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Article

Surface Family Pair with Bertrand Pair as Common Geodesic Curves in Galilean 3-Space $\mathbb{G}_3$

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Abstract: This paper is about deriving the necessary and sufficient conditions of a surface family pair with a Bertrand pair as common geodesic curves in Galilean 3-space $\mathbb{G}_3$. Thereafter, the consequence for the ruled surface family pair is also deduced. Meanwhile, some examples are provided to show the surfaces family with common Bertrand geodesic curves.

Keywords: isotropic normal; Serret–Frenet formulae; Bertrand pair; marching-scale functions

MSC: 53A04; 53A05; 53A17

1. Introduction

Traditional study on curves and surfaces focus on how to realize specific curves, such as asymptotic curve, geodesic curve, principal curve, etc., on a display surface. However, the reciprocal problem, that is, acquired surfaces having a distinct curve, is considerably more motivating. The design of surfaces with a given distinct curve is a new study subject that entices the attention of many scholars. The first work in this subject of design was presented by Wang et al. [1]. They created a surface family over a common geodesic. Stimulated by Wang et al. [1], researchers established restrictions for a prescribed curve to be a distinct curve on designed surfaces [2–12].

In the theory of distinct curves, the congruous correlation through the curves is a good problem. One of the traditional distinct curves is the Bertrand curve. If the principal normal vectors of two curves are linearly correlated at their matching points, the two curves are said to be a Bertrand pair [13–18]. In the 3$\mathbb{D}$ (three-dimensional) Galilean space $\mathbb{G}_3$, extra properties and descriptions of the Bertrand pair have been elaborated in a number of works; for example Abdel-Aziz and Khalifa considered a location vector of a random curve [19]. In addition, they imposed several conditions on the random curve's curvatures in order to investigate specific curves and their Smarandache curves. The parametrization of a set of surfaces over a specific geodesic curve has been investigated by Yuzbas and Bektas. On the parametric surfaces, they constructed the necessary and sufficient conditions for this curve to be an iso-geodesic curve [20]. The problem of designing a hypersurface family with common geodesic curve in 4D Galilean space $\mathbb{G}_4$ has been addressed in [21–23].

However, to our knowledge, no further work has been done to create surface family pairs with curve pairs that are geodesic curves. In order to cover this need, we investigate Bertrand pairs as geodesic curves and construct a surface family pair with a Bertrand pair as common geodesic curves. Furthermore, the extension to the ruled surfaces family is also described. Meanwhile, some examples are shown to construct the surfaces family and ruled surfaces family with common Bertrand geodesic curves.

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2. Basic Concepts

The Galilean 3-space $\mathbb{G}_3$ is a Cayley–Klein geometry provided with the projective metric of signature $(0,0,+,+)$ [16,17]. The absolute figure of the Galilean space depends on the organized triple $\{\varpi, L, I\}$, where ϖ is the (absolute) plane in the real 3-dimensional projective space $\mathbb{P}^3(\mathbb{R})$, L is the line (absolute line) in ϖ, and L is the stationary elliptic involution of points of L. Homogeneous coordinates in $\mathbb{G}_3$ are endowed in such a manner that the absolute plane ϖ is given by $x_0 = 0$, the absolute line L by $x_0 = x_1 = 0$, and the elliptic involution is given by $(0 : 0 : x_2 : x_3) \to (0 : 0 : x_3 : -x_2)$. A plane is named Euclidean if it includes L, otherwise it is named isotropic; that is, planes x = const are Euclidean, and so is the plane ω. Other planes are isotropic. In other words, an isotropic plane has no isotropic orientation.

For any $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, and $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3) \in \mathbb{G}_3$, their scalar product is

$$< \boldsymbol{\alpha}, \boldsymbol{\beta} >= \begin{cases} \alpha_1\beta_1, \text{ if } \alpha_1 \neq 0 \vee \beta_1 \neq 0, \\ \alpha_2\beta_2 + \alpha_3\beta_3, \text{ if } \alpha_1 = 0 \wedge \beta_1 = 0, \end{cases} \tag{1}$$

and their vector product is

$$\boldsymbol{\alpha} \times \boldsymbol{\beta} = \begin{cases} \begin{vmatrix} \mathbf{0} & \mathbf{e}_2 & \mathbf{e}_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix}, \text{ if } \alpha_1 \neq 0 \vee \beta_1 \neq 0, \\ \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & \alpha_2 & \alpha_3 \\ 0 & \beta_2 & \beta_3 \end{vmatrix}, \text{ if } \alpha_1 = 0 \wedge \beta_1 = 0. \end{cases} \tag{2}$$

where $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$ are the standard basis vectors in $\mathbb{G}_3$.

A curve $\boldsymbol{\varphi}(u) = (\varphi_1(u), \varphi_2(u), \varphi_3(u))$; $u \in I \subseteq \mathbb{R}$, is named allowable curve if it has no inflection points, that is, $\dot{\boldsymbol{\varphi}} \times \dddot{\boldsymbol{\varphi}} \neq \mathbf{0}$ and no isotropic tangents $\dot{\varphi}_1 \neq 0$. An allowable curve is similar to a smooth curve in Euclidean space. For an allowable curve $\boldsymbol{\varphi}$: $I \subseteq \mathbb{R} \to \mathbb{G}_3$ represented by the Galilean invariant arc-length s, we have:

$$\boldsymbol{\varphi}(s) = (s, \varphi_2(s), \varphi_3(s)). \tag{3}$$

The curvature $\kappa(s)$ and torsion $\tau(s)$ of the curve $\varphi(s)$ are

$$\begin{aligned} \kappa(s) &= \|\boldsymbol{\varphi}''(s)\| = \sqrt{(\varphi_2''(s))^2 + (\varphi_3''(s))^2}, \\ \tau(s) &= \frac{1}{\kappa^2(s)} \det\left(\boldsymbol{\varphi}', \boldsymbol{\varphi}'', \boldsymbol{\varphi}'''\right). \end{aligned} \tag{4}$$

Note that an allowable curve has $\kappa(s) \neq 0$. The Serret–Frenet vectors are:

$$\begin{aligned} \mathbf{t}(s) &= \boldsymbol{\varphi}'(s) = \left(1, \varphi_2'(s), \varphi_3'(s)\right), \\ \mathbf{n}(s) &= \frac{1}{\kappa(s)}\boldsymbol{\varphi}'(s) = \frac{1}{\kappa(s)}\left(0, \varphi_2''(s), \varphi_3''(s)\right), \\ \mathbf{b}(s) &= \frac{1}{\tau(s)}\left(0, \left(\frac{1}{\kappa(s)}\varphi_2''(s)\right)', \left(\frac{1}{\kappa(s)}\varphi_3''(s)\right)'\right), \end{aligned} \tag{5}$$

where $\mathbf{t}(s)$, $\mathbf{n}(s)$, and $\mathbf{b}(s)$, respectively, are the tangent, principal normal, and binormal vectors. For every point of $\boldsymbol{\varphi}(s)$, the Serret–Frenet formulae read:

$$\begin{pmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ 0 & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \tag{6}$$

The planes that match the subspaces Sp{**t**, **n**}, Sp{**n**, **b**}, and Sp{**b**, **t**}, respectively, are named the osculating plane, normal plane, and rectifying plane.

Definition 1 ([13–15,24]). *Let $\boldsymbol{\varphi}(s)$ and $\widehat{\boldsymbol{\varphi}}(s)$ be two allowable curves in $\mathbb{G}^3$; $\mathbf{n}(s)$ and $\widehat{\mathbf{n}}(s)$ are principal normal vectors of them, respectively; the pair $\{\widehat{\boldsymbol{\varphi}}(s), \boldsymbol{\varphi}(s)\}$ is named a Bertrand pair if $\mathbf{n}(s)$ and $\widehat{\mathbf{n}}(s)$ are linearly dependent at the corresponding points; $\boldsymbol{\varphi}(s)$ is named the Bertrand mate of $\widehat{\boldsymbol{\varphi}}(s)$; and*

$$\widehat{\boldsymbol{\alpha}}(s) = \boldsymbol{\alpha}(s) + f\mathbf{n}(s). \tag{7}$$

where f is a constant.

We indicate a surface M in $\mathbb{G}_3$ by

$$M : \mathbf{y}(s,t) = (y_1(s,t), y_2(s,t), y_3(s,t)), \ \ (s,t) \in \mathbb{D} \subseteq \mathbb{R}^2. \tag{8}$$

If $\mathbf{y}_j(s,t) = \frac{\partial \mathbf{y}}{\partial j}$, the isotropic surface normal is

$$\boldsymbol{\xi}(s,t) = \mathbf{y}_s \wedge \mathbf{y}_t, \tag{9}$$

which is orthogonal to each of the vectors $\mathbf{y}_s$ and $\mathbf{y}_t$.

Definition 2 ([1–24]). *A curve on a surface is geodesic if and only if the surface normal is everywhere parallel to the principal normal vector of the curve.*

An isoparametric curve is a curve $\boldsymbol{\varphi}(s)$ on a surface $\mathbf{y}(s,t)$ that has a constant s or t-parameter value. In other terms, there exists a parameter t_0 such that $\boldsymbol{\varphi}(s) = \mathbf{y}(s,t_0)$ or $\boldsymbol{\varphi}(t) = \mathbf{y}(s_0,t)$. Given a parametric curve $\boldsymbol{\varphi}(s)$, we call it an isogeodesic of the surface $\mathbf{y}(s,t)$ if it is both a geodesic and a parameter curve on $\mathbf{y}(s,t)$.

3. Main Results

This section presents a new approach for constructing a surface family pair interpolating a Bertrand pair as mutual geodesic curves in $\mathbb{G}_3$. To do this, we take into account a Bertrand pair such that the surface's tangent planes are coincident with the curve's rectifying planes.

Let $\boldsymbol{\varphi}(s)$ be an allowable curve, $\widehat{\boldsymbol{\varphi}}(s)$ is Bertrand mate of $\boldsymbol{\varphi}(s)$, and $\{\widehat{\mathbf{t}}(s), \widehat{\mathbf{n}}(s), \widehat{\mathbf{b}}(s)\}$ is the Frenet–Serret frame of $\widehat{\boldsymbol{\varphi}}(s)$ as in Equation (6). The surface family M interpolating $\boldsymbol{\varphi}(s)$ can be written as [18]

$$M : \mathbf{y}(s,t) = \boldsymbol{\varphi}(s) + u(s,t)\mathbf{t}(s) + v(s,t)\mathbf{b}(s); \ \ 0 \le t \le T. \tag{10}$$

Similarly, the surface $\widehat{M}$ is specified by

$$\widehat{M} : \widehat{\mathbf{y}}(s,t) = \widehat{\boldsymbol{\varphi}}(s) + u(s,t)\widehat{\mathbf{t}}(s) + v(s,t)\widehat{\mathbf{b}}(s); \ \ 0 \le t \le T. \tag{11}$$

Here, $u(s,t)$, and $v(s,t)$ are named directed marching-scale functions.

In order to show that $\boldsymbol{\varphi}(s)$ is a geodesic curve on M, according to Equation (10), we discuss what the marching-scale functions should satisfy. Therefore, we have

$$\left.\begin{array}{l} \mathbf{y}_s(s,t) = (1+u_s)\mathbf{t} + (u\kappa - \tau v)\mathbf{n} + \tau v_s \mathbf{b}, \\ \mathbf{y}_t(s,t) = u_t\mathbf{t} + v_t\mathbf{b}, \end{array}\right\} \tag{12}$$

and

$$\boldsymbol{\xi}(s,t) := \mathbf{y}_s \times \mathbf{y}_t = [-(1+u_s)v_t + v_s u_t]\mathbf{n} - (u\kappa - \tau v)u_t\mathbf{b}. \tag{13}$$

Since $\boldsymbol{\varphi}(s)$ is iso-parametric on M, there exists a value $t = t_0 \in [0,T]$ such that $\mathbf{y}(s,t_0) = \boldsymbol{\varphi}(s)$; that is,

$$u(s,t_0) = v(s,t_0) = 0, \ u_s(s,t_0) = v_s(s,t_0) = 0. \tag{14}$$

Hence, when $t = t_0$—i.e., over $\boldsymbol{\varphi}(s)$, we have

$$\boldsymbol{\xi}(s, t_0) = -v_t(s, t_0)\mathbf{n}(s). \tag{15}$$

The coincidence of the principal normal $\mathbf{n}(s)$ with the surface normal $\boldsymbol{\xi}$ recognizes $\boldsymbol{\varphi}(s)$ as a geodesic curve. We let $\{\widehat{M}, M\}$ denote the surface family pair. Hence, we have the following theorem:

Theorem 1. *$\{\widehat{M}, M\}$ interpolate $\{\widehat{\boldsymbol{\varphi}}(s), \boldsymbol{\varphi}(s)\}$ as common geodesic curves if and only if the following conditions*

$$\left.\begin{array}{l} u(s, t_0) = v(s, t_0) = 0, \\ u_s(s, t_0) = v_s(s, t_0) = 0, \\ v_t(s, t_0) \neq 0, \ \ 0 \leq t_0 \leq T, \ \ 0 \leq s \leq L, \end{array}\right\} \tag{16}$$

are satisfied.

For the above conditions in Theorem 1, $u(s, t)$ and $v(s, t)$ can be written as:

$$\begin{array}{c} u(s, t) = l(s)U(t), \\ v(s, t) = m(s)V(t). \end{array} \tag{17}$$

Here, $l(s), m(s), U(t)$, and $v(t)$ are nowhere vanishing C^1 functions. Hence, from Theorem 1, we gain:

Corollary 1. *If $u(s, t)$ and $v(s, t)$ as in Equations (17), the sufficient and necessary condition is*

$$\left.\begin{array}{l} U(t_0) = V(t_0) = 0, \ l(s) = const., \ m(s) = const. \neq 0, \\ \frac{dV(t_0)}{dt} = const. \neq 0, \ 0 \leq t_0 \leq T, \ \ 0 \leq s \leq L. \end{array}\right\} \tag{18}$$

For suitability in performance, $u(s, t)$ and $v(s, t)$ can be chosen in two special forms:

(1) If

$$\left\{\begin{array}{l} u(s, t) = \sum\limits_{k=1}^{p} x_{1k} l(s)^k U(t)^k, \\ v(s, t) = \sum\limits_{k=1}^{p} x_{2k} m(s)^k V(t)^k, \end{array}\right. \tag{19}$$

then,

$$\left\{\begin{array}{c} U(t_0) = V(t_0) = 0, \\ a_{21} \neq 0, \ m(s) \neq 0, \text{and } \frac{dV(t_0)}{dt} \neq 0, \end{array}\right. \tag{20}$$

where $U(t)$ and $V(t)$ are C^1 functions, $a_{ij} \in \mathbb{R}$ $(i = 1, 2; j = 1, 2, ..., p)$ and $l(s)$, and $m(s)$ are nowhere vanishing C^1 functions.

(2) If

$$\left\{\begin{array}{l} u(s, t) = f(\sum\limits_{k=1}^{p} x_{1k} l^k(s) U^k(t)), \\ v(s, t) = g(\sum\limits_{k=1}^{p} x_{2k} m^k(s) V^k(t)), \end{array}\right. \tag{21}$$

then

$$\left\{\begin{array}{c} U(t_0) = V(t_0) = v(t_0) = f(0) = g(0) = 0, \\ x_{21} \neq 0, \ \frac{dV(t_0)}{dt} = const \neq 0, \ m(s) \neq 0, \ g'(0) \neq 0, \end{array}\right. \tag{22}$$

where $l(s), m(s), U(t), V(t), f$, and g are C^1 functions. Since there are no restrictions attached to the given curve in Equations (18), (20), or (22), the set $\{\widehat{M}, M\}$ interpolates $\{\widehat{\boldsymbol{\varphi}}(s), \boldsymbol{\varphi}(s)\}$ as common geodesic curves and can constantly be specified by choosing suitable marching-scale functions.

Example 1. *Let $\boldsymbol{\varphi}(s)$ be an allowable helix specified by*

$$\boldsymbol{\varphi}(s) = (s, \sin s, \cos s),\ 0 \leq s \leq 2\pi.$$

Then,

$$\begin{array}{rcl} \boldsymbol{\varphi}'(s) & = & (1, \cos s, -\sin s), \\ \boldsymbol{\varphi}''(s) & = & (0, -\sin s, -\cos s), \\ \boldsymbol{\varphi}'''(s) & = & (0, -\cos s, \sin s). \end{array}$$

Using Equations (3)–(5) to gain $\kappa(s) = -\tau(s) = 1$, and

$$\begin{array}{rcl} \mathbf{t}(s) & = & (1, \cos s, -\sin s), \\ \mathbf{n}(s) & = & (0, -\sin s, -\cos s), \\ \mathbf{b}(s) & = & (0, \cos s, -\sin s). \end{array}$$

Let $f = 2$ in Equation (7); we obtain $\widehat{\boldsymbol{\varphi}}(s) = (s, -\sin s, -\cos s)$, and

$$\begin{array}{rcl} \widehat{\mathbf{t}}(s) & = & (1, -\cos s, \sin s), \\ \widehat{\mathbf{n}}(s) & = & (0, \sin s, \cos s), \\ \widehat{\mathbf{b}}(s) & = & (0, -\cos s, \sin s). \end{array}$$

According to Corollary 1, we have:

(1) If $u(s,t) = t$, $v(s,t) = 2t$, $t_0 = 0$, then Equation (18) is satisfied. Then, the set $\{\widehat{M}, M\}$ interpolates $\{\widehat{\boldsymbol{\varphi}}(s), \boldsymbol{\varphi}(s)\}$ as common geodesic curves as in (Figure 1):

$$\begin{cases} M : \mathbf{y}(s,t) = (s, \sin s, \cos s) + t(1, 3\cos s, -3\sin s), \\ \widehat{M} : \widehat{\mathbf{y}}(s,t) = (s, -\sin s, -\cos s) + t(1, -3\cos s, 3\sin s), \end{cases}$$

where the blue curve represents $\boldsymbol{\varphi}(s)$, the green curve is $\widehat{\boldsymbol{\varphi}}(s)$, $-1.5 \leq t \leq 1.5$, and $0 \leq s \leq 2\pi$.

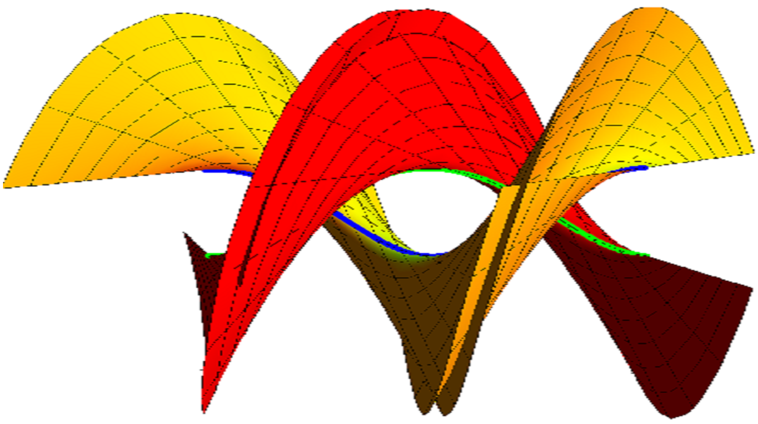

Figure 1. M (yellow) and $\widehat{M}$ (red).

(2) If $u(s,t) = 1 - \cot t$, $v(s,t) = \sin t$, $t_0 = 0$, then Equation (16) is satisfied. Then, the set $\{\widehat{M}, M\}$ interpolates $\{\widehat{\boldsymbol{\varphi}}(s), \boldsymbol{\varphi}(s)\}$ as common geodesic curves as in (Figure 2):

$$M : \mathbf{y}(s,t) = \begin{pmatrix} s + 1 - \cot t \\ \sin s + (\sin t + 1 - \cot t)\cos s \\ \cos s - (\sin t + 1 - \cot(t))\sin s \end{pmatrix},$$

and

$$\widehat{M}:\widehat{\mathbf{y}}(s,t)=\begin{pmatrix} s+1-\cot t \\ -\sin s-(\sin t+1-\cot t)\cos s \\ \cos s-(-\sin t+1-\cot(t))\sin s \end{pmatrix},$$

where the blue curve represents $\boldsymbol{\varphi}(s)$ and the green curve is $\widehat{\boldsymbol{\varphi}}(s)$ where $0 \leq s,t \leq 2\pi$.

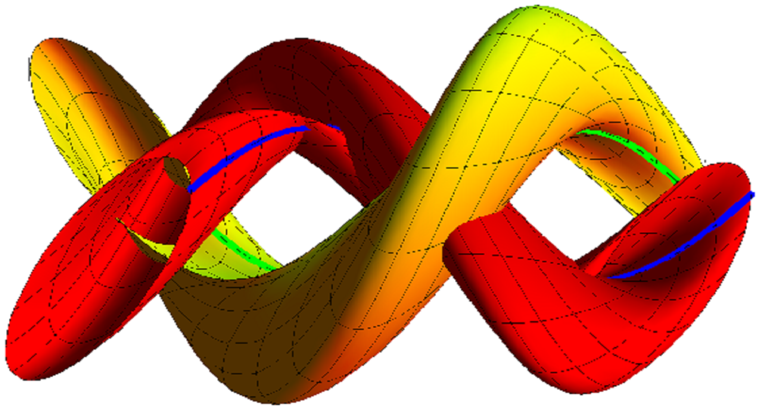

Figure 2. M (yellow) and $\widehat{M}$ (red).

Ruled Surfaces Family with Common Bertrand Geodesic Curves

Ruled surfaces are simple and common surfaces in geometric designs. Suppose $\mathbf{y}_i(s,t)$ is a ruled surface with the directrix $\boldsymbol{\varphi}_i(s)$, and $\boldsymbol{\varphi}_i(s)$ is also an isoparametric curve of $\mathbf{y}_i(s,t)$, then there exists t_0 such that $\mathbf{y}_i(s,t_0)=\boldsymbol{\varphi}_i(s)$. Consequently, the surface can be represented as

$$M_i:\mathbf{y}_i(s,t)-\mathbf{y}_i(s,t_0)=(t-t_0)\mathbf{e}_i(s), 0\leq s\leq L, \text{ with } t,\ t_0\in[0,T], \tag{23}$$

where $\mathbf{y}_i(s,t_0)=\boldsymbol{\varphi}_i(s)$ $(i=1,2,3)$, and $\mathbf{e}_i(s)$ defines the direction of the rulings. In view of Equation (10), we have

$$(t-t_0)\mathbf{e}_i(s)=u(s,t)\mathbf{t}_i(s)+v(s,t)\mathbf{b}_i(s),\ 0\leq s\leq L, \text{ with } t,\ t_0\in[0,T],$$

which is a system of equations in two unknown functions $u(s,t)$ and $v(s,t)$. For $u(s,t)$ and $v(s,t)$, we have

$$\begin{aligned} u(s,t)&=(t-t_0)<\mathbf{e}_i(s),\mathbf{t}_i(s)>, \\ v(s,t)&=(t-t_0)<\mathbf{e}_i(s),\mathbf{b}_i(s)>. \end{aligned} \tag{24}$$

The necessary and sufficient conditions for $\mathbf{y}_i(s,t)$ to be a ruled surface with a directrix $\boldsymbol{\varphi}_i(s)$; $i=1,2,3$ are represented in Equation (24).

In Galilean 3-space $\mathbb{G}_3$, it is demonstrated there exist only three types of ruled surfaces realized as follows [17]:

Type I. Non-conoidal or conoidal ruled surfaces with striction curve do not lie in a Euclidean plane.

Type II. Ruled surfaces with striction curve in a Euclidean plane.

Type III. Conoidal ruled surfaces with absolute line as the oriented line in infinity.

We now check if the curve $\boldsymbol{\varphi}_i(s)$ is also geodesic on these three types:

Type I: $\boldsymbol{\varphi}_1(s) = (s, y(s), z(s))$ does not lie in a Euclidean plane, and $\mathbf{e}_1(s) = (1, e_2(s), e_3(s))$ is non-isotropic. Then,

$$\begin{aligned} \mathbf{t}_1(s) &= (1, y'(s), z'(s)), \\ \mathbf{n}_1(s) &= \frac{1}{\kappa(s)}(0, y''(s), z''(s)), \\ \mathbf{b}_1(s) &= \frac{1}{\kappa(s)}(0, -z''(s), y''(s)), \end{aligned} \tag{25}$$

where $\kappa(s) = \sqrt{(y''(s))^2 + (z''(s))^2}$. From Equations (1), (24), and (25), we have:

$$u(s,t) = (t - t_0),\ v(s,t) = 0, \tag{26}$$

which does not satisfy Theorem 1.

Type II: $\boldsymbol{\varphi}_2(s) = (0, y(s), z(s))$ lie in a Euclidean plane, and $\mathbf{e}_2(s) = (1, e_2(s), e_3(s))$ is non-isotropic. Then,

$$\begin{aligned} \mathbf{t}_2(s) &= (0, y'(s), z'(s)), \\ \mathbf{n}_2(s) &= \frac{1}{\kappa(s)}(0, y''(s), z''(s)), \\ \mathbf{b}_2(s) &= \frac{1}{\kappa(s)}(0, 0, 0), \end{aligned} \tag{27}$$

where $\kappa(s) = \sqrt{(y''(s))^2 + (z''(s))^2}$. From Equations (1), (24), and (27), we have:

$$u(s,t) = v(s,t) = 0, \tag{28}$$

which does not satisfy Theorem 1.

Corollary 2. *There is no ruled surface $\{\widehat{M}, M\}$ of type I and II that interpolate the Bertrand pair as common geodesic curves in $\mathbb{G}_3$.*

Type III: $\boldsymbol{\varphi}_3(s) = (s, y(s), 0)$ does not lie in a Euclidean plane, and $\mathbf{e}_3(s) = (0, e_2(s), e_3(s))$ is non-isotropic. Then,

$$\begin{aligned} \mathbf{t}_3(s) &= (1, y'(s), 0), \\ \mathbf{n}_3(s) &= \frac{1}{\kappa(s)}(0, y''(s)0), \\ \mathbf{b}_3(s) &= \frac{1}{\kappa(s)}(0, 0, y''(s)), \end{aligned} \tag{29}$$

where $\kappa(s) = \sqrt{(y''(s))^2} = y''(s)$. From Equations (1), (24), and (29), we have:

$$\left.\begin{aligned} &u(s,t) = 0,\ v(s,t) = \epsilon(t - t_0)e_3(s), \\ &e_3(s) \neq 0,\ t - t_0 \neq 0, \end{aligned}\right\} \tag{30}$$

where

$$\epsilon = \begin{cases} 1, \text{ if } y''(s) > 0. \\ -1, \text{ if } y''(s) < 0. \end{cases} \tag{31}$$

Equation (30) satisfies Theorem 1. Thus, at all points on $\boldsymbol{\varphi}_3(s)$, the ruling $\mathbf{e}_3(s) \in Sp\{\mathbf{t}_3(s), \mathbf{b}_3(s)\}$. Further, the ruling $\mathbf{e}_3(s)$ and the vector $\mathbf{t}_3(s)$ should not be parallel. Thus,

$$\mathbf{e}_3(s) = \alpha(s)\mathbf{t}_3(s) + \gamma(s)\mathbf{b}_3(s),\ 0 \leq s \leq L, \tag{32}$$

for functions $\alpha(s)$, and $\gamma(s) \neq 0$. Replacing it into Equation (24), we get

$$\alpha(s)t = u(s,t),\ \gamma(s)t = v(s,t),\ 0 \leq s \leq L. \tag{33}$$

Hence, the ruled surface family with the common geodesic base curve $\boldsymbol{\varphi}_3(s)$ can be written as

$$M_3 : \mathbf{y}_3(s,t) = \boldsymbol{\varphi}_3(s) + t(\alpha(s)\mathbf{t}_3(s) + \gamma(s)\mathbf{b}_3(s)),\ 0 \leq s \leq L,\ 0 \leq t \leq T, \tag{34}$$

where $\alpha(s)$ and $\gamma(s) \neq 0$ can control the form of the surface family. It is clear that

$$\boldsymbol{\xi}_3(s,t) = t\gamma(\gamma\tau - \kappa\alpha')\mathbf{t}_3 + [\alpha\gamma' - (1+t\alpha')\gamma]\mathbf{n}_3 + (\gamma\tau - \alpha\kappa)\mathbf{b}_3. \tag{35}$$

Thus, when $t = 0$, that is, along $\boldsymbol{\varphi}_3(s)$, the surface normal is

$$\boldsymbol{\xi}_3(s,0) = -\gamma(s)\mathbf{n}_3.$$

Theorem 2. *The sufficient and necessary condition for M_3 being a ruled surface with $\boldsymbol{\varphi}_3(s)$ as a geodesic is that there exists a parameter $t_0 \in [0,T]$, as well as the functions $\alpha(s)$ and $\gamma(s) \neq 0$, so that M_3 can be specified by*

$$M_3 : \mathbf{y}_3(s,t) = \boldsymbol{\varphi}_3(s) + t\mathbf{e}_3(s),\ 0 \leq s \leq L, \tag{36}$$

where $\mathbf{e}_3(s) = \alpha(s)\mathbf{t}_3(s) + \gamma(s)\mathbf{b}_3(s)$.

It must pointed out that, in this family, there exist two geodesic curves crossing through each point on $\boldsymbol{\varphi}_3(s)$: one is $\boldsymbol{\varphi}_3(s)$ itself and the other is a non-isotropic line in the orientation $\mathbf{e}_3(s)$ as in Equation (32). All components of the isogeodesic ruled surfaces are specified by the two functions $\alpha(s)$ and $\gamma(s) \neq 0$, that is, by the orientation non-isotropic vector function $\mathbf{e}_3(s)$. Similarity, the ruled surfaces $\widehat{M}$ of type III has also have $\widehat{\boldsymbol{\varphi}}_3(s)$ as an isogeodesic curve.

Corollary 3. *The only ruled surfaces $\{\widehat{M}_3, M_3\}$ of type III interpolate the Bertrand pair as common geodesic curves.*

Now, we research the correlations of the ruled surface family of type III. Let $\boldsymbol{\varphi}_3(s) = (s, y(s), 0),\ \ 0 \leq s \leq L$ be a curve with $\boldsymbol{\varphi}_3''(s) \neq 0$, from Equations (7), (29) and (30); we have $\widehat{\boldsymbol{\varphi}}_3(s) = (s, y(s) + \epsilon f, 0)$. From Equations (10), (11) and (31), the ruled surfaces family of type III that interpolate the Bertrand pair as common geodesic curves is

$$\begin{cases} M_3 : \mathbf{y}_3(s,t) = (s, y(s), 0) + (t - t_0)e_3(s)(0,0,1). \\ \widehat{M}_3 : \widehat{\mathbf{y}}_3(s,t) = (s, y(s) + f, 0) + \epsilon(t - t_0)e_3(s)(0,0,1) \end{cases} \tag{37}$$

where f is a constant, ϵ satisfies Equation (31), $e_3 \neq 0$, and $t_0 \neq 0$.

Example 2. *In view of Example 1, we have:*

(1) If $u(s,t) = 0$, $v(s,t) = \sin t$, the ruled surfaces family $\{\widehat{M}_3, M_3\}$ interpolates $\{\widehat{\boldsymbol{\varphi}}_3(s), \boldsymbol{\varphi}_3(s)\}$ as common geodesic curves as in (Figure 3):

$$\begin{cases} M_3 : \mathbf{y}_3(s,t) = (s, \sin s, \cos s) + (0, \cos, -\sin s)\sin t, \\ \widehat{M}_3 : \widehat{\mathbf{y}}_3(s,t) = (s, -\sin s, -\cos s) + (0, -\cos, \sin s)\sin t, \end{cases}$$

where the blue curve represents $\boldsymbol{\varphi}_3(s)$, the green curve is $\widehat{\boldsymbol{\varphi}}_3(s)$, $-3 \leq t \leq 3$, and $0 \leq s \leq 2\pi$.

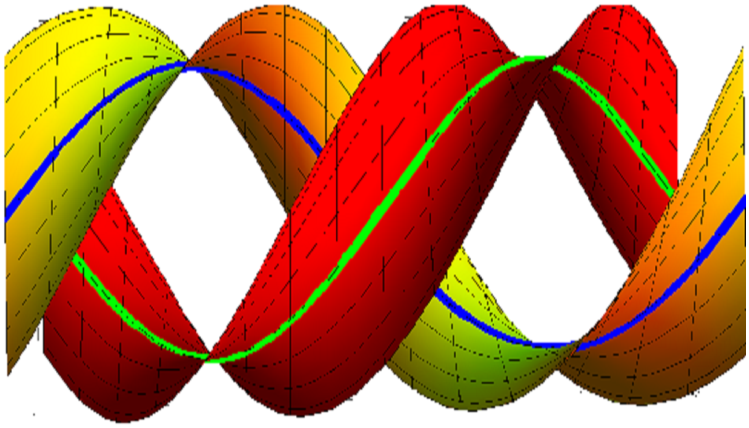

Figure 3. M (yellow) and $\widehat{M}$ (red).

(2) *If* $u(s,t) = 0$, $v(s,t) = t$, *the ruled surfaces family* $\{\widehat{M}_3, M_3\}$ *interpolates* $\{\widehat{\boldsymbol{\varphi}}_3(s), \boldsymbol{\varphi}_3(s)\}$ *as common geodesic curves as in (Figure 4):*

$$\begin{cases} M : \mathbf{y}(s,t) = (s, \sin s, \cos s) + t(0, \cos, -\sin s), \\ \widehat{M} : \widehat{\mathbf{y}}(s,t) = (s, -\sin s, -\cos s) + t(0, -\cos, \sin s), \end{cases}$$

where the blue curve represents $\boldsymbol{\varphi}(s)$, *the green curve is* $\widehat{\boldsymbol{\varphi}}(s)$, $-3 \leq t \leq 3$, *and* $0 \leq s \leq 2\pi$.

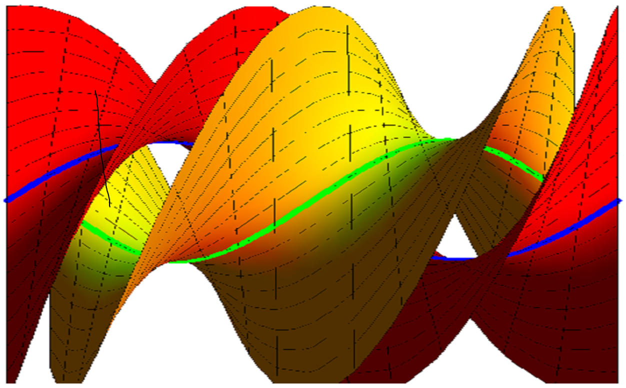

Figure 4. M (yellow) and $\widehat{M}$ (red).

(3) *If* $u(s,t) = 0$, $v(s,t) = \cos t$, *the ruled surfaces family* $\{\widehat{M}_3, M_3\}$ *interpolates* $\{\widehat{\boldsymbol{\varphi}}_3(s), \boldsymbol{\varphi}_3(s)\}$ *as common geodesic curves as in (Figure 5):*

$$\begin{cases} M : \mathbf{y}(s,t) = (s, \sin s, \cos s) + (0, \cos, -\sin s)\cos t, \\ \widehat{M} : \widehat{\mathbf{y}}(s,t) = (s, -\sin s, -\cos s) + (0, -\cos, \sin s)\cot t, \end{cases}$$

where the blue curve represents $\boldsymbol{\varphi}(s)$, *the green curve is* $\widehat{\boldsymbol{\varphi}}(s)$, $-3 \leq t \leq 3$, *and* $0 \leq s \leq 2\pi$.

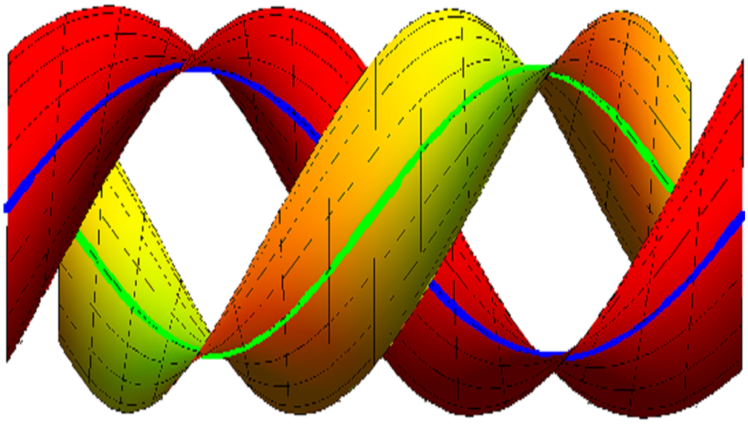

Figure 5. M (yellow) and $\widehat{M}$ (red).

4. Conclusions

In this work, we constructed the surfaces family and ruled surfaces family having Bertrand curves as common geodesic curves in Galilean space $\mathbb{G}_3$. For any allowable curve, there only exists the ruled surfaces family of type III having the same curve as common geodesic curves. Meanwhile, some curves were selected to organize the surfaces family and ruled surfaces family that have common Bertrand geodesic curves.

Hopefully, these results will be advantageous to physicists and those exploring general relativity theory. There are numerous opportunities for additional work; for example, consider the pseudo-Galilean geometry as a counterpart to the problem presented in the current study.

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