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Single-Server Queuing-Inventory Systems with Negative Customers and Catastrophes in the Warehouse

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Abstract: In this paper, we studied single-server models of queuing-inventory systems (QIS) with catastrophes in the warehouse part and negative customers (n-customers) in service facility. Consumer customers (c-customers) that arrived to buy inventory can be queued in an infinite buffer. Under catastrophes, all inventory of the system is destroyed but customers in the system (on server or in buffer) are still waiting for replenishment of stocks. Upon arrival of n-customer one c-customer is pushed out, if any. One of two replenishment policies (RP) can be used in the system: either (s, S) or randomized. In the investigated QISs, a hybrid service scheme was used: if upon arrival of the c-customer, the inventory level is zero, then according to the Bernoulli scheme, this customer is either lost (lost sale scheme) or joining the queue (backorder scheme). Mathematical models of the investigated QISs were constructed as two-dimensional Markov chains (2D MC). Ergodicity conditions of the investigated QISs were obtained, and the matrix-analytic method (MAM) was used to calculate the steady-state probabilities of the constructed 2D MCs. Formulas for performance measures were found and the results of numerical experiments are presented.

Keywords: queuing-inventory system; catastrophes; replenishment policies; matrix-analytic method

MSC: 60J28; 60K25; 90B05; 90B22

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1. Introduction

Queuing systems (QS), in which to service the customer, along with an idle server, certain items are also required, are called queuing-inventory systems (QIS), see [1,2]. In other words, QISs simultaneously possess the properties of classical QS and inventory control systems (ICS). In classical QS, only an idle server is enough to service a customer (in multi-rate QS, several idle servers will be required at the same time), and in classical ICS, the inventory is released to customers instantly, i.e., in classical ICS, there are no servers for customer service. However, in many real ICS, delivery of the inventory to customers is carried out using certain devices (servers), and this process will require some positive time to complete. Since the flow of customers is a random one, and the service time (i.e., the process of issuing stocks to customers) is a random variable, a queue of customers is formed to receive stocks. In other words, in QISs, it is necessary to manage both service and inventory control processes simultaneously, i.e., it is necessary to organize the process of servicing of customers and manage the inventory of the system.

The first work devoted to the study of QISs models are the works [3,4]. After these works, models of QISs were intensively studied by various authors over the past three decades. A detailed overview of known results is set out in the work [5].

In each QIS model, it is necessary to make certain assumptions about the type of distribution functions (d.f.) of random variables that form the model under study, i.e., d.f. of input flow, service time, the lifetime of inventory, etc. In addition, it is necessary to define the replenishment policy (RP) used. Usually, this d.f. and RP are the basis for classifying of QISs models. Based on the purpose of the considered paper, here, we will use the classification of QISs based on the lifetime of the stocks. According to this indicator, all QISs can be divided into two classes: QISs with an infinite life of stocks (i.e., the stocks of the system never deteriorate) and QISs with a finite life of stocks (i.e., the stocks of the system deteriorate after a finite time). Models of QISs with an infinite life of stocks are studied in detail in the available literature, see [5].

In models of perishable QISs, stock deterioration occurs within a certain positive time interval. In the class of perishable QISs, two sub-classes of systems are distinguished: (1) QIS with individual lifetime (ILT) in which each item can perish independently of the others, and (2) QIS with common lifetime (CLT) where all items perish together, e.g., foods with the same expiry date, medicines manufactured with the same expiry date and so on. Note that models of perishable QIS with ILT were intensively investigated, see, e.g., [6–9] and their reference lists. However, models of perishable QIS with CLT were little studied, see [10–13].

It is important to note that, in practice, there are QISs in which items can be destroyed instantly due to various reasons, e.g., due to the negligence of warehouse workers, as a result of a sudden power outage, etc. Despite their importance, such models of QISs were hardly studied, see [14–17]. Note that in the indicated papers, it was assumed that upon accident, the inventory level was instantly reduced only by one. In the present work, QIS models with catastrophes in the warehouse part of the system are studied. This means that all stocks of the system are destroyed at the same time. At first glance, it may seem that the models of QISs with catastrophes are similar to models of QISs with CLT, but these models differ from each other. Indeed, in models of QISs with CLT, it is required that, at any given time, all stocks in the warehouse have the same age, i.e., it is considered that all stocks arrived as a result of execution of one batch of orders. One can be achieved as follows: any items remaining in the inventory at the time of replenishment will be removed to accommodate the new batch of S items, where S is a maximum inventory capacity, see [10]. However, in the model of QIS with catastrophe, this rather rigid assumption is removed.

Note that the classical models of QS with catastrophes were studied in detail in the available literature, see [18–24]. These papers considered that, as a result of a catastrophe, the servers of the system fail, while the customers are not affected and they are waiting for the servers to be repaired. In other words, QS models with catastrophes are a useful tool for studying systems with unreliable servers.

Another feature of the QIS models studied here is that in addition to consumer customers, c-customers (i.e., customers that arrived to purchase the inventory), negative customers (n-customers) also enter the system. Negative customers do not require the stocks, but they force one C-customer out of the system. At the same time, n-customers will not affect the stocks of the system. For more details about the QS with n-customers, readers can refer to the pioneering work [25], as well as the review paper [26].

Despite their importance, QIS models with n-consumers almost were not studied in the available literature. To the best of our knowledge, for the first time, the Markovian model of single-server perishable QIS with finite waiting room under (s,Q), $Q = S - s > s + 1$, replenishment policy was considered in [27]. It is assumed that both types of customers, consumer and negative, arrive according to a Markovian arrival process (MAP). Authors considered the following removal rule: an n-customer at an arrival epoch removes one or more waiting c-customers and the number of removals is a random variable depending on the number of waiting c-customers in the system. The joint probability distribution of the number of c-customers in the system and the inventory level is obtained and various performance measures of the system are computed as well as the total expected cost rate is calculated. This paper showed also examples of real-life situations in which QIS models with n-customers can be applied. For instance, some people who promote the goods of other sellers may advertise their goods among the customers of this system, and thus, the customers of this system may leave the system without receiving its goods. In a recent paper [28], n-customers were taken into account for the perishable QIS model with double sources for replenishments.

Note that catastrophes and n-customers make models of QIS more realistic. To our best knowledge, no existing works on QIS management considered these two features simultaneously. It was also unclear whether known RPs still work well in this setting. As a result, it is desirable to develop models of such QISs and deeply analyze this problem. On the other hand, considering these realistic features simultaneously increases the difficulty of the constructed mathematical models as well as their computational complexity. The considered work is the first attempt in this direction.

Most of the existing literature on the QIS assumed that one of the following schemes is applied: (i) lost sales where any customer that faces a zero inventory is lost or (ii) backorder sales where each customer joins the queue if upon its arrival there is no inventory. However, in real QISs, some customers may join the queue or be lost according to the Bernoulli scheme if the inventory level is zero at the time of their arrival. We will call these schemes hybrid sales. Note that, to the best of our knowledge, QIS models with hybrid sales were hardly studied.

The next step after the description of the model was the choice of the appropriate mathematical tool. In this regard, we note that the matrix-analytic method (MAM) [29] is an effective tool. A modern exposition of the basis of the theory and practice of MAM can be found in monographs [30–34].

An analysis of the available literature showed that they studied QIS models under several unrealistic assumptions. For instance, in known works, it was assumed that, in the warehouse, there were no accidents that lead to the destruction of the entire inventory and the system used a unique sales scheme. In addition, most known works did not take into account the possibility of negative customers. Therefore, we summarize the main contributions of this work as follows:

- Our model simultaneously captures three important and realistic features of QISs: catastrophes in a warehouse, negative customers in a service facility, and hybrid sales;
- The investigated QISs can operate under two RPs: (s, S) policy or randomized replenishment policy;
- We obtain the easily checkable stability conditions of the investigated systems and show that in special cases, they do not depend on the storage size, the rate of catastrophes as well as the replenishment rate;
- Simple formulas for steady-state probability vectors as well as for performance measures of our systems are developed;
- The developed formulas allow analyzing of the effect of the initial parameters on performance measures of the studied QISs as well as on expected total cost (ETC) and appropriately select the optimal RPs parameters so that the ETC is minimized.

This paper is organized as follows. In Section 2, we describe the QIS models, clarifying the assumptions of the d.f. random variables that form the models. Stability conditions for both models are established and MAM is used for steady-state analysis of the models are given in Section 3. Explicit formulas for key performance measures are obtained in Section 4. The results of numerical examples are shown in Section 5. Concluding remarks are given in Section 6.

2. Describing the Models

The block diagram of the investigated single-server QIS of infinite capacity is shown in Figure 1. The homogeneous c-customers arrive at the service facility according to Poisson process with rate λ^+. The service times of the c-customers are assumed to be exponentially distributed with parameter μ. The service requires an idle server along with items (one for each c-customer) that are stored in an inventory of maximum capacity S.

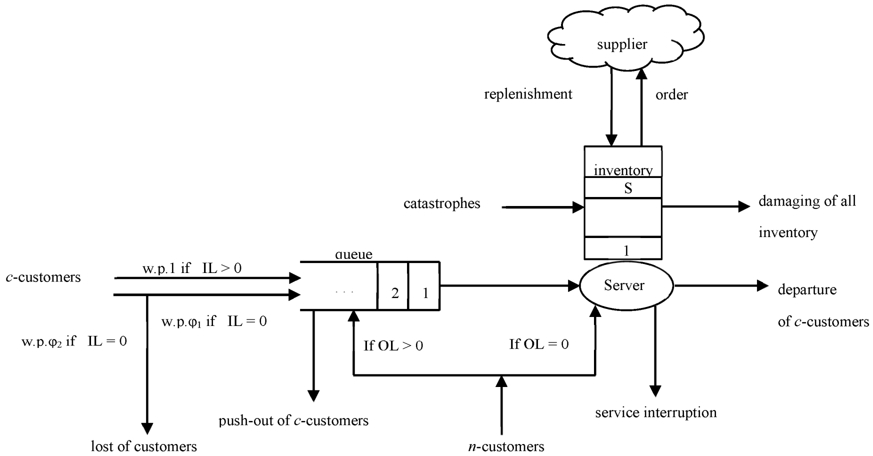

Figure 1. Block diagram of the system under study.

In the system, hybrid sales scheme is used, i.e., some part of c-customers is serviced according to the backorder sale scheme, while the other part is serviced according to the lost sale scheme. This means the following: if there are no stocks in the system upon arrival of c-customer, then, in accordance to the Bernoulli trials, it either, with probability (w.p.), φ_1 joins the queue of infinite length (backorder sale scheme), or w.p. φ_2 leaves the system unserved (lost sale scheme), where $\varphi_1 + \varphi_2 = 1$.

The system also receives n-customers with a rate λ^-. When a n-customer arrives, one c-customer force out of the system. A n-customer can force out of the system even a c-customer, which is in the server, while the inventory level does not change, since it is assumed that stocks are released after the completion of servicing a c-customer. If there is a queue of c-customers at the time an n-customer arrives, then only the c-customer is pushed out from the queue (i.e., the service of the c-customer, which is in the server, continues); if there are no c-customers in the system, then the received n-customer does not affect the operation of the system.

In the system, catastrophic events can occur only in its warehouse part. The flow of catastrophic events is Poisson one with the parameter κ, and at the moment of arrival of such an event, all the reserves of the system are instantly destroyed. As a result of the catastrophes, even the stock, which is at the status of release to the c-customer, is destroyed. In the latter case, the c-customer whose service was interrupted due to a catastrophe is returned to the queue; in other words, the catastrophe only destroys the stocks of the system and does not force c-customers out of the system. If the inventory level is zero, then the disaster does not affect the operation of the system warehouse.

Here, two inventory replenishment policies were considered. The first RP was according to a (s, S)-type policy (sometimes this policy is called "Up to S"). In this policy, when the inventory level drops to the reorder point s, where $0 \leq s < S$, an order was placed for replenishment and upon replenishment, the inventory level was restocked to level S, no matter how many items are still present in the inventory. Second RP is randomized (randomized replenishment policy, RRP), see [35]. In RRP, an order is placed only when the system's warehouse is completely empty and the volume of the supplied stock is a random variable with a known distribution; in other words, w.p. α_m, the volume of incoming stock is equal to m, where $\sum_{m=1}^{S} \alpha_m = 1$, $\alpha_S > 0$. In both RPs, the parameter ν indicates the reorder rate per order.

The task is to find the joint distribution of the number of c-customers in the system and the inventory level of the system, as well as to calculate the key performance measures of the system.

3. Stationary Distributions

First consider the computation of the steady-state probabilities of the system under (s, S) policy. Let X_t be the number of customers at time t and Y_t be the inventory level at time t. Then, the process $Z_t = \{(X_t, Y_t), t \geq 0\}$ forms a continuous time Markov chain (CTMC) with state space

$$E = \bigcup_{n=0}^{\infty} L(n),$$

where $L(n) = \{(n, 0), (n, 1), \ldots, (n, S)\}$ is the subset of state space E with $X_t = n$ called the level n.

Let $q((n_1, m_1), (n_2, m_2))$ denote the transition rate from state $(n_1, m_1) \in E$ to state $(n_2, m_2) \in E$. So, by noting the assumptions made in Section 2, we conclude that the investigated CTMC has a generator $G = (q((n_1, m_1), (n_2, m_2)))$, $(n_1, m_1), (n_2, m_2) \in E$, with the following transition rates for $(n_1, m_1) \in E$:

$$q((n_1, m_1), (n_1 + 1, 0)) = \lambda^+ \varphi_1 \cdot \chi(m_1 = 0); \tag{1}$$

$$q((n_1, m_1), (n_1 + 1, m_1)) = \lambda^+ \cdot \chi(m_1 > 0); \tag{2}$$

$$q((n_1, m_1), (n_1 - 1, m_1)) = \lambda^- \cdot \chi(n_1 > 0); \tag{3}$$

$$q((n_1, m_1), (n_1 - 1, m_1 - 1)) = \mu \cdot \chi(n_1 > 0) \cdot \chi(m_1 > 0); \tag{4}$$

$$q((n_1, m_1), (n_1, 0)) = \kappa \cdot \chi(m_1 > 0); \tag{5}$$

$$q((n_1, m_1), (n_1, S)) = \nu \cdot \chi(m_1 \leq s). \tag{6}$$

Hereinafter, $\chi(A)$ is the indicator function of the event A, which is 1 if A is true and 0 otherwise.

By re-numbering the states of the system in a lexicographic way, from relations (1)–(6) we conclude that the process $Z_t, t \geq 0$, is a level independent quasi birth–death (LIQBD) process and its generator G might be represented as follows:

$$G = \begin{pmatrix} B & A_0 & O & \ldots & O & \ldots \\ A_2 & A_1 & A_0 & \cdots & O & \ldots \\ O & A_2 & A_1 & A_0 & O & \ldots \\ O & O & A_2 & A_1 & A_0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix}, \tag{7}$$

where O denotes zero square matrix with dimension $S + 1$, and all other block matrices are square matrices of the same dimension. Entities of the block matrices $B = \|b_{ij}\|$ and $A_k = \|a_{ij}^{(k)}\|, i, j = 0, 1, \ldots, S$, are determined as follows:

$$b_{ij} = \begin{cases} \nu & \text{if } 0 \leq i \leq \text{s}, j = S, \\ \kappa & \text{if } i > 0, j = 0, \\ -(\nu + \lambda^+ \varphi_1) & \text{if } i = j = 0, \\ -(\nu + \kappa + \lambda^+) & \text{if } 0 < i \leq \text{s}, i = j, \\ -(\kappa + \lambda^+) & \text{if } s < i \leq S, i = j, \\ 0 & \text{in other cases;} \end{cases} \tag{8}$$

$$a_{ij}^{(0)} = \begin{cases} \lambda^+ \varphi_1 & \text{if } i = j = 0, \\ \lambda^+ & \text{if } i \neq 0, i = j, \\ 0 & \text{in other cases;} \end{cases} \tag{9}$$

$$a_{ij}^{(1)} = \begin{cases} \nu & \text{if } 0 \le i \le s, j = S, \\ \kappa & \text{if } i > 0, j = 0, \\ -(\lambda^- + \nu + \lambda^+ \varphi_1) & \text{if } i = j = 0, \\ -(\nu + \kappa + \mu + \lambda^+ + \lambda^-) & \text{if } 0 < i \le s, i = j, \\ -(\kappa + \mu + \lambda^+ + \lambda^-) & \text{if } i > s, i = j, \\ 0 & \text{in other cases;} \end{cases} \tag{10}$$

$$a_{ij}^{(2)} = \begin{cases} \lambda^- & \text{if } i = j, \\ \mu & \text{if } i > 0, j = i - 1, \\ 0 & \text{in other cases.} \end{cases} \tag{11}$$

The entities of the generator $A = A_0 + A_1 + A_2$ are determined as follows:

$$a_{ij} = \begin{cases} -\nu & \text{if } i = j = 0, \\ \nu & \text{if } 0 \le i \le s, j = S, \\ \mu + \kappa & \text{if } i = 1, j = 0, \\ \kappa & \text{if } i > 1, j = 0, \\ -\mu & \text{if } i > 0, j = i, \\ \mu & \text{if } i \ge 2, j = i - 1. \end{cases} \tag{12}$$

The stationary probability vector that corresponds to the generator A is denoted by $\boldsymbol{\pi} = (\pi(0), \pi(1), \ldots, \pi(S))$. In other words, we have the balance equations:

$$\boldsymbol{\pi} A = 0, \boldsymbol{\pi} \boldsymbol{e} = 1, \tag{13}$$

where $\mathbf{0}$ is the null row vector of dimension $S + 1$ and $\boldsymbol{e}$ is the column vector of dimension $S + 1$ that contains only 1's.

By using the recursive procedure, we obtained that Equation (13) had the following solution:

$$\pi(0) = \frac{1 + bc}{1 + dc}, \pi(1) = d\pi(0) - b; \pi(m) = a_m \pi(1), 2 \le m \le S, \tag{14}$$

where $d = \frac{\nu+\kappa}{\mu}$, $b = \frac{\kappa}{\mu}$, $c = \sum_{m=1}^{S} a_m$, $a_m = \begin{cases} (1+d)^{m-1}, & \text{if } 1 \le m \le s+1, \\ (1+d)^s (1+b)^{m-s-1}, & \text{if } s+1 < m \le S. \end{cases}$

Using the stationary probability vector of the generator A given by (14), we can derive the ergodicity (stability) condition of the process $Z_t, t \ge 0$.

Proposition 1. *Under (s, S) policy, the process $Z_t, t \ge 0$, is ergodic if and only if the following condition is fulfilled:*

$$\lambda^+ (1 - \varphi_2 \pi(0)) < \lambda^- + \mu(1 - \pi(0)). \tag{15}$$

Proof of Proposition 1. In accordance with [29] (pp. 81–83), the process $Z_t, t \ge 0$, is ergodic if and only if

$$\pi A_0 e < \pi A_2 e. \tag{16}$$

By using relations (14), from the matrices A_0 and A_2, we have

$$A_0 e = \lambda^+ \varphi_1 \pi(0) + \lambda^+ \sum_{m=1}^{S} \pi(0) = \lambda^+ \varphi_1 \pi(0) + \lambda^+ (1 - \pi(0)) = \lambda^+ (1 - \varphi_2 \pi(0))$$

and

$$\pi A_2 e = \lambda^- \sum_{m=0}^{S} \pi(m) + \mu \sum_{m=1}^{S} \pi(m) = \lambda^- + \mu(1 - \pi(0)).$$

Thus, relation (16) is equivalent to the inequality (15). □

Note 1. *The established ergodicity condition (15) has a probabilistic meaning, i.e., it indicates that the rate of c-customers entering the system must be less than the total rate of negative customers and the rate of served c-customers. We find from (15) that in general case stability condition for the present model is dependent on the storage size of system, the rate of catastrophes, and the replenishment rate.*

Note 2. *Consider the following special cases.*

(i) If $\varphi_2 = 1$ (i.e., when a pure lost sale scheme is used) and $\lambda^- = 0$ (i.e., when there are not negative customers) from (9), we find the ergodicity condition for the single-server Markovian queuing system, i.e., $\lambda^+ < \mu$. In other words, under such assumptions, the ergodicity condition of the system does not depend on the storage size of system, the rate of catastrophes, and the replenishment rate. Similar results for other models were obtained in [9,36,37].

(ii) If $\varphi_2 = 1$ and $\lambda^- > 0$, the ergodicity condition is depending on all indicated parameters of the system, see Formula (14).

(iii) If $\varphi_2 = 0$ (pure backorder scheme is used), the ergodicity condition is dependent on all indicated parameters of the system even for case $\lambda^- = 0$, see Formula (14).

A steady-state probability that corresponds to the generator matrix G, we denote by $p = (p_0,\ p_1,\ p_2, \cdots)$, where $p_n = (p(n,0), p(n,1), \ldots, p(n,S)), n = 0, 1, \cdots$. Under the ergodicity condition (15), desired steady-state probabilities are determined from the following equations:

$$p_n = p_0 R^n, n \geq 1, \tag{17}$$

where R is the nonnegative minimal solution of the following quadratic matrix equation:

$$R^2 A_2 + R A_1 + A_0 = 0.$$

From (8)–(11), it was concluded that bound probabilities p_0 are determined from the following system of equations with normalizing conditions:

$$p_0(B + R A_2) = 0,$$

$$p_0(I - R)^{-1} e = 1. \tag{18}$$

where I indicate the identity matrix of dimension $S + 1$.

Now consider the computation of the steady-state probabilities under RRP. In this case, parameters $q((n_1, m_1), (n_2, m_2))$ are calculated via relations (1)–(5) but relation (6) should be substituted by the following equations:

$$q((n_1, 0), (n_1, m)) = \nu_m \cdot \chi(1 \leq m \leq S),$$

where $\nu_m = \nu \alpha_m$, $1 \leq m \leq S$.

Therefore, for this policy the generator matrix of the process $Z_t, t \geq 0$, has the following form:

$$\widetilde{G} = \begin{pmatrix} \widetilde{B} & A_0 & O & \ldots & O & \ldots \\ A_2 & \widetilde{A}_1 & A_0 & \cdots & O & \ldots \\ O & A_2 & \widetilde{A}_1 & A_0 & O & \ldots \\ O & O & A_2 & \widetilde{A}_1 & A_0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix},$$

Here, entities of matrices $\widetilde{B}$ and $\widetilde{A}_1$ are calculated as follows:

$$\widetilde{b}_{ij} = \begin{cases} \nu_j & \text{if } i = 0, j > 0, \\ \kappa & \text{if } i > 0, j = 0, \\ -(\nu + \lambda^+ \varphi_1) & \text{if } i = j = 0, \\ -(\kappa + \lambda^+) & \text{if } 0 < i \leq S, i = j, \\ 0 & \text{in other cases ;} \end{cases} \tag{19}$$

$$\widetilde{a}_{ij}^{(1)} = \begin{cases} \nu_j & \text{if } i = 0, j > 0, \\ \kappa & \text{if } i > 0, j = 0, \\ -(\lambda^- + \nu + \lambda^+ \varphi_1) & \text{if } i = j = 0, \\ -(\kappa + \mu + \lambda^+ + \lambda^-) & \text{if } i > 0, i = j, \\ 0 & \text{in other cases.} \end{cases} \tag{20}$$

In this model, entities of the generator $\widetilde{A} = A_0 + \widetilde{A}_1 + A_2$ are determined as

$$\widetilde{a}_{ij} = \begin{cases} -\nu & \text{if } i = j = 0, \\ \nu_j & \text{if } i = 0, j > 0, \\ \mu + \kappa & \text{if } i = 1, j = 0, \\ \kappa & \text{if } i > 1, j = 0, \\ -\mu & \text{if } i > 0, j = i, \\ \mu & if\ i \geq 2, j = i - 1. \end{cases} \tag{21}$$

Again, using the recursive procedure, we found that the balance Equation (13), where the matrix A is replaced by $\widetilde{A}$, the following solution was used

$$\pi(m) = r_m \pi(0),\ 0 \leq m \leq S, \tag{22}$$

where r_m are calculated from the following reverse recursive relations

$$r_0 = 1,$$

$$r_S = \frac{\nu_S}{\mu + \kappa},$$

$$r_m = \frac{1}{\mu + \kappa}(\mu r_{m+1} + \nu_m),\ 1 \leq m \leq S - 1.$$

Here, the unknown parameter $\pi(0)$ is found from the normalizing condition, i.e.,

$$\pi(0) = \left(\sum_{r=0}^{S} r_m \right)^{-1}. \tag{23}$$

In analogy with Proposition 1, it is easy to show that the following fact is true.

Proposition 2. *Under RRP policy, the process $Z_t, t \geq 0$, is ergodic if and only if the condition (15) is fulfilled where $\pi(0)$ is defined as in (23).*

Furthermore, by using a system of Equations (17) and (18), the steady-state probabilities for this model were calculated.

4. Performance Measures

In this section, we are interested in the key performance measures of the investigated system related to both inventory and queuing under each RP. Having determined the steady-state probabilities under both RPs, we can compute the key performance measures of the investigated models explicitly.

Performance measures related to inventory are the following:

- Average inventory level (S_{av}) under both policy

$$S_{av} = \sum_{m=1}^{S} m \sum_{n=0}^{\infty} p(n,m); \tag{24}$$

- Average order size under (s,S) policy

$$V_{av} = \sum_{m=S-s}^{S} m \sum_{n=0}^{\infty} p(n,S-m); \tag{25}$$

under RRP

$$V_{av} = \left(\sum_{m=1}^{S} m\alpha_m\right)\left(\sum_{n=0}^{\infty} p(n,0)\right); \tag{26}$$

- Average reorder rate (RR) under (s, S) policy

$$RR = \mu \sum_{n=1}^{\infty} p(n,s+1) + \kappa\left(1 - \sum_{n=0}^{\infty} p(n,0)\right); \tag{27}$$

under RRP

$$RR = \mu \textstyle\sum_{n=1}^{\infty} p(n,1) + \kappa\Big(1 - \sum_{n=0}^{\infty} p(n,0)\Big). \tag{28}$$

Performance measures related to queuing are the following:

- Average length of the queue (L_{av}) under both policies

$$L_{av} = \sum_{n=1}^{\infty} n \sum_{m=0}^{S} p(n,m). \tag{29}$$

- Loss rate (LR) of customers under both policies

$$LR = \lambda^{+}\varphi_2 \sum_{n=0}^{\infty} p(n,0) + \lambda^{-}\left(1 - \sum_{m=0}^{S} p(0,m)\right). \tag{30}$$

5. Numerical Results

Here, we consider the results of numerical experiments for both models. These experiments were generated using a Fortran 90 code due to the authors' years of experience with this software. The running time (i.e., time from compiling the program to the time results appear) was only a few seconds.

Below, hypothetical models were considered for both policies, i.e., the values of initial parameters were chosen arbitrarily. Note that in realistic applications, these values can be changed.

First, consider the results for the model with "Up to S" policy. For this RP, we considered the behavior of performance measures versus s as well as the finding the optimal value of s to minimize the expected total cost (ETC) that was defined as follows:

$$ETC(s) = (K + c_r V_{av})RR + c_h S_{av} + c_{ps}\kappa S_{av} + c_l LR + c_w L_{av}, \tag{31}$$

where K is the fixed price of one order, c_r is the unit price of the order size, c_h is the unit inventory storage price per unit of time, c_{ps} is the price of unit inventory damaging, c_l is the cost for a single c-customer loss, c_w is the price per unit time of queuing delay for a single c-customer.

For this policy, it was assumed that values of all parameters of the QIS were fixed except the parameter s. In other words, here, numerical experiments were processed to analyze the effect of parameter s on the performance measures.

Let us consider $S = 50$ and that values of load parameters are selected as follows: $\lambda^+ = 6,\ \lambda^- = 1,\ \kappa = 1, \mu = 8, \varphi_1 = 0.6, \nu = 1$. The coefficients in the expression for functional in ETC (see (31)) were chosen as follows: $K = 10, c_r = 15, c_h = 10, c_l = 450, c_w = 400, c_{ps} = 15$.

The impact of reorder points s on performance measures, ETC, are shown in Table 1. From this table, we conclude that the rate of change of all performance measures was very low and ETC was a unimodal function; its minimal value is indicated in bold.

Table 1. Impact of reorder point s to performance measures and ETC.

s	S_{av}	V_{av}	L_{av}	RR	LR	ETC
1	21.4427	25.0148	14.1234	0.5004	2.4279	8176.89
2	21.4447	25.0169	14.1208	0.5005	2.4278	8175.77
3	21.4470	25.0193	14.1183	0.5006	2.4277	8174.74
4	21.4495	25.0219	14.1161	0.5008	2.4276	8173.85
5	21.4524	25.0249	14.1142	0.5009	2.4276	8173.12
6	21.4557	25.0282	14.1124	0.5011	2.4275	8172.47
7	21.4593	25.0319	14.1109	0.5013	2.4274	8171.96
8	21.4646	25.0348	14.1097	0.5015	2.4274	8171.64
9	21.4681	25.0407	14.1083	0.5018	2.4274	8171.22
10	21.4732	25.0459	14.1072	0.5021	2.4273	8171.00
11	21.4825	25.0480	14.1061	0.5025	2.4273	8170.92
12	21.4855	25.0583	14.1053	0.5028	2.4273	8170.81
13	21.4948	25.0655	14.1045	0.5032	2.4272	8170.84
14	21.5021	25.0724	14.1039	0.5036	2.4272	8171.02
15	21.5099	25.0827	14.1032	0.5043	2.4272	8171.19
16	21.5200	25.0929	14.1026	0.5049	2.4272	8171.50
17	21.5318	25.1054	14.1020	0.5058	2.4272	8172.04
18	21.5348	25.1166	14.1016	0.5066	2.4272	8172.45
19	21.5577	25.1305	14.1012	0.5076	2.4271	8173.11
20	21.5731	25.1459	14.1009	0.5087	2.4271	8173.99
21	21.5913	25.1618	14.1006	0.5101	2.4271	8174.78
22	21.6091	25.1820	14.1002	0.5116	2.4271	8175.86
23	21.6300	25.2029	14.1000	0.5133	2.4271	8177.14
24	21.6554	25.2233	14.0998	0.5154	2.4271	8178.81
25	21.6785	25.2514	14.0996	0.5177	2.4271	8180.22
26	21.6971	25.2764	14.0994	0.5194	2.4271	8182.75
27	21.7322	25.3014	14.0992	0.5218	2.4271	8184.44
28	21.7708	25.3438	14.0991	0.5271	2.4271	8186.67
29	21.8121	25.3939	14.0989	0.5329	2.4271	8189.58
30	21.8532	25.4310	14.0988	0.5399	2.4271	8192.77

The goals of the numerical experiments for the model with RRP were the investigation of the behavior of performance measures versus initial parameters for three schemas of changing of probabilities $\alpha_m,\ 1 \le m \le S$: (1) when $\alpha_m,\ 1 \le m \le S$ are constants, (2) when $\alpha_m,\ 1 \le m \le S$ are increasing ones, and (3) when $\alpha_m,\ 1 \le m \le S$ are decreasing ones.

Here, we again assumed that $S = 50$ and $\varphi_1 = 0.6$. Additionally, in the first schema, we set $\alpha_m = \frac{1}{50},\ 1 \le m \le 50$; in the second schema, we set $\alpha_1 = 0.01755,\ \alpha_m = \alpha_{m-1} + 0.0001,\ 2 \le m \le 50$; in the third schema, we set $\alpha_1 = 0.02245,\ \alpha_m = \alpha_{m-1} - 0.0001,\ 2 \le m \le 50$;

Values of other parameters are shown in the title of the appropriate Tables 2–5. In these tables, the first row corresponds to schema (1), the second row corresponds to schema (2), and the third row corresponds to schema (3).

Table 2. Performance measures vs. λ^+ under RRP, $\lambda^- = 1,\ \mu = 15,\ \nu = 1, \kappa = 1$.

λ^+	S_{av}	V_{av}	L_{av}	RR	LR
5	10.4293 10.8777 9.9749	13.6926 13.5965 13.7905	2.7998 2.7612 2.8397	0.5370 0.5332 0.5408	1.7367 1.7260 1.7475
5.2	10.3387 10.7845 9.8870	13.7382 13.6384 13.8400	3.0565 3.0111 3.1036	0.5388 0.5348 0.5427	1.8006 1.7892 1.8122
5.4	10.2490 10.6920 9.7998	13.7845 13.6810 13.8901	3.3380 3.2847 3.3936	0.5406 0.5365 0.5447	1.8645 1.8524 1.8769
5.6	10.1600 10.6003 9.7133	13.8315 13.7243 13.9409	3.6481 3.5854 3.7138	0.5424 0.5382 0.5467	1.9286 1.9156 1.9417
5.8	10.0718 10.5095 9.6276	13.8792 13.7685 13.9923	3.9915 3.9175 4.0692	0.5443 0.5399 0.5487	1.9926 1.9789 2.0067
6	9.9845 10.4197 9.5428	13.9276 13.8134 14.0445	4.3739 4.2864 4.4661	0.5462 0.5417 0.5508	2.0568 2.0423 2.0717
6.2	9.8981 10.3308 9.4588	13.9769 13.8591 14.0975	4.8024 4.6985 4.9122	0.5481 0.5435 0.5528	2.1212 2.1058 2.1370
6.4	9.8127 10.2430 9.3757	14.0269 13.9056 14.1511	5.2859 5.1620 5.4175	0.5501 0.5453 0.5549	2.1858 2.1696 2.2024
6.6	9.7283 10.1562 9.2930	14.0777 13.9530 14.2056	5.8360 5.6875 5.9926	0.5521 0.5472 0.5571	2.2506 2.2335 2.2681
6.8	9.6450 10.0706 9.2125	14.1294 14.0013 14.2608	6.4678 6.2886 6.6603	0.5541 0.5491 0.5592	2.3156 2.2976 2.3341
7	9.5627 9.9861 9.1325	14.1819 14.0504 14.3168	7.2012 6.9829 7.4371	0.5562 0.5510 0.5614	2.3809 2.3620 2.4003

Table 3. Performance measures vs. λ^- under RRP; $\lambda^+ = 5,\ \mu = 15,\ \nu = 1,\ \kappa = 1$.

λ^-	S_{av}	V_{av}	L_{av}	RR	LR
1	10.4293 10.8777 9.9749	13.6926 13.5965 13.7905	2.7998 2.7612 2.8397	0.5370 0.5332 0.5408	1.7367 1.7260 1.7475
1.2	10.5029 10.9531 10.0469	13.6569 13.5640 13.7514	2.5348 2.5032 2.5673	0.5356 0.5319 0.5393	1.8460 1.8353 1.8570
1.4	10.5277 11.0247 10.1150	13.6237 13.5338 13.7150	2.3027 2.2768 2.3294	0.5343 0.5307 0.5378	1.9491 1.9383 1.9601
1.6	10.7387 11.0924 10.1794	13.5928 13.5058 13.6812	2.0989 2.0775 2.1208	0.5330 0.5296 0.5365	2.0460 2.0352 2.0570

Table 3. *Cont.*

λ^-	S_{av}	V_{av}	L_{av}	RR	LR
1.8	10.7410 11.1563 10.2401	13.5640 13.4797 13.6497	1.9193 1.9016 1.9375	0.5319 0.5286 0.5353	2.1369 2.1262 2.1479
2	10.7595 11.2146 10.2972	13.5374 13.4555 13.6204	1.7608 1.7460 1.7759	0.5309 0.5277 0.5341	2.2221 2.2115 2.2330
2.2	10.8145 11.2729 10.3507	13.5126 13.4331 13.5932	1.6206 1.6082 1.6332	0.5299 0.5268 0.5331	2.3018 2.2913 2.3136
2.4	10.8660 11.3259 10.4009	13.4897 13.4124 13.5680	1.4963 1.4859 1.5069	0.5290 0.5260 0.5321	2.3763 2.3659 2.3869
2.6	10.9142 11.3755 10.4478	13.4684 13.3931 13.5446	1.3859 1.3771 1.3949	0.5282 0.5252 0.5312	2.4459 2.4356 2.4563
2.8	10.9594 11.4220 10.4917	13.4486 13.3752 13.5229	1.2877 1.2802 1.2954	0.5274 0.5245 0.5303	2.5108 2.5007 2.5210
3	11.0016 11.4655 10.5328	13.4303 13.3586 13.5028	1.2002 1.1937 1.2068	0.5267 0.5239 0.5295	2.5713 2.5614 2.5814

Table 4. Performance measures vs. ν under RRP; $\lambda^+ = 5,\ \lambda^- = 1,\ \mu = 15,\ \kappa = 1$.

ν	S_{av}	V_{av}	L_{av}	RR	LR
1	10.4293 10.8777 9.9749	13.6926 13.5965 13.7905	2.7998 2.7612 2.8397	0.5370 0.5332 0.5408	1.7367 1.7260 1.7475
1.2	11.5715 12.0587 11.0774	12.4789 12.3874 12.5721	2.1985 2.1724 2.2255	0.5872 0.5829 0.5916	1.5968 1.5869 1.6070
1.4	12.5297 13.0489 12.0029	11.4631 11.3761 11.5518	1.8188 1.7996 1.8385	0.6293 0.6246 0.6342	1.4813 1.4720 1.4908
1.6	13.3451 13.8910 12.7907	10.6004 10.5177 10.6847	1.5601 1.5452 1.5753	0.6651 0.6599 0.6704	1.3844 1.3757 1.3933
1.8	14.0472 14.6140 13.4694	9.8585 9.7797 9.9387	1.3742 1.3623 1.3865	0.6959 0.6903 0.7016	1.3020 1.2937 1.3103
2	14.6581 15.2465 14.0602	9.2136 9.1385 9.2900	1.2352 1.2254 1.2454	0.7226 0.7167 0.7286	1.2310 1.2232 1.2389
2.2	15.1945 15.8000 14.5790	8.6478 8.5762 8.7208	1.1281 1.1198 1.1367	0.7461 0.7399 0.7524	1.1693 1.1619 1.1767
2.4	15.6692 16.2898 15.0383	8.1476 8.0791 8.2173	1.0435 1.0363 1.0508	0.7668 0.7604 0.7734	1.1151 1.1082 1.1222

Table 4. *Cont.*

ν	S_{av}	V_{av}	L_{av}	RR	LR
2.6	16.0924 16.7261 15.4478	7.7020 7.6365 7.7687	0.9752 0.9690 0.9816	0.7853 0.7786 0.7921	1.0673 1.0607 1.0740
2.8	16.4718 17.1174 15.8151	7.3026 7.2398 7.3665	0.9193 0.9137 0.9249	0.8018 0.7950 0.8089	1.0247 1.0184 1.0311
3	16.8141 17.4703 16.1464	6.9425 6.8822 7.0039	0.8727 0.8678 0.8777	0.8168 0.8097 0.8240	0.9865 0.9805 0.9926

Table 5. Performance measures vs. κ under RRP; $\lambda^+ = 5,\ \lambda^- = 1,\ \mu = 15,\ \nu = 1.$

κ	S_{av}	V_{av}	L_{av}	RR	LR
1	10.4293 10.8777 9.9749	13.6926 13.5965 13.7905	2.7998 2.7612 2.8397	0.5370 0.5332 0.5408	1.7367 1.7260 1.7475
1.2	9.6443 10.0636 9.2205	14.7246 14.6395 14.8111	3.1521 3.1099 3.1958	0.5774 0.5741 0.5808	1.8486 1.8390 1.8583
1.4	8.9548 9.3468 8.5592	15.5927 15.5167 15.6699	3.5168 3.4702 3.5650	0.6115 0.6085 0.6145	1.9440 1.9353 1.9528
1.6	8.3492 8.7164 7.9791	16.3323 16.2637 16.4019	3.9015 3.8496 3.9553	0.6405 0.6378 0.6432	2.0264 2.0184 2.0345
1.8	7.8156 8.1604 7.4684	16.9696 16.9072 17.0327	4.3143 4.2560 4.3748	0.6655 0.6630 0.6680	2.0985 2.0911 2.1060
2	7.3432 7.6679 7.0166	17.5242 17.4672 17.5820	4.7640 4.6979 4.8327	0.6872 0.6850 0.6875	2.1622 2.1554 2.1692
2.2	6.9229 7.2294 6.6147	18.0112 17.9587 18.0644	5.2610 5.1852 5.3398	0.7063 0.7043 0.7084	2.2191 2.2127 2.2256
2.4	6.5469 6.8371 6.2553	18.4422 18.3936 18.4913	5.8178 5.7302 5.9090	0.7232 0.7213 0.7251	2.2704 2.2643 2.2765
2.6	6.2089 6.4842 5.9323	18.8262 18.7810 18.8719	6.4504 6.3481 6.5573	0.7383 0.7365 0.7401	2.3168 2.3111 2.3227
2.8	5.9035 6.1655 5.6405	19.1705 19.1284 19.2132	7.1797 7.0586 7.3065	0.7518 0.7501 0.7535	2.3593 2.3538 2.3648
3	5.6264 5.8762 5.3758	19.4810 19.4415 19.5210	8.0339 7.8884 8.1865	0.7640 0.7624 0.7655	2.3983 2.3931 2.4036

Now, we present the effect of initial parameters as well as considered schemas of changing probabilities $\alpha_m,\ 1 \le m \le S$ on the performance measures of the investigated RRP as follows:

- An analysis of data in Tables 2–5 showed that the second schema was favorable for all performance measures, except for the average inventory level. For the average inventory level, the third schema was favorable. It is interesting to note that the first schema was always intermediate between the three schemes.
- Table 2 shows that for all schemas, except for the average inventory level, performance measures increased versus the rate of consumer customers. These findings were expected.
- From Table 3, we can see that the average inventory level as well as the rate of loss of consumer customers increased when the rate of negative customers increased. However, the main performance measures decreased as the rate of negative customers increased. These findings were true for all schemas, and they were also expected.
- From Table 4, we can notice that the average inventory level as well as the reorder rate increased when the replenishment rate increased. A first observation concerning the behavior of reorder rate was unexpected. This phenomenon was explained as follows: when the replenishment rate increased, the probability that the inventory level was positive also increased and, hence, the catastrophe rate increased (see the second term in Formula (28)). Here, the rest of the performance measures were decreased versus replenishment rate. These findings were true for all schemas, and they were also expected.
- Table 5 shows that for all schemas, excluding the average inventory level, performance measures increased versus the rate of catastrophes. These findings were true for all schemas, and they were also expected.

  Note that the values of all performance measures in all Tables 2–5 changed smoothly.

6. Conclusions

A new model of the single-server QIS with negative customers and catastrophes in the warehouse under two replenishment policies was proposed. One of the replenishment policies was "Up to S" and the other was a randomized policy. The combination of lost sale scheme and backorder scheme was used, i.e., if the inventory level upon arrival of consumer customer was zero, then, in accordance to the Bernoulli trials, it either joined the queue of infinite length (backorder sale scheme), or left the system unserved (lost sale scheme). Negative customers required neither service nor inventory, i.e., upon the arrival of such a customer, one of the consumer customers was pushed out of the system. In the case of a catastrophe, all items in the warehouse, as well as items that were at the status of release to the consumer customer, were instantly destroyed but catastrophes did not force the consumer customers out of the system.

The mathematical models of the investigated system under both policies were two-dimensional Markov chains with different three-diagonal generators. The ergodicity conditions for the constructed 2D MC were found and probabilistic means of the obtained conditions were given. It was shown that some known results were special cases of the developed conditions. Formulas for calculating the performance measures were proposed. The results of numerical experiments were analyzed.

The direction of future work should be the investigation of the models with MAP flows of c-customers and/or n-customers as well as with PH distribution of service times for c-customers.

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List of Acronyms

ETC	Expected Total Cost
ICS	Inventory Control System
ILT	Individual Life Time
CLT	Common Life Time
MAM	Matrix-Analytic Method
MAP	Markovian Arrival Process
QS	Queuing System
QIS	Queuing-Inventory System
RP	Replenishment Policy
RRP	Randomized Replenishment Policy
IL	Inventory Level
QL	Queue Level

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