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Norm Estimates of the Pre-Schwarzian Derivatives for Functions with Conic-like Domains

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Abstract: The pre-Schwarzianand Schwarzian derivatives of analytic functions f are defined in $\mathbb{U}$, where $\mathbb{U}$ is the open unit disk. The pre-Schwarzian as well as Schwarzian derivatives are popular tools for studying the geometric properties of analytic mappings. These can also be used to obtain either necessary or sufficient conditions for the univalence of a function f. Because of the computational difficulty, the pre-Schwarzian norm has received more attention than the Schwarzian norm. It has applications in the theory of hypergeometric functions, conformal mappings, Teichmüller spaces, and univalent functions. In this paper, we find sharp norm estimates of the pre-Schwarzian derivatives of certain subfamilies of analytic functions involving some conic-like image domains. These results may also be extended to the families of strongly starlike, convex, as well as to functions with symmetric and conjugate symmetric points.

Keywords: Möbius functions; upper half-plane; Schwarz function

MSC: 30C45; 30C80

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1. Introduction and Definitions

Let $\mathcal{H}$ be the class of holomorphic or analytic functions f defined in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ and have series representation of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in \mathbb{U}). \tag{1}$$

From the above series representation, it is obvious that $f(0) = f'(0) - 1 = 0$. Let $\mathcal{P}$ denote the class of Carathéodory functions p such that $p(0) = 1, \Re(p(z)) > 0$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

The Möbius function $p_0(z) = \frac{1+z}{1-z}$ is a special case of the known bilinear fractional transformations. This function, or its rotation, acts as an extremal function for the class $\mathcal{P}$ and it maps the open unit disk to the right half-plane. For functions $p \in \mathcal{P}$, we have

$$\frac{1-|z|}{1+|z|} \leq |p(z)| \leq \frac{1+|z|}{1-|z|}. \tag{2}$$

Furthermore, recall the class $\mathcal{P}(\gamma) \subset \mathcal{P}$, $0 \leq \gamma < 1$, consisting of functions $p \in \mathcal{P}(\gamma)$ such that $\mathrm{Re}(p(z)) > \gamma$, $(z \in \mathbb{U})$. For holomorphic or analytic mappings f and h in $\mathbb{U}$, f is subordinate to h, and we write $f \prec h$, and if there is a Schwarz mapping or function ω : $f(z) = g(\omega(z))$, where $z \in \mathbb{U}$. For a univalent function h, we see that

$$f(z) \prec h(z) \text{ if and only if } f(\mathbb{U}) \subseteq h(\mathbb{U}) \text{ and } f(0) = h(0).$$

Additionally, $\mathcal{S} \subset \mathcal{H}$ represents the class of univalent functions, whereas $\mathcal{S}^*$ and $\mathcal{C}$ denote the subclasses of univalent starlike and univalent convex functions, respectively. For more detail and further explanation, see [1]. These families have various applications as seen in [2], and are related with the change in argument of the radius vector and tangent vector of the image of $re^{i\varphi}$ as non-decreasing functions of the angle φ, respectively. We recall that a function $f \in \mathcal{S}^*(\gamma)$, $(0 \leq \gamma < 1)$ if and only if

$$\mathrm{Re}\{zQ(f', f)(z)\} > \gamma, \qquad (z \in \mathbb{U}),$$

where $Q(f', f)(z)$ is such that

$$Q(f', f)(z) = \frac{f'(z)}{f(z)}, \quad z \in \mathbb{U}. \tag{3}$$

A function $f \in \mathcal{SS}^*(v)$ is strongly starlike of order v if and only if

$$\left|\arg\{zQ(f', f)(z)\}\right| < \frac{\pi v}{2} \quad (z \in \mathbb{U}), \tag{4}$$

where $0 < v \leq 1$ and $Q(f', f)$ is defined above by (3). For reference, see [3]. A function $f \in \mathcal{SS}^*(v)$ is strongly starlike with symmetric points if and only if

$$\left|\arg\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\}\right| < \frac{\pi v}{2} \quad (z \in \mathbb{U}).$$

Let $f \in \mathcal{A}$, $0 < \gamma_1, \gamma_2 \leq 1$. Then, $f \in \mathcal{S}_l^*(\gamma_1, \gamma_2)$ if and only if

$$zQ(f', f)(z) \prec l(z),$$

where $Q(f', f)$ is defined above by (3) and

$$l(z) = l(\gamma_1, \gamma_2, e^{\pi i\left(\frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2}\right)})(z) = \left(1 + \frac{\left(1 + e^{\pi i\left(\frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2}\right)}\right)z}{1 - z}\right)^{\frac{\gamma_1 + \gamma_2}{2}}, \qquad (l(0) = 1). \tag{5}$$

Remark 1. *Since*

$$\left(\frac{1 + e^{\pi i\left(\frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2}\right)}z}{1 - z}\right)^{\frac{\gamma_1 + \gamma_2}{2}} = 1 + \sum_{k=1}^{\infty} \binom{\frac{\gamma_1 + \gamma_2}{2}}{k} \left(1 + e^{\pi i\left(\frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2}\right)}\right)^k \left(\frac{z}{1 - z}\right)^k,$$

so by using binomial expansion, we write

$$l(z) = 1 + \sum_{n=1}^{\infty} \Lambda_n z^n,$$

where

$$\Lambda_n = \Lambda_n(\gamma_1, \gamma_2, e^{\pi i\left(\frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2}\right)}) = \sum_{k=1}^{n} \binom{n-1}{k-1} \binom{\frac{\gamma_1 + \gamma_2}{2}}{k} \left(1 + e^{\pi i\left(\frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2}\right)}\right)^k, \ n \geq 1.$$

We note that

$$\Lambda_n = \frac{(\gamma_1 + \gamma_2)\left(1 + e^{\pi i\left(\frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2}\right)}\right)}{2} \, {}_2F_1\left(1 - n, 1 - \frac{\gamma_1 + \gamma_2}{2}; 2; 1 + e^{\pi i\left(\frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2}\right)}\right), \ n \geq 1,$$

where ${}_2F_1$ denotes the Guass hypergeometric function.

For detail, we refer to [4]. The family $\mathcal{S}_t^*(\gamma_1,\gamma_2)$ is introduced in [5]. Obviously, $\mathcal{S}_t^*(\gamma_1,\gamma_2) \subset \mathcal{S}^*$ and also $\mathcal{S}_t^*(\gamma_1,\gamma_2) \subset \mathcal{SS}^*(\gamma), \gamma = \max\{\gamma_1,\gamma_2\}$. A function $f \in \mathcal{S}_t^*(\gamma_1,\gamma_2)$ if f satisfies the inequalities

$$-\frac{\pi\gamma_1}{2} < \arg\{zQ(f',f)(z)\} < \frac{\pi\gamma_2}{2}$$

We note that l as defined above is convex and it maps $\mathbb{U}$ onto $\Omega_{\gamma_1,\gamma_2}$, where

$$\Omega_{\gamma_1,\gamma_2} = \left\{w \in \mathbb{C} : -\frac{\pi\gamma_1}{2} < \arg\{w\} < \frac{\pi\gamma_2}{2}\right\}.$$

Let $f \in \mathcal{A}$ and $\gamma \in \left[\frac{\pi}{2}, \pi\right)$. Then, $f \in \mathcal{N}(\gamma)$ if and only if

$$\left(zQ(f',f)(z) - 1\right) \prec l_\gamma(z),$$

where

$$l_\gamma(z) = \frac{1}{2i\sin\gamma}\log\left(\frac{1+ze^{i\gamma}}{1+ze^{-i\gamma}}\right). \tag{6}$$

For detail, we refer to [6]. We note that

$$\Omega_\gamma \in \left\{w \in \mathbb{C} : 1 + \frac{\gamma - \pi}{2\sin\gamma} < \operatorname{Re}\{w\} < 1 + \frac{\gamma}{2\sin\gamma},\ \frac{\pi}{2} \le \gamma < \pi\right\}.$$

The function $l_\gamma(z)$ defined above is convex univalent in $\mathbb{U}$ with $l_\gamma(0) = l'_\gamma(0) - 1 = 0$ and it maps onto Ω_γ or onto the convex hull of three points (one of which may be that point at infinity) on the boundary of Ω_γ. Thus, $\mathcal{N}(\gamma)$ is a subfamily of starlike functions of order γ, where $\gamma \in [0.2146, 0.5)$, see [7].

In dynamics, we examine how a system behaves under a certain iterative scheme. Asymptotic behaviour under these iteration is of great interest. We concentrate on models that are continuous mappings of actual data. The Schwarzian and pre-Schwarzian derivatives were introduced to one-dimensional dynamics for any sufficiently smooth map. We may categorize maps based on the sign of their Schwarzian derivative. Now, let $\mathcal{UL}$ denote the subfamily of $\mathcal{H}$ such that

$$\mathcal{UL} = \{f \in \mathcal{H} : f'(z) \neq 0\}. \tag{7}$$

For a function $f \in \mathcal{UL}$, the pre-Schwarzian along with Schwarzian derivatives of f are defined as

$$T_f(z) = \frac{zf''(z)}{f'(z)} \text{ and } S_f(z) = T_f(z) - \frac{1}{2}T_f^2(z), z \in \mathbb{U}$$

respectively. The Schwarzian derivative of any Möbius transformation $g(z) = \frac{az+b}{cz+d}$ is zero. Conversely, the Schwarzian derivative is the only derivative which measures the degree to which a function fails to be a Möbius transformation. The second-order ordinary differential equation and Schwarzian derivative can be used to determine the Riemann mapping between any bounded polygon and the upper half-plane or unit circle. This reduces to the Schwarz–Christoffel mapping for polygons with straight edges, which can be derived directly without using the Schwarzian derivative. The pre-Schwarzian as well as Schwarzian derivatives are popular tools for studying the geometric properties of these mappings. They can, for example, be used to obtain either necessary or sufficient conditions for overall univalence, or to obtain specific geometric conditions on the image domain of such mappings. Many researchers have investigated estimates of the pre-Schwarzian as well as Schwarzian norms for a family of injective or univalent functions. Because of the computational difficulty, the pre-Schwarzian norm has received more attention than the Schwarzian norm.

Obradović [8] found bounds on the first- and second-order derivatives of starlike and convex functions respectively. Same results were then investigated by Tuneski [9]. He, along with Irmak [10], studied conditions for some other subfamilies of analytic functions. Singh and Tuneski [11] further extended these results for Janowski-type functions.

Historically, certain differential operators were first known to Riemann, but the first person who actually studied these operators extensively was Schwarz. He investigated and found that such differential operators were invariant with respect to Mobius transformations. These were then known as the Schwarzians. Much later, Lavie [12] showed that under the assumption that $f'(z) \neq 0$, all differential operators of order n on f are invariant with respect to M̉obius transformation, written as $S(f,z)$ and its derivative of orders up to $n-3$. In complex function theory, there are several branches depending on the Schwarzian or pre-Schwarzian derivatives. For a univalent or injective function f, it is obvious that $\left\|T_f\right\| \leq 6$, $\left\|S_f\right\| \leq 6$ and these results are the best possible. In [13], Fait et al. proved that every function $f \in \mathcal{S}^*(\gamma)$ is generalized to a $\sin\left(\frac{\pi\gamma}{2}\right)$-quasiconformal automorphism of $\mathbb{C}$. Thus, we have $\left\|T_f\right\| \leq 6\sin\left(\frac{\pi\gamma}{2}\right)$ and $\left\|S_f\right\| \leq 6\sin\left(\frac{\pi\gamma}{2}\right)$. Moreover, Chiang [14] determined that $\left\|T_f\right\| \leq 6\gamma$. However, such structures are slightly different. We see that T_f and S_f are analytic when f is analytic and meromorphic and $f'(z) \neq 0$ in $\mathbb{U}$. The family $\mathcal{UL}$ defined above by (7) is a vector space as seen in [15], so we define the norm on $f \in \mathcal{UL}$ by

$$||f|| = \sup_{z \in \mathbb{U}} (1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right|.$$

The $\|\ \ \|$ has a significance for Teichḿuller spaces and it is assumed as an element of a Banach space, see [16]. It is obvious that $||f|| < +\infty$ if and only if $f'(z) \neq 0$ in $\mathbb{U}$. Thus, there is a $\varepsilon = \varepsilon(f) > 0$ such that $f \in \mathcal{S}$ for each set

$$\left\{z \in \mathbb{C} : \left|\frac{z-\sigma}{1-\overline{\sigma}}\right| < \varepsilon, |\sigma| < 1\right\},$$

as seen in [17]. The norm $||f||$ also leads to the univalence of a meromorphic function f in $\mathbb{U}$. In fact, if $||f|| \leq 1$, then $f \in \mathcal{S}$ in $\mathbb{U}$ and for $f \in \mathcal{S}$ in $\mathbb{U}$, $||f|| \leq 6$ and $||f|| = 6$ for $e^{-i\theta}k\left(e^{i\theta}z\right)$, $\theta \in \mathbb{R}$ and k is the known Koebe function, see [18]. Additionally, if $f \in \mathcal{S}^*$ of order $\gamma \in [0,1)$, then we have the sharp estimate $||f|| \leq 6-4\gamma$ as found in [17]. For more details on the norm $||f||$, see [19]. Moreover, others have norm estimates as seen in [3,20–22].

Bazilevič, as studied in [23], introduced a family of functions $f(z)$ defined by the following integral representation:

$$f(z) = \left\{\frac{\beta}{1+\delta^2}\int_0^z (h(t)-\delta i)t^{\left(-i\delta\beta/1+\delta^2\right)-1} g(t)^{\frac{\beta}{1+\delta^2}}\,dt\right\}^{\frac{1+\delta i}{\beta}}, \tag{8}$$

where $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n : \operatorname{Re} h(z) > 0$, $g \in \mathcal{S}^*$ in $\mathbb{U}$, δ is real and $\beta > 0$. If we put $\delta = 0$ in (8), then

$$f(z) = \left(\beta \int_0^z h(t)t^{-1}[g(t)]^{\beta} dt\right)^{\frac{1}{\beta}}.$$

From this expression, we note that

$$\operatorname{Re}\left[zQ(f',f)(z)[Q(f,g)(z)]^{\beta}\right] = \operatorname{Re}(h(z)) > 0,\ \ z \in \mathbb{U}. \tag{9}$$

A function f observing the condition given by (9) is called a Bazilevic function of type β. For more detail, see [24] and others. Another important and well-known subclass of $\mathcal{S}$ which has been highly investigated in recent years, as seen in [25–31] and references therein, is the class $\mathcal{SU}$ defined subsequently.

Definition 1. *For f given by* (1), *$f \in \mathcal{SU}$, if and only if*

$$\left|zQ(z,f)(z)Q(f',f)(z) - 1\right| < 1, \tag{10}$$

where $z \in \mathbb{U}$ *and* $Q(.,.)$ *is defined above in* (3).

Although the above class is not strictly related to either $\mathcal{S}^*$ or $\mathcal{C}$, its definition resembles the class of non-Bazilevič functions. Like other fundamental subclasses of $\mathcal{S}$, this class is rotationally invariant.

We now see this class in a more general setting.

Definition 2. *For f given by* (1), $f \in \mathcal{SU}(\lambda)$, *if and only if*

$$\left|zQ(z,f)(z)Q(f',f)(z) - 1\right| < \lambda, \tag{11}$$

where $\lambda \in (0,1]$, $z \in \mathbb{U}$ *and* $Q(.,.)$ *is defined above in* (3).

For various properties of the class of $\mathcal{SU}(\lambda)$, we refer to [28,31–33] and others. It is noted that the class $\mathcal{SU}(\lambda)$ is not invariant under the n^{th} root transformation but preserved under rotation, dilation, omitted-value transformations and conjugation. In the following, we move to another known family as seen in [34].

Definition 3. *For f given by* (1), $f \in \mathcal{SU}(\beta,\lambda)$, *if and only if*

$$\left|zQ(f',f)(z)[Q(f,z)(z)]^{\beta} - 1\right| < \lambda, \tag{12}$$

where $\lambda \in (0,1]$, $\beta > 0$, $z \in \mathbb{U}$ *and* $Q(.,.)$ *is defined above in* (3).

In view of the above structure, we have the following:

Definition 4. *For f given by* (1), *we say* $f \in \mathcal{B}_g^*(\beta)$, *if and only if*

$$\left|zQ(f',f)(z)[Q(f,g)(z)]^{\beta} - 1\right| < 1, \tag{13}$$

where g is starlike, $\beta > 0, z \in \mathbb{U}$ *and* $Q(.,.)$ *is defined above as seen in* (3).

Definition 5. *For f given by* (1), *we say* $f \in \mathcal{B}_g^*(\beta,\lambda)$, *if and only if*

$$\left|zQ(f',f)(z)[Q(f,g)(z)]^{\beta} - 1\right| < \lambda, \tag{14}$$

where g is starlike, $\lambda \in (0,1]$, $\beta > 0$ *and* $Q(.,.)$ *is defined as in* (3).

For details of the related work, we refer to [1,23,24,31].

2. Main Results

Based on the observation and motivation of our main discussion, we find the best norm estimates of pre-Schwarzian derivative for functions f in the subfamilies defined above.

Theorem 1. *Let* $\gamma \in \left[\frac{\pi}{2}, \pi\right)$. *If a function* $f \in \mathcal{SU}$ *such that* $zQ(f',f) - 1 \prec l_\gamma$, *where* l_γ *is given above in* (6), *then*

$$\|f\| \leq \frac{4}{\sin\gamma}\sqrt{\pi^2 + \sin^2\gamma}.$$

This result is sharp.

Proof. Suppose that $f \in \mathcal{SU}$ such that it satisfies the condition

$$\left|zQ(z,f)(z)Q(f',f)(z) - 1\right| < 1,$$

where $Q(.,.)$ is defined by (3). Then in view of (10), we consider a function $s : \mathbb{U} \to \mathbb{U}$ with $s(0) = 0$ such that

$$zQ(z,f)(z)Q(f',f)(z)) = 1 + s(z). \tag{15}$$

Logarithmically differentiating (15), we obtain

$$Q(f'',f')(z) = \frac{s'(z)}{s(z)+1} - \frac{2}{z} + 2Q(f',f)(z).$$

By using the condition $zQ(f',f) - 1 \prec l_\gamma$, where l_γ is given above in (6), we find that

$$zQ(f',f)(z) = 1 + \frac{1}{2i\sin\gamma}\log\left(\frac{s(z)[\cos\gamma + i\sin\gamma]+1}{s(z)[\cos\gamma - i\sin\gamma]+1}\right). \tag{16}$$

Using the Schwarz–Pick lemma as seen in [1], we have

$$|s'(z)| \le \frac{1-|s(z)|^2}{1-|z|^2} \tag{17}$$

and we further see that

$$Q(f'',f')(z) = \frac{s'(z)}{s(z)+1} - \frac{2}{z} + \frac{2}{z} + \frac{1}{zi\sin\gamma}\log\left(\frac{s(z)[\cos\gamma + i\sin\gamma]+1}{s(z)[\cos\gamma - i\sin\gamma]+1}\right).$$

We also note that

$$Q(f'',f')(z) = \frac{s'(z)}{s(z)+1} + \frac{1}{iz\sin\gamma}\log\left(\frac{s(z)[\cos\gamma + i\sin\gamma]+1}{s(z)[\cos\gamma - i\sin\gamma]+1}\right). \tag{18}$$

On the other hand,

$$\log z = \ln|z| + i\arg z, \qquad (z \in \mathbb{U}\backslash\{0\}, -\pi < \arg z \le \pi).$$

It is known that

$$|\log z| \le \begin{cases} \sqrt{|z-1|^2 + \pi^2}, & |z| \ge 1 \\ \sqrt{\left|\frac{z-1}{z}\right|^2 + \pi^2}, & 0 < |z| < 1 \end{cases}. \tag{19}$$

Now according to (19), we discuss the two cases subsequently:
(i) We consider that

$$\left|\frac{s(z)[\cos\gamma + i\sin\gamma]+1}{s(z)[\cos\gamma - i\sin\gamma]+1}\right| \ge 1.$$

Therefore, we obtain

$$\begin{aligned} \left|\log\left(\frac{s(z)[\cos\gamma + i\sin\gamma]+1}{s(z)[\cos\gamma - \sin\gamma i]+1}\right)\right| &\le \sqrt{\left|\frac{s(z)[\cos\gamma + i\sin\gamma]+1}{s(z)[\cos\gamma - i\sin\gamma]+1} - 1\right|^2 + \pi^2} \\ &\le \frac{\sqrt{(\pi^2 + 4\sin^2\gamma)|z|^2 + \pi^2(1+2|z|)}}{1-|z|} \end{aligned} \tag{20}$$

By using (17), (18) and (20), we note that

$$|Q(f'',f')(z)| \le \frac{1}{1+|z|} + \frac{1}{|z|\sin\gamma}\frac{\sqrt{(\pi^2 + 4\sin^2\gamma)|z|^2 + \pi^2(1+2|z|)}}{1-|z|}.$$

Thus, the pre-Schwarzian norm $\|f\|$ of function f takes the form

$$\|f\| = \sup_{z\in\mathbb{U}}\left(1-|z|^2\right)|Q(f'',f')(z)| \le \frac{4}{\sin\gamma}\sqrt{\pi^2 + \sin^2\gamma}.$$

(ii) We also consider that

$$\left|\frac{s(z)[\cos\gamma + i\sin\gamma]+1}{s(z)[\cos\gamma - i\sin\gamma]+1}\right| < 1.$$

Using (19), we observe that

$$\left|\log\left(\frac{1+s(z)[\cos\gamma+i\sin\gamma]}{s(z)[\cos\gamma-i\sin\gamma]+1}\right)\right| \leq \sqrt{\left|\frac{s(z)[\cos\gamma+i\sin\gamma]-s(z)[\cos\gamma-i\sin\gamma]}{s(z)[\cos\gamma+i\sin\gamma]+1}\right|^2+\pi^2}.$$

After simplification, we note that

$$\left|\log\left(\frac{s(z)[\cos\gamma+i\sin\gamma]+1}{s(z)[\cos\gamma-i\sin\gamma]+1}\right)\right| \leq \frac{\sqrt{(\pi^2+4\sin^2\gamma)|z|^2+\pi^2(1+2|z|)}}{1-|z|}.$$

Therefore, we have the same estimates. So, in both cases, we have the desired sharp result. □

Corollary 1. *As an application of the Theorem 1, we define*

$$m(\gamma)=\frac{4}{\sin\gamma}\sqrt{\pi^2+\sin^2\gamma}, \gamma\in\left[\frac{\pi}{2},\pi\right).$$

We note that $\lim_{\gamma\to\pi^-} m(\gamma)=\infty$. *Additionally,*

$$m'(\gamma)=\frac{-4\pi^2\cos\gamma}{\sin^2\gamma\sqrt{\pi^2+\sin^2\gamma}},$$

such that $m'(\gamma)>0$ *for* $\gamma\in\left[\frac{\pi}{2},\pi\right)$. *So,* $m(\gamma)$ *is increasing and thus we find that*

$$13.1876... \approx m\left(\frac{\pi}{2}\right)\leq m(\gamma)<\infty.$$

This estimate shows that if $f\in\mathcal{SU}$, *then it is uniformly locally univalent.*

Theorem 2. *Let* $\gamma_1,\gamma_2\in(0,1]$. *If* $f\in\mathcal{SU}$ *such that* $zQ(f',f)\prec l$, l *is given above in* (5), *then*

$$\|f\|\leq 10, \gamma_1+\gamma_2\longrightarrow 2,$$

where $\mathcal{SU}$ *is defined above by* (10). *These bounds are sharp.*

Proof. Let $f\in\mathcal{SU}$, where $\mathcal{SU}$ is defined above by (10) as

$$\left|zQ(z,f)(z)Q(f',f)(z)-1\right|<1,$$

where $Q(.,.)$ is given by (3). Then, in view of (10), we consider a function $s:\mathbb{U}\to\mathbb{U}$ with $s(0)=0$ such that

$$zQ(z,f)(z)Q(f',f)(z)=1+s(z),$$

which on simplification leads to

$$Q(f'',f')(z)=\frac{s'(z)}{s(z)+1}-\frac{2}{z}+2Q(f',f)(z). \tag{21}$$

By using the condition $zQ(f',f)\prec l$ as given above in the statement of the theorem, we find that

$$zQ(f',f)(z)=\left(1+\frac{\left(1+e^{\pi i\left(\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}\right)}\right)s(z)}{1-s(z)}\right)^{\frac{\gamma_1+\gamma_2}{2}}, \tag{22}$$

where $\left|e^{\pi i\left(\frac{\gamma_2-\gamma_1}{\gamma_1+\alpha_2}\right)}\right|=1$. By using (22) in (21), we have

$$Q(f'',f')(z)=\frac{s'(z)}{s(z)+1}-\frac{2}{z}+2\left(1+\frac{\left(1+e^{\pi i\left(\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}\right)}\right)s(z)}{1-s(z)}\right)^{\frac{\gamma_1+\gamma_2}{2}}.$$

Using the inequality (17) and the identity $\left|e^{\pi i\left(\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}\right)}\right| = 1$, we see that

$$\left|Q(f'',f')(z)\right| = \frac{1+|s(z)|}{1-|z|^2} + \frac{2}{|z|} + 2\left(1 + \frac{2|s(z)|}{1-|s(z)|}\right)^{\frac{\gamma_1+\gamma_2}{2}}.$$

Since s is a Schwarz function and $|s(z)| \leq |z|$, thus the last inequality yields

$$\left|Q(f'',f')(z)\right| \leq \frac{1+|z|}{1-|z|^2} + \frac{2}{|z|} + 2\left(\frac{1+|z|}{1-|z|}\right)^{\frac{\gamma_1+\gamma_2}{2}}.$$

By the definition of the pre-Schwarzian norm, we note that

$$\begin{aligned}
\|f\| &= \sup_{z\in\mathbb{U}}\left(1-|z|^2\right)\left|Q(f'',f')(z)\right| \\
&\leq \sup_{z\in\mathbb{U}}\left(1+|z| + \frac{2\left(1-|z|^2\right)}{|z|} + 2\left(1-|z|^2\right)\left(\frac{|z|+1}{1-|z|}\right)^{\frac{\gamma_1+\gamma_2}{2}}\right) \\
&\leq 10, \gamma_1+\gamma_2 \longrightarrow 2.
\end{aligned}$$

These bounds are sharp □

In light of the above theorem and condition $zQ(f',f) - 1 \prec l_\gamma$, we find the following estimates:

Theorem 3. *Let $\gamma \in \left[\frac{\pi}{2},\pi\right)$. If a function $f \in \mathcal{SU}(\lambda)$ such that $zQ(f',f) - 1 \prec l_\gamma$, where l_γ is given above in (6), then*

$$\|f\| \leq \frac{4}{\sin\gamma}\sqrt{\pi^2+\sin^2\gamma}.$$

where $\lambda \in (0,1]$ and $\mathcal{SU}(\lambda)$ is defined above by (11). This result is sharp.

Proof. Let $f \in \mathcal{SU}(\lambda)$. Then, by rewriting (11), we have

$$\left|zQ(z,f)(z)Q(f',f)(z) - 1\right| < \lambda,$$

where $Q(.,.)$ is represented in (3). In view of (11), there exists a function $s : \mathbb{U} \to \mathbb{U}$ such that $s(0) = 0$ and note that

$$zQ(z,f)(z)Q(f',f)(z) = 1 + \lambda s(z).$$

On logarithmic differentiation, we see that

$$Q(f'',f')(z) = \frac{\lambda s'(z)}{1+\lambda s(z)} - \frac{2}{z} + 2Q(f',f)(z). \tag{23}$$

By using the condition $zQ(f',f) - 1 \prec l_\gamma$ as described above, we have

$$zQ(f',f)(z) = 1 + \frac{1}{2i\sin\gamma}\log\left(\frac{s(z)[\cos\gamma + i\sin\gamma]+1}{s(z)[\cos\gamma - i\sin\gamma]+1}\right). \tag{24}$$

Substituting (24) in (23), we note that

$$Q(f'',f')(z) = \frac{\lambda s'(z)}{1+\lambda s(z)} - \frac{2}{z} + 2\left\{\frac{1}{z} + \frac{1}{2z\sin\gamma}\log\left(\frac{s(z)[\cos\gamma + i\sin\gamma]+1}{s(z)[\cos\gamma - i\sin\gamma]+1}\right)\right\}.$$

On simplification, we write

$$Q(f'',f')(z) = \frac{\lambda s'(z)}{1+\lambda s(z)} + \frac{1}{iz\sin\gamma}\log\left(\frac{s(z)[\cos\gamma + i\sin\gamma]+1}{s(z)[\cos\gamma - i\sin\gamma]+1}\right). \tag{25}$$

In view of (19), we discuss two cases subsequently:
(i) We consider that

$$\left|\frac{s(z)[\cos\gamma + i\sin\gamma] + 1}{s(z)[\cos\gamma - i\sin\gamma] + 1}\right| \geq 1.$$

Therefore, we obtain

$$\left|\log\left(\frac{s(z)[\cos\gamma + i\sin\gamma] + 1}{s(z)[\cos\gamma - i\sin\gamma] + 1}\right)\right| \leq \sqrt{\left|\frac{s(z)[\cos\gamma + i\sin\gamma] + 1}{s(z)[\cos\gamma - i\sin\gamma] + 1} - 1\right|^2 + \pi^2}$$

or it takes the form

$$\left|\log\left(\frac{s(z)[\cos\gamma + i\sin\gamma] + 1}{s(z)[\cos\gamma - i\sin\gamma] + 1}\right)\right| = \frac{\sqrt{(\pi^2 + 4\sin^2\gamma)|z|^2 + \pi^2(1+2|z|)}}{1-|z|}.$$

By using (25), we note that

$$|Q(f'', f')(z)| \leq \frac{\lambda}{1+\lambda|s(z)|}\frac{1-|s(z)|^2}{1-|z|^2} + \frac{\sqrt{(\pi^2 + 4\sin^2\gamma)|z|^2 + \pi^2(1+2|z|)}}{(|z|\sin\gamma)(1-|z|)}.$$

or we write

$$|Q(f'', f')(z)| \leq \frac{\lambda}{1+\lambda|z|} + \frac{\sqrt{(\pi^2 + 4\sin^2\gamma)|z|^2 + \pi^2(1+2|z|)}}{(|z|\sin\gamma)(1-|z|)}.$$

Thus, the pre-Schwarzian norm $\|f\|$ takes the form

$$\|f\| = \sup_{z\in\mathbb{U}}\left(1-|z|^2\right)|Q(f'', f')(z)| \leq \frac{4}{\sin\gamma}\sqrt{\pi^2 + \sin^2\gamma}.$$

(ii) Moreover, we consider that

$$\left|\frac{s(z)[\cos\gamma + i\sin\gamma] + 1}{s(z)[\cos\gamma - i\sin\gamma] + 1}\right| < 1,\ s(0) = 0, |s(0)| < 1.$$

Using (19) and on simplification, we note that

$$\begin{aligned}
&\left|\log\left(\frac{s(z)[\cos\gamma + i\sin\gamma] + 1}{s(z)[\cos\gamma - i\sin\gamma] + 1}\right)\right| \\
&\leq \sqrt{\frac{|s(z)[\cos\gamma + i\sin\gamma] + 1 - (s(z)[\cos\gamma - i\sin\gamma] + 1)|^2}{|s(z)[\cos\gamma + i\sin\gamma] + 1|^2} + \pi^2} \\
&\leq \frac{\sqrt{(\pi^2 + 4\sin^2\gamma)|z|^2 + \pi^2(1+2|z|)}}{1-|z|}.
\end{aligned}$$

Therefore, in these cases we have the same estimates. So, in view of these cases, we obtain the required sharp result. □

By using the condition $zQ(f', f) \prec l$ and in view of (11), we have the following estimates:

Theorem 4. *Let $\gamma_1, \gamma_2 \in (0,1]$. If $f \in \mathcal{SU}(\lambda)$ such that $zQ(f', f) \prec l, l$ is given above in* (5)*, then*

$$\|f\| \leq 8, \gamma_1 + \gamma_2 \longrightarrow 2,$$

where $\mathcal{SU}(\lambda)$ is defined above by (11)*. These bounds are sharp.*

Proof. For $\gamma_1, \gamma_2 \in (0,1]$, suppose $f \in \mathcal{SU}(\lambda)$ such that

$$|zQ(z, f)(z)Q(f', f)(z) - 1| < \lambda,$$

where $Q(.,.)$ is defined above as in (3). Then there exists a function $s : \mathbb{U} \to \mathbb{U}$ such that $s(0) = 0$ such that

$$zQ(z,f)(z)Q(f',f)(z) = 1 + \lambda s(z). \tag{26}$$

On differentiating (26), we note that

$$Q(f'',f')(z) = \frac{\lambda s'(z)}{1+\lambda s(z)} - \frac{2}{z} + 2Q(f',f)(z). \tag{27}$$

By using the condition $zQ(f',f) \prec l$ as given above in the statement of the theorem, we find that

$$Q(f'',f')(z) = -\frac{2}{z} + \frac{\lambda s'(z)}{1+\lambda s(z)} + 2\left(1 + \frac{\left(1+e^{\pi i\left(\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}\right)}\right)s(z)}{1-s(z)}\right)^{\frac{\gamma_1+\gamma_2}{2}}.$$

By applying both the inequality (17) and the identity $\left|e^{\pi i\left(\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}\right)}\right| = 1$, the above equation leads to

$$|Q(f'',f')(z)| \leq \frac{2}{|z|} + \frac{\lambda|s'(z)|}{1+\lambda|s(z)|} + 2\left(1+\frac{2|s(z)|}{1-|s(z)|}\right)^{\frac{\gamma_1+\gamma_2}{2}}$$

or we can write

$$|Q(f'',f')(z)| \leq \left(\frac{1-|s(z)|^2}{1-|z|^2}\right)\left(\frac{\lambda}{1+\lambda|s(z)|}\right) + \frac{2}{|z|} + 2\left(1+\frac{2|s(z)|}{1-|s(z)|}\right)^{\frac{\gamma_1+\gamma_2}{2}}.$$

We use $|s(z)| \leq |z|$ in the last inequality to obtain

$$|Q(f'',f')(z)| \leq \frac{2}{|z|} + \frac{\lambda}{1+\lambda|z|} + 2\left(\frac{1+|z|}{1-|z|}\right)^{\frac{\gamma_1+\gamma_2}{2}}.$$

Thus,

$$\begin{aligned}
\|f\| &= \sup_{z\in\mathbb{U}}\left(1-|z|^2\right)|Q(f'',f')(z)| \\
&\leq \sup_{z\in\mathbb{U}}\left(\frac{2\left(1-|z|^2\right)}{|z|} + \frac{\lambda\left(1-|z|^2\right)}{1+\lambda|z|} + 2\left(1-|z|^2\right)\left(\frac{1+|z|}{1-|z|}\right)^{\frac{\gamma_1+\gamma_2}{2}}\right) \\
&\leq 8,\ \gamma_1+\gamma_2 \longrightarrow 2.
\end{aligned}$$

This leads to the desired proof. □

In view of the condition $zQ(f',f) - 1 \prec l_\gamma$, we find the following estimates for functions in the class $\mathcal{B}_g^*(\beta)$.

Theorem 5. *Let $\gamma \in \left[\frac{\pi}{2}, \pi\right)$. If a function $f \in \mathcal{B}_g^*(\beta)$ such that $zQ(f',f) - 1 \prec l_\gamma$, where l_γ is given above in (6), then*

$$\|f\| \leq \frac{2}{\sin\gamma}\left(\sin\gamma + \sqrt{\pi^2 + \sin^2\gamma}\right) + 4\beta,$$

where $\lambda \in (0,1], \beta > 0$ and $\mathcal{B}_g^*(\beta)$ is defined above by (13). These bounds are sharp.

Proof. Let $f \in \mathcal{B}_g^*(\beta)$ be defined above by (13) such that

$$\left|zQ(f',f)(z)[Q(f,g)(z)]^\beta - 1\right| < 1,$$

where $Q(.,.)$ is given above in (3). Then in view of Definition 4, there exists a function $s : \mathbb{U} \to \mathbb{U}$ with $s(0) = 0$ such that

$$zQ(f', f)(z)[Q(f, g)(z)]^{\beta} = 1 + s(z).$$

Logarithmically differentiating the above expression, we have

$$Q(f'', f')(z) = \frac{s'(z)}{s(z)+1} - \frac{1}{z} + (1-\beta)Q(f', f)(z) + \beta Q(g', g)(z).$$

By using the condition $zQ(f', f) - 1 \prec l_{\gamma}$ as described above in (16), we write

$$Q(f'', f')(z)$$
$$= \frac{s'(z)}{s(z)+1} - \frac{1}{z} + \frac{1-\beta}{z} + \frac{1-\beta}{2zi\sin\gamma} \log \frac{s(z)[\cos\gamma + i\sin\gamma] + 1}{s(z)[\cos\gamma - ei\sin\gamma] + 1} + \beta Q(g', g)(z)$$

or we see that

$$Q(f'', f')(z)$$
$$= \frac{s'(z)}{s(z)+1} - \frac{\beta}{z} + \frac{1-\beta}{2iz\sin\gamma} \log \frac{s(z)[\cos\gamma + i\sin\gamma] + 1}{s(z)[\cos\gamma - i\sin\gamma] + 1} + \beta Q(g', g)(z). \tag{28}$$

In view of (19), we discuss the following cases subsequently:

(i) Assume that

$$\left| \frac{s(z)[\cos\gamma + i\sin\gamma] + 1}{s(z)[\cos\gamma - i\sin\gamma] + 1} \right| \geq 1.$$

Therefore, we note that

$$\left| \log\left(\frac{s(z)[\cos\gamma + i\sin\gamma] + 1}{s(z)[\cos\gamma - i\sin\gamma] + 1} \right) \right| \leq \sqrt{\left| \frac{e^{i\gamma}s(z) \ + 1}{e^{-i\gamma}s(z) \ + 1} - 1 \right|^2 + \pi^2},$$

which leads to

$$\left| \log\left(\frac{s(z)[\cos\gamma + i\sin\gamma] + 1}{s(z)[\cos\gamma - i\sin\gamma] + 1} \right) \right| = \frac{\sqrt{(\pi^2 + 4\sin^2\gamma)|z|^2 + \pi^2(1+2|z|)}}{1-|z|} \tag{29}$$

In view of (29) and (28), we note that

$$|Q(f'', f')(z)|$$
$$\leq \frac{1+|z|}{1-|z|^2} + \frac{\beta}{|z|} + \frac{(1+\beta)\sqrt{(\pi^2 + 4\sin^2\gamma)|z|^2 + \pi^2(1+2|z|)}}{2|z|\sin\gamma(1-|z|)} + \beta|Q(g', g)(z)|$$

By using the upper bounds for $p \in \mathcal{P}$ as given by (2), the pre-Schwarzian norm $\|f\|$ of a function f takes the form

$$\|f\| = \sup\left(1-|z|^2\right)|Q(f'', f')(z)| \leq \frac{2}{\sin\gamma}\left(\sin\gamma + \sqrt{\pi^2 + \sin^2\gamma}\right) + 4\beta.$$

(ii) Assume that

$$\left| \frac{s(z)[\cos\gamma + i\sin\gamma] + 1}{s(z)[\cos\gamma - i\sin\gamma] + 1} \right| < 1.$$

Using (19) and then simplifying, we write

$$
\begin{aligned}
&\left|\log\left(\frac{1+s(z)[\cos\gamma+i\sin\gamma]}{s(z)[\cos\gamma-i\sin\gamma]+1}\right)\right| \\
&\leq \sqrt{\left|\frac{s(z)[\cos\gamma+i\sin\gamma]+1-(s(z)[\cos\gamma-i\sin\gamma]+1)}{s(z)[\cos\gamma+i\sin\gamma]+1}\right|^2+\pi^2} \\
&= \frac{\sqrt{(\pi^2+4\sin^2\gamma)|z|^2+\pi^2(1+2|z|)}}{1-|z|}.
\end{aligned}
$$

Continuing as above, again we have

$$
\|f\| = \sup\left(1-|z|^2\right)|Q(f'',f')(z)| \leq \frac{2}{\sin\gamma}\left(\sin\gamma+\sqrt{\pi^2+\sin^2\gamma}\right)+4\beta.
$$

Therefore, in these cases, we have equal sharp estimates. □

In view of the condition $zQ(f',f) \prec l$, where l is given above in (5), we find the following estimates for functions in the class $\mathcal{B}_g^*(\beta)$.

Theorem 6. *If $f \in \mathcal{B}_g^*(\beta)$ such that $zQ(f',f) \prec l, l$ is given above in (5), then*

$$
\|f\| \leq 10+4\beta,\ \gamma_1+\gamma_2 \longrightarrow 2,
$$

where $\mathcal{B}_g^*(\beta)$ is defined above by (13). These bounds are sharp.

Proof. We assume that $f \in \mathcal{B}_g^*(\beta)$. In view of Definition 4, f satisfies the condition

$$
\left|zQ(f',f)(z)[Q(f,g)(z)]^\beta-1\right|<1,
$$

where $Q(.,.)$ is defined above as in (3). Then, in view of Definition 4, there exists a function $s:\mathbb{U}\to\mathbb{U}$ with $s(0)=0$ such that

$$
zQ(f',f)(z)[Q(f,g)(z)]^\beta = 1+s(z).
$$

Logarithmically differentiating the above expression, we have

$$
Q(f'',f')(z) = \frac{s'(z)}{s(z)+1}-\frac{1}{z}+(1-\beta)Q(f',f)(z)+\beta Q(g',g)(z) \tag{30}
$$

By using (22) in (30), we obtain

$$
Q(f'',f')(z) = \frac{s'(z)}{s(z)+1}-\frac{1}{z}+(1-\beta)\left(1+\frac{\left(1+e^{\pi i\left(\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}\right)}\right)s(z)}{1-s(z)}\right)^{\frac{\gamma_1+\gamma_2}{2}}+\beta Q(g',g)(z)
$$

By using the inequality (17) and the identity $\left|e^{\pi i\left(\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}\right)}\right|=1$ in the above equation, we note that

$$
|Q(f'',f')(z)| = \frac{|s(z)|+1}{1-|z|^2}+\frac{1}{|z|}+(1+\beta)\left(\frac{|s(z)|+1}{1-|s(z)|}\right)^{\frac{\gamma_1+\gamma_2}{2}}+\beta|Q(g',g)(z)|.
$$

In view of the inequality $|s(z)| \leq |z|$, the last expression leads to

$$
|Q(f'',f')(z)| \leq \frac{1+|z|}{1-|z|^2}+\frac{2}{|z|}+2\left(\frac{1+|z|}{1-|z|}\right)^{\frac{\gamma_1+\gamma_2}{2}}+\beta|Q(g',g)(z)|.
$$

As $\|f\| = \sup_{z\in\mathbb{U}}\left(1-|z|^2\right)|Q(f'',f')(z)|$, by using the bounds for $p \in \mathcal{P}$ given above in (2), we have

$$\begin{aligned}\|f\| &\leq \sup_{z\in\mathbb{U}}\left(1+|z|-2\frac{|z|^2-1}{|z|}+2\left(1-|z|^2\right)\left(\frac{1+|z|}{1-|z|}\right)^{\frac{\gamma_1+\gamma_2}{2}}+\beta\left(1-|z|^2\right)\frac{1+|z|}{1-|z|}\right)\\ &\leq 10+4\beta, \quad \gamma_1+\gamma_2 \longrightarrow 2.\end{aligned}$$

□

Remark 2. *In view of the subfamilies $\mathcal{SU}(\beta,\lambda)$ and $\mathcal{B}_g^*(\beta,\lambda)$ defined by (12) and (14), we can add some more problems by using similar procedures as seen in the above theorems.*

3. Concluding Remarks

Some differential operators were known to Riemann. However, Schwarz extensively studied these operators and found them invariant under Mobius transformations. These were then known as the Schwarzians. Much later, Lavie along with others extended these results. There are several branches depending on the Schwarzian or pre-Schwarzian derivatives. For univalent, starlike and other function classes, similar results were extensively studied in the literature. We studied the pre-Schwarzian derivatives in the w-plane and found sharp norm estimates for some known classes of analytic functions. Our findings can be related to the existing literature and these results may also be extended to the families of strongly starlike, convex as well as functions with symmetric and conjugate symmetric points.

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