}$$

Using Lemma 1 and (6) , we obtain

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \alpha\frac{\partial}{\partial t}(\Delta_\phi u) - \beta\frac{\partial}{\partial t}|\nabla u|^2 + a\frac{\partial S}{\partial t} - b\frac{1}{t}\frac{\partial u}{\partial t} + b\frac{u}{t^2} + d\frac{n}{t^2} + c\frac{\phi}{t^2} - \frac{c}{t}\frac{\partial\phi}{\partial t}\\ &= \Delta_\phi Q - (\alpha-\beta)\Delta_\phi|\nabla u|^2 - a\Delta_\phi S + \frac{c}{t}\Delta_\phi\phi + \alpha\gamma(t)\Delta_\phi(u) - \alpha\delta(t)\Delta_\phi S\\ &\quad + 2\alpha S^{ij}\nabla_i\nabla_j u + 2\alpha(\operatorname{div}\mathcal{S} - \frac{1}{2}\nabla S)\nabla u - 2\alpha\mathcal{S}(\nabla\phi, \nabla u) - \alpha\nabla\phi_t\nabla u\\ &\quad - 2\beta S^{ij}\nabla_i u\nabla_j u - 2\beta\nabla(\Delta_\phi u)\nabla u + 2\beta\nabla(|\nabla u|^2)\nabla u - 2\beta\gamma(t)|\nabla u|^2\\ &\quad + 2\beta\delta(t)\nabla S\nabla u + a\frac{\partial}{\partial t}S + \frac{b}{t}|\nabla u|^2 - \frac{b}{t}\gamma(t)u + \frac{b}{t}\delta(t)S + b\frac{u}{t^2}\\ &\quad + d\frac{n}{t^2} + c\frac{\phi}{t^2} - \frac{c}{t}\phi_t.\end{aligned} \tag{17}$$

Using (15) and (16) in (17) and applying the weighted Bochner formula (Lemma 2), we obtain

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \Delta_\phi Q - 2(\alpha-\beta)(|\nabla^2 u|^2 + \nabla\Delta_\phi u\nabla u + Ric_\phi(\nabla u, \nabla u)) - a\Delta_\phi S + \frac{c}{t}\Delta_\phi\phi\\ &\quad + \alpha\gamma(t)\Delta_\phi(u) - \alpha\delta(t)\Delta_\phi S + 2\alpha S^{ij}\nabla_i\nabla_j u + 2\alpha(\operatorname{div}\mathcal{S} - \frac{1}{2}\nabla S)\nabla u\\ &\quad - 2\alpha\mathcal{S}(\nabla\phi, \nabla u) - \alpha\nabla\phi_t\nabla u - 2\beta S^{ij}\nabla_i u\nabla_j u - 2\beta\nabla(\Delta_\phi u)\nabla u\\ &\quad + 2\beta\nabla(|\nabla u|^2)\nabla u - 2\beta\gamma(t)|\nabla u|^2 + 2\beta\delta(t)\nabla S\nabla u + a\frac{\partial}{\partial t}S + \frac{b}{t}|\nabla u|^2\\ &\quad - \frac{b}{t}\gamma(t)u + \frac{b}{t}\delta(t)S + b\frac{u}{t^2} + d\frac{n}{t^2} + c\frac{\phi}{t^2} - \frac{c}{t}\phi_t.\end{aligned} \tag{18}$$

From (12), we obtain

$$\begin{aligned}\mathcal{D}_{a,\alpha,\beta}(S_{ij},-\nabla u) &= a(\frac{\partial}{\partial t}S-\Delta_\phi S-2|S_{ij}|^2)+\alpha(2\mathrm{div}\,\mathcal{S}-\nabla S)\nabla u\\ &+ 2\beta(Ric_\phi-\mathcal{S})(\nabla u,\nabla u).\end{aligned}\tag{19}$$

Using (19) in (18) we obtain our result (14). □

Theorem 2. *The expression for the evolution of Q derived in (14) has an alternative form for $\alpha \neq \beta$, $\alpha \neq 0$ and $\lambda \in \mathbb{R}$, given by*

$$\begin{aligned}\frac{\partial}{\partial t}Q &= \Delta_\phi Q-2\nabla Q\nabla u-2(\alpha-\beta)\left|\nabla_i\nabla_j u-\frac{\alpha}{2(\alpha-\beta)}S_{ij}-\frac{\lambda}{2t}g_{ij}\right|^2\\ &+ 2(a+\beta\delta(t))\nabla S\nabla u-2\frac{\alpha-\beta}{\alpha}\frac{\lambda}{t}Q+\frac{(\alpha-\beta)\lambda^2 n}{2t^2}\\ &- \left(b+\frac{2(\alpha-\beta)\lambda\beta}{\alpha}\right)\frac{|\nabla u|^2}{t}+\left(2a+\frac{\alpha^2}{2(\alpha-\beta)}\right)|S_{ij}|^2\\ &+ \left(\alpha\lambda+b\delta(t)+\frac{2(\alpha-\beta)\lambda a}{\alpha}\right)\frac{S}{t}+\left(1-\frac{2(\alpha-\beta)\lambda}{\alpha}\right)\left(\frac{bu}{t^2}+\frac{dn}{t^2}+\frac{c\phi}{t^2}\right)\\ &- \alpha\delta(t)\Delta_\phi S-2\alpha Ric_\phi(\nabla u,\nabla u)+\mathcal{D}_{a,\alpha,\beta}(S_{ij},-\nabla u)+\alpha\gamma(t)\Delta_\phi u\\ &- 2\beta\gamma(t)|\nabla u|^2-\frac{b}{t}\gamma(t)u+\frac{c}{t}\Delta_\phi\phi-2\alpha\mathcal{S}(\nabla\phi,\nabla u)-\alpha\langle\nabla\phi_t,\nabla u\rangle\\ &- \frac{c}{t}\phi_t-\frac{2c}{t}\nabla\phi\nabla u-2(\alpha-\beta)\frac{\lambda}{t}\nabla\phi\nabla u.\end{aligned}\tag{20}$$

Proof. It is easy to show that

$$\begin{aligned}&-2(\alpha-\beta)\left|\nabla_i\nabla_j u-\frac{\alpha}{2(\alpha-\beta)}S_{ij}-\frac{\lambda}{2t}g_{ij}\right|^2-2\frac{(\alpha-\beta)\lambda}{t}\nabla\phi\nabla u\\ =\;& -2(\alpha-\beta)|\nabla^2 u|^2+2\alpha S^{ij}\nabla_i\nabla_j u+2(\alpha-\beta)\frac{\lambda}{t}\Delta_\phi u-\frac{\lambda\alpha S}{t}-\frac{\alpha^2}{2(\alpha-\beta)}|S_{ij}|^2\\ &-\frac{(\alpha-\beta)\lambda^2 n}{2t^2}.\end{aligned}\tag{21}$$

From (13), we also have

$$\begin{aligned}&- 2(\alpha-\beta)\frac{\lambda}{t}(\Delta_\phi u-\frac{\alpha S}{2(\alpha-\beta)})-\frac{b|\nabla u|^2}{t}-\frac{b\delta(t)S}{t}+\frac{bu}{t^2}+\frac{dn}{t^2}+\frac{c\phi}{t^2}\\ =\;& -\frac{2(\alpha-\beta)}{\alpha}\frac{\lambda}{t}Q-\left(b+\frac{2(\alpha-\beta)\lambda\beta}{\alpha}\right)\frac{|\nabla u|^2}{t}+\left(\alpha\lambda-b\delta(t)+\frac{2(\alpha-\beta)\lambda a}{\alpha}\right)\frac{S}{t}\\ &+ \left(1-\frac{2(\alpha-\beta)\lambda}{\alpha}\right)\left(\frac{bu}{t^2}+\frac{dn}{t^2}+\frac{c\phi}{t^2}\right).\end{aligned}\tag{22}$$

Using (21) and (22) in (14), we obtain the result, i.e., (20). □

Theorem 2 is a concise form of Theorem 1. This will help us to gather similar terms and to suitably assemble them so that they can be eliminated or estimated later. Next, we derive the evolution equation for a specific Harnack quantity with a suitable choice of $\delta(t)$ and the constants $\alpha, \beta, a, b, \lambda, \sigma$.

Theorem 3. *If u satisfies the equation*

$$\frac{\partial u}{\partial t}=\Delta_\phi u-|\nabla u|^2-(a+2\sigma)S+\gamma(t)u,\tag{23}$$

where σ is a real constant, then the Harnack quantity

$$Q = 2\Delta_\phi u - |\nabla u|^2 + aS - \frac{dn}{t} - \frac{c\phi}{t} \tag{24}$$

evolves by

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \Delta_\phi Q - 2\nabla Q\nabla u - 2|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t}g_{ij}|^2 - \left(\frac{2}{t} - \gamma(t)\right)Q + (2-d)\frac{n}{t^2}\\ &+ \left(-\frac{2}{t} - \gamma(t)\right)|\nabla u|^2 + 2(2-\sigma)\frac{S}{t} - \frac{c\phi}{t^2} - aS\gamma(t) + d\gamma(t)\frac{n}{t}\\ &+ c\gamma(t)\frac{\phi}{t} + (a+\sigma)(2\mathcal{H}(S_{ij}, -\nabla u) + \mathcal{D}(S_{ij}, -\nabla u)) - 4\mathcal{S}(\nabla u, \nabla u)\\ &- 4Ric_\phi(\nabla u, \nabla u) + 2(a+\sigma+1)|S_{ij}|^2 + \mathcal{P}_{a,\sigma}(S_{ij}, -\nabla u) + W_{c,2,1}\end{aligned} \tag{25}$$

under the flow (1), where $W_{a,\alpha,\beta}$ and $\mathcal{P}_{a,\sigma}(S_{ij}, -\nabla u)$ are, respectively, defined in (11) and (12).

Proof. Putting $\alpha = 2,\ \beta = 1,\ b = 0,\ \lambda = 2,\ \delta(t) = a + 2\sigma$ in (20), we obtain

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \Delta_\phi Q - 2\nabla Q\nabla u - 2\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t}g_{ij}\right|^2 + 2(a+a+2\sigma)\nabla S\nabla u - \frac{2}{t}Q\\ &+ \frac{2n}{t^2} - 2\frac{|\nabla u^2|}{t} + (2a+2)|S_{ij}|^2 + (4+2a)\frac{S}{t} - \left(\frac{dn}{t^2} + \frac{c\phi}{t^2}\right)\\ &- 2(a+2\sigma)\Delta_\phi S - 4Ric_\phi(\nabla u, \nabla u) + a\frac{\partial S}{\partial t} - a\Delta_\phi S\\ &- 2a|S_{ij}|^2 + 2(2\ \mathrm{div}\ \mathcal{S} - \nabla S)\nabla u + 2(Ric_\phi - \mathcal{S})(\nabla u, \nabla u)\\ &+ 2\gamma(t)(\Delta_\phi u - |\nabla u|^2) + W_{c,2,1}.\end{aligned} \tag{26}$$

From (24), we have

$$2(\Delta_\phi u - |\nabla u|^2) = Q - |\nabla u|^2 - aS + \frac{dn}{t} + \frac{c\phi}{t}. \tag{27}$$

Adjusting (26) with $(a+\sigma)(2\mathcal{H}(S_{ij}, -\nabla u) + \mathcal{D}(S_{ij}, -\nabla u))$ and using Equations (10) and (27) on (26), we obtain our desired result (25). □

Corollary 1. *Under the following assumptions*

1. $-\frac{2}{t} \le \gamma(t) \le 0,\ \forall t \in (0, T)$
2. $\mathcal{P}_{a,\sigma}(S_{ij}, -\nabla u) \le 0$
3. $(2\mathcal{H} + \mathcal{D})(S_{ij}, -\nabla u) \ge 0$
4. $S > 0$
5. $a + \sigma \le -1$
6. $\sigma \ge 2$
7. $(Ric_\phi + \mathcal{S})(\nabla u, \nabla u) \ge 0$
8. $d \ge 2$

the Harnack quantity Q defined in (24) satisfies

$$\frac{\partial Q}{\partial t} \le \Delta_\phi Q - 2\nabla Q\nabla u - \left(\frac{2}{t} - \gamma(t)\right)Q - \left(\frac{1}{t} - \gamma(t)\right)\frac{c\phi}{t} + W_{c,2,1}, \tag{28}$$

where $W_{c,2,1} = \frac{c}{t}\Delta_\phi\phi - 4\mathcal{S}(\nabla\phi, \nabla u) - 2\langle\nabla u, \nabla\phi_t\rangle - \frac{c}{t}\phi_t - \frac{2(c+2)}{t}\nabla\phi\nabla u$.

Proof. Applying the above conditions (1) to (8) in (25), we have (28). □

Corollary 2. *If we assume $c \geq 0$, $\phi(t) \geq 0$, $\forall t \in \mathbb{R}$ in Corollary 1, then we have the following diffusion-reaction equation for Q*

$$\frac{\partial Q}{\partial t} \leq \Delta_\phi Q - 2\nabla Q \nabla u - \left(\frac{2}{t} - \gamma(t)\right) Q + W'_{c,2,1}, \tag{29}$$

where $W'_{c,2,1} = W_{c,2,1} + 4\mathcal{S}(\nabla\phi, \nabla u)$. In this case, $Q \leq 0$ for all $t \in (0, T)$.

Proof. Using $\phi(t) \geq 0$, $\forall t \in \mathbb{R}$ and $c \geq 0$ along with $S(X, X) \geq 0$, $\forall X \in \chi(M)$ in (28), we obtain (29).

Since Q satisfies (29), for sufficiently small values of t, we have $Q < 0$. Hence, by the maximum principle on parabolic equations, $Q \leq 0$, $\forall t \in (0, T)$. □

We have assumed some necessary conditions in the previous two corollaries for confirming that the Harnack quantity $Q \leq 0$ on M for all $t \in (0, T)$. Now, we show some classical applications using this result we just derived.

Corollary 3. *Under the same assumptions defined in Corollary 2, we have the following Li–Yau-type estimate*

$$\frac{|\nabla f|^2}{f^2} \leq \frac{2\Delta_\phi f}{f} - aS + \frac{dn}{t} + \frac{c\phi}{t} \tag{30}$$

and gradient estimate

$$\frac{|\nabla f|^2}{f^2} \leq \frac{2\partial_t f}{f} - 2(\gamma(t)\log f + \delta(t)S) - aS + \frac{dn}{t} + \frac{c\phi}{t}. \tag{31}$$

Proof. Putting $u = -\log f$ in (24), we obtain

$$\begin{aligned} Q - aS + \frac{dn}{t} + \frac{c\phi}{t} &= 2\Delta_\phi(-\log f) - |\nabla(-\log f)|^2 \\ &= -\frac{2\Delta_\phi f}{f} + \frac{|\nabla f|^2}{f^2}. \end{aligned} \tag{32}$$

From Corollaries 1 and 2, we have

$$\begin{aligned} Q \leq 0 \quad &\Longrightarrow \quad -\frac{2\Delta_\phi f}{f} + \frac{|\nabla f|^2}{f^2} + aS - \frac{dn}{t} - \frac{c\phi}{t} \leq 0 \\ &\Longrightarrow \quad \frac{|\nabla f|^2}{f^2} \leq \frac{2\Delta_\phi f}{f} - aS + \frac{dn}{t} + \frac{c\phi}{t} \text{ for all } t \in (0, T). \end{aligned} \tag{33}$$

This is a Li–Yau-type estimate. Putting the value of $\Delta_\phi f$ from Equation (4) into Equation (33), we obtain (31). □

The next corollary is a consequence of the Li–Yau-type estimate derived from the Harnack estimation. This is known as a Harnack-type inequality, which allows us to compare the amount of heat between two distinct points on two distinct times.

Corollary 4. *Let $\Gamma : [0, T) \to M$ be any spacetime path joining the points $(t_1, x_1), (t_2, x_2) \in [0, T) \times M$ with $0 \leq t_1 < t_2 < T$. If we assume all the conditions stated in Corollary 2, then any positive solution f of (23) along flow (1) satisfies the Harnack-type inequality*

$$f(t_1, x_1) \leq f(t_2, x_2)\left(\frac{t_2}{t_1}\right)^{\frac{dn}{2}} \exp\left\{\frac{1}{2}\inf_{\Gamma}\left(\int_{t_1}^{t_2}(|\dot{\Gamma}(t)|^2 - (3a + 4\sigma)S + 2\gamma(t)u + \frac{c\phi}{t})dt\right)\right\}. \tag{34}$$

Proof. We know that Q defined in (24) can be written as

$$Q = 2\frac{\partial}{\partial t}u + |\nabla u|^2 + (3a + 4\sigma)S - 2\gamma(t)u - \frac{dn + c\phi}{t}.$$

Thus, using Corollary 2, we obtain

$$\frac{\partial}{\partial t}u \leq -\frac{1}{2}|\nabla u|^2 - \frac{1}{2}(3a + 4\sigma)S + \gamma(t)u + \frac{dn + c\phi}{2t}. \tag{35}$$

Let $\Gamma : [0, T) \to M$ be any spacetime path joining $x_1, x_2 \in M$ such that $\Gamma(t_i) = x_i$, $i = 1, 2$ and $0 \leq t_1 < t_2 < T$. Differentiating u with respect to t , we have

$$\begin{aligned} \frac{du}{dt} &= \frac{\partial u}{\partial t} + \langle \nabla u, \dot{\Gamma}(t) \rangle \\ &\leq \frac{1}{2}|\dot{\Gamma}(t)|^2 - \frac{1}{2}(3a + 4\sigma)S + \gamma(t)u + \frac{dn + c\phi}{2t}. \end{aligned} \tag{36}$$

Here, we used an elementary inequality $\frac{1}{2}|\nabla u|^2 - \langle \nabla u, \dot{\Gamma}(t) \rangle \geq -\frac{1}{2}|\dot{\Gamma}(t)|^2$ and Equation (35). Next, we integrate (36) between $(t_1, x_1), (t_2, x_2)$, and taking infimum over all possible paths Γ, we obtain

$$\begin{aligned} u(t_2, x_2) - u(t_1, x_1) &\leq \frac{1}{2}\inf_{\Gamma}\left(\int_{t_1}^{t_2} (|\dot{\Gamma}(t)|^2 - (3a + 4\sigma)S + 2\gamma(t)u + \frac{c\phi}{t})dt \right) \\ &+ \frac{dn}{2}\log\left(\frac{t_2}{t_1}\right). \end{aligned} \tag{37}$$

Putting $u = -\log f$ in (37), we obtain (34). □

Monotonicity formulas are important applications of gradient estimation. In this technique, we wish to find bounds for some energy functional (or entropy functional) and its derivative with respect to time. The next theorem provides the monotonicity of an entropy functional for this estimation along the given flow.

Theorem 4. *If ϕ is a decreasing function of t, then the entropy functional*

$$\mathcal{F} = \int_M t^2 Q e^{-u} d\eta \tag{38}$$

is non-positive, i.e., $\mathcal{F} \leq 0$ and

$$\frac{d\mathcal{F}}{dt} \leq \int_M e^{-u} t^2 W'_{c,2,1} d\eta, \tag{39}$$

where $d\eta = e^{-\phi} d\mu$ and $W'_{c,2,1} = \frac{c}{t}\Delta_\phi \phi - 2\langle \nabla u, \nabla \phi_t \rangle - \frac{c}{t}\phi_t - \frac{2(c+2)}{t}\nabla\phi\nabla u$, provided that the integral exists.

Proof. From Corollary 2, we know that $Q \leq 0$; thus, it is trivial that $\mathcal{F} \leq 0$.

Differentiating (38) with respect to t and using Corollary 2 and (23), we obtain

$$\begin{aligned}\frac{d\mathcal{F}}{dt} &= \frac{\partial}{\partial t}\int_M t^2 Q e^{-u} d\eta \\ &= \int_M (t^2 e^{-u}\partial_t Q + 2tQe^{-u} - t^2 Q e^{-u} u_t - (S+\phi_t)t^2 Q e^{-u}) d\eta \\ &\leq \int_M (\Delta_\phi (t^2 e^{-u} Q) + t^2 e^{-u} Q(a+2\sigma-1)S - t^2 e^{-u} Q\phi_t + t^2 e^{-u} W'_{c,2,1}) d\eta \\ &\leq \int_M (t^2 e^{-u} Q(\sigma-2)S - t^2 e^{-u} Q\phi_t + t^2 e^{-u} W'_{c,2,1}) d\eta \\ &\leq \int_M (-t^2 e^{-u} Q\phi_t + t^2 e^{-u} W'_{c,2,1}) d\eta. \end{aligned} \tag{40}$$

Since $\phi_t \leq 0$, (40) trivially implies (39). □

Remark 1. *If we take $c = 0$, and ϕ is a nonconstant function, then (39) reduces to*

$$\frac{d\mathcal{F}}{dt} \leq -\int_M 2te^{-u}\langle t\nabla\phi_t - 2\nabla\phi, \nabla u\rangle d\eta.$$

4. Application to Specific Flows

In this section, we restate our results for Ricci flow and extended Ricci flow.

4.1. Hamilton's Ricci Flow

The Ricci flow by Hamilton [6] is defined by the equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \tag{41}$$

where $\mathcal{S} = Ric$ and $S = tr(\mathcal{S}) = R$ denote the Ricci tensor and the scalar curvature. If u is a positive solution of (23), ϕ is a nonconstant function, and c is any (nonzero) real number, then under the Hamilton's Ricci flow, the expression of $\mathcal{H}$ and $\mathcal{D}$ becomes

$$\mathcal{D}(R_{ij}, X) = 0, \tag{42}$$

$$\mathcal{H}(R_{ij}, X) = \frac{\partial}{\partial t}R + \frac{R}{t} - 2\nabla_i R X^i + 2R^{ij}X_i X_j. \tag{43}$$

Hence, the Harnack quantity defined in (24) becomes

$$Q = \alpha\Delta_\phi u - \beta|\nabla u|^2 + aR - b\frac{u}{t} - d\frac{n}{t} - c\frac{\phi}{t} \tag{44}$$

and it evolves by

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \Delta_\phi Q - 2\nabla Q\nabla u - 2|\nabla_i\nabla_j u - R_{ij} - \frac{1}{t}g_{ij}|^2 - \left(\frac{2}{t} - \gamma(t)\right)Q + (2-d)\frac{n}{t^2} \\ &+ \left(-\frac{2}{t} - \gamma(t)\right)|\nabla u|^2 + 2(2-\sigma)\frac{R}{t} - \frac{c\phi}{t^2} - aR\gamma(t) + d\gamma(t)\frac{n}{t} \\ &+ c\gamma(t)\frac{\phi}{t} + 2(a+\sigma)\left(\frac{\partial}{\partial t}R + \frac{R}{t} - 2\nabla_i R X^i + 2R^{ij}X_i X_j\right) - 4Ric(\nabla u, \nabla u) \\ &- 4Ric_\phi(\nabla u, \nabla u) + 2(a+\sigma+1)|R_{ij}|^2 + \mathcal{P}_{a,\sigma}(R_{ij}, -\nabla u) + W_{c,2,1}. \end{aligned} \tag{45}$$

The Li–Yau-type estimate (30) becomes

$$\frac{|\nabla f|^2}{f^2} \leq \frac{2\Delta_\phi f}{f} - aR + \frac{dn}{t} + \frac{c\phi}{t}. \tag{46}$$

The Harnack-type inequality (34) becomes

$$f(t_1,x_1) \leq f(t_2,x_2)\left(\frac{t_2}{t_1}\right)^{\frac{dn}{2}} \exp\left\{\frac{1}{2}\inf_{\Gamma}\left(\int_{t_1}^{t_2}(|\dot{\Gamma}(t)|^2-(3a+4\sigma)R+2\gamma(t)u+\frac{c\phi}{t})dt\right)\right\}. \tag{47}$$

To obtain the results involving the Laplace-Beltrami operator, we need to assume $c=0$ and ϕ as a constant function.

4.2. Extended Ricci Flow

The extended Ricci flow [58] is defined by the flow

$$\frac{\partial}{\partial t}g = -2Ric+2\alpha\nabla\psi\otimes\nabla\psi, \tag{48}$$

where α is a time-dependent constant called the coupling constant and $\psi\equiv\psi(t)$ is a smooth function on M. Such a flow has also been studied in [59]. In this flow, we have

$$\mathcal{S} = Ric-\alpha\nabla\psi\otimes\nabla\psi \tag{49}$$
$$S = R-\alpha|\nabla\psi|^2. \tag{50}$$

The expression of $\mathcal{D}$ and $\mathcal{H}$ becomes

$$\mathcal{D} = \alpha(\langle\nabla\psi,X\rangle+\Delta\psi)^2 \tag{51}$$
$$\mathcal{H} = \frac{\partial}{\partial t}R-\frac{R}{t}-2\nabla_i RX^i+2R^{ij}X_iX_j. \tag{52}$$

Assuming ϕ is a constant function and $c=0$, the Harnack quantity Q in (24) reduces to

$$Q=\alpha\Delta u-\beta|\nabla u|^2+a(R-\alpha|\nabla\psi|^2)-b\frac{u}{t}-d\frac{n}{t}. \tag{53}$$

The evolution of Q becomes

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \Delta Q-2\nabla Q\nabla u-2\left|\nabla_i\nabla_j u-R_{ij}+\alpha\nabla_i\psi\nabla_j\psi-\frac{1}{t}g_{ij}\right|^2-\left(\frac{2}{t}-\gamma(t)\right)Q\\
&+ (2-d)\frac{n}{t^2}+\left(-\frac{2}{t}-\gamma(t)\right)|\nabla u|^2+2(2-\sigma)\frac{R-\alpha|\nabla\psi|^2}{t}-a(R-\alpha|\nabla\psi|^2)\gamma(t)\\
&+ d\gamma(t)\frac{n}{t}+2(a+\sigma)\left(\frac{\partial}{\partial t}R+\frac{R}{t}-2\nabla_i RX^i+2R^{ij}X_iX_j\right)\\
&- 2(a+\sigma)\left(\frac{\partial}{\partial t}(\alpha|\nabla\psi|^2)+\frac{\alpha|\nabla\psi|^2}{t}-2\nabla_i(\alpha|\nabla\psi|^2)X^i+2\alpha\nabla_i\psi\nabla_j\psi X_iX_j\right)\\
&- 4(2Ric-\alpha\nabla\psi\otimes\nabla\psi)(\nabla u,\nabla u)+2(a+\sigma+1)|R_{ij}-\alpha\nabla_i\psi\nabla_j\psi|^2\\
&+ \mathcal{P}_{a,\sigma}(R_{ij}-\alpha\nabla_i\psi\nabla_j\psi,-\nabla u).\end{aligned} \tag{54}$$

The Li–Yau-type estimate (30) becomes

$$\frac{|\nabla f|^2}{f^2}\leq\frac{2\Delta f}{f}-a(R-\alpha|\nabla\psi|^2)+\frac{dn}{t}. \tag{55}$$

The Harnack-type inequality (34) becomes

$$f(t_1,x_1)\leq f(t_2,x_2)\left(\frac{t_2}{t_1}\right)^{\frac{dn}{2}}\exp\left\{\frac{1}{2}\inf_{\Gamma}\left(\int_{t_1}^{t_2}(|\dot{\Gamma}(t)|^2-(3a+4\sigma)(R-\alpha|\nabla\psi|^2)+2\gamma(t)u)dt\right)\right\}. \tag{56}$$

5. Conclusions

As mentioned in the introduction, the Harnack estimation is a powerful tool used to understand the nature of a solution for a certain heat-type equation. It is known that in a general way, it is not always possible to solve higher-dimensional PDEs. Hence, the method of gradient estimation allows us to derive intrinsic properties of solutions for such PDEs without analytically solving the equation. Not only in a time-independent space but also in a space that is evolving along a flow (i.e., the space being dependent on time) can we derive the estimations.

In this section, we mention some significance of the results derived in the article. Theorems 1 and 2 give the evolution formula for the Harnack quantity Q defined by (13), along the geometric flow (1). As a specific case, (45) shows the evolution formula for Q along the Ricci flow, and in a similar manner, Equation (54) shows the evolution formula for an extended Ricci flow. Theorem 3 deals with the Harnack quantity associated with Equation (23). Corollary 1 produces the reaction diffusion equation for Q under some restrictions, where we use the maximum principle to deduce the Harnack estimation. Next, Corollary 2 gives the Harnack estimation for a positive solution of (4) along (1) on a weighted Riemannian manifold. Corollary 3 deals with the Li–Yau-type gradient estimation, and Corollary 4 deals with the Harnack-type inequality as shown in (30) and (34). Theorem 4 gives the monotonicity of the entropy functional (38) with certain restrictions. Finally, Section 4 is devoted to the applications of our results, in particular, geometric flows.

To establish some specific cases, we suitably choose the parameters, e.g., whenever

1. ϕ is a constant function, $c = 0$, $\alpha = 2$, $\beta = 1$, $a = -3$, $b = 0$ and $d = 2$, our results reduce to [15],
2. ϕ is a constant function, $c = 0$, $\alpha = 2$, $\beta = -1$, $a = 3$, $b = 0$ and $d = -2$, we find the results in [16],
3. ϕ is a constant function, $c = 0$, $\alpha = 2$, $\beta = 1$, $a = -3$, $b = 0$ and d is a free variable, then we obtain the results in [20], etc.

Thus, we have derived a Harnack-type estimation for Equation (4) along the flow (1) on $(M^n, g, e^{-\phi}d\mu)$ and thus extended the works of [15,16,20].

Future Aspect

Our results can be further extended to even more complex cases by introducing the weighted p–Laplace operator (for a detailed study, see [59,60] and the references therein) to the heat equation. One can also improve the estimation by decreasing the number of restrictions. Moving forward, we are going to explore the practical implications of our primary findings by incorporating elements from singularity theory, submanifold theory, and other pertinent disciplines [61–82]. These can be considered as a future aspect of this article.

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