



# Article Characterization of Positive Invariance of Quadratic Convex Sets for Discrete-Time Systems Using Optimization Approaches

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Abstract: A positively invariant set is an important concept in dynamical systems. The study of positively invariant set conditions for discrete-time systems is one interesting topic in both theoretical studies and practical applications research. Different methods for characterizing the invariance of different types of sets have been established. For example, the ellipsoidal and the Lorenz cone, which are quadratic convex sets, have different properties from a polyhedral set. This paper presents an optimization method and a dual optimization method to characterize the positive invariance of the ellipsoidal and the Lorenz cone. The proposed methods are applicable to both linear and nonlinear discrete-time systems. Using nonlinear programming and an induced norm, the positive invariance condition problems are transformed into optimization problems, and the dual optimization method is also used to give equivalent dual forms. Fewer results on the positive invariance condition of Lorenz cones can be found than for the other type of set; this paper fulfills the results of this problem. In addition, the proposed methods in this paper provide more options for checking the positive invariance of quadratic convex sets from the perspective of optimization and dual optimization. The effectiveness of this method is demonstrated by numerical examples.

Keywords: discrete-time dynamical systems; positive invariance; ellipsoid; Lorenz cone; optimization

MSC: 90C25; 90C46; 93C05; 93C10

## 1. Introduction

A positively invariant set is an important concept in dynamical systems. As long as the initial state and the subsequent trajectory of the system are always in a particular positively invariant set, the state quantity of the system can be guaranteed to remain in the positively invariant set. Due to their good properties, positively invariant sets play an important role in the study of system stability analysis and feedback controller design [1–5]. Lyapunov's stability theory provides theoretical support for the study of the stability of dynamical systems. The characterization of the positively invariant set is one of the main topics in the study of positively invariant sets. Bitsoris first proposed in [6] the sufficient and necessary conditions for the polyhedral set to be the positively invariant set of a linear discrete-time system. Ellipsoids and Lorenz cones are general convex sets, both of which have quadratic forms, and the Lorenz cone itself has a constraint, which makes the study of its positive invariance more difficult than those of polyhedral sets. The Riccati equation approach is proposed in [7] for the positive invariance of an ellipsoidal set of a linear discrete-time system. The maximum invariant ellipsoids for discrete-time systems are also obtained in [8,9] using linear matrix inequalities and bilinear matrix inequalities. A special class of Lorenz cones is constructed using Dickin ellipsoids and some hyperplanes in [10], and the invariant cones of a given system are studied; the



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**Copyright:** © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). structure of the constructed cones is studied by the eigenvalue structure of the matrix in the ellipsoidal expression. The ellipsoidal positively invariant set of the Lorenz system has been estimated in [11] for all positive values of the parameters of the Lorenz system, and the minimum volume value of the ellipsoid has been obtained. There are many ways to determine whether a set is a positively invariant set of the dynamical system. The Lyapunov function is a classical method for studying the stability of systems, and a new method for solving the positively invariant set of Lorenz chaotic systems was obtained in [12] by constructing the Lyapunov function. The problem of the finite-time stability of the closed invariant sets of a class of nonlinear systems is discussed in [13] using Lyapunov functions. Ref. [14] is an excellent review paper about the conditions for positive invariance, or, from an algebraic perspective, the sufficient and necessary conditions for ellipsoidal sets to be the positively invariant sets of linear systems. Ref. [15] investigated sufficient conditions for the existence of robust positively invariant sets for switching systems with an average dwell time based on a novel sequence-based technique. In [16], a sufficient and necessary condition for an ordered class of sets to be a positively invariant set for nonlinear discrete-time systems is proposed from the point of view of the existence of monotone mappings. The condition for ellipsoidal invariance is proposed in [17,18] by the method of linear matrix inequality (LMI), and the optimal solution to the LMI problem is combined to determine whether the maximum ellipsoid is obtained. The S-procedure is an effective method for studying the positive invariance conditions for quadratic convex sets, and the sufficient and necessary conditions for ellipsoidal sets and Lorenz cones to be the positively invariant sets of linear systems are given in [19] based on the Lyapunov stability and S-procedure, but the invariance conditions for Lorenz cones are more complicated. The nonlinear programming techniques in [20,21] provide an alternative way to study positive invariance. The problem of estimating the maximum robust invariant set for discrete-time nonlinear regenerative systems in an optimal control framework is considered in [22]. The study of polyhedral sets' positive invariances via optimization is investigated in [23,24]. In [25], Nagumo's theorem is used to study the sufficient and necessary positive invariance conditions of the ellipsoid and Lorenz cone for continuous-time systems using optimization techniques. The existence of the solution to the optimization problem is discussed by the Karush–Kuhn–Tucker (KKT) condition.

There are many results on the positive invariance set of a dynamical system, though most of the studies are about polyhedral sets, and there are fewer discussions on the positive invariance of Lorenz cones, which contain a constraint of their own, thereby making the problem more difficult. The main contribution of this paper is the presentation of the positive invariance condition of ellipsoids and Lorenz cones by virtue of the optimization approach and dual form. For convex sets of quadratic forms, such as Lorenz cones, three positive invariance conditions are proposed in this paper using optimization methods. The proposed Lagrange and Wolfe dual optimization methods in [26–28] can sometimes simplify the primal problem. The method proposed in this paper establishes a connection among positive invariance conditions, optimization, dual optimization, and the induced norm, which provides an optional method for the positive invariance characterization of quadratic convex sets for nonlinear and linear discrete-time dynamical systems.

The optimization method and dual form have been successfully used in our previous work [24]. Additionally, in [24], three sufficient and necessary positive invariance conditions are presented for polyhedral sets and polyhedral cones of discrete-time systems. In this paper, the optimization approach and dual form are also used, however, they are used for ellipsoidal sets combined with the induced norm parametrization. The positive invariance condition for the Lorenz cone is also studied. To check the positive invariance, one only needs to check the sign of the optimization problem's maximum value, which makes it simpler. Moreover, in numerical examples, this paper avoids the complexity of checking the positive invariance of the max–min, as in [24].

The rest of this paper is organized as follows: Section 2 provides some preliminary knowledge and definitions. In Section 3, the sufficient and necessary conditions for the

positive invariances of ellipsoids in nonlinear and linear discrete-time systems are studied, respectively. In Section 4, the conditions of the positive invariance of the Lorenz cone are studied. An illustrative example is given in Section 5. The conclusions of this paper are summarized in Section 6.

**Notations:**  $x_k$ ,  $x_{k+1} \in \mathbb{R}^n$  denotes the state vector, where  $k \in \mathbb{N} \cup \{0\}$ . The set of real numbers is given by  $\mathbb{R}$ .  $\mathbb{R}^n$  denotes a column vector in dimension  $n \times 1$ .  $\mathbb{R}^{n \times n}$  denotes a real square matrix of an  $n \times n$  dimension.  $Q \succ 0 (\prec 0)$  means Q is a positive(negative) definite matrix.

## 2. Mathematical Preliminaries

2.1. Discrete-Time Dynamical Systems

Discrete-time systems are the systems mainly considered in this paper, and the forms of linear and nonlinear discrete-time dynamical systems are given in the following form:

$$x_{k+1} = A x_k. \tag{1}$$

$$x_{k+1} = f_d(x_k).$$
 (2)

where, *A* is an *n* by *n* dimensional matrix.  $x_k, x_{k+1} \in \mathbb{R}^n$  denotes the state vector, where  $k \in \mathbb{N} \cup \{0\}$ .  $f_d(x_k)$  denotes a continuous differentiable function on  $\mathbb{R}^n \to \mathbb{R}^n$ .

**Hypothesis 1.** Suppose  $f_d(x)$  in (2) is a continuous differentiable function mapping  $\mathbb{R}^n \to \mathbb{R}^n$ , and the ellipsoid and the Lorenz cone are nonempty.

**Definition 1** (Positively invariant set). *The set D is a positively invariant set of discrete-time systems if and only if*  $x_k \in D$  *implies*  $x_{k+1} \in D$  *for all*  $k \in \mathbb{N} \cup \{0\}$ .

### 2.2. Convex Sets

Convex sets in a quadratic form are the main study object of this paper, namely, ellipsoidal sets and Lorenz cones. The ellipsoidal set is defined as

$$S = \{ x \in \mathbb{R}^n \mid x^T Q x \le 1 \},\tag{3}$$

where  $Q \in \mathbb{R}^{n \times n}$ , and  $Q \succ 0$ . The ellipsoidal set is also often written as a unit ball in the form of a quadratic norm, namely,

$$S(Q,1) = \{ x \in \mathbb{R}^n \mid \sqrt{x^T Q x} \le 1 \}.$$
(4)

Let

then

$$S(Q,1) = \{ x \in \mathbb{R}^n \mid ||x||_Q \le 1 \}.$$
(5)

Any ellipsoid whose center is not at the origin can be transformed into an ellipsoid whose center is at the origin. Therefore, this paper discusses the invariance condition for the ellipsoidal set whose center is at the origin. The Lorenz cone is defined as

$$S_{L} = \{ x \in \mathbb{R}^{n} \mid x^{T} P x \le 0, x^{T} P U_{n} \le 0 \},$$
(6)

where  $P \in \mathbb{R}^{n \times n}$  is a symmetric nonsingular matrix with only one negative eigenvalue,  $\lambda_n$ .  $U_n$  is the eigenvector corresponding to the negative eigenvalue  $\lambda_n$ .

$$\|x\|_Q = \sqrt{x^T Q x},$$

## 2.3. Lagrange Function

The optimization problem is written in the following form:

$$\begin{array}{ll} \min & f(x) \\ s.t & g_i(x) \leq 0, \quad i = 1, 2, \dots, m, \\ & h_j(x) = 0, \quad j = 1, 2, \dots, n. \end{array}$$
 (7)

where the functions  $f : \mathbb{R}^n \to \mathbb{R}$ ,  $g_i(x) : \mathbb{R}^n \to \mathbb{R}$ , i = 1, 2, ..., m,  $h_j(x) : \mathbb{R}^n \to \mathbb{R}$ , j = 1, 2, ..., n are continuously differentiable. Additionally,  $g_i(x)$  denotes the inequality constraints, and  $h_j(x)$  denotes all of the equality constraints for all i, j. Define the Lagrange function as follows:

$$L(x,\lambda,\mu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{n} \mu_j h_j(x).$$
(8)

where  $\lambda_i$ ,  $\mu_i$  are called the Lagrange operator, and it is required that  $\lambda_i \ge 0$ , i = 1, 2, ..., m.

#### 2.4. Wolfe Dual Theory

Let f(x) be a convex differentiable function with respect to  $x \in \mathbb{R}^n$ , and  $g_i(x)$  is a differentiable convex function. Denote the gradient of the function g(x) by  $\nabla g(x)$ , i.e.,  $\nabla g(x) = (\frac{\partial g(x)}{\partial x_1}, \frac{\partial g(x)}{\partial x_2}, \dots, \frac{\partial g(x)}{\partial x_n})^T$ . The primal problem is in the following form:

$$\begin{array}{ll} \min & f(x) \\ s.t & g_i(x) \le 0, \quad i = 1, 2, \dots, m. \end{array}$$
(9)

The Wolfe dual form is:

$$max \quad f(x) + \sum_{i=1}^{m} c_i g_i(x)$$
  
s.t  $\nabla f(x) = -\sum_{i=1}^{m} c_i \nabla g_i(x), \quad c_i \ge 0.$  (10)

#### 2.5. Slater Condition

For the convex programming problem  $min\{f(x) \mid g(x) \le 0, h(x) = 0, x \in \mathbb{R}^n\}$ , if there exists a feasible point  $\bar{x}$  such that  $g(\bar{x}) < 0$ , the programming problem is said to satisfy the Slater constraint qualification, which is also known as the Slater condition.

**Hypothesis 2.** Suppose the Slater condition can be satisfied for the optimization problem in this paper, which means the inner point of the constraints is nonempty.

#### 3. Invariance Conditions for Ellipsoids

The necessary and sufficient conditions for checking the positive invariance of ellipsoids sets for linear and nonlinear discrete-time systems are investigated in this section. First, by virtue of the Lagrange duality and the Wolfe duality, three equivalent optimization models for the invariance conditions of nonlinear discrete-time systems are given. Then, the positive invariance condition of the ellipsoid is studied when  $f_d(x_k) = Ax_k$ .

## 3.1. Formulation of Positive Invariance Conditions

The invariance conditions for discrete-time systems, including linear and nonlinear systems, are discussed, respectively, in this section.

**Theorem 1.** For the ellipsoidal set (3) and the nonlinear discrete-time system (2), the sufficient and necessary condition for the ellipsoidal set to be a positively invariant set of the nonlinear discrete-time system is that the optimal value of the following optimization problem is non-negative.

$$\min_{x} \quad 1 - f_d^T(x)Qf_d(x) 
s.t \quad x^TQx - 1 \le 0.$$
(11)

**Proof.** The set *S* is a positively invariant set of the system (2) if and only if  $x_k \in S$ , and  $x_{k+1} \in S$ . It is necessary to satisfy

$$x_k^T Q x_k - 1 \le 0, \tag{12}$$

$$1 - f_d(x)^T Q f_d(x) \ge 0,$$
(13)

where  $x_k$ ,  $x_{k+1} \in \mathbb{R}^n$  denotes the state vector, and  $k \in \mathbb{N} \cup \{0\}$ . Equation (12) holds, which implies that (13) also holds. It is necessary that all function values of (13) are greater than or equal to zero, i.e., the minimum value of (13) is satisfied by being non-negative. Then, an optimization is formulated as

$$\min_{x} \quad 1 - f_d^T(x)Qf_d(x) \\
s.t \quad x^TQx - 1 \le 0.$$
(14)

When the system is a linear discrete-time system, i.e., with  $f_d(x) = Ax$ , the invariance condition is presented by Theorem 2.

**Theorem 2.** *The ellipsoidal set S is a positively invariant set of the linear discrete-time system* (1) *if and only if the optimal value of the following optimization problem is non-negative.* 

$$\min_{x} \quad 1 - x^{T} A^{T} Q A x$$

$$s.t \quad x^{T} Q x - 1 \le 0.$$
(15)

Since the proof procedure of Theorem 2 is similar to that of Theorem 1, it is not repeated here.

For the ellipsoids in (5), the sufficient and necessary conditions using the induced norm to check the positive invariance of the linear discrete-time systems are given below.

**Lemma 1.** A vector norm  $\| \bullet \|$  on  $\mathbb{C}^n$  is known, and, for any square matrix  $M \in \mathbb{C}^{n \times n}$ , let  $\| M \| = \sup_{\|x\|=1} \| Mx \|$ . Then, M is said to be the induced norm of the vector norm  $\| x \|$ .

**Theorem 3.** The ellipsoidal set S(Q, 1) is a positively invariant set of the linear discrete-time system  $x_{k+1} = Ax_k$  if and only if the optimal value of the optimization problem below is positive.

$$\min_{x} - x^{T} (A^{T} Q A - Q) x$$

$$s.t \quad x^{T} Q x - 1 = 0.$$
(16)

**Proof.** From the definition of the induced norm in Lemma 1, a sufficient and necessary condition for the ellipsoidal set S(Q, 1) to be a positively invariant set of a linear discrete-time system is that (17) holds for all x.

$$||A||_Q = \sup_{\sqrt{x^T Q x = 1}} \sqrt{(Ax)^T Q(Ax)} < 1.$$
(17)

That is,

$$(Ax)^{T}Q(Ax) - 1 < 0 \Rightarrow x^{T}A^{T}QAx - x^{T}Qx < 0,$$
  
$$\Rightarrow x^{T}(A^{T}QA - Q)x < 0,$$
  
$$\Rightarrow A^{T}QA - Q \prec 0.$$
 (18)

Transform (18) into an optimization problem with the following constraints:

$$\begin{aligned}
\min_{x} & -x^{T}(A^{T}QA - Q)x \\
s.t & x^{T}Qx - 1 = 0.
\end{aligned}$$
(19)

From Theorems 1 and 2, the positive invariance problem is formulated as an optimization problem. In practice, the dual approach sometimes simplifies the solving of the primal problem. Next, the Lagrange dual and the Wolfe dual forms of (11), (15), and (16) are discussed, respectively.

#### 3.2. Lagrange Dual

The Lagrange dual is a convex optimization problem regardless of whether the primal problem is convex or not, and the dual gives at least a lower bound on the optimal value of the primal problem. In this section, the Lagrange dual form of the primal problem is considered. Theorem 4 gives the positive invariance conditions in the Lagrange dual form of the ellipsoidal set for the nonlinear dynamical systems.

**Theorem 4.** Consider the nonlinear discrete-time system  $x_{k+1} = f_d(x_k)$  and the ellipsoidal set be  $S = \{x \in \mathbb{R}^n \mid x^T Qx \le 1\}$ , where  $Q \in \mathbb{R}^{n \times n}$ , and  $Q \succ 0$ . Let  $1 - f_d^T(x)Qf_d(x)$  be a continuous differentiable function with respect to x. The ellipsoidal set S is a positively invariant set of the nonlinear discrete-time system (2) if and only if there exists  $\lambda \ge 0$  such that the optimal value of the following optimization problem is non-negative.

$$\max_{\lambda>0} \min_{x} \quad 1 - f_d^T(x)Qf_d(x) + \lambda(x^TQx - 1)$$
(20)

**Proof.** Let (11) be the primal problem. Next, we derive its dual form M(x). Let the Lagrange multiplier be  $\lambda$  and  $\lambda \ge 0$ . Then, the Lagrange function is

$$L(x,\lambda) = 1 - f_d^T(x)Qf_d(x) + \lambda(x^TQx - 1).$$
(21)

Define

$$\max_{\lambda \ge 0} \quad 1 - f_d^T Q f_d(x) + \lambda (x^T Q x - 1) = \begin{cases} \infty, & otherwise \\ M(x), & x^T Q x \le 1 \end{cases}$$

The,  $\min_{x \in D^{H}} \max_{\lambda \geq 0} L(x, \lambda)$  is equivalent to the primal problem that satisfies the constraint, i.e.,

$$\min_{x \in \mathbb{R}^n} \max_{\lambda \ge 0} L(x, \lambda) = \min_{x \in \mathbb{R}^n} \{ 1 - f_d^T Q f_d(x) | x^T Q x - 1 \le 0 \}$$
(22)

the Lagrange dual of the primal problem (11) is

$$\max_{\lambda \ge 0} \min_{x \in \mathbb{R}^n} 1 - f_d^T Q f_d(x) + \lambda (x^T Q x - 1)$$
(23)

in te reason that the Lagrange dual satisfies

$$\min_{x \in \mathbb{R}^n} \max_{\lambda \ge 0} L(x, \lambda) \ge \max_{\lambda \ge 0} \min_{x \in \mathbb{R}^n} L(x, \lambda).$$
(24)

As a result, the optimal value of the primal problem must be non-negative when the optimal value of the dual problem (20) is non-negative.  $\Box$ 

**Remark 1.** Lagrange dual forms are valid only when the optimal value function of the inner optimization problem can be reduced to an analytic formula. Therefore, in Theorem 4, it is necessary to satisfy the requirement that  $1 - f_d^T(x)Qf_d(x)$  is a continuous function differentiable with respect to x.

Similarly, sufficient conditions are given for the ellipsoidal sets to be positively invariant sets of linear discrete-time systems.

**Theorem 5.** Let the linear discrete-time system be  $x_{k+1} = Ax_k$  and the ellipsoidal set be  $S = \{x \in \mathbb{R}^n \mid x^TQx \leq 1\}$ , where  $Q \in \mathbb{R}^{n \times n}$ , and  $Q \succ 0$ . Then, the ellipsoidal set is a positively invariant set of the linear discrete-time system if and only if there exists  $\lambda \in [0, 1]$  such that the optimal value of the following optimization problem is non-negative.

$$\max_{0 \le \lambda \le 1} \min_{x} \quad 1 - x^{T} A^{T} Q A x + \lambda (x^{T} Q x - 1)$$
(25)

**Proof.** The proof of the Lagrange dual form is similar to Theorem 3, and, here, the focus is on the range of values of the Lagrange multipliers. Let the Lagrange function of the primal problem be

$$L(x,\lambda) = 1 - x^T A^T Q A x + \lambda (x^T Q x - 1), \quad \lambda \ge 0.$$
(26)

Take *x* = 0, the positive invariance condition should also satisfy  $L(0, \lambda) \ge 0$ , i.e.,

$$1-\lambda \geq 0$$
,

that is

 $0 \le \lambda \le 1. \tag{27}$ 

**Theorem 6** ([19]). The ellipsoidal set *S* is a positively invariant set of the linear discrete-time system  $x_{k+1} = Ax_k$  if and only if there exists  $\lambda \in [0, 1]$  such that  $A^TQA - \lambda Q \leq 0$ .

**Proof.** From the proof of Theorem 5, it follows that if the ellipsoidal set *S* is a positively invariant set of the linear discrete-time system (1), then there exists  $\lambda \in [0, 1]$  such that the optimal value of the optimization problem (25) is non-negative. Then

$$1 - x^T A^T Q A x + \lambda (x^T Q x - 1) \ge 0$$

which means,

$$x^T A^T Q A x - \lambda x^T Q x \le 1 - \lambda$$

that is,

$$x^T (A^T Q A - \lambda Q) x \le 0$$

i.e.,

$$A^T Q A - \lambda Q \preceq 0.$$

The proof is completed.  $\Box$ 

**Remark 2.** Theorem 5 was once proved in [19] using an S-procedure, and, in this paper, a novel proof is given from an optimization point of view that shows a direct connection between positively invariant sets and optimization.

**Theorem 7.** The linear discrete-time system is  $x_{k+1} = Ax_k$ , and, when the ellipsoid is given by a quadratic norm of the form (4), then the ellipsoid is a positively invariant set of the linear discrete-time system (1) if and only if there exists  $\lambda \leq 0$  such that the optimal value of the following optimization problem is positive.

$$\max_{\lambda < 0} \min_{x} -x^{T} (A^{T} Q A - Q) x + \lambda (x^{T} Q x - 1)$$
(28)

**Remark 3.** The proof of Theorem 7 is similar to the proof in Theorem 4; the proof is omitted here.

By comparing the primal optimization problem with the Lagrange dual optimization problem, it can be seen that the feasible domain of the dual optimization problem is simpler than that of the primal problem. Subsequently, the case in which the primal problem is a convex optimization problem is discussed, and the Wolfe dual method's application to the primal optimization problem is discussed as well.

# 3.3. Wolfe Dual Forms

When the objective function is convex and the constraint is also convex, the primal optimization problem can be solved through the Wolfe dual form. The Wolfe dual form of the ellipsoid's positive invariance condition for the discrete-time systems is studied in this section.

**Remark 4.** Since the primal problem is a convex optimization problem, and the Slater condition is satisfied, the strong duality theorem holds. Therefore, the optimal value of the dual problem is equal to that of the primal problem. Then, whether the ellipsoidal set is a positively invariant set of the discrete-time system can be determined by the sign of the optimal value of the optimization problem.

**Theorem 8.** The ellipsoidal set is expressed as  $S = \{x \in \mathbb{R}^n \mid x^TQx \leq 1\}$ , where  $Q \in \mathbb{R}^{n \times n}$ , and  $Q \succ 0$ . The nonlinear discrete-time system is given by (2). Let  $1 - f_d^T(x)Qf_d(x)$  be a convex function differentiable with respect to x, and let  $x^TQx - 1$  be a convex function. Then, the ellipsoidal set S is a positively invariant set of nonlinear discrete-time systems (2) if and only if there exists  $\lambda \geq 0$  such that the optimal value of the following problem is non-negative.

$$\max_{x \in \mathbb{R}^n} \quad 1 - f_d^T Q f_d(x) + \lambda(x^T Q x - 1)$$
  
s.t 
$$\nabla (f_d^T Q f_d(x)) = \lambda \nabla (x^T Q x).$$
 (29)

**Proof.** From the primal problem (11), both the objective function and the constraint are convex functions. Then, the Wolfe dual theorem can be applied, i.e.,

$$\max_{x \in \mathbb{R}^n} \quad 1 - f_d^T Q f_d(x) + \lambda (x^T Q x - 1) s.t \quad \nabla (f_d^T Q f_d(x)) = \lambda \nabla (x^T Q x).$$
(30)

**Theorem 9.** Let the linear discrete-time system be (1) and the ellipsoidal set be  $S = \{x \in \mathbb{R}^n \mid x^TQx \le 1\}$ , where  $Q \in \mathbb{R}^{n \times n}$ , and  $Q \succ 0$ . Assume that  $1 - x^TA^TQAx$  is a convex function and that  $x^TQx - 1$  is a convex function; then, the ellipsoidal set S is a positively invariant set of linear discrete-time systems (1) if and only if there exists  $\lambda \in [0, 1]$  such that the optimal value of the following problem is non-negative.

$$\max_{x \in \mathbb{R}^n} \quad 1 - x^T A^T Q A x + \lambda (x^T Q x - 1)$$
  
s.t  $\lambda Q - A^T Q A = 0.$  (31)

**Proof.** From the assumptions, it is clear that (15) is a convex optimization problem, so the Wolfe dual theory can be applied. In particular, when x = 0, there is  $1 - \lambda \ge 0$ , i.e.,  $0 \le \lambda \le 1$ . The Wolfe dual of (15) is

$$\max_{x \in R^n} \quad 1 - x^T A^T Q A x + \lambda (x^T Q x - 1)$$
  
s.t  $\lambda Q - A^T Q A = 0.$ 

**Remark 5.** By comparing Theorems 4 and 5 with Theorems 8 and 9, it can be seen that the Wolfe dual applies to the case in which the primal problem is a convex optimization problem, while the Lagrange dual applies to the more general case.

#### 4. Positive Invariance Conditions for Lorenz Cone

The Lorenz cone is a convex set in quadratic form, but the positive invariance is more complicated because the Lorenz cone itself contains the constraint  $x^T P U_n \leq 0$ . The positive invariance condition for the Lorenz cone proposed in this paper makes it simpler. Similar to the ellipsoid, three equivalent sufficient conditions for the positive invariance of the Lorenz cone for nonlinear discrete-time systems are given.

**Theorem 10.** Lorenz cone (6) is a positively invariant set for a nonlinear discrete-time system (2) *if and only if the optimal value of the following optimization problem is non-negative.* 

$$\min_{x \in \mathbb{R}^n} -f_d(x)^T P f_d(x)$$
s.t  $f_d(x)^T P U_n \leq 0$ , (32)  
 $x^T P x \leq 0$ ,  
 $x^T P U_n < 0$ .

When  $f_d(x) = Ax$ , and the Lorenz cone  $S_L$  is a positively invariant set of linear discrete-time systems, then the optimal value of the following optimization problem needs to be non-negative.

$$\min_{x \in \mathbb{R}^n} -x^T A^T P A x$$
s.t  $x^T A^T P U_n \leq 0$ , (33)
 $x^T P x \leq 0$ ,
 $x^T P U_n \leq 0$ .

**Proof.** First, the Lorenz cone is a positively invariant set of the nonlinear discrete-time system (2) if and only if  $x_k \in S_L$ , and  $x_{k+1} \in S_L$ ; i.e., it needs to satisfy

 $x_k^T P x_k \le 0, x_k^T P U_n \le 0,$  $f(x_k)^T P f(x_k) \le 0, f(x_k)^T P U_n \le 0.$ 

and

When formulate as an optimization problem, it is expressed, i.e., as follows:

$$\min_{x \in \mathbb{R}^n} \quad -f_d(x)^T P f_d(x)$$
s.t  $f_d(x)^T P U_n \le 0,$ 
 $x^T P x \le 0,$ 
 $x^T P U_n < 0.$ 

If the discrete-time system is linear, i.e., when  $f(x_k) = Ax$ , the positive invariance condition is

$$\min_{x \in \mathbb{R}^n} - x^T A^T P A x \\ s.t \quad x^T A^T P U_n \le 0, \\ x^T P x \le 0, \\ x^T P U_n \le 0.$$

Next, the Lagrange dual optimization form for the optimization problem in (32) and (33) is studied.

**Theorem 11.** Consider that the nonlinear discrete-time systems are  $x_{k+1} = f_d(x_k)$ , and the Lorenz cone is given by (6). Let  $-f_d^T(x)Pf_d(x)$  be a continuous differentiable function with respect to x. Then the Lorenz cone  $S_L$  is a positively invariant set of the nonlinear discrete-time system (2) if and only if there exists  $\lambda, \mu, \eta \ge 0$  such that the optimal value of the following optimization problem is non-negative.

$$\max_{\lambda,\mu,\eta\geq 0} \min_{x\in R^n} \quad -f_d^T(x)Pf_d(x) + \lambda(f_d(x)^T P U_n) + \mu(x^T P x) + \eta(x^T P U_n)$$
(34)

If  $f_d(x) = Ax$ , and the Lorenz cone  $S_L$  is a positively invariant set of linear discrete-time systems, then the optimal value of the following optimization problem needs to be non-negative.

$$\max_{\lambda,\mu,\eta\geq 0}\min_{x\in\mathbb{R}^n} \quad -x^T A^T P A x + \lambda (x^T A^T P U_n) + \mu (x^T P x) + \eta (x^T P U_n)$$
(35)

**Proof.** Taking (32) as the primary problem and introducing the multiplier  $\lambda$ ,  $\mu$ ,  $\eta \ge 0$ , one can write its Lagrange function as follows:

$$-f_d^T(x)Pf_d(x) + \lambda(f_d(x)^T P U_n) + \mu(x^T P x) + \eta(x^T P U_n).$$

let the primary problem be P(x). There exists

$$\begin{aligned} & \max_{\lambda,\mu,\eta\geq 0} \quad -f_d^T(x) P f_d(x) + \lambda (f_d(x)^T P U_n) + \mu (x^T P x) + \eta (x^T P U_n) \\ &= \begin{cases} & \infty, & otherwise \\ & P(x), & f_d(x)^T P U_n \leq 0, x^T P x \leq 0, x^T P U_n \leq 0. \end{cases} \end{aligned}$$

Therefore,  $\min_{x \in \mathbb{R}^n} \max_{\lambda,\mu,\eta \ge 0} L(x,\lambda,\mu,\eta)$  is equivalent to (32), and the Lagrange dual of (32) is

$$\max_{\lambda,\mu,\eta\geq 0}\min_{x\in \mathbb{R}^n} \quad -x^T A^T P A x + \lambda (x^T A^T P U_n) + \mu (x^T P x) + \eta (x^T P U_n).$$

Since the optimal value of the primal problem must be greater than or equal to the optimal value of its Lagrange dual problem, when the optimal value of the Lagrange dual problem

is non-negative, the optimal value of the primal problem must be non-negative. When  $f(x_k) = Ax$ , the Lagrange dual problem of (33) is

$$\max_{\lambda,\mu,\eta\geq 0} \min_{x\in \mathbb{R}^n} \quad -x^T A^T P A x + \lambda (x^T A^T P U_n) + \mu (x^T P x) + \eta (x^T P U_n).$$

When the optimization problem in (32) and (33) is convex optimization, it can be solved by the Wolfe dual form optimization.

**Theorem 12.** The Lorenz cone  $S_L = \{x \in \mathbb{R}^n \mid x^T P x \leq 0, x^T P U_n \leq 0\}$ , where  $P \in \mathbb{R}^{n \times n}$ . Let (32) be a convex optimization problem. Then, the Lorenz cone  $S_L$  is a positively invariant set of nonlinear discrete-time systems (2) if and only if there exists  $\lambda, \mu, \eta \geq 0$  such that the optimal value of the following problem is non-negative.

$$\max_{\mathbf{x}\in\mathbb{R}^n} -f_d^T P f_d(\mathbf{x}) + \lambda (f_d(\mathbf{x})^T P U_n) + \mu (\mathbf{x}^T P \mathbf{x}) + \eta (\mathbf{x}^T P U_n)$$
  
s.t  $\nabla (f_d(\mathbf{x})^T P f_d(\mathbf{x})) = \lambda \nabla (f_d(\mathbf{x})^T P U_n) + \mu \nabla (\mathbf{x}^T P \mathbf{x}) + \eta \nabla (\mathbf{x}^T P U_n).$  (36)

If  $f_d(x) = Ax$ , when (33) is convex optimization, the Lorenz cone  $S_L$  is a positively invariant set of linear discrete-time systems if and only if there exists  $\lambda$ ,  $\mu$ ,  $\eta \ge 0$  such that the optimal value of the following optimization problem needs to be non-negative.

$$\max_{x \in \mathbb{R}^n} -x^T A^T P A x + \lambda (x^T A^T P U_n) + \mu (x^T P x) + \eta (x^T P U_n)$$
  
s.t  $2(A^T P A - \mu P) x = \lambda A^T P U_n + \eta P U_n.$  (37)

The proof of Theorem 12 is similar to the proof of Theorem 8; it is omitted here.

**Remark 6.** *Our above conclusions can also be generalized to additional systems, such as Markovian jump systems* [29].

#### 5. Numerical Examples

1

In this section, some examples are given to illustrate the theorems given in Sections 3 and 4. The numerical examples in this paper are mainly solved by Matlab and Lingo. Note that the change in the initial state does not affect the final determination of whether the set is a positively invariant set or not.

**Remark 7.** In all of the examples in this paper, the superscript k in  $x^{(k)}$  represents the time index corresponding to the state of discrete-time systems (1) and (2).

**Example 1.** The ellipsoidal set is  $S = \{(x_1^{(k)}, x_2^{(k)}) \mid (x_1^{(k)})^2 + (x_2^{(k)})^2 \le 1\}$ ; the nonlinear discrete-time system is  $x_1^{(k+1)} = \frac{\sqrt{x_1^{(k)} + x_2^{(k)}}}{2}, x_2^{(k+1)} = \frac{\sqrt{x_1^{(k)} - 3x_2^{(k)}}}{2}$ .

*In method 1, the positive invariance of this ellipsoidal set is checked by using (11) of Theorem 1. We transform the invariance problem into an optimization problem, i.e.,* 

$$\min_{x} \quad 1 - f_d^T(x)Qf_d(x)$$
  
s.t  $x^TQx \le 1.$ 

*Firstly, simplify the objective function.* 

$$\begin{split} \min_{x} & 1 - (\frac{\sqrt{x_{1}^{(k)} + x_{2}^{(k)}}}{2} - \frac{\sqrt{x_{1}^{(k)} - 3x_{2}^{(k)}}}{2}) \\ &= \min_{x} & 1 - [\frac{x_{1}^{(k)} + x_{2}^{(k)}}{4} + \frac{x_{1}^{(k)} - 3x_{2}^{(k)}}{4}] \\ &= \min_{x} & 1 - [\frac{x_{1}^{(k)} - x_{2}^{(k)}}{2}] \\ &= \min_{x} & 1 - \frac{x_{1}^{(k)}}{2} + \frac{x_{2}^{(k)}}{2}. \end{split}$$

Next, the constraint function is simplified.

$$\begin{bmatrix} x_1^{(k)} & x_2^{(k)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} & x_1^{(k)} \\ & x_2^{(k)} \end{bmatrix}$$
$$= (x_1^{(k)})^2 + (x_2^{(k)})^2 \le 1.$$

The simplified optimization problem is

$$\min_{x} \quad 1 - \frac{x_{1}^{(k)}}{2} + \frac{x_{2}^{(k)}}{2} \\ s.t \quad (x_{1}^{(k)})^{2} + (x_{2}^{(k)})^{2} \le 1 .$$

Solve nonlinear programming problems in Matlab by applying the fmincon function. A  $2 \times 1$ -dimensional matrix is generated as the initial state by using rand(2, 1). The optimal value of this optimization problem is obtained by MATLAB as 0.5 > 0. Then, the ellipsoidal set S is a positively invariant set of the nonlinear discrete-time system.

In method 2, apply (20) in Theorem 4 to check whether the ellipsoidal set is a positively invariant set of this nonlinear discrete-time system. The optimization problem in method 1 is transformed into its Lagrange dual optimization form, i.e.,

$$\max_{\lambda \ge 0} \min_{x} \quad 1 - f_d^T(x)Qf_d(x) + \lambda(x^TQx - 1)$$
  
$$\Rightarrow \max_{\lambda \ge 0} \min_{x} \quad 1 - \frac{x_1^{(k)}}{2} + \frac{x_2^{(k)}}{2} + \lambda[(x_1^{(k)})^2 + (x_2^{(k)})^2 - 1]$$

Firstly, let the inner optimization problem be  $g(x_1^{(k)}, x_2^{(k)})$ , and take the partial derivatives of  $x_1^{(k)}$ ,  $x_2^{(k)}$  in  $g(x_1^{(k)}, x_2^{(k)})$  to be equal to zero, respectively, i.e.,

$$\frac{\partial g}{\partial x_1^{(k)}} = -\frac{1}{2} + 2\lambda x_1^{(k)} = 0 \Rightarrow x_1^{(k)} = \frac{1}{4\lambda},$$
$$\frac{\partial g}{\partial x_2^{(k)}} = \frac{1}{2} + 2\lambda x_2^{(k)} = 0 \Rightarrow x_2^{(k)} = -\frac{1}{4\lambda}.$$

By substituting  $x_1^{(k)}$  and  $x_2^{(k)}$  into the optimization function, one can obtain the function that is only related to  $\lambda$ . When  $\lambda = 0.5$ , the optimal value of the function is 1 > 0. Therefore, the ellipsoidal set is the positively invariant set of the nonlinear discrete-time system.

In method 3, since the optimization problem in method 1 is a convex optimization problem, (29) in Theorem 8 can be applied to solve this problem. That is, it is necessary to find a u that satisfies u > 0 such that the optimal value of the objective function is non-negative.

$$\max_{x \in \mathbb{R}^n} \quad 1 - f_d^T Q f_d(x) + \lambda (x^T Q x - 1)$$
  
s.t 
$$\nabla (f_d^T Q f_d(x)) = \lambda \nabla (x^T Q x).$$

In this example, let  $\lambda = 0.25$ , and substitute the objective function to obtain the value 0.25 > 0. *Therefore, the ellipsoidal set is the positively invariant set of this nonlinear discrete-time system.* 

**Remark 8.** Due to the complexity of nonlinear systems, it is more complicated to theoretically derive the conditions in which quadratic convex sets are the positively invariant sets of nonlinear systems. The effectiveness and advantage of the optimization method proposed in this paper for determining the quadratic convex set to be a positively invariant set of a nonlinear system is obviously simple.

**Example 2.** Let the linear discrete-time system  $x_{k+1} = Ax_k$ , where  $A = \begin{bmatrix} 0 & -3.2 \\ -0.1 & 0.4 \end{bmatrix}$  and the set of ellipsoids is  $S = \{(x_1^{(k)})^2 + (x_2^{(k)})^2 \le 1\}$ . Next, apply (15) in Theorem 2 to check it. Substituting the data in this example into (15) yields an optimization problem.

$$\min_{x} 1 - \begin{bmatrix} x_{1}^{(k)} & x_{2}^{(k)} \end{bmatrix} \begin{bmatrix} 0 & -0.1 \\ -3.2 & 0.4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -3.2 \\ -0.1 & 0.4 \end{bmatrix} \begin{bmatrix} x_{1}^{(k)} \\ x_{2}^{(k)} \end{bmatrix}$$
  
s.t  $(x_{1}^{(k)})^{2} + (x_{2}^{(k)})^{2} - 1 \le 0.$ 

Firstly, the objective function is simplified.

$$\begin{split} &1 - \begin{bmatrix} x_1^{(k)} & x_2^{(k)} \end{bmatrix} \begin{bmatrix} 0 & -0.1 \\ -3.2 & 0.4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -3.2 \\ -0.1 & 0.4 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} \\ &= &1 - \begin{bmatrix} 0.01x_1^{(k)} - 0.04x_2^{(k)} & -0.04x_1^{(k)} + 10.4x_2^{(k)} \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} \\ &= &1 - 0.01(x_1^{(k)})^2 + 0.08x_1^{(k)}x_2^{(k)} - 10.4(x_2^{(k)})^2. \end{split}$$

The constraint function is  $(x_1^{(k)})^2 + (x_2^{(k)})^2 - 1 \le 0$ . The optimization function is finally reduced to

$$\begin{split} \min_{x} & 1 - 0.01(x_{1}^{(k)})^{2} + 0.08x_{1}^{(k)}x_{2}^{(k)} - 10.4(x_{2}^{(k)})^{2} \\ s.t & (x_{1}^{(k)})^{2} + (x_{2}^{(k)})^{2} - 1 \leq 0. \end{split}$$

We can solve nonlinear optimization problems using Matlab with the fmincon function. A  $2 \times 1$ -dimensional random matrix is generated as the initial state by using rand(2,1) and the optimal value of the optimization function as -9.4002 < 0. Then, the ellipsoidal set is not a positively invariant set of this linear discrete-time system.

**Remark 9.** In the positive invariance condition given in [19], for the ellipsoid to be a linear discretetime system, it can be verified that the ellipsoid is a positively invariant set of the discrete-time system, but it is more complex in this example.

**Example 3.** The linear discrete-time system is  $x_{k+1} = Ax_k$ , where  $A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ . Addition-

ally, the ellipsoidal set is  $(x_1^{(k)})^2 + (x_2^{(k)})^2 \leq 1$ .

In method 1, using (15) in Theorem 2, the problem of checking that the ellipsoidal set is a positively invariant set of a linear discrete-time system is transformed into an optimization problem.

$$\min_{x} \quad 1 - x^{T} A^{T} Q A x$$
  
s.t  $x^{T} Q x - 1 \le 0$ ,

where

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, the matrices A, Q are substituted into the objective function to obtain

$$\min_{x} \left[ x_{1}^{(k)} \quad x_{2}^{(k)} \right] \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_{1}^{(k)} \\ x_{2}^{(k)} \end{bmatrix}$$
$$= 1 - 0.25(x_{1}^{(k)})^{2} - 0.25(x_{2}^{(k)})^{2}.$$

Then, the optimization function is

$$\begin{split} \min_{x} & 1 - 0.25 (x_1^{(k)})^2 - 0.25 (x_2^{(k)})^2 \\ s.t & (x_1^{(k)})^2 + (x_2^{(k)})^2 - 1 \leq 0. \end{split}$$

Solve nonlinear optimization problems using Matlab with the f mincon function. The initial state is set as rand(2, 1) in MATLAB, and the optimal value of the objective function is 0.75 > 0. Therefore, the ellipsoidal set S is the positively invariant set of the linear discrete-time system.

In method 2, the ellipsoidal set is written in the form of (4), and the invariance condition of the ellipsoidal set is transformed into an optimization problem by using (16) in Theorem 3.

$$\min_{x} - x^{T} (A^{T} Q A - Q) x$$
  
s.t  $x^{T} O x - 1 = 0.$ 

where

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then, the matrix A and Q are substituted and simplified to obtain the following optimization problem, i.e.,

$$\min_{x} \quad 0.75(x_1^{(k)})^2 + 0.75(x_2^{(k)})^2 \\ s.t \quad (x_1^{(k)})^2 + (x_2^{(k)})^2 - 1 = 0.$$

Solve nonlinear optimization problems using Matlab with the fmincon function. The initial state is set as rand(2, 1) in MATLAB, and the optimal value of the objective function is 0.75 > 0. Therefore, the ellipsoidal set S is the positively invariant set of the linear discrete-time system.

*In method 3, use* (25) *in Theorem 5 to determine whether the ellipsoidal set is a positively invariant set of the linear discrete-time system.* 

$$\max_{0 \le \lambda \le 1} \min_{x} \quad 1 - x^{T} A^{T} Q A x + \lambda (x^{T} Q x - 1)$$

That is

$$\max_{0 \le \lambda \le 1} \min_{x} \quad 1 - 0.25(x_1^{(k)})^2 - 0.25(x_2^{(k)})^2 + \lambda((x_1^{(k)})^2 + (x_2^{(k)})^2 - 1)$$

Let the inner optimization function be  $g(x_1^{(k)}, (x_2^{(k)}))$ , and take the partial derivatives with respect to  $x_1^{(k)}, x_2^{(k)}$  to be equal to zero, respectively, i.e.,

$$\frac{\partial g}{x_1^{(k)}} = 2(\lambda - 0.25)x_1^{(k)} = 0,$$
$$\frac{\partial g}{x_2^{(k)}} = 2(\lambda - 0.25)x_2^{(k)} = 0.$$

Take  $x_1 = x_2 = 0$ , the optimal value is 1 > 0 by substituting it into the objective function. When  $x_1, x_2 \neq 0$ ,  $\lambda = 0.25$  can be obtained, and the optimal value is 0.75 > 0 by substituting it into

the objective function. Therefore, the set of ellipsoids is the positively invariant set of this linear discrete-time system.

In method 4, apply (28) in Theorem 7 to determine the positive invariance of the ellipsoidal set. That is, in this example, it is necessary to find a  $\lambda < 0$  such that the optimal value of the optimization problem is positive. Substituting the data in this example into (28) and simplifying gives

$$\max_{\lambda \le 0} \min_{x} \quad 0.75(x_1^{(k)})^2 + 0.75(x_2^{(k)})^2 + \lambda((x_1^{(k)})^2 + (x_2^{(k)})^2) - 1)$$

The optimal value is 0.75 when  $\lambda = 0.75$ . Therefore, this ellipsoidal set is the positively invariant set of this linear discrete-time system.

In method 5, apply (29) in Theorem 8 to determine the positive invariance of the ellipsoidal set. This method needs to check whether a  $\lambda \ge 0$  can be found such that the optimal value of the optimization problem is non-negative.

$$\max_{x \in R^n} \quad 1 - x^T A^T Q A x + \lambda (x^T Q x - 1)$$
  
s.t  $\lambda Q - A^T Q A = 0.$ 

where

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The constraint condition is satisfied when  $\lambda = 0.25$ , and the value of 0.75 > 0 can be obtained by substituting  $\lambda$  into the objective function. Therefore, the ellipsoidal set is the positively invariant set of the linear discrete-time system.

Examples 4 and 5 apply Lingo to solve the optimization problems.

**Example 4.** The Lorenz cone is represented by  $S_L = \{(x_1^{(k)})^2 - (x_2^{(k)})^2 \le 0, x_2^{(k)} \ge 0\}$ , and the nonlinear discrete-time system is  $x_1^{(k+1)} = [-(x_1^{(k)})^2 + 2x_2^{(k)} - x_1^{(k)}]^{\frac{1}{2}}, x_2^{(k+1)} = [(x_2^{(k)})^2 - x_2^{(k)} - 2x_1^{(k)}]^{\frac{1}{2}}$ .

*In method 1, (32) in Theorem 10 is applied to determine whether the Lorenz cone is a positively invariant set of this nonlinear discrete-time system.* 

$$\min_{x \in \mathbb{R}^n} \quad -f_d(x)^T P f_d(x)$$
s.t  $f_d(x)^T P U_n \le 0$ 
 $x^T P x \le 0$ 
 $x^T P U_n \le 0$ ,

where

$$P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, U_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Substitute P,  $U_n$  into the optimization problem and simplifying yields

$$\min_{x \in \mathbb{R}^n} \quad (x_1^{(k)})^2 + (x_2^{(k)})^2 - x_1^{(k)} - 4x_2^{(k)} \\
s.t \quad -\sqrt{(x_2^{(k)})^2 - x_2^{(k)} - 2x_1^{(k)}} \le 0 \\
\quad (x_1^{(k)})^2 - (x_2^{(k)})^2 \le 0 \\
\quad -x_2^{(k)} \le 0.$$

According to (32) in Theorem 10, when the optimal value is great than/equal 0, the Lorenz cone is the positively invariant set of the nonlinear discrete-time system.

*In method 2, apply (34) in Theorem 11.* 

*Since the optimization problem in method* 1 *is a convex optimization, the Wolfe dual can be used to solve it. Substituting the data in this example into* (34) *and simplifying yields* 

$$\begin{split} \max_{x \in \mathbb{R}^n} & (x_1^{(k)})^2 + (x_2^{(k)})^2 - x_1^{(k)} - 4x_2^{(k)} + \lambda(-\sqrt{(x_2^{(k)})^2 - x_2^{(k)} - 2x_1^{(k)}}) \\ & + \mu((x_1^{(k)})^2 - (x_2^{(k)})^2) + \eta(-x_2^{(k)}) \\ s.t & \begin{bmatrix} 2x_1^{(k)} - 1 \\ 2x_2^{(k)} - 4 \end{bmatrix} = \lambda \begin{bmatrix} \frac{1}{\sqrt{(x_2^{(k)})^2 - x_2^{(k)} - 2x_1^{(k)}}} \\ -\frac{2x_2^{(k)} - 1}{2\sqrt{(x_2^{(k)})^2 - x_2^{(k)} - 2x_1^{(k)}}} \end{bmatrix} + \mu \begin{bmatrix} 2x_1^{(k)} \\ -2x_2^{(k)} \end{bmatrix} + \eta \begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{split}$$

Take  $\lambda = 0, \mu = 0.75$ , and  $\eta = 2.5$ , the value of the objective function is 1.625 > 0. So, the Lorenz cone is the positively invariant set of this nonlinear discrete-time system.

**Remark 10.** The verification given in [19] for determining that the Lorenz cone is a positively invariant set of a linear discrete-time system is computationally intensive; the optimization method in this paper is less computationally intensive for matrix simplification, and Lingo provides a good aid for solving the optimization problem in this example.

**Example 5.** The Lorenz cone is  $S_L = \{(x_1^{(k)})^2 + (x_2^{(k)})^2 - (x_3^{(k)})^2 \le 0, (x_3^{(k)}) \ge 0\}$ , and the linear discrete-time system is represented by  $x_{k+1} = Ax_k$ , where A = [0.5, 0, 0; 0, 0.5, 0; 0, 0, 1].

In method 1, (33) in Theorem 10 is applied to check it. Then, the sufficient and necessary condition for the Lorenz cone to be a positively invariant set of this linear discrete-time system is that the optimal value of the following optimization problem is non-negative, i.e.,

$$\min_{x \in \mathbb{R}^n} -x^T A^T P A x$$
s.t  $x^T A^T P U_n \le 0$ ,  $x^T P x \le 0$ ,  $x^T P U_n \le 0$ ,

where

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, u_n = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Substitute the values of A, P, and  $U_n$ , into the optimization framework yields the optimal value of 0, which is non-negative. According to (33) in Theorem 10, when the optimal value is 0, then the Lorenz cone is a positively invariant set of this linear discrete-time system.

In method 2, apply (37) in Theorem 12 to determine whether the Lorenz cone is a positively invariant set of this linear discrete-time system. It is necessary to find the parameters  $\lambda$ ,  $\mu$ , and  $\eta \ge 0$  such that the optimal value of the optimization problem is non-negative.

$$\max_{\lambda,\mu,\eta\geq 0} \min_{x\in R^n} \quad -x^T A^T P A x + \lambda (x^T A^T P U_n) + \mu (x^T P x) + \eta (x^T P U_n)$$

For the inner optimization function, substituting A, P, and  $U_n$  and simplifying yields

$$\min_{x} g(x_{1}^{(k)}, x_{2}^{(k)}, x_{3}^{(k)})$$

$$= \min_{x} (\mu - 0.25)(x_{1}^{(k)})^{2} + (\mu - 0.25)(x_{2}^{(k)})^{2} + (1 - \mu)(x_{3}^{(k)})^{2} - (\lambda + \eta)x_{3}^{(k)}$$

By taking the partial derivatives of each of the variables in the function  $g(x_1^{(k)}, x_2^{(k)}, x_3^{(k)})$  and making them equal to zero, i.e.,

$$\begin{aligned} \frac{\partial g}{\partial x_1^{(k)}} &= 2(\mu - 0.25)x_1^{(k)} = 0,\\ \frac{\partial g}{\partial x_2^{(k)}} &= 2(\mu - 0.25)x_2^{(k)} = 0,\\ \frac{\partial g}{\partial x_1^{(k)}} &= 2(1 - \mu)x_3^{(k)} - (\lambda + \eta) = 0. \end{aligned}$$

Then it is obtained that  $\mu = 0.25$  and  $x_3^{(k)} = \frac{2}{3}(\lambda + \eta)$ . By substituting it into the inner optimization function, we then obtain

$$\max_{\lambda \ge 0, \eta \ge 0} -\frac{1}{3} (\lambda + \eta)^2$$

The optimal solution of the optimization problem is zero, so this Lorenz cone is a positively invariant set of this linear discrete-time system.

When using the Wolfe duality, it is necessary to determine whether the original problem is a convex optimization. In this case, the objective function is not convex, so the Wolfe dual cannot be used.

#### 6. Conclusions

In this paper, the sufficient and necessary conditions for determining the positive invariance of the ellipsoidal set and the Lorenz cone for discrete-time dynamical systems are presented by virtue of the optimization and the dual optimization method. Both the linear discrete-time system and the nonlinear discrete-time system cases are discussed in this paper. The positive invariance conditions for ellipsoidal sets and Lorenz cones are formulated as optimization problems. For ellipsoids, novel methods are proposed to check the positive invariance conditions using the induced norm and optimization techniques. For the Lorenz cone, the optimization method is also used, and the proposed method in this paper only needs to check the sign of the optimal value of the optimization problem. It is also interesting that the invariance conditions obtained in this paper relate algebraic problems to optimization problems and provide additional alternative methods for the calculation of the positive invariance sets of ellipsoids and Lorenz cones for nonlinear and linear discrete-time dynamical systems.

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