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Local Solvability for a Compressible Fluid Model of Korteweg Type on General Domains

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Abstract: In this paper, we consider a compressible fluid model of the Korteweg type on general domains in the N-dimensional Euclidean space for $N \geq 2$. The Korteweg-type model is employed to describe fluid capillarity effects or liquid–vapor two-phase flows with phase transition as a diffuse interface model. In the Korteweg-type model, the stress tensor is given by the sum of the standard viscous stress tensor and the so-called Korteweg stress tensor, including higher order derivatives of the fluid density. The local existence of strong solutions is proved in an L_p-in-time and L_q-in-space setting, $p \in (1, \infty)$ and $q \in (N, \infty)$, with additional regularity of the initial density on the basis of maximal regularity for the linearized system.

Keywords: compressible fluid; viscous fluid; capillarity; Korteweg type; local solvability; general domain; maximal regularity

MSC: Primary 35Q35; Secondary 35M12

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1. Introduction

This paper is concerned with a compressible fluid model of the Korteweg type presented in (1) below. In order to model fluid capillarity effects, Korteweg formulated a constitutive equation in 1901 for stress tensors

that included gradients of the fluid density ρ. The Korteweg stress tensor $\mathbf{K}(\rho)$, see (2) or (3) below, was introduced by Dunn and Serrin [1] (p. 107) on the basis of the thermodynamics of interstitial workings.

The Korteweg-type model was employed to analyze not only fluid capillarity effects but also a liquid–vapor phase transition; see, e.g., Liu, Landis, Gomez, and Hughes [2].

Let us introduce a short history of mathematical studies of the Korteweg-type model.

There are many studies of the Korteweg-type model such as the existence of weak solutions, the local and global well-posedness for strong solutions, large time decay of solutions, time periodic solutions, the vanishing capillarity limit, and maximal regularity; see, e.g., ref. [3] and references therein for more details. Concerning strong and weak solutions for other kind of fluids, we refer, e.g., to [4–7]. We focus on well-posedness results for strong solutions of the Korteweg-type model in what follows.

Let us start with problems in the whole space. Hattori and Li [8,9] proved local and global unique existence theorems on smooth solutions in L_2-based Sobolev spaces. On the other hand, Dancian and Desjardins [10] used critical Besov spaces to relax the regularity of initial data and proved unique existence theorems on local and global strong solutions. Furthermore, Murata and Shibata [11] proved the global well-posedness in an L_p-in-time and L_q-in-space setting by means of maximal regularity and time decay estimates of an analytic C_0-semigroup associated with a linearized system. Let $P(\rho)$ be the pressure function on $[0, \infty)$ and let $P'(\rho)$ be the derivative of $P(\rho)$ with respect to ρ. Recently, the asymptotic stability of the constant steady state $(\rho, \mathbf{u}) = (\rho_*, 0)$ satisfying $\rho_* > 0$ and $P'(\rho_*) = 0$ is actively studied by Kobayashi and his collaborators; see, e.g., [12–14].

Boundary value problems of the Korteweg-type model can be found in Bresch, Desjardins, and Lin [15]. Kotschote [16] considered (1) below for $\Gamma_S = \emptyset$ in the case where Ω is a bounded domain or an exterior domain, and proved the local well-posedness for strong solutions in an L_p setting with $p > N+2$ for both space and time. This result was extended to a non-isothermal case in [17] and to a non-Newtonian case in [18]. Notice that [18] considered not only the Dirichlet boundary condition but also the slip boundary condition and that [17,18] treated inhomogeneous boundary data. Furthermore, Kotschote [19] proved the asymptotic stability of non-trivial steady states when Ω is a bounded domain with $\Gamma_S = \emptyset$.

The present paper aims to extend the local well-posedness result given by [16] to the case where $\Gamma_S \neq \emptyset$ and Ω is a general domain, which is also called a uniform C^3 domain, see Definition 1 below.

Furthermore, our result is in an L_p-in-time and L_q-in-space setting, $p \in (1, \infty)$ and $q \in (N, \infty)$, with additional regularity of the initial density, which also gives us an extension of [16]; see Theorem 1 and Remark 2 below for more details. Theorem 1 is the main result of this paper, and is proved by the contraction mapping theorem with the help of the maximal regularity stated in Section 5.2, below.

This paper is organized as follows. The next section introduces our problem setting. Section 3 first introduces the notation used throughout this paper, and then the main result of this paper is stated. Section 4 treats resolvent problems in the whole space and in the half space, and then one introduces the existence of $\mathcal{R}$-bounded solution operator families, also called $\mathcal{R}$-solvers, for the resolvent problems. Based on these results, we next demonstrate that a resolvent problem in a general domain admits an $\mathcal{R}$-solver. Section 5 demonstrates our linear theory, i.e., the generation of an analytic C_0-semigroup and maximal regularity for some linearized system, which is obtained from the $\mathcal{R}$-solver in a general domain given by Section 4. Section 6 proves the main result of this paper.

2. Problem Setting

Let Ω be a domain in the N-dimensional Euclidean space $\mathbf{R}^N$, $N \geq 2$, and let the boundary of Ω consist of two hypersurfaces Γ_D and Γ_S. Throughout this paper, we assume

$$\mathrm{dist}(\Gamma_D, \Gamma_S) = \inf\{|x-y| : x \in \Gamma_D, y \in \Gamma_S\} \geq d > 0,$$

provided that $\Gamma_D \neq \emptyset$ and $\Gamma_S \neq \emptyset$. Notice that $\Gamma_D = \emptyset$ or $\Gamma_S = \emptyset$ is admissible in the present paper. Let $\mathbf{n} = \mathbf{n}(x) = (n_1(x), \ldots, n_N(x))^{\mathsf{T}}$ be the unit outward normal vector on $\Gamma_D \cup \Gamma_S$, where $\mathbf{M}^{\mathsf{T}}$ denotes the transpose of $\mathbf{M}$.

We consider the motion of a compressible barotropic viscous fluid of the Korteweg type in Ω with the Dirichlet boundary condition on Γ_D and the slip boundary condition on Γ_S. Such a motion is governed by the following set of equations:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0 & \text{in } \Omega \times (0,T), \\ \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = \mathrm{Div}(\mathbf{S}(\mathbf{u}) + \mathbf{K}(\rho) - P(\rho)\mathbf{I}) + \rho \mathbf{b} & \text{in } \Omega \times (0,T), \\ \mathbf{n} \cdot \nabla \rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma_D \times (0,T), \\ \mathbf{n} \cdot \nabla \rho = 0, \quad (\mathbf{D}(\mathbf{u})\mathbf{n})_\tau = 0, \quad \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \Gamma_S \times (0,T), \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0 + \rho_\infty, \mathbf{u}_0) & \text{in } \Omega, \end{cases} \tag{1}$$

where T is a positive constant. Throughout this paper, we assume that $\rho_\infty > 0$ denotes a constant reference density. The initial data

$$\rho_0 = \rho_0(x), \quad \mathbf{u}_0 = \mathbf{u}_0(x) = (u_{01}(x), \ldots, u_{0N}(x))^{\mathsf{T}}$$

are given functions of $x \in \Omega$, and also the body force $\mathbf{b} = \mathbf{b}(x,t) = (b_1(x,t), \ldots, b_N(x,t))^\top$ is a given function of $(x,t) \in \Omega \times (0,T)$.

Here, $\rho = \rho(x,t)$ and $\mathbf{u} = \mathbf{u}(x,t) = (u_1(x,t), \ldots, u_N(x,t))^\top$ are, respectively, the density of the fluid and the velocity of the fluid at position $x = (x_1, \ldots, x_N) \in \Omega$ and time $t > 0$. Let $\partial_t = \partial/\partial t$ and $\partial_j = \partial/\partial x_j$ for $j = 1, \ldots, N$. The doubled deformation rate tensor is denoted by $\mathbf{D}(\mathbf{u})$, i.e., $\mathbf{D}(\mathbf{u}) = \nabla \mathbf{u} + (\nabla \mathbf{u})^\top$ for

$$\nabla \mathbf{u} = \begin{pmatrix} \partial_1 u_1 & \ldots & \partial_N u_1 \\ \vdots & \ddots & \vdots \\ \partial_1 u_N & \ldots & \partial_N u_N \end{pmatrix},$$

while $\mathbf{I}$ is the $N \times N$ identity matrix. The pressure $P : (0, \infty) \to \mathbf{R}$ is a given smooth function. For $\mathbf{a} = (a_1, \ldots, a_N)^\top$ and $\mathbf{b} = (b_1, \ldots, b_N)^\top$, we set $\mathbf{a} \cdot \mathbf{b} = \sum_{j=1}^N a_j b_j$ and $\mathbf{a} \otimes \mathbf{b} = (a_i b_j)_{1 \le i,j \le N}$. In addition,

$$\mathbf{a}_\tau = \mathbf{a} - \mathbf{n}(\mathbf{n} \cdot \mathbf{a}).$$

Let $\mathbf{v} = (v_1(x), \ldots, v_N(x))^\top$ and $\mathbf{w} = (w_1(x), \ldots, w_N(x))^\top$. Then

$$\mathbf{v} \cdot \nabla \mathbf{w} = \left(\sum_{j=1}^N v_j \partial_j w_1, \ldots, \sum_{j=1}^N v_j \partial_j w_N \right)^\top$$

and

$$\nabla^2 \mathbf{v} = \{\partial_i \partial_j v_k : i, j, k = 1, \ldots, N\}.$$

For an $N \times N$ matrix-valued function $\mathbf{M} = (M_{ij}(x))_{1 \le i,j \le N}$, we set

$$\operatorname{Div} \mathbf{M} = \left(\sum_{j=1}^N \partial_j M_{1j}, \ldots, \sum_{j=1}^N \partial_j M_{Nj} \right)^\top.$$

Let us introduce two stress tensors $\mathbf{S}(\mathbf{u})$ and $\mathbf{K}(\rho)$. One denotes the standard viscous stress tensor by $\mathbf{S}(\mathbf{u})$, i.e.,

$$\mathbf{S}(\mathbf{u}) = \mu \mathbf{D}(\mathbf{u}) + (\nu - \mu) \operatorname{div} \mathbf{u} \mathbf{I}$$

for the viscosity coefficients $\mu = \mu(x,t)$, $\nu = \mu(x,t)$ and $\operatorname{div} \mathbf{u} = \sum_{j=1}^N \partial_j u_j$. On the other hand, $\mathbf{K}(\rho)$ is called the Korteweg stress tensor and given by

$$\mathbf{K}(\rho) = \frac{\kappa}{2}(\Delta \rho^2 - |\nabla \rho|^2)\mathbf{I} - \kappa \nabla \rho \otimes \nabla \rho \tag{2}$$

for the capillary coefficient $\kappa = \kappa(x,t)$, where

$$\nabla \rho = (\partial_1 \rho, \ldots, \partial_N \rho)^\top, \quad |\nabla \rho|^2 = \nabla \rho \cdot \nabla \rho = \sum_{j=1}^N (\partial_j \rho)^2.$$

Since

$$\Delta \rho^2 = \sum_{j=1}^N \partial_j^2 \rho^2 = 2 \sum_{j=1}^N \left((\partial_j \rho)^2 + \rho \partial_j^2 \rho \right) = 2|\nabla \rho|^2 + 2\rho \Delta \rho,$$

(2) is equivalent to

$$\mathbf{K}(\rho) = \kappa \left(\rho \Delta \rho + \frac{|\nabla \rho|^2}{2} \right) \mathbf{I} - \kappa \nabla \rho \otimes \nabla \rho. \tag{3}$$

3. Notation and Main Result

This section first introduces the notation used throughout this paper, and then the main result of this paper is stated.

3.1. Notation

Let $\mathbf{N}$ be the set of all positive integers and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Define $\mathbf{R}_+ = (0, \infty)$, $\mathbf{C}_{+,\delta} = \{z \in \mathbf{C} : \Re z > \delta\}$ for $\delta \in \mathbf{R}$, and $\mathbf{C}_+ = \mathbf{C}_{+,0}$.

Let $p \in [1, \infty]$ and G be a domain in $\mathbf{R}^N$. Then $L_p(G)$ and $H_p^m(G)$, $m \in \mathbf{N}$, stands for the Lebesgue space on G and the Sobolev space on G, respectively. The norm of $L_p(G)$ is denoted by $\| \cdot \|_{L_p(G)}$, while the norm of $H_p^m(G)$ is denoted by $\| \cdot \|_{H_p^m(G)}$. Let $H_p^0(G) = L_p(T)$. In addition, $B_{q,p}^s(G)$ is the Besov space on G for $q \in (1, \infty)$ and $s > 0$, and its norm is denoted by $\| \cdot \|_{B_{q,p}^s(G)}$.

Let X be a Banach space. Then X^M denotes the M-product space of X for $M \in \mathbf{N}$, while the norm of X^M is usually denoted by $\| \cdot \|_X$ instead of $\| \cdot \|_{X^M}$ for short. Let Y be another Banach space, and then $\mathcal{L}(X, Y)$ stands for the Banach space of all bounded linear operators from X to Y. In addition, $\mathcal{L}(X)$ is the abbreviation of $\mathcal{L}(X, X)$. For a domain U in $\mathbf{C}$, $\mathrm{Hol}(U, \mathcal{L}(X, Y))$ is the set of all $\mathcal{L}(X, Y)$-valued holomorphic functions on U.

Let $p \in [1, \infty]$ and I be an interval of $\mathbf{R}$. Then $L_p(I, X)$ and $H_p^1(I, X)$ are the X-valued Lebesgue space on I and the X-valued Sobolev space on I, respectively. The norm of $L_p(I, X)$ is given by

$$\|f\|_{L_p(I,X)} = \begin{cases} \left(\int_I \|f(t)\|_X^p \, dt \right)^{1/p} & \text{for } p \in [1, \infty), \\ \operatorname{ess\,sup}_{t \in I} \|f(t)\|_X & \text{for } p = \infty, \end{cases}$$

while the norm of $H_p^1(I, X)$ is given by

$$\|f\|_{H_p^1(I,X)} = \left(\|f\|_{L_p(I,X)}^p + \|\partial_t f\|_{L_p(I,X)}^p \right)^{1/p}.$$

We denote the set of all continuous functions $f : I \to X$ by $C(I, X)$. Furthermore, we set for $T > 0$ or $T = \infty$

$${}_0H_p^1((0,T), X) = \{f \in H_p^1((0,T), X) : f|_{t=0} = 0 \text{ in } X\}$$

with the norm $\| \cdot \|_{{}_0H_p^1((0,T),X)} := \| \cdot \|_{H_p^1((0,T),X)}$.

We now introduce the definition of uniform C^3 domains.

Definition 1. *Let D be a domain in $\mathbf{R}^N$ with boundary ∂D. Then D is called a uniform C^3 domain, if there exist positive constants α, β, and K such that the following assertions hold: for any $x_0 = (x_{01}, \ldots, x_{0N}) \in \partial D$ there exist a coordinate number j and a C^3 function $h(x')$ ($x' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N)$) on $B'_\alpha(x'_0)$, with $x'_0 = (x_{01}, \ldots, x_{0j-1}, x_{0j+1}, \ldots, x_{0N})$,*

$$B'_\alpha(x'_0) = \{x' \in \mathbf{R}^{N-1} : |x' - x'_0| < \alpha\}, \quad \|h\|_{H_\infty^3(B'_\alpha(x'_0))} \le K,$$

such that

$$D \cap B_\beta(x_0) = \{x \in \mathbf{R}^N : x_j > h(x'),\ x' \in B'_\alpha(x'_0)\} \cap B_\beta(x_0),$$
$$\partial D \cap B_\beta(x_0) = \{x \in \mathbf{R}^N : x_j = h(x'),\ x' \in B'_\alpha(x'_0)\} \cap B_\beta(x_0).$$

Here $B_\beta(x_0) = \{x \in \mathbf{R}^N : |x - x_0| < \beta\}$.

Example 1.

(1) *If Ω is a bounded domain or an exterior domain in $\mathbf{R}^N$, $N \geq 2$, whose boundary is of class C^3, then Ω is a uniform C^3 domain.*

(2) *Let $h_+(x')$ and $h_-(x')$, $x' = (x_1, \ldots, x_{N-1})$ be of class C^3 and have compact supports with $\|h_\pm\|_{L_\infty(\mathbf{R}^{N-1})} \leq 1/2$. Then*

$$\Omega = \{(x', x_N) : x' \in \mathbf{R}^{N-1}, -1 + h_-(x') \leq x_N \leq 1 + h_+(x')\}$$

becomes a uniform C^3 domain with boundary $\Gamma_S \cup \Gamma_D$, where

$$\Gamma_D = \{(x', x_N) : x' \in \mathbf{R}^{N-1}, x_N = -1 + h_-(x')\},$$
$$\Gamma_S = \{(x', x_N) : x' \in \mathbf{R}^{N-1}, x_N = 1 + h_+(x')\}.$$

Let Ω be a uniform C^3 domain and let $C^{0,1}(\overline{\Omega})$ be the Banach space of all bounded and uniformly Lipschitz continuous functions on $\overline{\Omega} = \Omega \cup \Gamma_D \cup \Gamma_S$ with the norm:

$$\|f\|_{C^{0,1}(\overline{\Omega})} = \|f\|_{L_\infty(\Omega)} + \sup_{x,y\in\Omega, x\neq y} \frac{|f(x) - f(y)|}{|x-y|}.$$

Remark 1. *It follows from [20] (Theorem 3.14) that $C^{0,1}(\overline{\Omega}) = H^1_\infty(\Omega)$. This fact is often used throughout this paper.*

Let $T > 0$ or $T = \infty$, and let $p, q \in (1, \infty)$. Define

$$K^1_{p,q;T} = H^1_p((0,T), H^1_q(\Omega)) \cap L_p((0,T), H^3_q(\Omega)),$$
$$\|\rho\|_{K^1_{p,q;T}} = \|\rho\|_{H^1_p((0,T), H^1_q(\Omega))} + \|\rho\|_{L_p((0,T), H^3_q(\Omega))},$$

and also

$$K^2_{p,q;T} = H^1_p((0,T), L_q(\Omega)^N) \cap L_p((0,T), H^2_q(\Omega)^N),$$
$$\|\mathbf{u}\|_{K^2_{p,q;T}} = \|\mathbf{u}\|_{H^1_p((0,T), L_q(\Omega)^N)} + \|\mathbf{u}\|_{L_p((0,T), H^2_q(\Omega)^N)}.$$

Furthermore,

$${}_0K^1_{p,q;T} = {}_0H^1_p((0,T), H^1_q(\Omega)) \cap L_p((0,T), H^3_q(\Omega)),$$
$${}_0K^2_{p,q;T} = {}_0H^1_p((0,T), L_q(\Omega)^N) \cap L_p((0,T), H^2_q(\Omega)^N).$$

We now set

$$Z_{p,q;T} = Z^1_{p,q;T} \times Z^2_{p,q;T} \quad \text{for } Z \in \{K, {}_0K\}$$

and

$$\|(\rho, \mathbf{u})\|_{K_{p,q;T}} = \|\rho\|_{K^1_{p,q;T}} + \|\mathbf{u}\|_{K^2_{p,q;T}}.$$

Let $L > 0$. Then, ${}_0K_{p,q;T}(L)$ is defined by

$${}_0K_{p,q;T}(L) = \{(\rho, \mathbf{u}) \in {}_0K_{p,q;T} : \|(\rho, \mathbf{u})\|_{K_{p,q;T}} \leq L\}.$$

3.2. Main Result

To state our main result for (1), we write (1) as an equivalent system in what follows. We replace ρ by $\rho + \rho_\infty$ in (1) in order to obtain

$$\begin{cases} \partial_t \rho + \operatorname{div}((\rho + \rho_\infty)\mathbf{u}) = 0 & \text{in } \Omega \times (0,T), \\ (\rho + \rho_\infty)(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = \operatorname{Div}(\mathbf{S}(\mathbf{u}) + \mathbf{K}(\rho + \rho_\infty) - P(\rho + \rho_\infty)\mathbf{I}) & \\ \qquad + (\rho + \rho_\infty)\mathbf{b} & \text{in } \Omega \times (0,T), \\ \mathbf{n} \cdot \nabla \rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma_D \times (0,T), \\ \mathbf{n} \cdot \nabla \rho = 0, \quad (\mathbf{D}(\mathbf{u})\mathbf{n})_\tau = 0, \quad \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \Gamma_S \times (0,T), \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega. \end{cases} \tag{4}$$

Let us define

$$\begin{aligned} \mu_0(x) &= \mu(x,t)|_{t=0}, \quad \nu_0(x) = \nu(x,t)|_{t=0}, \\ \kappa_0(x) &= \kappa(x,t)|_{t=0}, \quad r_0(x) = \rho_0(x) + \rho_\infty, \\ \mathbf{S}_0(\mathbf{u}) &= \mu_0(x)\mathbf{D}(\mathbf{u}) + (\nu_0(x) - \mu_0(x)) \operatorname{div} \mathbf{u}\mathbf{I}. \end{aligned} \tag{5}$$

The first equation of (4) is then written as

$$\partial_t \rho + r_0 \operatorname{div} \mathbf{u} = -\mathbf{u} \cdot \nabla \rho - (\rho - \rho_0) \operatorname{div} \mathbf{u} =: \mathsf{D}(\rho, \mathbf{u}).$$

We next consider the second equation of (4). Recalling (3), we observe that

$$\begin{aligned} \mathbf{K}(\rho + 1) &= (\kappa - \kappa_0)(\rho + 1)\Delta\rho\mathbf{I} + \kappa_0(\rho - \rho_0)\Delta\rho\mathbf{I} + \kappa_0 r_0 \Delta\rho\mathbf{I} \\ &\quad + \kappa\frac{|\nabla\rho|^2}{2}\mathbf{I} - \kappa\nabla\rho \otimes \nabla\rho. \end{aligned}$$

The second equation of (4) is thus written as

$$\begin{aligned} & r_0 \partial_t \mathbf{u} - \operatorname{Div}(\mathbf{S}_0(\mathbf{u}) + \kappa_0 r_0 \Delta\rho\mathbf{I}) \\ & = -(\rho - \rho_0)\partial_t \mathbf{u} - (\rho + \rho_\infty)\mathbf{u} \cdot \nabla \mathbf{u} + \operatorname{Div}(\mathbf{S}(\mathbf{u}) - \mathbf{S}_0(\mathbf{u})) \\ & - P'(\rho + \rho_\infty)\nabla\rho + \operatorname{Div}\Big((\kappa - \kappa_0)(\rho + \rho_\infty)\Delta\rho\mathbf{I} + \kappa_0(\rho - \rho_0)\Delta\rho\mathbf{I} \\ & + \kappa\frac{|\nabla\rho|^2}{2}\mathbf{I} - \kappa\nabla\rho \otimes \nabla\rho\Big) + (\rho + \rho_\infty)\mathbf{b} \\ & =: \widetilde{\mathsf{F}}(\rho, \mathbf{u}), \end{aligned} \tag{6}$$

where $P'(\rho) = (dP/d\rho)(\rho)$. Furthermore, since

$$\begin{aligned} & \operatorname{Div}(\mathbf{S}_0(\mathbf{u}) + \kappa_0 r_0 \Delta\rho\mathbf{I}) \\ & = \kappa_0 \operatorname{Div}(\kappa_0^{-1}\mathbf{S}_0(\mathbf{u}) + r_0\Delta\rho\mathbf{I}) + (\kappa_0^{-1}\mathbf{S}_0(\mathbf{u}) + r_0\Delta\rho\mathbf{I})\nabla\kappa_0, \end{aligned}$$

(6) is reduced to

$$\begin{aligned} & \partial_t \mathbf{u} - r_0^{-1}\kappa_0 \operatorname{Div}(\kappa_0^{-1}\mathbf{S}_0(\mathbf{u}) + r_0\Delta\rho\mathbf{I}) \\ & = r_0^{-1}\widetilde{\mathsf{F}}(\rho, \mathbf{u}) + r_0^{-1}(\kappa_0^{-1}\mathbf{S}_0(\mathbf{u}) + r_0\Delta\rho\mathbf{I})\nabla\kappa_0 =: \mathsf{F}(\rho, \mathbf{u}). \end{aligned}$$

Summing up the above calculations, we have achieved the following equivalent system of (1):

$$
\begin{cases}
\partial_t\rho + r_0\operatorname{div}\mathbf{u} = \mathsf{D}(\rho,\mathbf{u}) & \text{in } \Omega\times(0,T),\\
\partial_t\mathbf{u} - r_0^{-1}\kappa_0\operatorname{Div}(\kappa_0^{-1}\mathbf{S}_0(\mathbf{u}) + r_0\Delta\rho\mathbf{I}) = \mathsf{F}(\rho,\mathbf{u}) & \text{in } \Omega\times(0,T),\\
\mathbf{n}\cdot\nabla\rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma_D\times(0,T),\\
\mathbf{n}\cdot\nabla\rho = 0, \quad (\mathbf{D}(\mathbf{u})\mathbf{n})_\tau = 0, \quad \mathbf{u}\cdot\mathbf{n} = 0 & \text{on } \Gamma_S\times(0,T),\\
(\rho,\mathbf{u})|_{t=0} = (\rho_0,\mathbf{u}_0) & \text{in } \Omega.
\end{cases} \tag{7}
$$

We further reduce (7) to some system with $(\rho_0,\mathbf{u}_0) = (0,0)$. Let $(\widehat{\rho},\widehat{\mathbf{u}})$ be a unique solution to the following linear system:

$$
\begin{cases}
\partial_t\widehat{\rho} + r_0\operatorname{div}\widehat{\mathbf{u}} = 0 & \text{in } \Omega\times(0,T),\\
\partial_t\widehat{\mathbf{u}} - r_0^{-1}\kappa_0\operatorname{Div}(\kappa_0^{-1}\mathbf{S}_0(\widehat{\mathbf{u}}) + r_0\Delta\widehat{\rho}\,\mathbf{I}) = 0 & \text{in } \Omega\times(0,T),\\
\mathbf{n}\cdot\nabla\widehat{\rho} = 0, \quad \widehat{\mathbf{u}} = 0 & \text{on } \Gamma_D\times(0,T),\\
\mathbf{n}\cdot\nabla\widehat{\rho} = 0, \quad (\mathbf{D}(\widehat{\mathbf{u}})\mathbf{n})_\tau = 0, \quad \widehat{\mathbf{u}}\cdot\mathbf{n} = 0 & \text{on } \Gamma_S\times(0,T),\\
(\widehat{\rho},\widehat{\mathbf{u}})|_{t=0} = (\rho_0,\mathbf{u}_0) & \text{in } \Omega,
\end{cases} \tag{8}
$$

see Section 5.1 below for more details on $(\widehat{\rho},\widehat{\mathbf{u}})$. Replace $(\rho,\mathbf{u})$ by $(\rho+\widehat{\rho},\mathbf{u}+\widehat{\mathbf{u}})$ in (7), and the resultant system becomes

$$
\begin{cases}
\partial_t\rho + r_0\operatorname{div}\mathbf{u} = \mathsf{D}(\rho+\widehat{\rho},\mathbf{u}+\widehat{\mathbf{u}}) & \text{in } \Omega\times(0,T),\\
\partial_t\mathbf{u} - r_0^{-1}\kappa_0\operatorname{Div}(\kappa_0^{-1}\mathbf{S}_0(\mathbf{u}) + r_0\Delta\rho\mathbf{I}) = \mathsf{F}(\rho+\widehat{\rho},\mathbf{u}+\widehat{\mathbf{u}}) & \text{in } \Omega\times(0,T),\\
\mathbf{n}\cdot\nabla\rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma_D\times(0,T),\\
\mathbf{n}\cdot\nabla\rho = 0, \quad (\mathbf{D}(\mathbf{u})\mathbf{n})_\tau = 0, \quad \mathbf{u}\cdot\mathbf{n} = 0 & \text{on } \Gamma_S\times(0,T),\\
(\rho,\mathbf{u})|_{t=0} = (0,0) & \text{in } \Omega.
\end{cases} \tag{9}
$$

Our main result of this paper is then stated as follows.

Theorem 1. *Let $N \geq 2$ and Ω be a uniform C^3 domain in $\mathbf{R}^N$. Let $p \in (1,\infty)$ and $q \in (N,\infty)$. Suppose that R, R_1, R_2, T_0, and ρ_∞ are positive constants with $R_1 \leq R_2$. Then, there exist constants $L \geq 1$ and $T \in (0,T_0]$ such that (9) admits a unique solution $(\rho,\mathbf{u})$ in ${}_0K_{p,q;T}(L)$ if ρ_0, $\mathbf{u}_0$, $\mathbf{b}$, P, μ, ν, and κ satisfy the following conditions:*

(a) *$(\rho_0,\mathbf{u}_0) \in D_{q,p}(\Omega)$ with $\|(\rho_0,\mathbf{u}_0)\|_{D_{q,p}(\Omega)} \leq R$, where $D_{q,p}(\Omega)$ is given by Section 5.1, below, as well as a subspace of $B^{3-2/p}_{q,p}(\Omega)\times B^{2-2/p}_{q,p}(\Omega)^N$;*

(b) *$r_0 = \rho_0 + \rho_\infty \in C^{0,1}(\overline{\Omega})$ with $\|r_0\|_{C^{0,1}(\overline{\Omega})} \leq R$ and*

$$\frac{\rho_\infty}{2} \leq r_0(x) \leq 2\rho_\infty \quad (x\in\overline{\Omega});$$

(c) *$\mathbf{b} \in L_p((0,T_0),L_q(\Omega)^N)$ with $\|\mathbf{b}\|_{L_p((0,T_0),L_q(\Omega)^N)} \leq R$;*

(d) *P is a C^1 function on $[\rho_\infty/4, 4\rho_\infty]$ with $P' \in C^{0,1}([\rho_\infty/4,4\rho_\infty])$ and $\|P'\|_{C^{0,1}([\rho_\infty/4,4\rho_\infty])} \leq R$, where $P'(s) = (dP/ds)(s)$;*

(e) *$\mu,\nu,\kappa \in C([0,T_0],C^{0,1}(\overline{\Omega}))$ with*

$$\sup_{t\in[0,T_0]}\|\mu(t)\|_{C^{0,1}(\overline{\Omega})} \leq R, \quad \sup_{t\in[0,T_0]}\|\nu(t)\|_{C^{0,1}(\overline{\Omega})} \leq R,$$
$$\sup_{t\in[0,T_0]}\|\kappa(t)\|_{C^{0,1}(\overline{\Omega})} \leq R;$$

(f) *$\mu_0(x) = \mu(x,t)|_{t=0}$, $\nu_0(x) = \nu(x,t)|_{t=0}$, and $\kappa_0(x) = \kappa(x,t)|_{t=0}$ satisfy*

$$R_1 \leq \mu_0(x) \leq R_2, \quad R_1 \leq \mu_0(x)+\nu_0(x) \leq R_2,$$
$$R_1 \leq \kappa_0(x) \leq R_2 \quad (x\in\overline{\Omega}).$$

Remark 2.

(1) *If p, q satisfy $2/p + N/q < 2$ additionally, then $B^{3-2/p}_{q,p}(\Omega)$ is continuously embedded into $H^1_\infty(\Omega)$, which is equivalent to $C^{0,1}(\overline{\Omega})$, as stated in Remark 1; see, e.g., Remark 1 (b) of Subsection 2.8.1 in [21]. In this case, the additional regularity of the initial density, i.e., $r_0 \in C^{0,1}(\overline{\Omega})$, may be removed.*

(2) *Our linear theory requires that r_0 belongs to $C^{0,1}(\overline{\Omega})$; see Section 5 below for more details.*

4. $\mathcal{R}$-Solvers for Resolvent Problems

In this section, we consider resolvent problems and prove the existence of $\mathcal{R}$-bounded solution operator families, also called $\mathcal{R}$-solvers, for the resolvent problems. The main result of this section, as shown in Theorem 2 below, gives us a generation of an analytic C_0-semigroup and maximal regularity for the linearized system of (9) in the next section.

4.1. $\mathcal{R}$-Solver in the Whole Space

Let us first introduce the definition of $\mathcal{R}$-boundedness.

Definition 2. *Let X and Y be Banach spaces, and let $r_n(t)$ be the Rademacher functions on [0, 1], i.e.,*

$$r_n(t) = \operatorname{sign}(\sin(2^n\pi t)) \quad (n \in \mathbf{N}, 0 \le t \le 1).$$

A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, if there exist constants $p \in [1,\infty)$ and $C > 0$ such that the following assertion holds: for each $m \in \mathbf{N}$, $\{T_j\}_{j=1}^m \subset \mathcal{T}$, and $\{f_j\}_{j=1}^m \subset X$,

$$\left(\int_0^1 \left\|\sum_{j=1}^m r_j(t)T_jf_j\right\|_Y^p dt\right)^{1/p} \le C\left(\int_0^1 \left\|\sum_{j=1}^m r_j(t)f_j\right\|_X^p dt\right)^{1/p}.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X,Y)$ and denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

Remark 3.

(1) *The constant C in Definition 2 may depend on p.*

(2) *It is known that $\mathcal{T}$ is $\mathcal{R}$-bounded for any $p \in [1,\infty)$, provided that $\mathcal{T}$ is $\mathcal{R}$-bounded for some $p \in [1,\infty)$. This fact follows from Kahane's inequality; see, e.g., [22] (Theorem 2.4).*

(3) *The $\mathcal{R}$-boundedness implies the uniform boundedness. In fact, taking $m = 1$ in the definition of the $\mathcal{R}$-boundedness yields that $\|Tf\|_Y \le C\|f\|_X$ for any $T \in \mathcal{T}$ and $f \in X$.*

This subsection considers the following resolvent problem in the whole space:

$$\begin{cases} \lambda\rho + \gamma_1 \operatorname{div}\mathbf{u} = d & \text{in } \mathbf{R}^N, \\ \lambda\mathbf{u} - \gamma_4^{-1}\operatorname{Div}(\gamma_2(\mathbf{D}(\mathbf{u}) + (\gamma_3-\gamma_2)\operatorname{div}\mathbf{u}\mathbf{I} + \gamma_1\Delta\rho\mathbf{I}) = \mathbf{f} & \text{in } \mathbf{R}^N. \end{cases} \tag{10}$$

Let $q \in (1,\infty)$. For the right member $(d,\mathbf{f})$ of (10), we set

$$\mathcal{X}^1_q(\mathbf{R}^N) = H^1_q(\mathbf{R}^N) \times L_q(\mathbf{R}^N)^N, \quad \mathfrak{X}^1_q(\mathbf{R}^N) = L_q(\mathbf{R}^N)^{N+1+N}$$

and set for $\mathbf{F}^1 = (d,\mathbf{f}) \in \mathcal{X}^1_q(\mathbf{R}^N)$ and $\lambda \in \mathbf{C}\setminus(-\infty,0]$

$$\mathcal{F}^1_\lambda\mathbf{F}^1 = (\nabla d, \lambda^{1/2}d, \mathbf{f}) \in \mathfrak{X}^1_q(\mathbf{R}^N).$$

On the other hand, for the solution $(\rho,\mathbf{u})$ of (10), we set

$$\begin{aligned} &\mathfrak{A}^0_q(G) = L_q(G)^{N^3+N^2+N+1}, \quad \mathcal{S}^0_\lambda\rho = (\nabla^3\rho, \lambda^{1/2}\nabla^2\rho, \lambda\nabla\rho, \lambda^{3/2}\rho), \\ &\mathfrak{B}_q(G) = L_q(G)^{N^3+N^2+N}, \qquad \mathcal{T}_\lambda\mathbf{u} = (\nabla^2\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \lambda\mathbf{u}), \end{aligned} \tag{11}$$

where G is a domain in $\mathbf{R}^N$. The following lemma then holds.

Lemma 1. *Let $q \in (1, \infty)$ and let γ_1, γ_2, γ_3, and γ_4 be constants satisfying*

$$\gamma_i > 0 \quad (i = 1, 2, 4), \quad \gamma_2 + \gamma_3 > 0. \tag{12}$$

Then, the following assertions hold.

(1) *For any $\lambda \in \mathbf{C}_+$ there exist operators $\mathcal{A}^1(\lambda)$, $\mathcal{B}^1(\lambda)$, with*

$$\mathcal{A}^1(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(\mathfrak{X}^1_q(\mathbf{R}^N), H^3_q(\mathbf{R}^N))),$$
$$\mathcal{B}^1(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(\mathfrak{X}^1_q(\mathbf{R}^N), H^2_q(\mathbf{R}^N)^N)),$$

such that for any $\mathbf{F}^1 = (d, \mathbf{f}) \in \mathcal{X}^1_q(\mathbf{R}^N)$

$$(\rho, \mathbf{u}) = (\mathcal{A}^1(\lambda)\mathcal{F}^1_\lambda \mathbf{F}^1, \mathcal{B}^1(\lambda)\mathcal{F}^1_\lambda \mathbf{F}^1)$$

is a unique solution to (10).

(2) *There exists a positive constant $C = C(N, q, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$, such that for $n = 0, 1$*

$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}^1_q(\mathbf{R}^N), \mathfrak{A}^0_q(\mathbf{R}^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n \left(\mathcal{S}^0_\lambda \mathcal{A}^1(\lambda)\right) : \lambda \in \mathbf{C}_+\right\}\right) \le C,$$
$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}^1_q(\mathbf{R}^N), \mathfrak{B}_q(\mathbf{R}^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n \left(\mathcal{T}_\lambda \mathcal{B}^1(\lambda)\right) : \lambda \in \mathbf{C}_+\right\}\right) \le C,$$

where $\mathfrak{A}^0_q(\mathbf{R}^N)$, $\mathfrak{B}_q(\mathbf{R}^N)$, $\mathcal{S}^0_\lambda$, and $\mathcal{T}_\lambda$ are given by (11) with $G = \mathbf{R}^N$.

Proof. The proof is similar to [23] (Theorem 2.1), so that the detailed proof may be omitted. □

4.2. $\mathcal{R}$-Solver in the Half Space

Let us first consider the following resolvent problem with the Dirichlet boundary condition for the fluid velocity:

$$\begin{cases} \lambda\rho + \gamma_1 \operatorname{div}\mathbf{u} = d & \text{in } \mathbf{R}^N_+, \\ \lambda\mathbf{u} - \gamma_4^{-1}\operatorname{Div}(\gamma_2(\mathbf{D}(\mathbf{u}) + (\gamma_3 - \gamma_2)\operatorname{div}\mathbf{u}\mathbf{I} + \gamma_1\Delta\rho\mathbf{I}) = \mathbf{f} & \text{in } \mathbf{R}^N_+, \\ \mathbf{n}\cdot\nabla\rho = g, \quad \mathbf{u} = \mathbf{h} & \text{on } \mathbf{R}^N_0, \end{cases} \tag{13}$$

where

$$\mathbf{R}^N_+ = \{(x', x_N) : x' = (x_1, \ldots, x_{N-1}) \in \mathbf{R}^{N-1}, x_N > 0\},$$
$$\mathbf{R}^N_0 = \{(x', x_N) : x' = (x_1, \ldots, x_{N-1}) \in \mathbf{R}^{N-1}, x_N = 0\}.$$

Let $q \in (1, \infty)$. For the right member $(d, \mathbf{f}, g, \mathbf{h})$, we set

$$\mathcal{X}^2_q(\mathbf{R}^N_+) = H^1_q(\mathbf{R}^N_+) \times L_q(\mathbf{R}^N_+)^N \times H^2_q(\mathbf{R}^N_+) \times H^2_q(\mathbf{R}^N_+)^N,$$
$$\mathfrak{X}^2_q(\mathbf{R}^N_+) = L_q(\mathbf{R}^N_+)^{(N+1)+N+(N^2+N+1)+(N^3+N^2+N)}$$

and set for $\mathbf{F}^2 = (d, \mathbf{f}, g, \mathbf{h}) \in \mathcal{X}^2_q(\mathbf{R}^N_+)$ and $\lambda \in \mathbf{C} \setminus (-\infty, 0]$

$$\mathcal{F}^2_\lambda \mathbf{F}^2 = (\nabla d, \lambda^{1/2} d, \mathbf{f}, \nabla^2 g, \lambda^{1/2}\nabla g, \lambda g, \nabla^2 \mathbf{h}, \lambda^{1/2}\nabla\mathbf{h}, \lambda\mathbf{h}) \in \mathfrak{X}^2_q(\mathbf{R}^N_+).$$

The following lemma then holds.

Lemma 2. *Let $q \in (1, \infty)$ and γ_i $(i = 1, 2, 3, 4)$ be constants satisfying (12). Then, the following assertions hold.*

(1) *For any $\lambda \in \mathbf{C}_+$, there exist operators $\mathcal{A}^2(\lambda)$, $\mathcal{B}^2(\lambda)$, with*

$$\mathcal{A}^2(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(\mathfrak{X}_q^2(\mathbf{R}_+^N), H_q^3(\mathbf{R}_+^N))),$$
$$\mathcal{B}^2(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(\mathfrak{X}_q^2(\mathbf{R}_+^N), H_q^2(\mathbf{R}_+^N)^N)),$$

such that for any $\mathbf{F}^2 = (d, \mathbf{f}, g, \mathbf{h}) \in \mathcal{X}_q^2(\mathbf{R}_+^N)$

$$(\rho, \mathbf{u}) = (\mathcal{A}^2(\lambda)\mathcal{F}_\lambda^2\mathbf{F}^2, \mathcal{B}^2(\lambda)\mathcal{F}_\lambda^2\mathbf{F}^2)$$

is a unique solution to (13).

(2) *There exists a positive constant $C = C(N, q, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ such that for $n = 0, 1$*

$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q^2(\mathbf{R}_+^N), \mathfrak{A}_q^0(\mathbf{R}_+^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{S}_\lambda^0\mathcal{A}^2(\lambda)\right) : \lambda \in \mathbf{C}_+\right\}\right) \leq C,$$
$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q^2(\mathbf{R}_+^N), \mathfrak{B}_q(\mathbf{R}_+^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{T}_\lambda\mathcal{B}^2(\lambda)\right) : \lambda \in \mathbf{C}_+\right\}\right) \leq C,$$

where $\mathfrak{A}_q^0(\mathbf{R}_+^N)$, $\mathfrak{B}_q(\mathbf{R}_+^N)$, $\mathcal{S}_\lambda^0$, and $\mathcal{T}_\lambda$ are given by (11) with $G = \mathbf{R}_+^N$.

Proof. This lemma was proved by [24] (Theorem 1.4) when $\mathbf{h} = 0$ and γ_i $(i = 1, 2, 3, 4)$ are positive constants. Define

$$\widetilde{\mathcal{X}}_q^2(\mathbf{R}_+^N) = H_q^1(\mathbf{R}_+^N) \times L_q(\mathbf{R}_+^N)^N \times H_q^2(\mathbf{R}_+^N),$$
$$\widetilde{\mathfrak{X}}_q^2(\mathbf{R}_+^N) = L_q(\mathbf{R}_+^N)^{(N+1)+N+(N^2+N+1)},$$

and set for $\widetilde{\mathbf{F}}^2 = (d, \mathbf{f}, g) \in \widetilde{\mathcal{X}}_q^2(\mathbf{R}_+^N)$ and $\lambda \in \mathbf{C} \setminus (-\infty, 0]$

$$\widetilde{\mathcal{F}}_\lambda^2\widetilde{\mathbf{F}}^2 = (\nabla d, \lambda^{1/2}d, \mathbf{f}, \nabla^2 g, \lambda^{1/2}\nabla g, \lambda g) \in \widetilde{\mathfrak{X}}_q^2(G).$$

Then, ref. [24] (Theorem 1.4) can be extended to the case where $\mathbf{h} = 0$ and γ_i $(i = 1, 2, 3, 4)$ are constants satisfying (12) by slightly modifying its proof, i.e., for any $\lambda \in \mathbf{C}_+$ there exist operators $\widetilde{\mathcal{A}}^2(\lambda)$, $\widetilde{\mathcal{B}}^2(\lambda)$, with

$$\widetilde{\mathcal{A}}^2(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(\widetilde{\mathfrak{X}}_q^2(\mathbf{R}_+^N), H_q^3(\mathbf{R}_+^N))),$$
$$\widetilde{\mathcal{B}}^2(\lambda) \in \mathrm{Hol}(\mathbf{C}_+, \mathcal{L}(\widetilde{\mathfrak{X}}_q^2(\mathbf{R}_+^N), H_q^2(\mathbf{R}_+^N)^N)),$$

such that for any $\widetilde{\mathbf{F}}^2 = (d, \mathbf{f}, g) \in \widetilde{\mathcal{X}}_q^2(\mathbf{R}_+^N)$

$$(\sigma, \mathbf{v}) = (\widetilde{\mathcal{A}}^2(\lambda)\widetilde{\mathcal{F}}_\lambda^2\widetilde{\mathbf{F}}^2, \widetilde{\mathcal{B}}^2(\lambda)\widetilde{\mathcal{F}}_\lambda^2\widetilde{\mathbf{F}}^2)$$

is a unique solution to

$$\begin{cases} \lambda\sigma + \gamma_1 \operatorname{div} \mathbf{v} = d & \text{in } \mathbf{R}^N_+, \\ \lambda\mathbf{v} - \gamma_4^{-1} \operatorname{Div}(\gamma_2(\mathbf{D}(\mathbf{v}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{v}\mathbf{I} + \gamma_1 \Delta\sigma\mathbf{I}) = \mathbf{f} & \text{in } \mathbf{R}^N_+, \\ \mathbf{n} \cdot \nabla\sigma = g, \quad \mathbf{v} = 0 & \text{on } \mathbf{R}^N_0, \end{cases}$$

where γ_i $(i = 1,2,3,4)$ are constants satisfying (12). In addition, for $n = 0,1$,

$$\begin{aligned} &\mathcal{R}_{\mathcal{L}(\widetilde{\mathfrak{X}}^2_q(\mathbf{R}^N_+),\mathfrak{A}^0_q(\mathbf{R}^N_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{S}^0_\lambda\widetilde{\mathcal{A}}^2(\lambda)\right) : \lambda \in \mathbf{C}_+\right\}\right) \le C, \\ &\mathcal{R}_{\mathcal{L}(\widetilde{\mathfrak{X}}^2_q(\mathbf{R}^N_+),\mathfrak{B}_q(\mathbf{R}^N_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{T}_\lambda\widetilde{\mathcal{B}}^2(\lambda)\right) : \lambda \in \mathbf{C}_+\right\}\right) \le C, \end{aligned} \tag{14}$$

with a positive constant $C = C(N, q, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$.

Let us now consider

$$\begin{cases} \lambda u - \Delta u = 0 & \text{in } \mathbf{R}^N_+, \\ u = h & \text{on } \mathbf{R}^N_0. \end{cases} \tag{15}$$

It is well-known that (15) admits an $\mathcal{R}$-solver, i.e., there exists an operator

$$\mathcal{U}(\lambda) \in \operatorname{Hol}(\mathbf{C}_+, \mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1}, H^2_q(\mathbf{R}^N_+)))$$

such that $u = \mathcal{U}(\lambda)\mathcal{T}_\lambda h$, $h \in H^2_q(\mathbf{R}^N_+)$ is a solution to (15) and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+)^{N^2+N+1})}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n(\mathcal{T}_\lambda\mathcal{U}(\lambda)) : \lambda \in \mathbf{C}_+\right\}\right) \le C$$

for $n = 0,1$ with a positive constant $C = C(N, q)$. This enables us to define

$$\mathcal{V}(\lambda) \in \operatorname{Hol}(\mathbf{C}_+, \mathcal{L}(L_q(\mathbf{R}^N_+)^{N^3+N^2+N}), H^2_q(\mathbf{R}^N_+)^N)$$

by $\mathcal{V}(\lambda)\mathcal{T}_\lambda\mathbf{h} = (\mathcal{U}(\lambda)\mathcal{T}_\lambda h_1, \dots, \mathcal{U}(\lambda)\mathcal{T}_\lambda h_N)$ for $\mathbf{h} = (h_1, \dots, h_N)^\mathsf{T}$, and then

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N_+)^{N^3+N^2+N})}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n(\mathcal{T}_\lambda\mathcal{V}(\lambda)) : \lambda \in \mathbf{C}_+\right\}\right) \le C \tag{16}$$

for $n = 0,1$ with a positive constant $C = C(N, q)$.

Let $\mathbf{u} = \mathbf{w} + \mathcal{V}(\lambda)\mathcal{T}_\lambda\mathbf{h}$ in (13). Then, (13) is reduced to

$$\begin{cases} \lambda\rho + \gamma_1 \operatorname{div} \mathbf{w} = \widetilde{d} & \text{in } \mathbf{R}^N_+, \\ \lambda\mathbf{w} - \gamma_4^{-1} \operatorname{Div}(\gamma_2(\mathbf{D}(\mathbf{w}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{w}\mathbf{I} + \gamma_1 \Delta\rho\mathbf{I}) = \widetilde{\mathbf{f}} & \text{in } \mathbf{R}^N_+, \\ \mathbf{n} \cdot \nabla\rho = g, \quad \mathbf{w} = 0 & \text{on } \mathbf{R}^N_0, \end{cases}$$

together with

$$\begin{aligned} &\widetilde{d} = d - \gamma_1 \operatorname{div} \mathcal{V}(\lambda)\mathcal{T}_\lambda\mathbf{h}, \\ &\widetilde{\mathbf{f}} = \mathbf{f} - \lambda\mathcal{V}(\lambda)\mathcal{T}_\lambda\mathbf{h} + \gamma_4^{-1}(\gamma_2\Delta\mathcal{V}(\lambda)\mathcal{T}_\lambda\mathbf{h} + \gamma_3\nabla \operatorname{div} \mathcal{V}(\lambda)\mathcal{T}_\lambda\mathbf{h}), \end{aligned}$$

where one has used the fact that

$$\begin{aligned} &\operatorname{Div}(\gamma_2\mathbf{D}(\mathcal{V}(\lambda)\mathcal{T}_\lambda\mathbf{h}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathcal{V}(\lambda)\mathcal{T}_\lambda\mathbf{h}\mathbf{I}) \\ &= \gamma_2\Delta\mathcal{V}(\lambda)\mathcal{T}_\lambda\mathbf{h} + \gamma_3\nabla \operatorname{div} \mathcal{V}(\lambda)\mathcal{T}_\lambda\mathbf{h}. \end{aligned}$$

From this viewpoint, we set for $\mathbf{H} = (H_1, \dots, H_9) \in \mathfrak{X}^2_q(\mathbf{R}^N_+)$

$$\mathcal{A}^2(\lambda)\mathbf{H} = \widetilde{\mathcal{A}}^2(\lambda)(H_1 - \gamma_1\nabla \operatorname{div} \mathcal{V}(\lambda)(H_7, H_8, H_9),$$

$$
\begin{aligned}
&\qquad\qquad H_2 - \gamma_1\lambda^{1/2}\operatorname{div}\mathcal{V}(\lambda)(H_7, H_8, H_9),\\
&\qquad\qquad H_3 - \lambda\mathcal{V}(\lambda)(H_7, H_8, H_9) + \gamma_4^{-1}\gamma_2\Delta\mathcal{V}(\lambda)(H_7, H_8, H_9)\\
&\qquad\qquad + \gamma_4^{-1}\gamma_3\nabla\operatorname{div}\mathcal{V}(\lambda)(H_7, H_8, H_9), H_4, H_5, H_6),\\
\mathcal{B}^2(\lambda)\mathbf{H} &= \widetilde{\mathcal{B}}^2(\lambda)(H_1 - \gamma_1\nabla\operatorname{div}\mathcal{V}(\lambda)(H_7, H_8, H_9),\\
&\qquad\qquad H_2 - \gamma_1\lambda^{1/2}\operatorname{div}\mathcal{V}(\lambda)(H_7, H_8, H_9),\\
&\qquad\qquad H_3 - \lambda\mathcal{V}(\lambda)(H_7, H_8, H_9) + \gamma_4^{-1}\gamma_2\Delta\mathcal{V}(\lambda)(H_7, H_8, H_9)\\
&\qquad\qquad + \gamma_4^{-1}\gamma_3\nabla\operatorname{div}\mathcal{V}(\lambda)(H_7, H_8, H_9), H_4, H_5, H_6)\\
&\quad + \mathcal{V}(\lambda)(H_7, H_8, H_9),
\end{aligned}
$$

where H_1, H_2, H_3, H_4, H_5, and H_6, H_7, H_8, and H_9 are, respectively, corresponding to ∇d, $\lambda^{1/2}d$, $\mathbf{f}$, $\nabla^2 g$, $\lambda^{1/2}\nabla g$, λg, $\nabla^2\mathbf{h}$, $\lambda^{1/2}\nabla\mathbf{h}$, and $\lambda\mathbf{h}$. It is then clear that $(\rho, \mathbf{u}) = (\mathcal{A}^2(\lambda)\mathcal{F}_\lambda^2\mathbf{F}^2, \mathcal{B}^2(\lambda)\mathcal{F}_\lambda^2\mathbf{F}^2)$ is a solution to (13) for $\mathbf{F}^2 = (d, \mathbf{f}, g, \mathbf{h}) \in \mathcal{X}_q^2(\mathbf{R}_+^N)$ and that (14), (16), and the definition of the $\mathcal{R}$-boundedness give us for $n = 0, 1$

$$
\begin{aligned}
\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q^2(\mathbf{R}_+^N), \mathfrak{A}_q^0(\mathbf{R}_+^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{S}_\lambda^0\mathcal{A}^2(\lambda)\right) : \lambda \in \mathbf{C}_+\right\}\right) &\le C,\\
\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q^2(\mathbf{R}_+^N), \mathfrak{B}_q(\mathbf{R}_+^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{T}_\lambda\mathcal{B}^2(\lambda)\right) : \lambda \in \mathbf{C}_+\right\}\right) &\le C,
\end{aligned}
$$

with a positive constant $C = C(N, q, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$. This completes the proof of Lemma 2. □

We next consider the following resolvent problem with the slip boundary condition for the fluid velocity:

$$
\begin{cases}
\lambda\rho + \gamma_1\operatorname{div}\mathbf{u} = d & \text{in } \mathbf{R}_+^N,\\
\lambda\mathbf{u} - \gamma_4^{-1}\operatorname{Div}(\gamma_2(\mathbf{D}(\mathbf{u}) + (\gamma_3 - \gamma_2)\operatorname{div}\mathbf{u}\mathbf{I} + \gamma_1\Delta\rho\mathbf{I}) = \mathbf{f} & \text{in } \mathbf{R}_+^N,\\
\mathbf{n}\cdot\nabla\rho = g, \quad (\mathbf{D}(\mathbf{u})\mathbf{n})_\tau = \mathbf{k}_\tau, \quad \mathbf{u}\cdot\mathbf{n} = l & \text{on } \mathbf{R}_0^N.
\end{cases}
\tag{17}
$$

Let $q \in (1, \infty)$. For the right member $(d, \mathbf{f}, g, \mathbf{k}, l)$, we set

$$
\begin{aligned}
\mathcal{X}_q^3(\mathbf{R}_+^N) &= H_q^1(\mathbf{R}_+^N) \times L_q(\mathbf{R}_+^N)^N \times H_q^2(\mathbf{R}_+^N) \times H_q^1(\mathbf{R}_+^N)^N \times H_q^2(\mathbf{R}_+^N),\\
\mathfrak{X}_q^3(\mathbf{R}_+^N) &= L_q(\mathbf{R}_+^N)^{(N+1)+N+(N^2+N+1)+(N^2+N)+(N^2+N+1)}
\end{aligned}
$$

and set for $\mathbf{F}^3 = (d, \mathbf{f}, g, \mathbf{k}, l) \in \mathcal{X}_q^3(\mathbf{R}_+^N)$ and $\lambda \in \mathbf{C} \setminus (-\infty, 0]$

$$
\mathcal{F}_\lambda^3\mathbf{F}^3 = (\nabla d, \lambda^{1/2}d, \mathbf{f}, \nabla^2 g, \lambda^{1/2}\nabla g, \lambda g, \nabla\mathbf{k}, \lambda^{1/2}\mathbf{k}, \nabla^2 l, \lambda^{1/2}\nabla l, \lambda l) \in \mathfrak{X}_q^3(\mathbf{R}_+^N).
$$

The following lemma then holds.

Lemma 3. *Let $q \in (1, \infty)$ and γ_i $(i = 1, 2, 3, 4)$ be constants satisfying (12). Then, the following assertions hold.*

(1) *For any $\lambda \in \mathbf{C}_+$ there exist operators $\mathcal{A}^3(\lambda)$, $\mathcal{B}^3(\lambda)$, with*

$$
\begin{aligned}
\mathcal{A}^3(\lambda) &\in \operatorname{Hol}(\mathbf{C}_+, \mathcal{L}(\mathfrak{X}_q^3(\mathbf{R}_+^N), H_q^3(\mathbf{R}_+^N))),\\
\mathcal{B}^3(\lambda) &\in \operatorname{Hol}(\mathbf{C}_+, \mathcal{L}(\mathfrak{X}_q^3(\mathbf{R}_+^N), H_q^2(\mathbf{R}_+^N)^N)),
\end{aligned}
$$

such that for any $\mathbf{F}^3 = (d, \mathbf{f}, g, \mathbf{k}, l) \in \mathcal{X}_q^3(\mathbf{R}_+^N)$

$$
(\rho, \mathbf{u}) = (\mathcal{A}^3(\lambda)\mathcal{F}_\lambda^3\mathbf{F}^3, \mathcal{B}^3(\lambda)\mathcal{F}_\lambda^3\mathbf{F}^3)
$$

is a unique solution to (17).

(2) *There exists a positive constant $C = C(N, q, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ such that for $n = 0, 1$*

$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}^3_q(\mathbf{R}^N_+), \mathfrak{A}^0_q(\mathbf{R}^N_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{S}^0_\lambda \mathcal{A}^3(\lambda)\right) : \lambda \in \mathbf{C}_+\right\}\right) \leq C,$$
$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}^3_q(\mathbf{R}^N_+), \mathfrak{B}_q(\mathbf{R}^N_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{T}_\lambda \mathcal{B}^3(\lambda)\right) : \lambda \in \mathbf{C}_+\right\}\right) \leq C,$$

where $\mathfrak{A}^0_q(\mathbf{R}^N_+)$, $\mathfrak{B}_q(\mathbf{R}^N_+)$, $\mathcal{S}^0_\lambda$, and $\mathcal{T}_\lambda$ are given by (11) with $G = \mathbf{R}^N_+$.

Proof. Let $\widetilde{\rho} = \rho/\gamma_1$, $\widetilde{d} = d/\gamma_1$, and $\widetilde{g} = g/\gamma_1$. Then, (17) is equivalent to

$$\begin{cases} \lambda\widetilde{\rho} + \operatorname{div}\mathbf{u} = \widetilde{d} & \text{in } \mathbf{R}^N_+, \\ \lambda\mathbf{u} - \mu\Delta\mathbf{u} - \nu\nabla\operatorname{div}\mathbf{u} - \kappa\nabla\Delta\widetilde{\rho} = \mathbf{f} & \text{in } \mathbf{R}^N_+, \\ \mathbf{n}\cdot\nabla\widetilde{\rho} = \widetilde{g}, \quad (\mathbf{D}(\mathbf{u})\mathbf{n})_\tau = \mathbf{k}_\tau, \quad \mathbf{u}\cdot\mathbf{n} = l & \text{on } \mathbf{R}^N_0, \end{cases} \tag{18}$$

where

$$\mu = \frac{\gamma_2}{\gamma_4}, \quad \nu = \frac{\gamma_3}{\gamma_4}, \quad \kappa = \frac{\gamma_1^2}{\gamma_4}.$$

By [25] (Theorem 1.3), we observe that (18) admits $\mathcal{R}$-solvers satisfying the desired properties under the condition that μ, ν, and κ are positive constants satisfying

$$\left(\frac{\mu+\nu}{2\kappa}\right)^2 - \frac{1}{\kappa} \neq 0 \quad \text{and} \quad \kappa \neq \mu\nu.$$

This result can be extended to the case where μ, ν, and κ are any constants satisfying $\mu > 0$, $\mu + \nu > 0$, and $\kappa > 0$ by direct calculations in the same manner as in [24]. The uniqueness of solutions follows from the existence of solutions; see, e.g., ([3] Subsection 3.3). This completes the proof of Lemma 3. □

4.3. $\mathcal{R}$-Solver in a General Domain

This subsection considers the resolvent problem in a uniform C^3 domain Ω:

$$\begin{cases} \lambda\rho + \gamma_1\operatorname{div}\mathbf{v} = d & \text{in } \Omega, \\ \lambda\mathbf{u} - \gamma_4^{-1}\operatorname{Div}(\gamma_2(\mathbf{D}(\mathbf{u}) + (\gamma_3 - \gamma_2)\operatorname{div}\mathbf{u}\mathbf{I} + \gamma_1\Delta\rho\mathbf{I}) = \mathbf{f} & \text{in } \Omega, \\ \mathbf{n}\cdot\nabla\rho = g_D, \quad \mathbf{u} = \mathbf{h} & \text{on } \Gamma_D, \\ \mathbf{n}\cdot\nabla\rho = g_S, \quad (\mathbf{D}(\mathbf{u})\mathbf{n})_\tau = \mathbf{k}_\tau, \quad \mathbf{u}\cdot\mathbf{n} = l & \text{on } \Gamma_S. \end{cases} \tag{19}$$

We introduce an assumption about the coefficients γ_1, γ_2, γ_3, and γ_4.

Assumption 1. *The coefficients $\gamma_i = \gamma_i(x)$, $i = 1, 2, 3, 4$, are real valued uniformly Lipschitz continuous functions on $\overline{\Omega} = \Omega \cup \Gamma_D \cup \Gamma_S$, i.e., there exists a positive constant γ_L, such that $|\gamma_i(x) - \gamma_i(y)| \leq \gamma_L|x - y|$ for any $x, y \in \overline{\Omega}$ and for $i = 1, 2, 3, 4$. In addition, there exist positive constants γ_*, γ^*, such that $\gamma_* \leq \gamma_i(x) \leq \gamma^*$ $(i = 1, 2, 4)$ and $\gamma_* \leq \gamma_2(x) + \gamma_3(x) \leq \gamma^*$ for any $x \in \overline{\Omega}$.*

Let $q \in (1, \infty)$. For the right member $(d, \mathbf{f}, g_D, \mathbf{h}, g_S, \mathbf{k}, l)$ of (19), we set

$$\mathcal{X}_q(\Omega) = H_q^1(\Omega) \times L_q(\Omega)^N \times H_q^2(\Omega) \times H_q^2(\Omega)^N \times H_q^2(\Omega) \times H_q^1(\Omega)^N \times H_q^2(\Omega)$$
$$\mathfrak{X}_q^0(\Omega) = L_q(\Omega)^{(N+1)+N+(N^2+N+1)+(N^3+N^2+N)+(N^2+N+1)+(N^2+N)+(N^2+N+1)}$$

and set for $\mathbf{F} = (d, \mathbf{f}, g_D, \mathbf{h}, g_S, \mathbf{k}, l) \in \mathcal{X}_q(\Omega)$ and $\lambda \in \mathbf{C} \setminus (-\infty, 0]$

$$\mathcal{F}_\lambda^0\mathbf{F} = (\nabla d, \lambda^{1/2}d, \mathbf{f}, \nabla^2 g_D, \lambda^{1/2}\nabla g_D, \lambda g_D, \nabla^2\mathbf{h}, \lambda^{1/2}\nabla\mathbf{h}, \lambda\mathbf{h},$$
$$\nabla^2 g_S, \lambda^{1/2}\nabla g_S, \lambda g_S, \nabla\mathbf{h}, \lambda^{1/2}\mathbf{h}, \nabla^2 l, \lambda^{1/2}\nabla l, \lambda l) \in \mathfrak{X}_q^0(\Omega).$$

By Lemmas 1–3, we can prove the following proposition on the basis of the standard localization technique; see, e.g., [3].

Proposition 1. *Let Ω be a uniform C^3 domain in $\mathbf{R}^N$. Let $q \in (1, \infty)$ and suppose that Assumption 1 holds. Then, there exists a constant $\lambda_1 \geq 1$, depending solely on N, q, γ_L, γ_*, and γ^*, such that the following assertions hold.*

(1) *For any $\lambda \in \mathbf{C}_{+,\lambda_1}$, there exist operators $\mathcal{A}^0(\lambda)$, $\mathcal{B}^0(\lambda)$, with*

$$\mathcal{A}^0(\lambda) \in \mathrm{Hol}(\mathbf{C}_{+,\lambda_1}, \mathcal{L}(\mathfrak{X}_q^0(\Omega), H_q^3(\Omega))),$$
$$\mathcal{B}^0(\lambda) \in \mathrm{Hol}(\mathbf{C}_{+,\lambda_1}, \mathcal{L}(\mathfrak{X}_q^0(\Omega), H_q^2(\Omega)^N)),$$

such that for any $\mathbf{F} = (d, \mathbf{f}, g_D, \mathbf{h}, g_S, \mathbf{k}, l) \in \mathcal{X}_q(\Omega)$

$$(\rho, \mathbf{u}) = (\mathcal{A}^0(\lambda)\mathcal{F}_\lambda^0\mathbf{F}, \mathcal{B}^0(\lambda)\mathcal{F}_\lambda^0\mathbf{F})$$

is a unique solution to (19).

(2) *There exists a positive constant C, depending solely on N, q, γ_L, γ_*, and γ^*, such that for $n = 0, 1$*

$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q^0(\Omega), \mathfrak{A}_q^0(\Omega))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{S}_\lambda^0\mathcal{A}^0(\lambda)\right) : \lambda \in \mathbf{C}_{+,\lambda_1}\right\}\right) \leq C,$$
$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q^0(\Omega), \mathfrak{B}_q(\Omega))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\mathcal{T}_\lambda\mathcal{B}^0(\lambda)\right) : \lambda \in \mathbf{C}_{+,\lambda_1}\right\}\right) \leq C,$$

where $\mathfrak{A}_q^0(\Omega)$, $\mathfrak{B}_q(\Omega)$, $\mathcal{S}_\lambda^0$, and $\mathcal{T}_\lambda$ are given by (11) with $G = \Omega$.

Proposition 1 is not enough to obtain our linear theory in the next section due to $\lambda^{3/2}\rho$ and $\lambda^{1/2}d$. To eliminate these terms, we construct another $\mathcal{R}$-solver for (19) based on $\mathcal{A}^0(\lambda)$, $\mathcal{B}^0(\lambda)$ in what follows. We start with

$$\begin{cases} \lambda R + \theta_1 \operatorname{div} \mathbf{U} = D & \text{in } \mathbf{R}^N, \\ \lambda\mathbf{U} - \theta_4^{-1}\operatorname{Div}(\theta_2\mathbf{D}(\mathbf{U}) + (\theta_3 - \theta_2)\operatorname{div}\mathbf{U}\mathbf{I} + \theta_1\Delta R\mathbf{I}) = \mathbf{F} & \text{in } \mathbf{R}^N. \end{cases} \tag{20}$$

Let us define for $q \in (1, \infty)$ and $\lambda \in \mathbf{C} \setminus (-\infty, 0]$

$$\mathfrak{A}_q(G) = L_g(G)^{N^3+N^2} \times H_q^1(G), \quad \mathcal{S}_\lambda\rho = (\nabla^3\rho, \lambda^{1/2}\nabla^2\rho, \lambda\rho), \tag{21}$$

where G is a domain in $\mathbf{R}^N$. The following proposition follows from [23] (Theorem 2.1) and the standard localization technique; see also [3] (Theorem 7.1).

Proposition 2. *Let $q \in (1, \infty)$, and let $\theta_i = \theta_i(x)$ $(i = 1, 2, 3, 4)$ be real valued uniformly Lipschitz continuous functions on $\mathbf{R}^N$, i.e., there exists a positive constant θ_L, such that $|\theta_i(x) - \theta_i(y)| \leq \theta_L |x - y|$ for any $x, y \in \mathbf{R}^N$ and for $i = 1, 2, 3, 4$. Assume that there exist positive constants θ_* and θ^*, such that for any $x \in \mathbf{R}^N$*

$$\theta_* \leq \theta_i(x) \leq \theta^* \quad (i = 1, 2, 4), \quad \theta_* \leq \theta_2(x) + \theta_3(x) \leq \theta^*.$$

Then, there exists $\lambda_2 \geq 1$, depending on at most N, q, θ_L, θ_*, and θ^*, such that the following assertions hold.

(1) *For any $\lambda \in \mathbf{C}_{+,\lambda_2}$, there exist operators $\Phi(\lambda)$ and $\Psi(\lambda)$, with*

$$\Phi(\lambda) \in \mathcal{L}(\mathbf{C}_{+,\lambda_2}, \mathcal{L}(H^1_q(\mathbf{R}^N) \times L_q(\mathbf{R}^N)^N, H^3_q(\mathbf{R}^N))),$$
$$\Psi(\lambda) \in \mathcal{L}(\mathbf{C}_{+,\lambda_2}, \mathcal{L}(H^1_q(\mathbf{R}^N) \times L_q(\mathbf{R}^N)^N, H^2_q(\mathbf{R}^N)^N)),$$

such that for any $(D, \mathbf{F}) \in H^1_q(\mathbf{R}^N) \times L_q(\mathbf{R}^N)^N$

$$(R, \mathbf{U}) = (\Phi(\lambda)(D, \mathbf{F}), \Psi(\lambda)(D, \mathbf{F}))$$

is a unique solution to (20).

(2) *There exists a positive constant C, depending on at most N, q, θ_L, θ_*, and θ^*, such that for $n = 0, 1$*

$$\mathcal{R}_{\mathcal{L}(H^1_q(\mathbf{R}^N) \times L_q(\mathbf{R}^N)^N, \mathfrak{A}_q(\mathbf{R}^N))}\left(\left\{\left(\lambda \frac{d}{d\lambda}\right)^n (\mathcal{S}_\lambda \Phi(\lambda)) : \lambda \in \mathbf{C}_{+,\lambda_2}\right\}\right) \leq C,$$
$$\mathcal{R}_{\mathcal{L}(H^1_q(\mathbf{R}^N) \times L_q(\mathbf{R}^N)^N, \mathfrak{B}_q(\mathbf{R}^N))}\left(\left\{\left(\lambda \frac{d}{d\lambda}\right)^n (\mathcal{T}_\lambda \Psi(\lambda)) : \lambda \in \mathbf{C}_{+,\lambda_2}\right\}\right) \leq C.$$

Here, $\mathfrak{A}_q(\mathbf{R}^N)$ and $\mathcal{S}_\lambda$ are given by (21) for $G = \mathbf{R}^N$, while $\mathfrak{B}_q(\mathbf{R}^N)$ and $\mathcal{T}_\lambda$ are given by (11) for $G = \mathbf{R}^N$.

To use Proposition 2, we extend the coefficients γ_i $(i = 1, 2, 3, 4)$ satisfying Assumption 1 to ones defined on $\mathbf{R}^N$ by the following lemma.

Lemma 4. *Let Ω be a uniform C^3 domain in $\mathbf{R}^N$, and let f be a real valued uniformly Lipschitz continuous function on $\overline{\Omega}$, i.e., there exists a positive constant L, such that $|f(x) - f(y)| \leq L|x - y|$ for any $x, y \in \overline{\Omega}$. Assume that there exist positive constants c_* and c^*, such that $c_* \leq f(x) \leq c^*$ for any $x \in \overline{\Omega}$. Then, there exists a real valued uniformly Lipschitz continuous function F on $\mathbf{R}^N$ and a positive constant M, depending solely on c^* and L, such that the following assertions hold.*

(1) *$F(x) = f(x)$ for any $x \in \overline{\Omega}$.*
(2) *$|F(x) - F(y)| \leq (M + (c_*/2))|x - y|$ for any $x, y \in \mathbf{R}^N$.*
(3) *$c_*/2 \leq F(x) \leq M + (c_*/2)$ for any $x \in \mathbf{R}^N$.*

Proof. See [3] (Lemma 7.2 and Appendix A). □

Let us define

$$\mathfrak{X}_q(\Omega) = H^1_q(\Omega) \times L_q(\Omega)^N \times L_q(\Omega)^{(N^2+N+1)+(N^3+N^2+N)+(N^2+N+1)+(N^2+N)+(N^2+N+1)}$$

and set for $\mathbf{F} = (d, \mathbf{f}, g_D, \mathbf{h}, g_S, \mathbf{k}, l) \in \mathcal{X}_q(\Omega)$ and $\lambda \in \mathbf{C} \setminus (-\infty, 0]$

$$\mathcal{F}_\lambda \mathbf{F} = (d, \mathbf{f}, \nabla^2 g_D, \lambda^{1/2} \nabla g_D, \lambda g_D, \nabla^2 \mathbf{h}, \lambda^{1/2} \nabla \mathbf{h}, \lambda \mathbf{h},$$
$$\nabla^2 g_S, \lambda^{1/2} \nabla g_S, \lambda g_S, \nabla \mathbf{h}, \lambda^{1/2} \mathbf{h}, \nabla^2 l, \lambda^{1/2} \nabla l, \lambda l) \in \mathfrak{X}_q(\Omega).$$

We are now in a position to construct a new $\mathcal{R}$-solver for (19).

Theorem 2. *Let Ω be a uniform C^3 domain in $\mathbf{R}^N$. Let $q \in (1,\infty)$ and suppose that Assumption 1 holds. Then, there exists a constant $\lambda_3 \geq 1$, depending solely on N, q, γ_L, γ_*, and γ^*, such that the following assertions hold.*

(1) *For any $\lambda \in \mathbf{C}_{+,\lambda_3}$, there exist operators $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$, with*

$$\mathcal{A}(\lambda) \in \mathrm{Hol}(\mathbf{C}_{+,\lambda_3}, \mathcal{L}(\mathfrak{X}_q(\Omega), H^3_q(\Omega))),$$
$$\mathcal{B}(\lambda) \in \mathrm{Hol}(\mathbf{C}_{+,\lambda_3}, \mathcal{L}(\mathfrak{X}_q(\Omega), H^2_q(\Omega)^N)),$$

such that for any $\mathbf{F} = (d, \mathbf{f}, g_D, \mathbf{h}, g_S, \mathbf{k}, l) \in \mathcal{X}_q(\Omega)$

$$(\rho, \mathbf{u}) = (\mathcal{A}(\lambda)\mathcal{F}_\lambda \mathbf{F}, \mathcal{B}(\lambda)\mathcal{F}_\lambda \mathbf{F})$$

is a unique solution to (19).

(2) *There exists a positive constant C, depending solely on N, q, γ_L, γ_*, and γ^*, such that for $n = 0, 1$*

$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q(\Omega), \mathfrak{A}_q(\Omega))}\left(\left\{\left(\lambda \frac{d}{d\lambda}\right)^n (\mathcal{S}_\lambda \mathcal{A}(\lambda)) : \lambda \in \mathbf{C}_{+,\lambda_3}\right\}\right) \leq C,$$
$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q(\Omega), \mathfrak{B}_q(\Omega))}\left(\left\{\left(\lambda \frac{d}{d\lambda}\right)^n (\mathcal{T}_\lambda \mathcal{B}(\lambda)) : \lambda \in \mathbf{C}_{+,\lambda_3}\right\}\right) \leq C.$$

Here, $\mathfrak{A}_q(\Omega)$ and $\mathcal{S}_\lambda$ are given by (21) for $G = \Omega$, while $\mathfrak{B}_q(\Omega)$ and $\mathcal{T}_\lambda$ are given by (11) for $G = \Omega$.

Proof. Define $\delta(x) = \gamma_2(x) + \gamma_3(x)$. By Lemma 4, we extend $\gamma_i(x)$ $(i = 1, 2, 4)$ and $\delta(x)$ on $\overline{\Omega}$ to $\widetilde{\gamma}_i(x)$ $(i = 1, 2, 4)$ and $\widetilde{\delta}(x)$ on $\mathbf{R}^N$, respectively. They are real valued uniformly Lipschitz continuous functions on $\mathbf{R}^N$ and satisfy

$$\frac{\gamma_*}{2} \leq \widetilde{\gamma}_i(x) \leq M + \frac{\gamma_*}{2} \quad (i = 1, 2, 4), \quad \frac{\gamma_*}{2} \leq \widetilde{\delta}(x) \leq M + \frac{\gamma_*}{2}$$

for any $x \in \mathbf{R}^N$ with a positive constant $M = M(\gamma^*, \gamma_L)$. Define $\widetilde{\gamma}_3(x) = \widetilde{\delta}(x) - \widetilde{\gamma}_2(x)$. This shows that $\widetilde{\gamma}_3(x) = \gamma_3(x)$ for $x \in \overline{\Omega}$ and that $\widetilde{\gamma}_3(x)$ is a real valued uniformly Lipschitz continuous function on $\mathbf{R}^N$ with

$$\frac{\gamma_*}{2} \leq \widetilde{\gamma}_2(x) + \widetilde{\gamma}_3(x) \leq M + \frac{\gamma_*}{2} \quad \text{for any } x \in \mathbf{R}^N.$$

Furthermore, the Lipschitz constants of $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$, $\widetilde{\gamma}_3$, and $\widetilde{\gamma}_4$ are bounded above by $2(M + \gamma_*/2)$.

We use Proposition 2 with $\theta_i = \widetilde{\gamma}_i$ for $i = 1, 2, 3, 4$. Let $(d, \mathbf{f}, g_D, \mathbf{h}, g_S, \mathbf{k}, l) \in \mathcal{X}_q(\Omega)$ in what follows. Let E be an extension operator from $H^1_q(\Omega)$ to $H^1_q(\mathbf{R}^N)$, while $E_0\mathbf{f}$ is the zero extension of $\mathbf{f}$, i.e., $E_0\mathbf{f} = \mathbf{f}$ in Ω and $E_0\mathbf{f} = 0$ in $\mathbf{R}^N \setminus \Omega$. We define

$$(R, \mathbf{U}) = (\Phi(\lambda)(Ed, E_0\mathbf{f}), \Psi(\lambda)(Ed, E_0\mathbf{f})).$$

Then, $(R, \mathbf{U})$ satisfies

$$\begin{cases} \lambda R + \widetilde{\gamma}_1 \operatorname{div} \mathbf{U} = Ed & \text{in } \mathbf{R}^N, \\ \lambda \mathbf{U} - \widetilde{\gamma}_4^{-1} \operatorname{Div}(\widetilde{\gamma}_2 \mathbf{D}(\mathbf{U}) + (\widetilde{\gamma}_3 - \widetilde{\gamma}_2) \operatorname{div} \mathbf{U}\mathbf{I} + \widetilde{\gamma}_1 \Delta R\mathbf{I}) = E_0\mathbf{f} & \text{in } \mathbf{R}^N. \end{cases}$$

Setting $\rho = R + \sigma$ in (19) yields

$$\begin{cases} \lambda\sigma + \gamma_1 \operatorname{div} \mathbf{u} = \widetilde{d} & \text{in } \Omega, \\ \lambda\mathbf{u} - \gamma_4^{-1} \operatorname{Div}(\gamma_2 \mathbf{D}(\mathbf{u}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{u}\mathbf{I} + \gamma_1 \Delta\sigma\mathbf{I}) = \widetilde{\mathbf{f}} & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla\sigma = \widetilde{g}_D, \quad \mathbf{u} = \mathbf{h} & \text{on } \Gamma_D, \\ \mathbf{n} \cdot \nabla\sigma = \widetilde{g}_S, \quad (\mathbf{D}(\mathbf{u}))_\tau = \mathbf{k}_\tau, \quad \mathbf{u} \cdot \mathbf{n} = l & \text{on } \Gamma_S, \end{cases} \tag{22}$$

where

$$\widetilde{d} = \gamma_1 \operatorname{div} \mathbf{U}, \quad \widetilde{\mathbf{f}} = \lambda\mathbf{U} - \gamma_4^{-1} \operatorname{Div}(\gamma_2 \mathbf{D}(\mathbf{U}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{U}\mathbf{I}),$$
$$\widetilde{g}_D = g_D - \widetilde{\mathbf{n}} \cdot \nabla R, \quad \widetilde{g}_S = g_S - \widetilde{\mathbf{n}} \cdot \nabla R.$$

Notice that $\widetilde{\mathbf{n}}$ is an extension of $\mathbf{n}$ with $\widetilde{\mathbf{n}} \in H^2_\infty(\mathbf{R}^N)$, see [26] (Corollary A.3) for more details. From (22) and Proposition 1, we observe that the solution $(\rho, \mathbf{u})$ of (19) can be written as

$$\begin{aligned} \rho &= R + \sigma = \Phi(\lambda)(Ed, E_0\mathbf{f}) + \mathcal{A}^0(\lambda)\mathcal{F}^0_\lambda(\widetilde{d}, \widetilde{\mathbf{f}}, \widetilde{g}_D, \mathbf{h}, \widetilde{g}_S, \mathbf{k}, l), \\ \mathbf{u} &= \mathcal{B}^0(\lambda)\mathcal{F}^0_\lambda(\widetilde{d}, \widetilde{\mathbf{f}}, \widetilde{g}_D, \mathbf{h}, \widetilde{g}_S, \mathbf{k}, l). \end{aligned} \tag{23}$$

Let us recall

$$\begin{aligned} &\mathcal{F}^0_\lambda(\widetilde{d}, \widetilde{\mathbf{f}}, \widetilde{g}_D, \mathbf{h}, \widetilde{g}_S, \mathbf{k}, l) \\ &\quad = (\nabla\widetilde{d}, \lambda^{1/2}\widetilde{d}, \widetilde{\mathbf{f}}, \nabla^2\widetilde{g}_D, \lambda^{1/2}\nabla\widetilde{g}_D, \lambda\widetilde{g}_D, \nabla^2\mathbf{h}, \lambda^{1/2}\nabla\mathbf{h}, \lambda\mathbf{h}, \\ &\qquad \nabla^2\widetilde{g}_S, \lambda^{1/2}\nabla\widetilde{g}_S, \lambda\widetilde{g}_S, \nabla\mathbf{k}, \lambda^{1/2}\mathbf{k}, \nabla^2 l, \lambda^{1/2}\nabla l, \lambda l). \end{aligned}$$

In view of this formula, for $\mathbf{H} = (H_1, \dots, H_{16}) \in \mathfrak{X}_q(\Omega)$ and $(\mathsf{Z}, \mathcal{Z}) \in \{(\mathsf{A}, \mathcal{A}), (\mathsf{B}, \mathcal{B})\}$, we set

$$\begin{aligned} \mathsf{Z}(\lambda)\mathbf{H} = \mathcal{Z}^0(\lambda)\Big(&\nabla\Big(\gamma_1 \operatorname{div} \Psi(\lambda)(EH_1, E_0H_2)\Big), \\ &\lambda^{1/2}\gamma_1 \operatorname{div} \Psi(\lambda)(EH_1, E_0H_2), \\ &\lambda\Psi(\lambda)(EH_1, E_0H_2) - \gamma_4^{-1} \operatorname{Div}\Big(\gamma_2 \mathbf{D}(\Psi(\lambda)(EH_1, E_0H_2)) \\ &+ (\gamma_3 - \gamma_2) \operatorname{div} \Psi(\lambda)(EH_1, E_0H_2)\mathbf{I}\Big), \\ &H_3 - \nabla^2\Big(\widetilde{\mathbf{n}} \cdot \nabla\Phi(\lambda)(EH_1, E_0H_2)\Big), \\ &H_4 - \lambda^{1/2}\nabla\Big(\widetilde{\mathbf{n}} \cdot \nabla\Phi(\lambda)(EH_1, E_0H_2)\Big), \\ &H_5 - \lambda\Big(\widetilde{\mathbf{n}} \cdot \nabla\Phi(\lambda)(EH_1, E_0H_2)\Big), \\ &H_6, H_7, H_8, \\ &H_9 - \nabla^2\Big(\widetilde{\mathbf{n}} \cdot \nabla\Phi(\lambda)(EH_1, E_0H_2)\Big), \\ &H_{10} - \lambda^{1/2}\nabla\Big(\widetilde{\mathbf{n}} \cdot \nabla\Phi(\lambda)(EH_1, E_0H_2)\Big), \\ &H_{11} - \lambda\Big(\widetilde{\mathbf{n}} \cdot \nabla\Phi(\lambda)(EH_1, E_0H_2)\Big), \\ &H_{12}, H_{13}, H_{14}, H_{15}, H_{16}\Big). \end{aligned}$$

We also set for $\mathbf{H} = (H_1, \dots, H_{16}) \in \mathfrak{X}_q(\Omega)$

$$\mathcal{A}(\lambda)\mathbf{H} = \Phi(\lambda)(EH_1, E_0H_2) + \mathsf{A}(\lambda)\mathbf{H}, \quad \mathcal{B}(\lambda)\mathbf{H} = \mathsf{B}(\lambda)\mathbf{H}.$$

It then follows from (23) that $(\rho, \mathbf{u}) = (\mathcal{A}(\lambda)\mathcal{F}_\lambda \mathbf{F}, \mathcal{B}(\lambda)\mathcal{F}_\lambda \mathbf{F})$. In addition, $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ satisfy the desired estimates from the definition of the $\mathcal{R}$-boundedness and Propositions 1 and 2. The uniqueness of solutions is already discussed in Proposition 1. This completes the proof of Theorem 2. □

5. Linear Theory

This section considers the following time-dependent linear system:

$$\begin{cases} \partial_t \rho + \gamma_1 \operatorname{div} \mathbf{u} = d & \text{in } \Omega \times \mathbf{R}_+, \\ \partial_t \mathbf{u} - \gamma_4^{-1} \operatorname{Div}(\gamma_2 \mathbf{D}(\mathbf{u}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{u} \mathbf{I} + \gamma_1 \Delta \rho \mathbf{I}) = \mathbf{f} & \text{in } \Omega \times \mathbf{R}_+, \\ \mathbf{n} \cdot \nabla \rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma_D \times \mathbf{R}_+, \\ \mathbf{n} \cdot \nabla \rho = 0, \quad (\mathbf{D}(\mathbf{u})\mathbf{n})_\tau = 0, \quad \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \Gamma_S \times \mathbf{R}_+, \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega, \end{cases} \tag{24}$$

where the coefficients $\gamma_i = \gamma_i(x)$, $i = 1, 2, 3, 4$, satisfy Assumption 1. In the following subsections, we first introduce an analytic C_0-semigroup associated with (24), and then we state the maximal regularity for (24) with $(\rho_0, \mathbf{u}_0) = (0, 0)$.

5.1. An Analytic C_0-Semigroup

Let us define for $q \in (1, \infty)$

$$X_q = H_q^1(\Omega) \times L_q(\Omega)^N, \quad \|(\rho, \mathbf{u})\|_{X_q} = \|\rho\|_{H_q^1(\Omega)} + \|\mathbf{u}\|_{L_q(\Omega)}.$$

Furthermore, the operator A_q is defined by

$$A_q(\rho, \mathbf{u}) = (-\gamma_1 \operatorname{div} \mathbf{u}, \gamma_4^{-1} \operatorname{Div}(\gamma_2 \mathbf{D}(\mathbf{u}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{u} \mathbf{I} + \gamma_1 \Delta \rho \mathbf{I}))$$

with the domain

$$\begin{aligned} D(A_q) = \{(\rho, \mathbf{u}) \in H_q^3(\Omega) \times H_q^2(\Omega)^N : \mathbf{n} \cdot \nabla \rho = 0, \, \mathbf{u} = 0 \text{ on } \Gamma_D, \\ \mathbf{n} \cdot \nabla \rho = 0, \, (\mathbf{D}(\mathbf{u})\mathbf{n})_\tau = 0, \, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma_S\}. \end{aligned}$$

Noting Remark 3(3) and following [3] (Remark 2.10 (1)), we observe from Theorem 2 that A_q generates an analytic C_0-semigroup $(e^{A_q t})_{t \geq 0}$ on X_q, as follows.

Proposition 3. *Suppose that Ω is a uniform C^3 domain in $\mathbf{R}^N$, and Assumption 1 holds. Let $q \in (1, \infty)$. Then, the following assertions hold.*

(1) *A_q is a densely defined closed operator on X_q.*

(2) *A_q generates an analytic C_0-semigroup $(e^{A_q t})_{t \geq 0}$ on X_q. In addition, there exist constants $\delta_1 = \delta_1(N, q, \gamma_L, \gamma_*, \gamma^*) \geq 1$ and $C = C(N, q, \gamma_L, \gamma_*, \gamma^*) > 0$, such that for any $t > 0$*

$$\begin{aligned} \|e^{A_q t}(\rho_0, \mathbf{u}_0)\|_{X_q} &\leq C e^{(\delta_1/2)t} \|(\rho_0, \mathbf{u}_0)\|_{X_q} && ((\rho_0, \mathbf{u}_0) \in X_q), \\ \|\partial_t e^{A_q t}(\rho_0, \mathbf{u}_0)\|_{X_q} &\leq C e^{(\delta_1/2)t} t^{-1} \|(\rho_0, \mathbf{u}_0)\|_{X_q} && ((\rho_0, \mathbf{u}_0) \in X_q), \\ \|\partial_t e^{A_q t}(\rho_0, \mathbf{u}_0)\|_{X_q} &\leq C e^{(\delta_1/2)t} \|(\rho_0, \mathbf{u}_0)\|_{D(A_q)} && ((\rho_0, \mathbf{u}_0) \in D(A_q)), \end{aligned}$$

where $\|\cdot\|_{D(A_q)}$ denotes the graph norm of A_q.

Let $(\cdot, \cdot)_{\theta, p}$ be the real interpolation functor for $\theta \in (0, 1)$ and $p \in (1, \infty)$; see, e.g., [27] (Definition 1.37). We set

$$D_{q,p}(\Omega) = (X_q, D(A_q))_{1-1/p, p}.$$

Then $D_{q,p}(\Omega) \subset B_{q,p}^{3-2/p}(\Omega) \times B_{q,p}^{2-2/p}(\Omega)^N$. The next proposition immediately follows from Proposition 3 in the same manner as in [28] (Theorem 3.9).

Proposition 4. *Suppose that Ω is a uniform C^3 domain in $\mathbf{R}^N$ and Assumption 1 holds. Let $p, q \in (1, \infty)$. Then, for any $(\rho_0, \mathbf{u}_0) \in D_{q,p}(\Omega)$, $(\rho, \mathbf{u}) = e^{A_q t}(\rho_0, \mathbf{u}_0)$ is a unique solution to (24) with $(d, \mathbf{f}) = (0, 0)$ and satisfies*

$$\begin{aligned}
&\|e^{-\delta_1 t}\partial_t \rho\|_{L_p(\mathbf{R}_+, H^1_q(\Omega))} + \|e^{-\delta_1 t}\rho\|_{L_p(\mathbf{R}_+, H^3_q(\Omega))} \\
&+ \|e^{-\delta_1 t}\partial_t \mathbf{u}\|_{L_p(\mathbf{R}_+, L_q(\Omega)^N)} + \|e^{-\delta_1 t}\mathbf{u}\|_{L_p(\mathbf{R}_+, H^2_q(\Omega)^N)} \\
&\leq C\|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)},
\end{aligned}$$

where δ_1 is given by Proposition 3 and $C = C(N, p, q, \gamma_L, \gamma_*, \gamma^*)$ is a positive constant.

Let us now consider (8). Recall that $\mathbf{S}_0(\mathbf{u})$, μ_0, ν_0, κ_0, and r_0 are given by (5). Define

$$\gamma_1 := r_0 = \rho_0 + \rho_\infty, \quad \gamma_2 := \frac{\mu_0}{\kappa_0}, \quad \gamma_3 := \frac{\nu_0}{\kappa_0}, \quad \gamma_4 := \frac{r_0}{\kappa_0} = \frac{\rho_0 + \rho_\infty}{\kappa_0}. \tag{25}$$

We assume that r_0, μ_0, ν_0, and κ_0 satisfy the conditions (b), (e), and (f) of Theorem 1. Then, γ_1, γ_2, γ_3, and γ_4 satisfy Assumption 1, and γ_L, γ_*, and γ^* in Assumption 1 become constants depending only on R, R_1, and R_2. We therefore obtain the following corollary of Proposition 4.

Corollary 1. *Let Ω be a uniform C^3 domain in $\mathbf{R}^N$. Let $p, q \in (1, \infty)$ and let R, R_1, R_2, and ρ_∞ be positive constants with $R_1 \leq R_2$. Suppose that $(\rho_0, \mathbf{u}_0) \in D_{q,p}(\Omega)$ and that $r_0 = \rho_0 + \rho_\infty$, μ_0, ν_0, and κ_0 satisfy (b), (e), and (f) of Theorem 1. Then, (8) admits a unique solution $(\widehat{\rho}, \widehat{\mathbf{u}})$, which satisfies*

$$\begin{aligned}
&\|e^{-\eta_1 t}\partial_t \widehat{\rho}\|_{L_p(\mathbf{R}_+, H^1_q(\Omega))} + \|e^{-\eta_1 t}\widehat{\rho}\|_{L_p(\mathbf{R}_+, H^3_q(\Omega))} \\
&+ \|e^{-\eta_1 t}\partial_t \widehat{\mathbf{u}}\|_{L_p(\mathbf{R}_+, L_q(\Omega)^N)} + \|e^{-\eta_1 t}\widehat{\mathbf{u}}\|_{L_p(\mathbf{R}_+, H^2_q(\Omega)^N)} \\
&\leq C\|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}
\end{aligned}$$

with positive constants $\eta_1 = \eta_1(N, q, R, R_1, R_2, \rho_\infty)$ and $C = C(N, p, q, R, R_1, R_2, \rho_\infty)$.

5.2. Maximal Regularity

From Proposition 5 to Corollary 2 below, we discuss the maximal regularity for (24) with $(\rho_0, \mathbf{u}_0) = (0, 0)$. Concerning the theory of maximal regularity in L_p-in-time and L_q-in-space settings, we refer to [29] (Chapter 3), written by Shibata.

Combining Theorem 2 with the operator-valued Fourier multiplier theorem introduced by Weis [30] yields the following proposition.

Proposition 5. *Suppose that Ω is a uniform C^3 domain in $\mathbf{R}^N$ and Assumption 1 holds. Let $p, q \in (1, \infty)$. Then, there exists a constant $\delta_2 = \delta_2(N, q, \gamma_L, \gamma_*, \gamma^*) \geq 1$, such that the following assertions hold.*

(1) *For any $e^{-\delta_2 t} d \in L_p(\mathbf{R}_+, H^1_q(\Omega))$ and $e^{-\delta_2 t}\mathbf{f} \in L_p(\mathbf{R}_+, L_q(\Omega)^N)$, (24) with $(\rho_0, \mathbf{u}_0) = (0, 0)$ admits a unique solution $(\rho, \mathbf{u})$ with*

$$\begin{aligned}
&\rho \in H^1_{p,\mathrm{loc}}(\mathbf{R}_+, H^1_q(\Omega)) \cap L_{p,\mathrm{loc}}(\mathbf{R}_+, H^3_q(\Omega)), \\
&\mathbf{u} \in H^1_{p,\mathrm{loc}}(\mathbf{R}_+, L_q(\Omega)^N) \cap L_{p,\mathrm{loc}}(\mathbf{R}_+, H^2_q(\Omega)^N).
\end{aligned}$$

(2) *The solution $(\rho, \mathbf{u})$ satisfies*

$$\begin{aligned}
&\|e^{-\delta_2 t}\partial_t\rho\|_{L_p(\mathbf{R}_+,H^1_q(\Omega))} + \|e^{-\delta_2 t}\rho\|_{L_p(\mathbf{R}_+,H^3_q(\Omega))}\\
&+ \|e^{-\delta_2 t}\partial_t\mathbf{u}\|_{L_p(\mathbf{R}_+,L_q(\Omega)^N)} + \|e^{-\delta_2 t}\mathbf{u}\|_{L_p(\mathbf{R}_+,H^2_q(\Omega)^N)}\\
&\le C\Big(\|e^{-\delta_2 t}d\|_{L_p(\mathbf{R}_+,H^1_q(\Omega))} + \|e^{-\delta_2 t}\mathbf{f}\|_{L_p(\mathbf{R}_+,L_q(\Omega)^N)}\Big)
\end{aligned}$$

for some positive constant $C = C(N, p, q, \gamma_L, \gamma_*, \gamma^*)$.

Proof. See [29] (Subsection 3.4.6) for the proof of the existence of solutions satisfying the desired estimate.

Let us prove the uniqueness of solutions in what follows. Let $(\rho, \mathbf{u})$ satisfy the regularity stated in (1) and the following homogeneous system:

$$\begin{cases}
\partial_t\rho + \gamma_1 \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega\times\mathbf{R}_+,\\
\partial_t\mathbf{u} - \gamma_4^{-1}\operatorname{Div}(\gamma_2\mathbf{D}(\mathbf{u}) + (\gamma_3-\gamma_2)\operatorname{div}\mathbf{u}\mathbf{I} + \gamma_1\Delta\rho\mathbf{I}) = 0 & \text{in } \Omega\times\mathbf{R}_+,\\
\mathbf{n}\cdot\nabla\rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma_D\times\mathbf{R}_+,\\
\mathbf{n}\cdot\nabla\rho = 0, \quad (\mathbf{D}(\mathbf{u})\mathbf{n})_\tau = 0, \quad \mathbf{u}\cdot\mathbf{n} = 0 & \text{on } \Gamma_S\times\mathbf{R}_+,\\
(\rho,\mathbf{u})|_{t=0} = (0,0) & \text{in } \Omega.
\end{cases} \tag{26}$$

Let $\varphi \in C_0^\infty(\Omega\times\mathbf{R}_+)^N$, where $C_0^\infty(\Omega\times\mathbf{R}_+)$ is the set of all C^∞ functions whose supports are compact and contained in $\Omega\times\mathbf{R}_+$. Let T be a positive constant such that $\operatorname{supp}\varphi \subset \Omega\times(0,T)$ and define $\varphi_T(x,t) = \varphi(x, T-t)$. Then, $\operatorname{supp}\varphi_T \subset \Omega\times(0,T)$ and $\varphi_T \in L_{p'}(\mathbf{R}_+, L_{q'}(\Omega)^N)$ for $p' = p/(p-1)$ and $q' = q/(q-1)$. Thus, there exists $(\sigma, \mathbf{v})$, with

$$\begin{aligned}
&\sigma \in H^1_{p',\mathrm{loc}}(\mathbf{R}_+, H^1_{q'}(\Omega)) \cap L_{p',\mathrm{loc}}(\mathbf{R}_+, H^3_{q'}(\Omega)),\\
&\mathbf{v} \in H^1_{p',\mathrm{loc}}(\mathbf{R}_+, L_{q'}(\Omega)^N) \cap L_{p',\mathrm{loc}}(\mathbf{R}_+, H^2_{q'}(\Omega)^N),
\end{aligned}$$

such that

$$\begin{cases}
\partial_t\sigma + \gamma_1 \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega\times\mathbf{R}_+,\\
\partial_t\mathbf{v} - \gamma_4^{-1}\operatorname{Div}(\gamma_2\mathbf{D}(\mathbf{v}) + (\gamma_3-\gamma_2)\operatorname{div}\mathbf{v}\mathbf{I} + \gamma_1\Delta\sigma\mathbf{I}) = \gamma_4^{-1}\varphi_T & \text{in } \Omega\times\mathbf{R}_+,\\
\mathbf{n}\cdot\nabla\sigma = 0, \quad \mathbf{v} = 0 & \text{on } \Gamma_D\times\mathbf{R}_+,\\
\mathbf{n}\cdot\nabla\sigma = 0, \quad (\mathbf{D}(\mathbf{v})\mathbf{n})_\tau = 0, \quad \mathbf{v}\cdot\mathbf{n} = 0 & \text{on } \Gamma_S\times\mathbf{R}_+,\\
(\sigma,\mathbf{v})|_{t=0} = (0,0) & \text{in } \Omega.
\end{cases} \tag{27}$$

Let $\mathbf{a} = (a_1(x), \dots, a_N(x))^\mathsf{T}$, $\mathbf{b} = (b_1(x), \dots, b_N(x))^\mathsf{T}$, $\mathbf{A} = (A_{ij}(x))_{1\le i,j\le N}$, and $\mathbf{B} = (B_{ij}(x))_{1\le i,j\le N}$. Define

$$\begin{aligned}
(\mathbf{a},\mathbf{b})_\Omega &= \sum_{j=1}^N \int_\Omega a_j(x)b_j(x)\,dx,\\
(\mathbf{A},\mathbf{B})_\Omega &= \sum_{i,j=1}^N \int_\Omega A_{ij}(x)B_{ij}(x)\,dx,
\end{aligned}$$

and set for $\Gamma = \Gamma_D$ or $\Gamma = \Gamma_S$

$$(\mathbf{a},\mathbf{b})_\Gamma = \sum_{i=1}^N \int_\Gamma a_j(x)b_j(x)\,dS,$$

where dS is the surface element of Γ. In addition, for $f = f(x,t)$, $g = g(x,t)$, $\mathbf{f} = (f_1(x,t),\ldots,f_N(x,t))^\mathsf{T}$, $\mathbf{g} = (g_1(x,t),\ldots,g_N(x,t))^\mathsf{T}$, $\mathbf{F} = (F_{ij}(x,t))_{1\le i,j\le N}$, and $\mathbf{G} = (G_{ij}(x,t))_{1\le i,j\le N}$,

$$\begin{aligned}(f,g)_{\Omega\times(0,T)} &= \int_0^T\int_\Omega f(x,t)g(x,t)\,dxdt,\\ (\mathbf{f},\mathbf{g})_{\Omega\times(0,T)} &= \sum_{j=1}^N\int_0^T\int_\Omega f_j(x,t)g_j(x,t)\,dxdt,\\ (\mathbf{F},\mathbf{G})_{\Omega\times(0,T)} &= \sum_{i,j=1}^N\int_0^T\int_\Omega F_{ij}(x,t)G_{ij}(x,t)\,dxdt.\end{aligned}$$

Let $\mathbf{M} = (M_{ij}(x))_{1\le i,j\le N}$ with $\mathbf{M}^\mathsf{T} = \mathbf{M}$. Integration by parts then shows

$$(\mathrm{Div}\,\mathbf{M},\mathbf{a})_\Omega = -\frac{1}{2}(\mathbf{M},\mathbf{D}(\mathbf{a}))_\Omega + (\mathbf{M}\mathbf{n},\mathbf{a})_{\Gamma_D\cup\Gamma_S},$$

which, combined with

$$\mathbf{M}\mathbf{n} = (\mathbf{M}\mathbf{n})_\tau + \mathbf{n}(\mathbf{n}\cdot\mathbf{M}\mathbf{n}),$$

furnishes

$$\begin{aligned}(\mathrm{Div}\,\mathbf{M},\mathbf{a})_\Omega &= -\frac{1}{2}(\mathbf{M},\mathbf{D}(\mathbf{a}))_\Omega + (\mathbf{M}\mathbf{n},\mathbf{a})_{\Gamma_D}\\ &\quad + ((\mathbf{M}\mathbf{n})_\tau,\mathbf{a})_{\Gamma_S} + (\mathbf{n}\cdot\mathbf{M}\mathbf{n},\mathbf{a}\cdot\mathbf{n})_{\Gamma_S}.\end{aligned}\tag{28}$$

Let us define for $(x,t)\in\Omega\times(0,T)$

$$\tau(x,t) = \sigma(x,T-t),\quad \mathbf{w}(x,t) = \mathbf{v}(x,T-t).$$

It then follows from the second equation of (27) that for $(x,t)\in\Omega\times(0,T)$

$$\begin{aligned}&\gamma_4\partial_t\mathbf{w} + \mathrm{Div}(\gamma_2\mathbf{D}(\mathbf{w}) + (\gamma_3-\gamma_2)\,\mathrm{div}\,\mathbf{w}\mathbf{I} + \gamma_1\Delta\tau\mathbf{I})\\ &= -\{\gamma_4\partial_t\mathbf{v} - \mathrm{Div}(\gamma_2\mathbf{D}(\mathbf{v}) + (\gamma_3-\gamma_2)\,\mathrm{div}\,\mathbf{v}\mathbf{I} + \gamma_1\Delta\sigma\mathbf{I})\}(x,T-t)\\ &= -\varphi(x,t).\end{aligned}\tag{29}$$

Since $\mathbf{u}|_{t=0} = 0$ and $\mathbf{w}|_{t=T} = 0$, one observes by integration by parts that

$$(\gamma_4\partial_t\mathbf{u},\mathbf{w})_{\Omega\times(0,T)} = -(\mathbf{u},\gamma_4\partial_t\mathbf{w})_{\Omega\times(0,T)}.\tag{30}$$

Together with the boundary condition of (26) and (27), we use (28) with $\mathbf{a} = \mathbf{w}$ and $\mathbf{M} = \gamma_2\mathbf{D}(\mathbf{u}) + (\gamma_3-\gamma_2)\,\mathrm{div}\,\mathbf{u}\mathbf{I} + \gamma_1\Delta\rho\mathbf{I}$ in order to obtain

$$\begin{aligned}&(\mathrm{Div}(\gamma_2\mathbf{D}(\mathbf{u}) + (\gamma_3-\gamma_2)\,\mathrm{div}\,\mathbf{u}\mathbf{I} + \gamma_1\Delta\rho\mathbf{I}),\mathbf{w})_{\Omega\times(0,T)}\\ &= -\frac{1}{2}(\gamma_2\mathbf{D}(\mathbf{u}) + (\gamma_3-\gamma_2)\,\mathrm{div}\,\mathbf{u}\mathbf{I} + \gamma_1\Delta\rho\mathbf{I},\mathbf{D}(\mathbf{w}))_{\Omega\times(0,T)}.\end{aligned}\tag{31}$$

It holds that

$$((\gamma_3-\gamma_2)\,\mathrm{div}\,\mathbf{u}\mathbf{I},\mathbf{D}(\mathbf{w}))_{\Omega\times(0,T)} = (\mathbf{D}(\mathbf{u}),(\gamma_3-\gamma_2)\,\mathrm{div}\,\mathbf{w}\mathbf{I})_{\Omega\times(0,T)}.\tag{32}$$

In addition,

$$(\gamma_1\Delta\rho\mathbf{I},\mathbf{D}(\mathbf{w}))_{\Omega\times(0,T)} = (\Delta\rho,\gamma_1\,\mathrm{div}\,\mathbf{w})_{\Omega\times(0,T)} =: (\mathrm{RHS})_1,$$

which, combined with $\partial_t \tau = \gamma_1 \operatorname{div} \mathbf{w}$, furnishes

$$(\text{RHS})_1 = (\Delta\rho, \partial_t \tau)_{\Omega\times(0,T)} =: (\text{RHS})_2.$$

Together with $\rho|_{t=0} = 0$, $\tau|_{t=T} = 0$, and the boundary condition of (26) and (27), we observe by integration by parts that

$$(\text{RHS})_2 = -(\partial_t \rho, \Delta\tau)_{\Omega\times(0,T)} = (\gamma_1 \operatorname{div} \mathbf{u}, \Delta\tau)_{\Omega\times(0,T)} = (\mathbf{D}(\mathbf{u}), \gamma_1 \Delta\tau \mathbf{I})_{\Omega\times(0,T)}.$$

Thus

$$(\gamma_1 \Delta\rho \mathbf{I}, \mathbf{D}(\mathbf{w}))_{\Omega\times(0,T)} = (\mathbf{D}(\mathbf{u}), \gamma_1 \Delta\tau \mathbf{I})_{\Omega\times(0,T)}.$$

Summing up this Equations (31) and (32), we have

$$\begin{aligned}&(\operatorname{Div}(\gamma_2 \mathbf{D}(\mathbf{u}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{u}\mathbf{I} + \gamma_1 \Delta\rho \mathbf{I}), \mathbf{w})_{\Omega\times(0,T)} \\ &= -\frac{1}{2}(\mathbf{D}(\mathbf{u}), \gamma_2 \mathbf{D}(\mathbf{w}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{w}\mathbf{I} + \gamma_1 \Delta\tau \mathbf{I})_{\Omega\times(0,T)}.\end{aligned}$$

Let us use (28) with $\mathbf{a} = \mathbf{u}$ and $\mathbf{M} = \gamma_2 \mathbf{D}(\mathbf{w}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{w}\mathbf{I} + \gamma_1 \Delta\tau \mathbf{I}$ together with the boundary condition of (26) and (27), and then

$$\begin{aligned}&(\operatorname{Div}(\gamma_2 \mathbf{D}(\mathbf{w}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{w}\mathbf{I} + \gamma_1 \Delta\tau \mathbf{I}), \mathbf{u})_{\Omega\times(0,T)} \\ &= -\frac{1}{2}(\gamma_2 \mathbf{D}(\mathbf{w}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{w}\mathbf{I} + \gamma_1 \Delta\tau \mathbf{I}, \mathbf{D}(\mathbf{u}))_{\Omega\times(0,T)}.\end{aligned}$$

The last two equations give us

$$\begin{aligned}&(\operatorname{Div}(\gamma_2 \mathbf{D}(\mathbf{u}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{u}\mathbf{I} + \gamma_1 \Delta\rho \mathbf{I}), \mathbf{w})_{\Omega\times(0,T)} \\ &= (\mathbf{u}, \operatorname{Div}(\gamma_2 \mathbf{D}(\mathbf{w}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{w}\mathbf{I} + \gamma_1 \Delta\tau \mathbf{I}))_{\Omega\times(0,T)},\end{aligned}$$

which, combined with (29) and (30), and the second equation of (26), shows that

$$\begin{aligned}0 &= (\gamma_4 \partial_t \mathbf{u} - \operatorname{Div}(\gamma_2 \mathbf{D}(\mathbf{u}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{u}\mathbf{I} + \gamma_1 \Delta\rho \mathbf{I}), \mathbf{w})_{\Omega\times(0,T)} \\ &= -(\mathbf{u}, \gamma_4 \partial_t \mathbf{w} + \operatorname{Div}(\gamma_2 \mathbf{D}(\mathbf{w}) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{w}\mathbf{I} + \gamma_1 \Delta\tau \mathbf{I}))_{\Omega\times(0,T)} \\ &= (\mathbf{u}, \varphi)_{\Omega\times(0,T)} = (\mathbf{u}, \varphi)_{\Omega\times\mathbf{R}_+}.\end{aligned}$$

Thus, $\mathbf{u} = 0$. It then follows from the first equation of (26) that $\partial_t \rho = 0$, which, combined with $\rho|_{t=0} = 0$, furnishes for $(x, t) \in \Omega \times \mathbf{R}_+$

$$0 = \int_0^t \partial_s \rho(x, s)\, ds = \rho(x, t) - \rho(x, 0) = \rho(x, t).$$

Thus $\rho = 0$. This shows the uniqueness of solutions to (24) and completes the proof of Proposition 5. □

Let T be a positive constant. We next consider the following time-dependent linear system on $(0, T)$:

$$\begin{cases} \partial_t \rho + r_0 \operatorname{div} \mathbf{u} = d & \text{in } \Omega \times (0,T), \\ \partial_t \mathbf{u} - r_0^{-1} \kappa_0 \operatorname{Div}(\kappa_0^{-1} \mathbf{S}_0(\mathbf{u}) + r_0 \Delta\rho \mathbf{I}) = \mathbf{f} & \text{in } \Omega \times (0,T), \\ \mathbf{n} \cdot \nabla\rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma_D \times (0,T), \\ \mathbf{n} \cdot \nabla\rho = 0, \quad (\mathbf{D}(\mathbf{u})\mathbf{n})_\tau = 0, \quad \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \Gamma_S \times (0,T), \\ (\rho, \mathbf{u})|_{t=0} = (0, 0) & \text{in } \Omega, \end{cases} \tag{33}$$

where $\mathbf{S}_0(\mathbf{u})$, μ_0, ν_0, κ_0, and r_0 are given by (5). As a corollary of Proposition 5, we obtain

Corollary 2. *Let Ω be a uniform C^3 domain in $\mathbf{R}^N$. Let $p, q \in (1, \infty)$ and let R, R_1, R_2, and ρ_∞ be positive constants with $R_1 \leq R_2$. Let $T_0 \in (0, \infty)$ and $T \in (0, T_0]$. Suppose that $r_0 = \rho_0 + \rho_\infty$, μ_0, ν_0, and κ_0 satisfy (b), (e), and (f) of Theorem 1. Then, the following assertions hold.*

(1) *For any $d \in L_p((0,T), H^1_q(\Omega))$ and $\mathbf{f} \in L_p((0,T), L_q(\Omega)^N)$, (33) admits a unique solution $(\rho, \mathbf{u})$ with*

$$\rho \in H^1_p((0,T), H^1_q(\Omega)) \cap L_p((0,T), H^3_q(\Omega)),$$
$$\mathbf{u} \in H^1_p((0,T), L_q(\Omega)^N) \cap L_p((0,T), H^2_q(\Omega)^N).$$

(2) *The solution $(\rho, \mathbf{u})$ satisfies*

$$\begin{aligned}
&\|\partial_t \rho\|_{L_p((0,T),H^1_q(\Omega))} + \|\rho\|_{L_p((0,T),H^3_q(\Omega))} \\
&+ \|\partial_t \mathbf{u}\|_{L_p((0,T),L_q(\Omega)^N)} + \|\mathbf{u}\|_{L_p((0,T),H^2_q(\Omega)^N)} \\
&\leq M_1 \Big(\|d\|_{L_p((0,T),H^1_q(\Omega))} + \|\mathbf{f}\|_{L_p((0,T),L_q(\Omega)^N)} \Big)
\end{aligned}$$

for some positive constant $M_1 = M_1(N, p, q, R, R_1, R_2, T_0, \rho_\infty)$. In particular, M_1 is independent of T.

Proof. We apply Proposition 5 with (25)–(33). Notice that Assumption 1 is satisfied by our assumption about r_0, μ_0, ν_0, and κ_0.

Let $\widetilde{d}$ and $\widetilde{\mathbf{f}}$ be the zero extensions of d and $\mathbf{f}$, respectively, i.e.,

$$\widetilde{d} = \begin{cases} d & \text{for } t \in (0,T), \\ 0 & \text{for } t \in (T, \infty), \end{cases} \qquad \widetilde{\mathbf{f}} = \begin{cases} \mathbf{f} & \text{for } t \in (0,T), \\ 0 & \text{for } t \in (T, \infty). \end{cases}$$

Then

$$e^{-\delta_2 t} \widetilde{d} \in L_p(\mathbf{R}_+, H^1_q(\Omega)), \quad e^{-\delta_2 t} \widetilde{\mathbf{f}} \in L_p(\mathbf{R}_+, L_q(\Omega)^N),$$

where δ_2 is given by Proposition 5, which yields the solution $(\widetilde{\rho}, \widetilde{\mathbf{u}})$ to

$$\begin{cases}
\partial_t \widetilde{\rho} + r_0 \operatorname{div} \widetilde{\mathbf{u}} = \widetilde{d} & \text{in } \Omega \times \mathbf{R}_+, \\
\partial_t \widetilde{\mathbf{u}} - r_0^{-1} \kappa_0 \operatorname{Div}(\kappa_0^{-1} \mathbf{S}_0(\widetilde{\mathbf{u}}) + r_0 \Delta \widetilde{\rho} \mathbf{I}) = \widetilde{\mathbf{f}} & \text{in } \Omega \times \mathbf{R}_+, \\
\mathbf{n} \cdot \nabla \widetilde{\rho} = 0, \quad \widetilde{\mathbf{u}} = 0 & \text{on } \Gamma_D \times \mathbf{R}_+, \\
\mathbf{n} \cdot \nabla \widetilde{\rho} = 0, \quad (\mathbf{D}(\widetilde{\mathbf{u}})\mathbf{n})_\tau = 0, \quad \widetilde{\mathbf{u}} \cdot \mathbf{n} = 0 & \text{on } \Gamma_S \times \mathbf{R}_+, \\
(\widetilde{\rho}, \widetilde{\mathbf{u}})|_{t=0} = (0, 0) & \text{in } \Omega.
\end{cases}$$

In addition, $(\widetilde{\rho}, \widetilde{\mathbf{u}})$ satisfies

$$\begin{aligned}
&\|e^{-\delta_2 t} \partial_t \widetilde{\rho}\|_{L_p(\mathbf{R}_+,H^1_q(\Omega))} + \|e^{-\delta_2 t} \widetilde{\rho}\|_{L_p(\mathbf{R}_+,H^3_q(\Omega))} \\
&+ \|e^{-\delta_2 t} \partial_t \widetilde{\mathbf{u}}\|_{L_p(\mathbf{R}_+,L_q(\Omega)^N)} + \|e^{-\delta_2 t} \widetilde{\mathbf{u}}\|_{L_p(\mathbf{R}_+,H^2_q(\Omega)^N)} \\
&\leq C \Big(\|e^{-\delta_2 t} \widetilde{d}\|_{L_p(\mathbf{R}_+,H^1_q(\Omega))} + \|e^{-\delta_2 t} \widetilde{\mathbf{f}}\|_{L_p(\mathbf{R}_+,H^1_q(\Omega))} \Big).
\end{aligned}$$

Combining this inequality with

$$\|e^{-\delta_2 t} \widetilde{d}\|_{L_p(\mathbf{R}_+,H^1_q(\Omega))} \leq \|d\|_{L_p((0,T),H^1_q(\Omega))},$$
$$\|e^{-\delta_2 t} \widetilde{\mathbf{f}}\|_{L_p(\mathbf{R}_+,L_q(\Omega)^N)} \leq \|\mathbf{f}\|_{L_p((0,T),L_q(\Omega)^N)},$$

shows that

$$\begin{aligned}
&\|e^{-\delta_2 t}\partial_t\widetilde{\rho}\|_{L_p(\mathbf{R}_+,H^1_q(\Omega))} + \|e^{-\delta_2 t}\widetilde{\rho}\|_{L_p(\mathbf{R}_+,H^3_q(\Omega))} \\
&+ \|e^{-\delta_2 t}\partial_t\widetilde{\mathbf{u}}\|_{L_p(\mathbf{R}_+,L_q(\Omega)^N)} + \|e^{-\delta_2 t}\widetilde{\mathbf{u}}\|_{L_p(\mathbf{R}_+,H^2_q(\Omega)^N)} \\
&\leq C\Big(\|d\|_{L_p((0,T),H^1_q(\Omega))} + \|\mathbf{f}\|_{L_p((0,T),L_q(\Omega)^N)}\Big),
\end{aligned} \tag{34}$$

where C is a positive constant independent of T.

Let $(\rho, \mathbf{u})$ be the restriction of $(\widetilde{\rho}, \widetilde{\mathbf{u}})$ to $(0, T)$. Then, $(\rho, \mathbf{u})$ becomes a solution to (33). In addition, since

$$\begin{aligned}
&\|\partial_t\rho\|_{L_p((0,T),H^1_q(\Omega))} + \|\rho\|_{L_p((0,T),H^3_q(\Omega))} \\
&\leq e^{\delta_2 T}\Big(\|e^{-\delta_2 t}\partial_t\rho\|_{L_p((0,T),H^1_q(\Omega))} + \|e^{-\delta_2 t}\rho\|_{L_p((0,T),H^3_q(\Omega))}\Big) \\
&\leq e^{\delta_2 T_0}\Big(\|e^{-\delta_2 t}\partial_t\widetilde{\rho}\|_{L_p(\mathbf{R}_+,H^1_q(\Omega))} + \|e^{-\delta_2 t}\widetilde{\rho}\|_{L_p(\mathbf{R}_+,H^3_q(\Omega))}\Big),
\end{aligned}$$

(34) gives us

$$\begin{aligned}
&\|\partial_t\rho\|_{L_p((0,T),H^1_q(\Omega))} + \|\rho\|_{L_p((0,T),H^3_q(\Omega))} \\
&\leq Ce^{\delta_2 T_0}\Big(\|d\|_{L_p((0,T),H^1_q(\Omega))} + \|\mathbf{f}\|_{L_p((0,T),L_q(\Omega)^N)}\Big).
\end{aligned}$$

Analogously,

$$\begin{aligned}
&\|\partial_t\mathbf{u}\|_{L_p((0,T),L_q(\Omega)^N)} + \|\mathbf{u}\|_{L_p((0,T),H^2_q(\Omega)^N)} \\
&\leq Ce^{\delta_2 T_0}\Big(\|d\|_{L_p((0,T),H^1_q(\Omega))} + \|\mathbf{f}\|_{L_p((0,T),L_q(\Omega)^N)}\Big).
\end{aligned}$$

The last two estimates demonstrate that $(\rho, \mathbf{u})$ satisfies the desired estimate. The uniqueness of solutions can be proved in the same manner as in Proposition 5. This completes the proof of Corollary 2. □

6. Local Solvability of the Nonlinear Problem

This section proves our main result of this paper, i.e., Theorem 1. To this end, we first introduce several embedding properties. We next estimate nonlinear terms. Finally, we prove Theorem 1. Throughout this section, we assume that Ω is a uniform C^3 domain in $\mathbf{R}^N$ for $N \geq 2$.

6.1. Embedding Properties

Recall that $(\cdot,\cdot)_{\theta,p}$ is the real interpolation functor for $\theta \in (0,1)$ and $p \in (1,\infty)$. We then have the following lemma; see, e.g., [20] (Section 1.4).

Lemma 5. *Let $p \in (1,\infty)$. Let X, Y be Banach spaces so that Y is a dense subspace of X and Y is continuously embedded into X. Then,*

$$H^1_p(\mathbf{R}_+, X) \cap L_p(\mathbf{R}_+, Y) \subset C([0,\infty), (X,Y)_{1-1/p,p})$$

and

$$\sup_{t\in[0,\infty)} \|f(t)\|_{(X,Y)_{1-1/p,p}} \leq \Big(\|f\|^p_{H^1_p(\mathbf{R}_+,X)} + \|f\|^p_{L_p(\mathbf{R}_+,Y)}\Big)^{1/p}$$

for any $f \in H^1_p(\mathbf{R}_+, X) \cap L_p(\mathbf{R}_+, Y)$.

Let us recall

$$(H^1_q(\Omega), H^3_q(\Omega))_{1-1/p,p} = B^{3-2/p}_{q,p}(\Omega),$$
$$(L_q(\Omega), H^2_q(\Omega))_{1-1/p,p} = B^{2-2/p}_{q,p}(\Omega).$$

Lemma 5 gives us

Lemma 6. *Let $p, q \in (1, \infty)$. Then, the following assertions hold.*

(1) *There holds*

$$H^1_p(\mathbf{R}_+, H^1_q(\Omega)) \cap L_p(\mathbf{R}_+, H^3_q(\Omega)) \subset C([0,\infty), B^{3-2/p}_{q,p}(\Omega))$$

and

$$\sup_{t\in[0,\infty)} \|f(t)\|_{B^{3-2/p}_{q,p}(\Omega)} \leq C\Big(\|f\|_{H^1_p(\mathbf{R}_+,H^1_q(\Omega))} + \|f\|_{L_p(\mathbf{R}_+,H^3_q(\Omega))}\Big)$$

for any $f \in H^1_p(\mathbf{R}_+, H^1_q(\Omega)) \cap L_p(\mathbf{R}_+, H^3_q(\Omega))$ with a positive constant C.

(2) *There holds*

$$H^1_p(\mathbf{R}_+, L_q(\Omega)) \cap L_p(\mathbf{R}_+, H^2_q(\Omega)) \subset C([0,\infty), B^{2-2/p}_{q,p}(\Omega))$$

and

$$\sup_{t\in[0,\infty)} \|f(t)\|_{B^{2-2/p}_{q,p}(\Omega)} \leq C\Big(\|f\|_{H^1_p(\mathbf{R}_+,L_q(\Omega))} + \|f\|_{L_p(\mathbf{R}_+,H^2_q(\Omega))}\Big)$$

for any $f \in H^1_p(\mathbf{R}_+, L_q(\Omega)) \cap L_p(\mathbf{R}_+, H^2_q(\Omega))$ with a positive constant C.

We next prove.

Lemma 7. *Let $p \in (1, \infty)$ and $q \in (N, \infty)$. Suppose that $T > 0$ or $T = \infty$. Then $L_p((0, T), H^1_q(\Omega)) \subset L_p((0, T), L_\infty(\Omega))$ and*

$$\begin{aligned}&\|f\|_{L_p((0,T),L_\infty(\Omega))}\\ &\leq C\Big(\|\nabla f\|^{\frac{N}{q}}_{L_p((0,T),L_q(\Omega))}\|f\|^{1-\frac{N}{q}}_{L_p((0,T),L_q(\Omega))} + \|f\|_{L_p((0,T),L_q(\Omega))}\Big)\end{aligned}$$

for any $f \in L_p((0, T), H^1_q(\Omega))$, where C is a positive constant independent of T.

Proof. Since $q > N$, it holds that $H^1_q(\Omega) \subset L_\infty(\Omega)$ and

$$\|f\|_{L_\infty(\Omega)} \leq C\Big(\|\nabla f\|^{\frac{N}{q}}_{L_q(\Omega)}\|f\|^{1-\frac{N}{q}}_{L_q(\Omega)} + \|f\|_{L_q(\Omega)}\Big)$$

for any $f \in H^1_q(\Omega)$. This inequality shows that for $\theta = N/q \in (0, 1)$

$$\begin{aligned}\|f\|^p_{L_p((0,T),L_\infty(\Omega))} &= \int_0^T \|f(t)\|^p_{L_\infty(\Omega)}\,d\,t\\ &\leq C\Big(\int_0^T \|\nabla f(t)\|^{p\theta}_{L_q(\Omega)}\|f(t)\|^{p(1-\theta)}_{L_q(\Omega)}\,dt + \int_0^T \|f(t)\|^p_{L_q(\Omega)}\,dt\Big).\end{aligned}$$

On the other hand, Hölder's inequality gives us

$$\int_0^T \|\nabla f(t)\|_{L_q(\Omega)}^{p\theta} \|f(t)\|_{L_q(\Omega)}^{p(1-\theta)}\,d\,t \le \left(\int_0^T \|\nabla f(t)\|_{L_q(\Omega)}^p\right)^\theta \left(\int_0^T \|f(t)\|_{L_q(\Omega)}^p\,dt\right)^{1-\theta}.$$

Summing up the last two inequalities, we have

$$\|f\|_{L_p((0,T),L_\infty(\Omega))}^p \le C\Big(\|\nabla f\|_{L_p((0,T),L_q(\Omega))}^{p\theta} \|f\|_{L_p((0,T),L_q(\Omega))}^{p(1-\theta)} + \|f\|_{L_p((0,T),L_q(\Omega))}^p\Big).$$

This yields the desired inequality and completes the proof of Lemma 7. □

Let us next prove:

Lemma 8. *Suppose that $T > 0$ or $T = \infty$. Then, the following assertions hold.*

(1) *Let $p, q \in (1, \infty)$. Then, for any $f \in L_p((0,T), H_q^2(\Omega))$*

$$\|f\|_{L_p((0,T),H_q^1(\Omega))} \le C\|f\|_{L_p((0,T),H_q^2(\Omega))}^{\frac{1}{2}} \|f\|_{L_p((0,T),L_q(\Omega))}^{\frac{1}{2}},$$

where C is a positive constant independent of T.

(2) *Let $p \in (1, \infty)$ and $q \in (N, \infty)$. Then, for any $f \in L_p((0,T), H_q^2(\Omega))$*

$$\|f\|_{L_p((0,T),H_\infty^1(\Omega))} \le C\|f\|_{L_p((0,T),H_q^2(\Omega))}^{\frac{1}{2}} \times \Big(\|f\|_{L_p((0,T),H_q^2(\Omega))}^{\frac{N}{2q}} \|f\|_{L_p((0,T),L_q(\Omega))}^{\frac{1}{2}\left(1-\frac{N}{q}\right)} + \|f\|_{L_p((0,T),L_q(\Omega))}^{\frac{1}{2}}\Big),$$

where C is a positive constant independent of T.

Proof. (1) Let $[\cdot,\cdot]_\theta$ be the complex interpolation functor for $\theta \in (0,1)$; see, e.g., [27] (Definition 1.38). It follows from Remark 2 (d) of Subsection 2.4.2 in [21] that $[L_q(\Omega), H_q^2(\Omega)]_{1/2} = H_q^1(\Omega)$, which, combined with Theorem 1.9.3 (f) in [21], demonstrates that for any $g \in H_q^2(\Omega)$

$$\|g\|_{H_q^1(\Omega)} \le C\|g\|_{H_q^2(\Omega)}^{\frac{1}{2}} \|g\|_{L_q(\Omega)}^{\frac{1}{2}}.$$

Using this inequality, we observe that

$$\begin{aligned}\|f\|_{L_p((0,T),H_q^1(\Omega))}^p &= \int_0^T \|f(t)\|_{H_q^1(\Omega)}^p\,d\,t \\ &\le C\int_0^T \|f(t)\|_{H_q^2(\Omega)}^{\frac{p}{2}} \|f(t)\|_{L_q(\Omega)}^{\frac{p}{2}}\,d\,t \\ &\le C\left(\int_0^T \|f(t)\|_{H_q^2(\Omega)}^p\,dt\right)^{\frac{1}{2}} \left(\int_0^T \|f(t)\|_{L_q(\Omega)}^p\,dt\right)^{\frac{1}{2}}.\end{aligned}$$

This yields the desired inequality.

(2) Let us first consider $\partial_j f$ for $j = 1, \dots, N$. Lemma 7 gives us

$$
\begin{aligned}
&\|\partial_j f\|_{L_p((0,T),L_\infty(\Omega))} \\
&\leq C\Big(\|\nabla\partial_j f\|^{\frac{N}{q}}_{L_p((0,T),L_q(\Omega))}\|\partial_j f\|^{1-\frac{N}{q}}_{L_p((0,T),L_q(\Omega))} + \|\partial_j f\|_{L_p((0,T),L_q(\Omega))}\Big) \\
&\leq C\Big(\|f\|^{\frac{N}{q}}_{L_p((0,T),H^2_q(\Omega))}\|f\|^{1-\frac{N}{q}}_{L_p((0,T),H^1_q(\Omega))} + \|f\|_{L_p((0,T),H^1_q(\Omega))}\Big),
\end{aligned}
$$

which, combined with (1), demonstrates

$$
\begin{aligned}
&\|\partial_j f\|_{L_p((0,T),L_\infty(\Omega))} \\
&\leq C\|f\|^{\frac{1}{2}}_{L_p((0,T),H^2_q(\Omega))}\Big(\|f\|^{\frac{N}{2q}}_{L_p((0,T),H^2_q(\Omega))}\|f\|^{\frac{1}{2}\left(1-\frac{N}{q}\right)}_{L_p((0,T),L_q(\Omega))} + \|f\|^{\frac{1}{2}}_{L_p((0,T),L_q(\Omega))}\Big).
\end{aligned}
$$

This estimate holds with $\|\partial_j f\|_{L_p((0,T),L_\infty(\Omega))}$ replaced by $\|f\|_{L_p((0,T),L_\infty(\Omega))}$. The desired estimate thus holds. This completes the proof of Lemma 8 $\square$

From Lemma 8, we obtain

Lemma 9. *Let $p \in (1,\infty)$ and $q \in (N,\infty)$, and let $T \in (0,\infty)$. Then there exists a positive constant C, independent of T, such that the following assertions hold.*

(1) *For any $f \in {}_0H^1_p((0,T),L_q(\Omega)) \cap L_p((0,T),H^2_q(\Omega))$*

$$
\|f\|_{L_p((0,T),H^1_q(\Omega))} \leq CT^{\frac{1}{2}}\Big(\|f\|_{H^1_p((0,T),L_q(\Omega))} + \|f\|_{L_p((0,T),H^2_q(\Omega))}\Big).
$$

(2) *For any $f \in {}_0H^1_p((0,T),L_q(\Omega)) \cap L_p((0,T),H^2_q(\Omega))$*

$$
\begin{aligned}
\|f\|_{L_p((0,T),H^1_\infty(\Omega))} &\leq C\Big(T^{\frac{1}{2}\left(1-\frac{N}{q}\right)} + T^{\frac{1}{2}}\Big) \\
&\quad \times \Big(\|f\|_{H^1_p((0,T),L_q(\Omega))} + \|f\|_{L_p((0,T),H^2_q(\Omega))}\Big).
\end{aligned}
$$

(3) *For any $g \in {}_0H^1_p((0,T),H^1_q(\Omega)) \cap L_p((0,T),H^3_q(\Omega))$*

$$
\|g\|_{L_p((0,T),H^2_q(\Omega))} \leq CT^{\frac{1}{2}}\Big(\|g\|_{H^1_p((0,T),H^1_q(\Omega))} + \|g\|_{L_p((0,T),H^3_q(\Omega))}\Big).
$$

(4) *For any $g \in {}_0H^1_p((0,T),H^1_q(\Omega)) \cap L_p((0,T),H^3_q(\Omega))$*

$$
\begin{aligned}
\|g\|_{L_p((0,T),H^2_\infty(\Omega))} &\leq C\Big(T^{\frac{1}{2}\left(1-\frac{N}{q}\right)} + T^{\frac{1}{2}}\Big) \\
&\quad \times \Big(\|g\|_{H^1_p((0,T),H^1_q(\Omega))} + \|g\|_{L_p((0,T),H^3_q(\Omega))}\Big).
\end{aligned}
$$

Proof. (1) Let $f \in {}_0H^1_p((0,T),L_q(\Omega)) \cap L_p((0,T),H^2_q(\Omega))$. Since

$$
\|f\|_{L_p((0,T),L_q(\Omega))} \leq T^{1/p}\|f\|_{L_\infty((0,T),L_q(\Omega))},
$$

Lemma 8(1) shows that

$$
\|f\|_{L_p((0,T),H^1_q(\Omega))} \leq T^{\frac{1}{2p}}\|f\|^{\frac{1}{2}}_{L_p((0,T),H^2_q(\Omega))}\|f\|^{\frac{1}{2}}_{L_\infty((0,T),L_q(\Omega))}.
$$

On the other hand, $f|_{t=0} = 0$ gives us

$$
f(x,t) = \int_0^t \partial_s f(x,s)\,ds, \tag{35}
$$

which implies

$$\|f\|_{L_\infty((0,T),L_q(\Omega))} \leq \int_0^T \|\partial_s f(s)\|_{L_q(\Omega)}\,ds \leq T^{1/p'}\|f\|_{H^1_p((0,T),L_q(\Omega))} \tag{36}$$

for $p' = p/(p-1)$. Combining this with the last estimate of $\|f\|_{L_p((0,T),H^1_q(\Omega))}$ yields the desired inequality.

(2) The desired inequality follows from Lemma 8(2) in the same manner as in the proof of (1), so that the detailed proof may be omitted.

(3), (4) Let $j = 1, \dots, N$. Since $\partial_j g, g \in {}_0H^1_p((0,T), L_q(\Omega)) \cap L_p((0,T), H^2_q(\Omega))$, the desired inequalities of (3) and (4) immediately follow from (1) and (2), respectively. This completes the proof of Lemma 9. □

Recall $K_{p,q;T} = K^1_{p,q;T} \times K^2_{p,q;T}$ and ${}_0K_{p,q;T} = {}_0K^1_{p,q;T} \times {}_0K^2_{p,q;T}$ given by Subsection 3.1. We then have

Proposition 6. *Let $p \in (1,\infty)$, $q \in (N,\infty)$, and $T \in (0,\infty)$. Then, there exists a positive constant C, independent of T, such that the following assertions hold.*

(1) $\|\rho\|_{L_p((0,T),H^2_q(\Omega))} \leq CT^{1/2}\|\rho\|_{K^1_{p,q;T}}$ *for any* $\rho \in {}_0K^1_{p,q;T}$.

(2) $\|\rho\|_{L_p((0,T),H^2_\infty(\Omega))} \leq C(T^{(1-N/q)/2} + T^{1/2})\|\rho\|_{K^1_{p,q;T}}$ *for any* $\rho \in {}_0K^1_{p,q;T}$.

(3) $\|\rho\|_{L_\infty((0,T),H^1_q(\Omega))} \leq CT^{1-1/p}\|\rho\|_{K^1_{p,q;T}}$ *for any* $\rho \in {}_0K^1_{p,q;T}$.

(4) $\|\rho\|_{L_\infty((0,T),L_\infty(\Omega))} \leq CT^{1-1/p}\|\rho\|_{K^1_{p,q;T}}$ *for any* $\rho \in {}_0K^1_{p,q;T}$.

(5) $\|\mathbf{u}\|_{L_p((0,T),H^1_q(\Omega)^N)} \leq CT^{1/2}\|\mathbf{u}\|_{K^2_{p,q;T}}$ *for any* $\mathbf{u} \in {}_0K^2_{p,q;T}$.

(6) $\|\mathbf{u}\|_{L_p((0,T),H^1_\infty(\Omega)^N)} \leq C(T^{(1-N/q)/2} + T^{1/2})\|\mathbf{u}\|_{K^2_{p,q;T}}$ *for any* $\mathbf{u} \in {}_0K^2_{p,q;T}$.

(7) $\|\mathbf{u}\|_{L_\infty((0,T),L_q(\Omega)^N)} \leq CT^{1-1/p}\|\mathbf{u}\|_{K^2_{p,q;T}}$ *for any* $\mathbf{u} \in {}_0K^2_{p,q;T}$.

Proof. The desired inequalities of (1), (2), (5), and (6) follow from Lemma 9 immediately. The proofs of (3) and (7) are similar to (35) and (36), so that the detailed proof may be omitted. Since $H^1_q(\Omega)$ is continuously embedded into $L_\infty(\Omega)$ by the assumption $q > N$, the desired inequality of (4) follows from (3). This completes the proof of Proposition 6. □

Let $T_0 \in (0,\infty)$ and $T \in (0, T_0]$. Since

$$T^{1/2} \leq T_0^{1/2}, \quad T^{(1-N/q)/2} \leq T_0^{(1-N/q)/2}, \quad T^{1-1/p} \leq T_0^{1-1/p}$$

for $p \in (1,\infty)$ and $q \in (N,\infty)$, the next proposition follows from Proposition 6 immediately.

Proposition 7. *Let $p \in (1,\infty)$ and $q \in (N,\infty)$. Let $T_0 \in (0,\infty)$ and $T \in (0, T_0]$. Then, there exists a positive constant C_{T_0} depending on T_0, but independent of T, such that the following assertions hold.*

(1) $\|\rho\|_{L_p((0,T),H^2_\infty(\Omega))} \leq C_{T_0}\|\rho\|_{K^1_{p,q;T}}$ *for any* $\rho \in {}_0K^1_{p,q;T}$.

(2) $\|\rho\|_{L_\infty((0,T),H^1_q(\Omega))} \leq C_{T_0}\|\rho\|_{K^1_{p,q;T}}$ *for any* $\rho \in {}_0K^1_{p,q;T}$.

(3) $\|\rho\|_{L_\infty((0,T),L_\infty(\Omega))} \leq C_{T_0}\|\rho\|_{K^1_{p,q;T}}$ *for any* $\rho \in {}_0K^1_{p,q;T}$.

(4) $\|\mathbf{u}\|_{L_p((0,T),H^1_\infty(\Omega)^N)} \leq C_{T_0}\|\mathbf{u}\|_{K^2_{p,q;T}}$ *for any* $\mathbf{u} \in {}_0K^2_{p,q;T}$.

(5) $\|\mathbf{u}\|_{L_\infty((0,T),L_q(\Omega)^N)} \leq C_{T_0}\|\mathbf{u}\|_{K^2_{p,q;T}}$ *for any* $\mathbf{u} \in {}_0K^2_{p,q;T}$.

We finally introduce embedding properties for the solution $(\widehat{\rho}, \widehat{\mathbf{u}})$ of (8).

Proposition 8. *Let $p \in (1, \infty)$ and $q \in (N, \infty)$, and let R, R_1, R_2, and ρ_∞ be positive constants with $R_1 \le R_2$. Let $T_0 \in (0, \infty)$ and $T \in (0, T_0]$. Suppose that $(\rho_0, \mathbf{u}_0) \in D_{q,p}(\Omega)$ and that $r_0 = \rho_0 + \rho_\infty$, μ_0, ν_0, and κ_0 satisfy (b), (e), and (f) of Theorem 1. Then, there exists a positive constant C_{T_0} depending on N, p, q, R, R_1, R_2, T_0, and ρ_∞, but independent of T, such that the following assertions hold.*

(1) $\|(\widehat{\rho}, \widehat{\mathbf{u}})\|_{K_{p,q;T}} \le C_{T_0} \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}$.

(2) $\|(\widehat{\rho}, \widehat{\mathbf{u}})\|_{L_\infty((0,T), B^{3-2/p}_{q,p}(\Omega) \times B^{2-2/p}_{q,p}(\Omega)^N)} \le C_{T_0} \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}$.

(3) $\|(\widehat{\rho}, \widehat{\mathbf{u}})\|_{L_\infty((0,T), H^1_q(\Omega) \times L_q(\Omega)^N)} \le C_{T_0} \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}$.

(4) $\|(\widehat{\rho}, \widehat{\mathbf{u}})\|_{L_p((0,T), H^2_\infty(\Omega) \times H^1_\infty(\Omega)^N)} \le C_{T_0} \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}$.

(5) $\|\widehat{\rho}\|_{L_\infty((0,T), L_\infty(\Omega))} \le C_{T_0} \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}$.

(6) $\|\widehat{\rho} - \rho_0\|_{L_\infty((0,T), H^1_q(\Omega))} \le C_{T_0} T^{1-1/p} \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}$.

(7) $\|\widehat{\rho} - \rho_0\|_{L_\infty((0,T), H^1_q(\Omega))} \le C_{T_0} \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}$.

(8) $\|\widehat{\rho} - \rho_0\|_{L_\infty((0,T), L_\infty(\Omega))} \le C_{T_0} T^{1-1/p} \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}$.

(9) $\|\widehat{\rho} - \rho_0\|_{L_\infty((0,T), L_\infty(\Omega))} \le C_{T_0} \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}$.

Proof. (1), (2) The desired inequalities follow from Corollary 1 and Lemma 6 in the same manner as in the proof of Corollary 2.

(3) Since $B^{3-2/p}_{q,p}(\Omega)$ and $B^{2-2/p}_{q,p}(\Omega)$ are continuously embedded into $H^1_q(\Omega)$ and $L_q(\Omega)$, respectively, the desired inequality follows from (2).

(4), (5) By the assumption $q > N$, we observe for $m \in \mathbf{N}$ that $H^m_q(\Omega)$ is continuously embedded into $H^{m-1}_\infty(\Omega)$. Thus, the desired inequality of (4) follows from (1), while one of (5) follows from (3).

(6) Since

$$\widehat{\rho}(x,t) - \rho_0(x) = \int_0^t \partial_s \widehat{\rho}(x,s)\, ds,$$

we observe by Hölder's inequality that

$$\begin{aligned} \|\widehat{\rho} - \rho_0\|_{L_\infty((0,T), H^1_q(\Omega))} &\le \int_0^T \|\partial_s \widehat{\rho}(s)\|_{H^1_q(\Omega)}\, d\, s \\ &\le T^{1-1/p} \|\partial_t \widehat{\rho}\|_{L_p((0,T), H^1_q(\Omega))}. \end{aligned}$$

Combining this inequality with (1) demonstrates the desired inequality.

(7) Since $T^{1-1/p} \le T_0^{1-1/p}$, the desired inequality follows from (6) immediately.

(8) and (9) Since $H^1_q(\Omega)$ is continuously embedded into $L_\infty(\Omega)$ as mentioned above, the desired inequalities of (8) and (9) follow from (6) and (7), respectively. This completes the proof of Proposition 8. □

6.2. Estimates of Nonlinear Terms

This subsection estimates the nonlinear terms $\mathsf{D}(\rho, \mathbf{u})$ and $\mathsf{F}(\rho, \mathbf{u})$ given by Section 3.2. Throughout this subsection, we assume

Assumption 2. *Let $p \in (1, \infty)$, $q \in (N, \infty)$, and $T_0 \in (0, \infty)$. Let R, R_1, R_2, and ρ_∞ be positive constants with $R_1 \le R_2$. Suppose that $(\rho_0, \mathbf{u}_0) \in D_{q,p}(\Omega)$ and that (b), (e), and (f) of Theorem 1 hold.*

Let us define for $T \in (0, T_0]$

$$\varphi(T) = T^{\frac{1}{2}\left(1-\frac{N}{q}\right)} + T^{\frac{1}{2}} + T^{1-\frac{1}{p}},$$
$$\psi(T) = \sup_{t\in[0,T]} \Big(\|\mu(t)-\mu_0\|_{H^1_\infty(\Omega)} + \|\nu(t)-\nu_0\|_{H^1_\infty(\Omega)} + \|\kappa(t)-\kappa_0\|_{H^1_\infty(\Omega)} \Big). \tag{37}$$

We then observe that
$$\lim_{T\searrow 0} \varphi(T) = 0, \quad \lim_{T\searrow 0} \psi(T) = 0, \tag{38}$$
where we note Remark 1.

Let us decompose the nonlinear terms into the lower order terms and the the highest order terms. Define
$$\mathsf{D}_1(\rho,\mathbf{u}) = \mathbf{u}\cdot\nabla\rho, \quad \mathsf{D}_2(\rho,\mathbf{u}) = (\rho-\rho_0)\operatorname{div}\mathbf{u}.$$
One then sees that
$$\mathsf{D}(\rho,\mathbf{u}) = -\mathsf{D}_1(\rho,\mathbf{u}) - \mathsf{D}_2(\rho,\mathbf{u}),$$
where $\mathsf{D}_1(\rho,\mathbf{u})$ and $\mathsf{D}_2(\rho,\mathbf{u})$ are corresponding to the lower order and the highest order, respectively.

We next consider $\mathsf{F}(\rho,\mathbf{u})$. Let us observe that
$$\begin{aligned}\operatorname{Div}(\mathbf{S}(\mathbf{u})-\mathbf{S}_0(\mathbf{u})) &= (\mu-\mu_0)\Delta\mathbf{u} + (\nu-\nu_0)\nabla\operatorname{div}\mathbf{u} + \mathbf{D}(\mathbf{u})\nabla(\mu-\mu_0) \\ &\quad + (\operatorname{div}\mathbf{u})\nabla((\nu-\nu_0)-(\mu-\mu_0))\end{aligned}$$
and that
$$\begin{aligned}&\operatorname{Div}\Big((\kappa-\kappa_0)(\rho+\rho_\infty)\Delta\rho\mathbf{I} + \kappa_0(\rho-\rho_0)\Delta\rho\mathbf{I} + \kappa\frac{|\nabla\rho|^2}{2}\mathbf{I} - \kappa\nabla\rho\otimes\nabla\rho\Big) \\ &= (\nabla(\kappa-\kappa_0))(\rho+\rho_\infty)\Delta\rho + (\kappa-\kappa_0)(\nabla\rho)\Delta\rho + (\kappa-\kappa_0)(\rho+\rho_\infty)\nabla\Delta\rho \\ &+ (\nabla\kappa_0)(\rho-\rho_0)\Delta\rho + \kappa_0(\nabla(\rho-\rho_0))\Delta\rho + \kappa_0(\rho-\rho_0)\nabla\Delta\rho \\ &+ (\nabla\kappa)\frac{|\nabla\rho|^2}{2} - (\nabla\rho\otimes\nabla\rho)\nabla\kappa - \kappa(\nabla\rho)\Delta\rho.\end{aligned}$$
Define
$$\begin{aligned}\mathsf{F}_1(\rho,\mathbf{u}) &= (\rho+\rho_\infty)\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{D}(\mathbf{u})\nabla(\mu-\mu_0) + (\operatorname{div}\mathbf{u})\nabla((\nu-\nu_0)-(\mu-\mu_0)) \\ &\quad + (\nabla(\kappa-\kappa_0))(\rho+\rho_\infty)\Delta\rho + (\kappa-\kappa_0)(\nabla\rho)\Delta\rho \\ &\quad + (\nabla\kappa_0)(\rho-\rho_0)\Delta\rho + \kappa_0(\nabla(\rho-\rho_0))\Delta\rho \\ &\quad + (\nabla\kappa)\frac{|\nabla\rho|^2}{2} - (\nabla\rho\otimes\nabla\rho)\nabla\kappa - \kappa(\nabla\rho)\Delta\rho \\ &\quad + (\rho+\rho_\infty)\mathbf{b} + (\kappa_0^{-1}(\nu_0-\mu_0)\operatorname{div}\mathbf{u}\mathbf{I} + r_0\Delta\rho\mathbf{I})\nabla\kappa_0\end{aligned}$$
and
$$\begin{aligned}\mathsf{F}_2(\rho,\mathbf{u}) &= -(\rho-\rho_0)\partial_t\mathbf{u} + (\mu-\mu_0)\Delta\mathbf{u} + (\nu-\nu_0)\nabla\operatorname{div}\mathbf{u} \\ &\quad + (\kappa-\kappa_0)(\rho+\rho_\infty)\nabla\Delta\rho + \kappa_0(\rho-\rho_0)\nabla\Delta\rho.\end{aligned}$$
These give us
$$\mathsf{F}(\rho,\mathbf{u}) = r_0^{-1}\mathsf{F}_1(\rho,\mathbf{u}) + r_0^{-1}\mathsf{F}_2(\rho,\mathbf{u}) - r_0^{-1}P'(\rho+\rho_\infty)\nabla\rho,$$
where $\mathsf{F}_1(\rho,\mathbf{u})$ and $\mathsf{F}_2(\rho,\mathbf{u})$ are corresponding to the lower order and the highest order, respectively.

Let us now estimate $\mathsf{D}_1(\rho,\mathbf{u})$.

Lemma 10. *Suppose that Assumption 2 holds. Then, there exists a positive constant* $C = C(N, p, q, R, R_1, R_2, T_0, \rho_\infty)$*, such that for any* $T \in (0, T_0]$ *and* $(\rho^i, \mathbf{u}^i) \in {}_0K_{p,q;T}$*,* $i = 1, 2$*,*

$$
\begin{aligned}
&\|\mathsf{D}_1(\rho^1 + \widehat{\rho}, \mathbf{u}^1 + \widehat{\mathbf{u}}) - \mathsf{D}_1(\rho^2 + \widehat{\rho}, \mathbf{u}^2 + \widehat{\mathbf{u}})\|_{L_p((0,T),H^1_q(\Omega))} \\
&\le C\varphi(T)\|(\rho^2 - \rho^1, \mathbf{u}^2 - \mathbf{u}^1)\|_{K_{p,q;T}} \\
&\times \Big(\|(\rho^1, \mathbf{u}^1)\|_{K_{p,q;T}} + \|(\rho^2, \mathbf{u}^2)\|_{K_{p,q;T}} + \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big).
\end{aligned}
$$

Proof. Let us write

$$
\begin{aligned}
&\mathsf{D}_1(\rho^1 + \widehat{\rho}, \mathbf{u}^1 + \widehat{\mathbf{u}}) - \mathsf{D}_1(\rho^2 + \widehat{\rho}, \mathbf{u}^2 + \widehat{\mathbf{u}}) \\
&= (\mathbf{u}^1 - \mathbf{u}^2) \cdot \nabla(\rho^1 + \widehat{\rho}) + (\mathbf{u}^2 + \widehat{\mathbf{u}}) \cdot \nabla(\rho^1 - \rho^2) \\
&=: I_1 + I_2.
\end{aligned}
$$

For $k = 1, \dots, N$

$$
\partial_k I_1 = \partial_k(\mathbf{u}^1 - \mathbf{u}^2) \cdot \nabla(\rho^1 + \widehat{\rho}) + (\mathbf{u}^1 - \mathbf{u}^2) \cdot \partial_k \nabla(\rho^1 + \widehat{\rho}),
$$

which gives us

$$
\begin{aligned}
&\|\partial_k I_1\|_{L_p((0,T),L_q(\Omega))} \\
&\le \|\partial_k(\mathbf{u}^1 - \mathbf{u}^2)\|_{L_p((0,T),L_\infty(\Omega)^N)}\|\nabla(\rho^1 + \widehat{\rho})\|_{L_\infty((0,T),L_q(\Omega)^N)} \\
&+ \|\mathbf{u}^1 - \mathbf{u}^2\|_{L_\infty((0,T),L_q(\Omega)^N)}\|\partial_k\nabla(\rho^1 + \widehat{\rho})\|_{L_p((0,T),L_\infty(\Omega)^N)} \\
&\le \|\mathbf{u}^1 - \mathbf{u}^2\|_{L_p((0,T),H^1_\infty(\Omega)^N)}\|\rho^1 + \widehat{\rho}\|_{L_\infty((0,T),H^1_q(\Omega))} \\
&+ \|\mathbf{u}^1 - \mathbf{u}^2\|_{L_\infty((0,T),L_q(\Omega)^N)}\|\rho^1 + \widehat{\rho}\|_{L_p((0,T),H^2_\infty(\Omega))}.
\end{aligned}
$$

By Propositions 6–8, we observe that

$$
\begin{aligned}
&\|\partial_k I_1\|_{L_p((0,T),L_q(\Omega))} \\
&\le C\varphi(T)\|\mathbf{u}^1 - \mathbf{u}^2\|_{K^2_{p,q;T}}\Big(\|\rho^1\|_{K^1_{p,q;T}} + \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big).
\end{aligned}
$$

Analogously, we obtain from Propositions 6–8

$$
\begin{aligned}
&\|I_1\|_{L_p((0,T),L_q(\Omega))} \\
&\le C\varphi(T)\|\mathbf{u}^1 - \mathbf{u}^2\|_{K^2_{p,q;T}}\Big(\|\rho^1\|_{K^1_{p,q;T}} + \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big), \\
&\|I_2\|_{L_p((0,T),L_q(\Omega))} \\
&\le C\varphi(T)\|\rho^1 - \rho^2\|_{K^1_{p,q;T}}\Big(\|\mathbf{u}^2\|_{K^2_{p,q;T}} + \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big), \\
&\|\partial_k I_2\|_{L_p((0,T),L_q(\Omega))} \\
&\le C\varphi(T)\|\rho^1 - \rho^2\|_{K^1_{p,q;T}}\Big(\|\mathbf{u}^2\|_{K^2_{p,q;T}} + \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big).
\end{aligned}
$$

Summing up the above estimates of I_1, I_2, $\partial_k I_1$, and $\partial_k I_2$, we complete the proof of Lemma 10. □

In the same manner as in Lemma 10, we have

Lemma 11. *Suppose that Assumption 2 holds. Then, for any $T \in (0, T_0]$*

$$\|\mathsf{D}_1(\widehat{\rho}, \widehat{\mathbf{u}})\|_{L_p((0,T),H^1_q(\Omega))} \le C\|(\rho_0, \mathbf{u}_0)\|^2_{D_{q,p}(\Omega)},$$

where $C = C(N, p, q, R, R_1, R_2, T_0, \rho_\infty)$ is a positive constant independent of T.

We next consider $\mathsf{F}_1(\rho, \mathbf{u})$.

Lemma 12. *Suppose that Assumption 2 holds and $\mathbf{b} \in L_p((0, T_0), L_q(\Omega)^N)$. Then, there exists a positive constant $C = C(N, p, q, R, R_1, R_2, T_0, \rho_\infty)$, such that for any $T \in (0, T_0]$ and $(\rho^i, \mathbf{u}^i) \in {}_0K_{p,q;T}$, $i = 1, 2$,*

$$\begin{aligned} &\|\mathsf{F}_1(\rho^1 + \widehat{\rho}, \mathbf{u}^1 + \widehat{\mathbf{u}}) - \mathsf{F}_1(\rho^2 + \widehat{\rho}, \mathbf{u}^2 + \widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)} \\ &\le C\varphi(T)\|(\rho^2 - \rho^1, \mathbf{u}^2 - \mathbf{u}^1)\|_{K_{p,q;T}} \Big\{ \|\mathbf{b}\|_{L_p((0,T_0),L_q(\Omega)^N)} \\ &+ \sum_{j=0}^{2} \Big(\|(\rho^1, \mathbf{u}^1)\|_{K_{p,q;T}} + \|(\rho^2, \mathbf{u}^2)\|_{K_{p,q;T}} + \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)} \Big)^j \Big\}. \end{aligned}$$

Proof. Notice that the quadratic terms of $\mathsf{F}_1(\rho, \mathbf{u})$:

$$\begin{aligned} &\mathbf{u} \cdot \nabla \mathbf{u}, && (\nabla(\kappa - \kappa_0))\rho\Delta\rho, && (\kappa - \kappa_0)(\nabla\rho)\Delta\rho, \\ &(\nabla\kappa)\frac{|\nabla\rho|^2}{2}, && (\nabla\rho \otimes \nabla\rho)\nabla\kappa, && \kappa(\nabla\rho)\Delta\rho \end{aligned}$$

can be treated in the same manner as in the proof of Lemma 10. In this proof, we focus on estimating the following terms:

$$\begin{aligned} &\mathcal{F}_1(\mathbf{u}) = \mathbf{D}(\mathbf{u})\nabla(\mu - \mu_0), && \mathcal{F}_2(\mathbf{u}) = (\operatorname{div} \mathbf{u})\nabla((\nu - \nu_0) - (\mu - \mu_0)), \\ &\mathcal{F}_3(\rho) = (\nabla(\kappa - \kappa_0))\Delta\rho, && \mathcal{F}_4(\rho) = (\nabla\kappa_0)(\rho - \rho_0)\Delta\rho, \\ &\mathcal{F}_5(\rho) = \kappa_0(\nabla(\rho - \rho_0))\Delta\rho, && \mathcal{F}_6(\rho) = (\rho + \rho_\infty)\mathbf{b}, \end{aligned}$$

and also

$$\mathcal{F}_7(\rho, \mathbf{u}) = (\kappa_0^{-1}(\nu_0 - \mu_0) \operatorname{div} \mathbf{u}\mathbf{I} + r_0\Delta\rho\mathbf{I})\nabla\kappa_0, \quad \mathcal{F}_8(\rho, \mathbf{u}) = \rho\mathbf{u} \cdot \nabla\mathbf{u}.$$

Let us first consider $\mathcal{F}_1(\mathbf{u})$. It can be written as

$$\mathcal{F}_1(\mathbf{u}^1 + \widehat{\mathbf{u}}) - \mathcal{F}_1(\mathbf{u}^2 + \widehat{\mathbf{u}}) = \mathbf{D}(\mathbf{u}^1 - \mathbf{u}^2)\nabla(\mu - \mu_0).$$

It thus follows from Proposition 6 that

$$\begin{aligned} &\|\mathcal{F}_1(\mathbf{u}^1 + \widehat{\mathbf{u}}) - \mathcal{F}_1(\mathbf{u}^2 + \widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)} \\ &\le \Big(\|\mu\|_{L_\infty((0,T_0),H^1_\infty(\Omega))} + \|\mu_0\|_{H^1_\infty(\Omega)} \Big) \|\mathbf{u}^1 - \mathbf{u}^2\|_{L_p((0,T),H^1_q(\Omega))} \\ &\le C\varphi(T)\|\mathbf{u}^1 - \mathbf{u}^2\|_{K^2_{p,q;T}}. \end{aligned}$$

Analogously,

$$\begin{aligned} \|\mathcal{F}_2(\mathbf{u}^1 + \widehat{\mathbf{u}}) - \mathcal{F}_2(\mathbf{u}^2 + \widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)} &\le C\varphi(T)\|\mathbf{u}^1 - \mathbf{u}^2\|_{K^2_{p,q;T}}, \\ \|\mathcal{F}_3(\rho^1 + \widehat{\rho}) - \mathcal{F}_4(\rho^2 + \widehat{\rho})\|_{L_p((0,T),L_q(\Omega)^N)} &\le C\varphi(T)\|\rho^1 - \rho^2\|_{K^1_{p,q;T}}, \end{aligned}$$

and

$$\|\mathcal{F}_7(\rho^1+\widehat{\rho},\mathbf{u}^1+\widehat{\mathbf{u}})-\mathcal{F}_7(\rho^2+\widehat{\rho},\mathbf{u}^2+\widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)}$$
$$\le C\varphi(T)\|(\rho^1-\rho^2,\mathbf{u}^1-\mathbf{u}^2)\|_{K_{p,q;T}}.$$

We next consider $\mathcal{F}_4(\rho)$. It can be written as

$$\begin{aligned}&\mathcal{F}_4(\rho^1+\widehat{\rho})-\mathcal{F}_4(\rho^2+\widehat{\rho})\\&=(\nabla\kappa_0)(\rho^1-\rho^2)\Delta(\rho^1+\widehat{\rho})+(\nabla\kappa_0)(\rho^2+\widehat{\rho}-\rho_0)\Delta(\rho^1-\rho^2)\\&=:I_1+I_2.\end{aligned}$$

Propositions 6–8 show that

$$\begin{aligned}&\|I_1\|_{L_p((0,T),L_q(\Omega)^N)}\\&\le\|\nabla\kappa_0\|_{L_\infty(\Omega)}\|\rho^1-\rho^2\|_{L_\infty((0,T),L_q(\Omega))}\|\Delta(\rho^1+\widehat{\rho})\|_{L_p((0,T),L_\infty(\Omega))}\\&\le C\varphi(T)\|\rho^1-\rho^2\|_{K^1_{p,q;T}}\Big(\|\rho^1\|_{K^1_{p,q;T}}+\|(\rho_0,\mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big),\\&\|I_2\|_{L_p((0,T),L_q(\Omega)^N)}\\&\le\|\nabla\kappa_0\|_{L_\infty(\Omega)}\Big(\|\rho^2\|_{L_\infty((0,T),L_q(\Omega))}+\|\widehat{\rho}-\rho_0\|_{L_\infty((0,T),L_q(\Omega))}\Big)\\&\times\|\Delta(\rho^1-\rho^2)\|_{L_p((0,T),L_\infty(\Omega))}\\&\le C\varphi(T)\|\rho^1-\rho^2\|_{K^1_{p,q;T}}\Big(\|\rho^2\|_{K^1_{p,q;T}}+\|(\rho_0,\mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big).\end{aligned}$$

Thus

$$\begin{aligned}&\|\mathcal{F}_4(\rho^1+\widehat{\rho})-\mathcal{F}_4(\rho^2+\widehat{\rho})\|_{L_p((0,T),L_q(\Omega)^N)}\\&\le C\varphi(T)\|\rho^1-\rho^2\|_{K^1_{p,q;T}}\Big(\|\rho^1\|_{K^1_{p,q;T}}+\|\rho^2\|_{K^1_{p,q;T}}+\|(\rho_0,\mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big).\end{aligned}$$

Analogously,

$$\begin{aligned}&\|\mathcal{F}_5(\rho^1+\widehat{\rho})-\mathcal{F}_5(\rho^2+\widehat{\rho})\|_{L_p((0,T),L_q(\Omega)^N)}\\&\le C\varphi(T)\|\rho^1-\rho^2\|_{K^1_{p,q;T}}\Big(\|\rho^1\|_{K^1_{p,q;T}}+\|\rho^2\|_{K^1_{p,q;T}}+\|(\rho_0,\mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big).\end{aligned}$$

We next consider $\mathcal{F}_6(\rho)$. Since

$$\mathcal{F}_6(\rho^1+\widehat{\rho})-\mathcal{F}_6(\rho^2+\widehat{\rho})=(\rho^1-\rho^2)\mathbf{b},$$

it follows from Proposition 6 that

$$\begin{aligned}&\|\mathcal{F}_6(\rho^1+\widehat{\rho})-\mathcal{F}_6(\rho^2+\widehat{\rho})\|_{L_p((0,T),L_q(\Omega)^N)}\\&\le\|\rho^1-\rho^2\|_{L_\infty((0,T),L_\infty(\Omega))}\|\mathbf{b}\|_{L_p((0,T_0),L_q(\Omega)^N)}\\&\le C\varphi(T)\|\rho^1-\rho^2\|_{K^1_{p,q;T}}\|\mathbf{b}\|_{L_p((0,T_0),L_q(\Omega)^N)}.\end{aligned}$$

We finally consider $\mathcal{F}_8(\rho,\mathbf{u})$. It can be written as

$$\begin{aligned}&\mathcal{F}_8(\rho^1+\widehat{\rho},\mathbf{u}^1+\widehat{\mathbf{u}})-\mathcal{F}_8(\rho^2+\widehat{\rho},\mathbf{u}^2+\widehat{\mathbf{u}})\\&=(\rho^1-\rho^2)(\mathbf{u}^1+\widehat{\mathbf{u}})\cdot\nabla(\mathbf{u}^1+\widehat{\mathbf{u}})+(\rho^2+\widehat{\rho})(\mathbf{u}^1-\mathbf{u}^2)\cdot\nabla(\mathbf{u}^1+\widehat{\mathbf{u}})\\&+(\rho^2+\widehat{\rho})(\mathbf{u}^2+\widehat{\mathbf{u}})\cdot\nabla(\mathbf{u}^1-\mathbf{u}^2)\\&=:I_3+I_4+I_5.\end{aligned}$$

By Propositions 6–8, we observe that

$$\begin{aligned}\|I_3\|_{L_p((0,T),L_q(\Omega)^N)} &\leq \|\rho^1-\rho^2\|_{L_\infty((0,T),L_\infty(\Omega))}\|\mathbf{u}^1+\widehat{\mathbf{u}}\|_{L_\infty((0,T),L_q(\Omega)^N)} \\ &\quad\times\|\nabla(\mathbf{u}^1+\widehat{\mathbf{u}})\|_{L_p((0,T),L_\infty(\Omega)^{N\times N})} \\ &\leq C\varphi(T)\|\rho^1-\rho^2\|_{K^1_{p,q;T}}\Big(\|\mathbf{u}^1\|_{K^2_{p,q;T}}+\|(\rho_0,\mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big)^2.\end{aligned}$$

Analogously,

$$\begin{aligned}&\|I_4\|_{L_p((0,T),L_q(\Omega)^N)} \\ &\leq C\varphi(T)\|\mathbf{u}^1-\mathbf{u}^2\|_{K^2_{p,q;T}}\Big(\|\rho^2\|_{K^1_{p,q;T}}+\|(\rho_0,\mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big) \\ &\times\Big(\|\mathbf{u}^1\|_{K^2_{p,q;T}}+\|(\rho_0,\mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big), \\ &\|I_5\|_{L_p((0,T),L_q(\Omega)^N)} \\ &\leq C\varphi(T)\|\mathbf{u}^1-\mathbf{u}^2\|_{K^2_{p,q;T}}\Big(\|\rho^2\|_{K^1_{p,q;T}}+\|(\rho_0,\mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big) \\ &\times\Big(\|\mathbf{u}^2\|_{K^2_{p,q;T}}+\|(\rho_0,\mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big).\end{aligned}$$

It thus holds that

$$\begin{aligned}&\|\mathcal{F}_8(\rho^1+\widehat{\rho},\mathbf{u}^1+\widehat{\mathbf{u}})-\mathcal{F}_8(\rho^2+\widehat{\rho},\mathbf{u}^2+\widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)} \\ &\leq C\varphi(T)\|(\rho^1-\rho^2,\mathbf{u}^1-\mathbf{u}^2)\|_{K_{p,q;T}} \\ &\times\Big(\|(\rho^1,\mathbf{u}^1)\|_{K_{p,q;T}}+\|(\rho^2,\mathbf{u}^2)\|_{K_{p,q;T}}+\|(\rho_0,\mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big)^2.\end{aligned}$$

Summing up the above estimates of the quadratic terms and $\mathcal{F}_1,\ldots,\mathcal{F}_8$, we have obtained the desired inequality. This completes the proof of Lemma 12. □

In the same manner as in Lemma 12, we have

Lemma 13. *Suppose that Assumption 2 holds and* $\mathbf{b}\in L_p((0,T_0),L_q(\Omega)^N)$*. Then, for any* $T\in(0,T_0]$

$$\begin{aligned}&\|\mathsf{F}_1(\widehat{\rho},\widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)} \\ &\leq C\Big\{\Big(\|(\rho_0,\mathbf{u}_0)\|_{D_{q,p}(\Omega)}+1\Big)\|\mathbf{b}\|_{L_p((0,T_0),L_q(\Omega)^N)}+\sum_{j=1}^{3}\|(\rho_0,\mathbf{u}_0)\|^j_{D_{q,p}(\Omega)}\Big\},\end{aligned}$$

where $C=C(N,p,q,R,R_1,R_2,T_0,\rho_\infty)$ *is a positive constant independent of* T*.*

We next consider the highest order terms $\mathsf{D}_2(\rho,\mathbf{u})$ and $\mathsf{F}_2(\rho,\mathbf{u})$.

Lemma 14. *Suppose that Assumption 2 holds. Then, there exists a positive constant* $C=C(N,p,q,R,R_1,R_2,T_0,\rho_\infty)$*, such that for any* $T\in(0,T_0]$ *and* $(\rho^i,\mathbf{u}^i)\in{}_0K_{p,q;T}$*,* $i=1,2$*,*

$$\begin{aligned}&\|\mathsf{D}_2(\rho^1+\widehat{\rho},\mathbf{u}^1+\widehat{\mathbf{u}})-\mathsf{D}_2(\rho^2+\widehat{\rho},\mathbf{u}^2+\widehat{\mathbf{u}})\|_{L_p((0,T),H^1_q(\Omega))} \\ &\leq C\varphi(T)\|(\rho^2-\rho^1,\mathbf{u}^2-\mathbf{u}^1)\|_{K_{p,q;T}} \\ &\times\Big(\|(\rho^1,\mathbf{u}^1)\|_{K_{p,q;T}}+\|(\rho^2,\mathbf{u}^2)\|_{K_{p,q;T}}+\|(\rho_0,\mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big), \\ &\|\mathsf{F}_2(\rho^1+\widehat{\rho},\mathbf{u}^1+\widehat{\mathbf{u}})-\mathsf{F}_2(\rho^2+\widehat{\rho},\mathbf{u}^2+\widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)} \\ &\leq C(\psi(T)+\varphi(T))\|(\rho^2-\rho^1,\mathbf{u}^2-\mathbf{u}^1)\|_{K_{p,q;T}}\end{aligned}$$

$$\times \sum_{j=0}^{1} \Big(\|(\rho^1, \mathbf{u}^1)\|_{K_{p,q;T}} + \|(\rho^2, \mathbf{u}^2)\|_{K_{p,q;T}} + \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)} \Big)^j.$$

Proof. Let us first consider $\mathsf{D}_2(\rho, \mathbf{u})$. It can be written as

$$\begin{aligned} &\mathsf{D}_2(\rho^1 + \widehat{\rho}, \mathbf{u}^1 + \widehat{\mathbf{u}}) - \mathsf{D}_2(\rho^2 + \widehat{\rho}, \mathbf{u}^2 + \widehat{\mathbf{u}}) \\ &\quad = (\rho^1 - \rho^2) \operatorname{div}(\mathbf{u}^1 + \widehat{\mathbf{u}}) + (\rho^2 + \widehat{\rho} - \rho_0) \operatorname{div}(\mathbf{u}^1 - \mathbf{u}^2). \end{aligned}$$

Since $H^1_q(\Omega)$ is a Banach algebra by the assumption $q > N$, we observe that

$$\begin{aligned} &\|\mathsf{D}_2(\rho^1 + \widehat{\rho}, \mathbf{u}^1 + \widehat{\mathbf{u}}) - \mathsf{D}_2(\rho^2 + \widehat{\rho}, \mathbf{u}^2 + \widehat{\mathbf{u}})\|_{L_p((0,T),H^1_q(\Omega))} \\ &\leq C\Big\{ \|\rho^1 - \rho^2\|_{L_\infty((0,T),H^1_q(\Omega))} \|\operatorname{div}(\mathbf{u}^1 + \widehat{\mathbf{u}})\|_{L_p((0,T),H^1_q(\Omega))} \\ &+ \Big(\|\rho^2\|_{L_\infty((0,T),H^1_q(\Omega))} + \|\widehat{\rho} - \rho_0\|_{L_\infty((0,T),H^1_q(\Omega))} \Big) \\ &\times \|\operatorname{div}(\mathbf{u}^1 - \mathbf{u}^2)\|_{L_p((0,T),H^1_q(\Omega))} \Big\}. \end{aligned}$$

Combining this inequality with Propositions 6 and 8 demonstrates the desired inequality of $\mathsf{D}_2(\rho, \mathbf{u})$.

We next consider $\mathsf{F}_2(\rho, \mathbf{u})$. To this end, we set

$$\begin{aligned} \mathcal{G}_1(\rho, \mathbf{u}) &= (\rho - \rho_0)\partial_t \mathbf{u}, & \mathcal{G}_2(\mathbf{u}) &= (\mu - \mu_0)\Delta \mathbf{u}, \\ \mathcal{G}_3(\mathbf{u}) &= (\nu - \nu_0)\nabla \operatorname{div} \mathbf{u}, & \mathcal{G}_4(\rho) &= (\kappa - \kappa_0)\rho \nabla \Delta \rho, \\ \mathcal{G}_5(\rho) &= (\kappa - \kappa_0)\nabla \Delta \rho, & \mathcal{G}_6(\rho) &= \kappa_0(\rho - \rho_0)\nabla \Delta \rho. \end{aligned}$$

We observe that

$$\begin{aligned} &\mathcal{G}_1(\rho^1 + \widehat{\rho}, \mathbf{u}^1 + \widehat{\mathbf{u}}) - \mathcal{G}_1(\rho^2 + \widehat{\rho}, \mathbf{u}^2 + \widehat{\mathbf{u}}) \\ &\quad = (\rho^1 - \rho^2)\partial_t(\mathbf{u}^1 + \widehat{\mathbf{u}}) + (\rho^2 + \widehat{\rho} - \rho_0)\partial_t(\mathbf{u}^1 - \mathbf{u}^2), \end{aligned}$$

which yields

$$\begin{aligned} &\|\mathcal{G}_1(\rho^1 + \widehat{\rho}, \mathbf{u}^1 + \widehat{\mathbf{u}}) - \mathcal{G}_1(\rho^2 + \widehat{\rho}, \mathbf{u}^2 + \widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)} \\ &\leq \|\rho^1 - \rho^2\|_{L_\infty((0,T),L_\infty(\Omega))} \|\partial_t(\mathbf{u}^1 + \widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)} \\ &+ \Big(\|\rho^2\|_{L_\infty((0,T),L_\infty(\Omega))} + \|\widehat{\rho} - \rho_0\|_{L_\infty((0,T),L_\infty(\Omega))} \Big) \\ &\times \|\partial_t(\mathbf{u}^1 - \mathbf{u}^2)\|_{L_p((0,T),L_q(\Omega)^N)}. \end{aligned}$$

Combining this inequality with Propositions 6 and 8 shows

$$\begin{aligned} &\|\mathcal{G}_1(\rho^1 + \widehat{\rho}, \mathbf{u}^1 + \widehat{\mathbf{u}}) - \mathcal{G}_1(\rho^2 + \widehat{\rho}, \mathbf{u}^2 + \widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)} \\ &\leq C\varphi(T)\|(\rho^1 - \rho^2, \mathbf{u}^1 - \mathbf{u}^2)\|_{K_{p,q;T}} \\ &\times \Big(\|\mathbf{u}^1\|_{K^2_{p,q;T}} + \|\rho^2\|_{K^1_{p,q;T}} + \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)} \Big). \end{aligned}$$

Analogously, we have for $j = 4, 6$

$$\begin{aligned} &\|\mathcal{G}_j(\rho^1 + \widehat{\rho}) - \mathcal{G}_j(\rho^2 + \widehat{\rho})\|_{L_p((0,T),L_q(\Omega)^N)} \\ &\leq C\varphi(T)\|\rho^1 - \rho^2\|_{K^1_{p,q;T}} \Big(\|\rho^1\|_{K^1_{p,q;T}} + \|\rho^2\|_{K^1_{p,q;T}} + \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)} \Big). \end{aligned}$$

On the other hand, it is clear that for $j = 2, 3$

$$\|\mathcal{G}_j(\mathbf{u}^1 + \widehat{\mathbf{u}}) - \mathcal{G}_j(\mathbf{u}^2 + \widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)} \leq C\psi(T)\|\mathbf{u}^1 - \mathbf{u}^2\|_{K^2_{p,q;T}},$$
$$\|\mathcal{G}_5(\rho^1 + \widehat{\rho}) - \mathcal{G}_5(\rho^2 + \widehat{\rho})\|_{L_p((0,T),L_q(\Omega)^N)} \leq C\psi(T)\|\rho^1 - \rho^2\|_{K^2_{p,q;T}}.$$

Summing up the above inequalities of $\mathcal{G}_1, \ldots, \mathcal{G}_6$, we have obtained the desired inequality of $\mathsf{F}_2(\rho, \mathbf{u})$. This completes the proof of Lemma 14. □

In the same manner as in Lemma 14, we have

Lemma 15. *Suppose that Assumption 2 holds. Then for any $T \in (0, T_0]$*

$$\|\mathsf{D}_2(\widehat{\rho}, \widehat{\mathbf{u}})\|_{L_p((0,T),H^1_q(\Omega))} \leq C\|(\rho_0, \mathbf{u}_0)\|^2_{D_{q,p}(\Omega)},$$
$$\|\mathsf{F}_2(\widehat{\rho}, \widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)} \leq C\sum_{j=1}^{2}\|(\rho_0, \mathbf{u}_0)\|^j_{D_{q,p}(\Omega)},$$

where $C = C(N, p, q, R, R_1, R_2, T_0, \rho_\infty)$ is a positive constant independent of T.

The pressure term is next estimated.

Lemma 16. *Suppose that Assumption 2 and (d) of Theorem 1 hold. Let $\|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)} \leq R$. Then, there exists a positive constant $T_1 \in (0, T_0]$, depending on N, p, q, R, R_1, R_2, T_0, and ρ_∞, such that for any $T \in (0, T_1]$*

$$\|P'(\widehat{\rho} + \rho_\infty)\nabla\widehat{\rho}\|_{L_p((0,T),L_q(\Omega)^N)} \leq C\|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)},$$

where $C = C(N, p, q, R, R_1, R_2, T_0, \rho_\infty)$ is a positive constant independent of T.

Proof. Since $\widehat{\rho} + \rho_\infty = \widehat{\rho} - \rho_0 + \rho_0 + \rho_\infty$, it holds that

$$\begin{aligned}
&\|\rho_0 + \rho_\infty\|_{L_\infty(\Omega)} - \|\widehat{\rho} - \rho_0\|_{L_\infty((0,T),L_\infty(\Omega))} \\
&\leq \|\widehat{\rho} + \rho_\infty\|_{L_\infty((0,T),L_\infty(\Omega))} \\
&\leq \|\rho_0 + \rho_\infty\|_{L_\infty(\Omega)} + \|\widehat{\rho} - \rho_0\|_{L_\infty((0,T),L_\infty(\Omega))},
\end{aligned}$$

which, combined with (b) of Theorem 1 and Propositions 8, furnishes

$$\frac{\rho_\infty}{2} - C_1 T^{1-1/p} R \leq \|\widehat{\rho} + \rho_\infty\|_{L_\infty((0,T),L_\infty(\Omega))} \leq 2\rho_\infty + C_2 T^{1-1/p} R$$

for any $T \in (0, T_0]$ with positive constants C_1 and C_2 depending on N, p, q, R, R_1, R_2, T_0, and ρ_∞, but independent of T. Choosing $T_1 \in (0, T_0]$ so small that

$$C_1 T_1^{1-1/p} R \leq \frac{\rho_\infty}{8}, \quad C_2 T_1^{1-1/p} R \leq \rho_\infty,$$

we obtain for any $T \in (0, T_1]$

$$\frac{3}{8}\rho_\infty \leq \widehat{\rho}(x, t) + \rho_\infty \leq 3\rho_\infty \quad (x, t) \in \overline{\Omega} \times [0, T].$$

It thus holds for any $T \in (0, T_1]$ that

$$\|P'(\widehat{\rho} + \rho_\infty)\nabla\widehat{\rho}\|_{L_p((0,T),L_q(\Omega)^N)} \leq \|P'\|_{L_\infty([\rho_\infty/4, 4\rho_\infty])}\|\nabla\widehat{\rho}\|_{L_p((0,T),L_q(\Omega)^N)},$$

which, combined with Propositions 8, furnishes the desired inequality. This completes the proof of Lemma 16. □

Summing up Lemmas 11, 13, 15 and 16, we obtain

Proposition 9. *Suppose that all the assumptions of Theorem 1 hold. Let T_1 be the positive constant given by Lemma 16. Then there exists a positive constant L, depending on N, p, q, R, R_1, R_2, T_0, and ρ_∞, such that for any $T \in (0, T_1]$*

$$\|\mathsf{D}(\widehat{\rho}, \widehat{\mathbf{u}})\|_{L_p((0,T),H_q^1(\Omega))} \leq \frac{L}{4M_1}, \quad \|\mathsf{F}(\widehat{\rho}, \widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)} \leq \frac{L}{4M_1},$$

where M_1 is the positive constant given by Corollary 2.

We continue to estimate the pressure term.

Lemma 17. *Suppose that Assumption 2 and (d) of Theorem 1 hold. Let $\|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)} \leq R$. Let T_1 and L be the positive constants given by Lemma 16 and Proposition 9, respectively. Then, the following assertions hold.*

(1) *There exists a constant $T_2 = T_2(N, p, q, R, R_1, R_2, T_0, \rho_\infty, L) \in (0, T_1]$ such that for any $T \in (0, T_2]$ and $\rho \in {}_0K^1_{p,q;T}$ with $\|\rho\|_{K^1_{p,q;T}} \leq L$*

$$\frac{\rho_\infty}{4} \leq \rho(x,t) + \widehat{\rho}(x,t) + \rho_\infty \leq 4\rho_\infty \quad \text{for } (x,t) \in \overline{\Omega} \times [0,T].$$

(2) *For any $T \in (0, T_2]$ and $\rho^i \in {}_0K^1_{p,q;T}$, $i = 1, 2$, with $\|\rho^i\|_{K^1_{p,q;T}} \leq L$*

$$\begin{aligned}
&\|P'(\rho^1 + \widehat{\rho} + \rho_\infty)\nabla(\rho^1 + \widehat{\rho}) - P'(\rho^2 + \widehat{\rho} + \rho_\infty)\nabla(\rho^2 + \widehat{\rho})\|_{L_p((0,T),L_q(\Omega)^N)} \\
&\leq C\varphi(T)\|\rho^1 - \rho^2\|_{K^1_{p,q;T}} \sum_{j=0}^{1} \Big(\|\rho^1\|_{K^1_{p,q;T}} + \|\rho^2\|_{K^1_{p,q;T}} + \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big)^j,
\end{aligned}$$

where $C = C(N, p, q, R, R_1, R_2, T_0, \rho_\infty)$ is a positive constant independent of T.

Proof. (1) The proof is similar to one of Lemma 16; thus, the detailed proof may be omitted.

(2) Let us write

$$\begin{aligned}
&P'(\rho^1 + \widehat{\rho} + \rho_\infty)\nabla(\rho^1 + \widehat{\rho}) - P'(\rho^2 + \widehat{\rho} + \rho_\infty)\nabla(\rho^2 + \widehat{\rho}) \\
&= (P'(\rho^1 + \widehat{\rho} + \rho_\infty) - P'(\rho^2 + \widehat{\rho} + \rho_\infty))\nabla(\rho^1 + \widehat{\rho}) + P'(\rho^2 + \widehat{\rho} + \rho_\infty)\nabla(\rho^1 - \rho^2) \\
&=: I_1 + I_2.
\end{aligned}$$

It holds by (1) that for any $T \in (0, T_2]$

$$\begin{aligned}
&\|P'(\rho^1 + \widehat{\rho} + \rho_\infty) - P'(\rho^2 + \widehat{\rho} + \rho_\infty)\|_{L_\infty((0,T),L_\infty(\Omega))} \\
&\leq \|P'\|_{C^{0,1}([\rho_\infty/4, 4\rho_\infty])}\|\rho^1 - \rho^2\|_{L_\infty((0,T),L_\infty(\Omega))},
\end{aligned}$$

which, combined with Propositions 6 and 8, demonstrates

$$\begin{aligned}
&\|I_1\|_{L_p((0,T),L_q(\Omega)^N)} \\
&\leq \|P'\|_{C^{0,1}([\rho_\infty/4, 4\rho_\infty])}\|\rho^1 - \rho^2\|_{L_\infty((0,T),L_\infty(\Omega))}\|\nabla(\rho^1 + \widehat{\rho})\|_{L_p((0,T),L_q(\Omega)^N)} \\
&\leq C\varphi(T)\|\rho^1 - \rho^2\|_{K^1_{p,q;T}}\Big(\|\rho^1\|_{K^1_{p,q;T}} + \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}\Big).
\end{aligned}$$

On the other hand, (1) and Proposition 6 shows that for any $T \in (0, T_2]$

$$\begin{aligned}
\|I_2\|_{L_p((0,T),L_q(\Omega)^N)} &\leq \|P'\|_{L_\infty([\rho_\infty/4, 4\rho_\infty])}\|\nabla(\rho^1 - \rho^2)\|_{L_p((0,T),L_q(\Omega)^N)} \\
&\leq C\varphi(T)\|\rho^1 - \rho^2\|_{K^1_{p,q;T}}.
\end{aligned}$$

The desired inequality thus holds. This completes the proof of Lemma 17. □

From Lemmas 10, 12, 14 and 17, we obtain the following proposition.

Proposition 10. *Suppose that all the assumptions of Theorem 1 hold. Let L and T_2 be the positive constants given by Proposition 9 and Lemma 17, respectively. Then, there exist a positive constant M_2, depending on N, p, q, R, R_1, R_2, T_0, ρ_∞, and L, such that for any $T \in (0, T_2]$ the following assertions hold.*

(1) *For any $(\rho^i, \mathbf{u}^i) \in {}_0K_{p,q;T}(L)$, $i = 1, 2$,*

$$\begin{aligned}
&\|\mathsf{D}(\rho^1 + \widehat{\rho}, \mathbf{u}^1 + \widehat{\mathbf{u}}) - \mathsf{D}(\rho^2 + \widehat{\rho}, \mathbf{u}^2 + \widehat{\rho})\|_{L_p((0,T),H^1_q(\Omega))} \\
&\leq M_2(\varphi(T) + \psi(T))\|(\rho^1 - \rho^2, \mathbf{u}^1 - \mathbf{u}^2)\|_{K_{p,q;T}}, \\
&\|\mathsf{F}(\rho^1 + \widehat{\rho}, \mathbf{u}^1 + \widehat{\mathbf{u}}) - \mathsf{F}(\rho^2 + \widehat{\rho}, \mathbf{u}^2 + \widehat{\rho})\|_{L_p((0,T),L_q(\Omega)^N)} \\
&\leq M_2(\varphi(T) + \psi(T))\|(\rho^1 - \rho^2, \mathbf{u}^1 - \mathbf{u}^2)\|_{K_{p,q;T}}.
\end{aligned}$$

(2) *For any $(\rho, \mathbf{u}) \in {}_0K_{p,q;T}(L)$*

$$\begin{aligned}
\|\mathsf{D}(\rho + \widehat{\rho}, \mathbf{u} + \widehat{\mathbf{u}})\|_{L_p((0,T),H^1_q(\Omega))} &\leq M_2(\varphi(T) + \psi(T))\|(\rho, \mathbf{u})\|_{K_{p,q;T}} + \frac{L}{4M_1}, \\
\|\mathsf{F}(\rho + \widehat{\rho}, \mathbf{u} + \widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)} &\leq M_2(\varphi(T) + \psi(T))\|(\rho, \mathbf{u})\|_{K_{p,q;T}} + \frac{L}{4M_1},
\end{aligned}$$

where M_1 is the positive constant given by Corollary 2.

Proof. (1) The desired inequalities follow from Lemmas 10, 12, 14 and 17, immediately.

(2) The desired inequalities follow from (1) and Proposition 9 immediately. This completes the proof of Proposition 10. □

6.3. Proof of Theorem 1

Throughout this subsection, we assume that all the assumptions of Theorem 1 hold. Let M_1 be the positive constant given by Corollary 2. In addition, L, T_2, and M_2 are the same positive constants as in the previous subsection. Recall that $\varphi(T)$ and $\psi(T)$ satisfy (37) and (38).

Let us choose T so small that

$$M_1 M_2(\varphi(T) + \psi(T)) \leq \frac{1}{4}.$$

Let $(\rho, \mathbf{u}) \in {}_0K_{p,q;T}(L)$. We consider

$$\begin{cases}
\partial_t \sigma + r_0 \operatorname{div} \mathbf{v} = \mathsf{D}(\rho + \widehat{\rho}, \mathbf{u} + \widehat{\mathbf{u}}) & \text{in } \Omega \times (0, T), \\
\partial_t \mathbf{v} - r_0^{-1}\kappa_0 \operatorname{Div}(\kappa_0^{-1}\mathbf{S}_0(\mathbf{v}) + r_0 \Delta\sigma \mathbf{I}) = \mathsf{F}(\rho + \widehat{\rho}, \mathbf{u} + \widehat{\mathbf{u}}) & \text{in } \Omega \times (0, T), \\
\mathbf{n} \cdot \nabla\sigma = 0, \quad \mathbf{v} = 0 & \text{on } \Gamma_D \times (0, T), \\
\mathbf{n} \cdot \nabla\sigma = 0, \quad (\mathbf{D}(\mathbf{v})\mathbf{n})_\tau = 0, \quad \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \Gamma_S \times (0, T), \\
(\sigma, \mathbf{v})|_{t=0} = (0, 0) & \text{in } \Omega.
\end{cases} \tag{39}$$

By Proposition 10(2), we observe that

$$\mathsf{D}(\rho + \widehat{\rho}, \mathbf{u} + \widehat{\mathbf{u}}) \in L_p((0, T), H^1_q(\Omega)), \quad \mathsf{F}(\rho + \widehat{\rho}, \mathbf{u} + \widehat{\mathbf{u}}) \in L_p((0, T), L_q(\Omega)^N)$$

and that

$$\|\mathsf{D}(\rho + \widehat{\rho}, \mathbf{u} + \widehat{\mathbf{u}})\|_{L_p((0,T),H^1_q(\Omega))} \leq \frac{L}{2M_1},$$

$$\|\mathsf{F}(\rho+\widehat{\rho},\mathbf{u}+\widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)} \le \frac{L}{2M_1}.$$

This enables us to apply Corollary 2 to (39), and then there exists a solution $(\sigma, \mathbf{v}) \in {}_0K_{p,q;T}$ of (39) such that

$$\|(\sigma,\mathbf{v})\|_{K_{p,q;T}} \le M_1\Big(\frac{L}{2M_1}+\frac{L}{2M_1}\Big) = L.$$

Thus the mapping $\Phi : {}_0K_{p,q;T}(L) \to {}_0K_{p,q;T}(L)$ can be defined by $\Phi(\rho,\mathbf{u}) := (\sigma,\mathbf{v})$.

From now on, we prove that Φ is a contraction mapping on ${}_0K_{p,q;T}(L)$. Let $(\rho^i,\mathbf{u}^i) \in {}_0K_{p,q;T}(L)$ for $i=1,2$ and set $(\sigma^i,\mathbf{v}^i) = \Phi(\rho^i,\mathbf{u}^i)$. Define

$$\tau = \sigma^1 - \sigma^2, \quad \mathbf{w} = \mathbf{v}^1 - \mathbf{v}^2.$$

Then $(\tau,\mathbf{w})$ satisfies

$$\begin{cases} \partial_t\tau + r_0 \operatorname{div}\mathbf{w} = \mathsf{D}(\rho^1+\widehat{\rho},\mathbf{u}^1+\widehat{\mathbf{u}}) \\ \qquad\qquad - \mathsf{D}(\rho^2+\widehat{\rho},\mathbf{u}^2+\widehat{\mathbf{u}}) & \text{in } \Omega\times(0,T), \\ \partial_t\mathbf{w} - r_0^{-1}\kappa_0 \operatorname{Div}(\kappa_0^{-1}\mathbf{S}_0(\mathbf{w}) + r_0\Delta\tau\mathbf{I}) = \mathsf{F}(\rho^1+\widehat{\rho},\mathbf{u}^1+\widehat{\mathbf{u}}) \\ \qquad\qquad - \mathsf{F}(\rho^2+\widehat{\rho},\mathbf{u}^2+\widehat{\mathbf{u}}) & \text{in } \Omega\times(0,T), \\ \mathbf{n}\cdot\nabla\tau = 0, \quad \mathbf{w} = 0 & \text{on } \Gamma_D\times(0,T), \\ \mathbf{n}\cdot\nabla\tau = 0, \quad (\mathbf{D}(\mathbf{w})\mathbf{n})_\tau = 0, \quad \mathbf{w}\cdot\mathbf{n} = 0 & \text{on } \Gamma_S\times(0,T), \\ (\tau,\mathbf{w})|_{t=0} = (0,0) & \text{in } \Omega. \end{cases} \tag{40}$$

By Proposition 10(1), we observe that

$$\begin{aligned} &\|\mathsf{D}(\rho^1+\widehat{\rho},\mathbf{u}^1+\widehat{\mathbf{u}}) - \mathsf{D}(\rho^2+\widehat{\rho},\mathbf{u}^2+\widehat{\mathbf{u}})\|_{L_p((0,T),H_q^1(\Omega))} \\ &\le \frac{1}{4M_1}\|(\rho^1-\rho^2,\mathbf{u}^1-\mathbf{u}^2)\|_{K_{p,q;T}}, \\ &\|\mathsf{F}(\rho^1+\widehat{\rho},\mathbf{u}^1+\widehat{\mathbf{u}}) - \mathsf{F}(\rho^2+\widehat{\rho},\mathbf{u}^2+\widehat{\mathbf{u}})\|_{L_p((0,T),L_q(\Omega)^N)} \\ &\le \frac{1}{4M_1}\|(\rho^1-\rho^2,\mathbf{u}^1-\mathbf{u}^2)\|_{K_{p,q;T}}. \end{aligned}$$

Applying Corollary 2 to (40) shows that

$$\begin{aligned} \|(\tau,\mathbf{w})\|_{K_{p,q;T}} &\le M_1\Big(\frac{1}{4M_1}+\frac{1}{4M_1}\Big)\|(\rho^1-\rho^2,\mathbf{u}^1-\mathbf{u}^2)\|_{K_{p,q;T}} \\ &\le \frac{1}{2}\|(\rho^1-\rho^2,\mathbf{u}^1-\mathbf{u}^2)\|_{K_{p,q;T}}. \end{aligned}$$

This guarantees that Φ is a contraction mapping on ${}_0K_{p,q;T}(L)$. The contraction mapping theorem thus yields a unique fixed point $(\rho_*,\mathbf{u}_*)$ of Φ in ${}_0K_{p,q;T}(L)$, i.e., $\Phi(\rho_*,\mathbf{u}_*) = (\rho_*,\mathbf{u}_*) \in {}_0K_{p,q;T}(L)$. Then, $(\rho_*,\mathbf{u}_*)$ becomes a unique solution of (9) in ${}_0K_{p,q;T}(L)$. This completes the proof of Theorem 1.

7. Conclusions and Future Works

In this paper, we have proved in Theorem 1 the local existence of strong solutions for the Navier–Stokes–Korteweg system in a general domain with the Dirichlet boundary condition or the slip boundary condition in an L_p-in-time and L_q-in-space setting, where $p \in (1,\infty)$ and $q \in (N,\infty)$, based on the theory of maximal regularity. Our result demonstrates an extension of Kotschote [16] in view of domains and boundary conditions, and also extends the exponents p and q.

We will consider time periodic solutions of the Navier–Stokes–Korteweg system in a forthcoming paper by means of results of the present paper.

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