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Some Conformal Transformations on Finsler Warped Product Manifolds

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Abstract: The conformal transformation, which preserves Einstein metrics on Finsler warped product manifolds, is studied in this paper. We obtain sufficient and necessary conditions of a conformal transformation preserving Einstein metrics. In addition, we provide nontrivial examples of conformal transformations. Furthermore, we completely classify Einstein Riemannian warped product metrics and obtain the existence of a nontrivial conformal transformation that preserves Einstein metrics.

Keywords: conformal transformations; Einstein metrics; Finsler warped product metrics

MSC: 53C30; 53C60

1. Introduction

In Finsler geometry, the Weyl theorem shows that the projective and conformal properties of a Finsler metric uniquely determine its geometrical properties. Therefore, the conformal properties of Finsler metrics have attracted the attention of many scholars. Bácsó and Cheng [1] characterized the conformal transformations that preserve Riemann curvatures, Ricci curvatures, Landsberg curvatures, mean Landsberg curvatures, or S-curvatures, respectively. Chen, Cheng, and Zou [2] characterized the conformal transformations between two (α, β)-metrics. Furthermore, they proved that if both conformally related (α, β)-metrics F and $\bar{F}$ were both of isotropic S-curvatures, then the conformal transformation between them is a homothety. Shen [3] noted that the conformal transformation, which preserves the S-curvature invariant, must be a homothety.

The warped product metric was first introduced by Bishop and O'Neil [4] to study Riemannian manifolds of negative curvature as a generalization of the Riemannian product metric. The warped product metric was later extended to the case of Finsler manifolds by the work of Kozma, Peter, and Varga and Chen, Shen, and Zhao [5,6], respectively. Recently, significant progress has been made in the study of Finsler warped product metrics. Liu and Mo [7] obtained all Finsler warped product Douglas metrics and explicitly constructed some new warped product Douglas metrics. Yang and Zhang [8] obtained the differential equations that characterize such metrics and provided some examples for Finsler warped product metrics with relatively isotropic Landsberg curvature.

The Einstein metric has significant importance in Riemannian geometry. Especially for $\mathbb{S}^m$, many Einstein metrics have been constructed. Jensen [9] discovered that $\mathbb{S}^{4k+3}$ admits another Einstein metric in 1973. In 1998, Böhm [10] constructed infinite sequences of non-isometric Einstein metrics on $\mathbb{S}^m (m = 5, 6, 7, 8, 9)$. In 2005, Boyer, Galicki, and Kollár [11] constructed many Sasakian—Einstein metrics on $\mathbb{S}^{4k+1}$. It is a natural problem to study Einstein Finsler metrics. Recently, there have been many studies on this problem. Bao and Robles [12] gave the complete classification of Einstein Randers metrics by the navigation data. Chen, Shen, and Zhao [6] completely determined the local structure of Ricci-flat (α, β)-metrics which are also of the Douglas type. Later in [13], Chen, Shen, and

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Zhao gave the formulae of the flag curvature and Ricci curvature of Finsler warped product metrics and obtained the characterization of such metrics to be Einsteinian.

In this article, we focus on conformally related Einstein metrics on Finsler warped product manifolds and obtain the characterization of the conformal transformation that preserves Einstein metrics. We obtain the following results.

Theorem 1. *Let F and $\bar{F}$ be two Finsler warped product metrics on an $n(\geq 3)$-dimensional manifold M. Then the conformal transformation $\bar{F} = e^{\rho(r)}F$ preserves Einstein metrics if and only if the Riemannian metric α has isotropic Ricci curvature χ, and the following equations hold:*

$$\begin{cases} \big[\lambda_s + (n-1)\mu_s - \nu_s\big]\omega = \big[\chi + \lambda + (n-1)\mu - \nu\big]\omega_s, \\ \Delta\rho_{rr} + \Theta\rho_r^2 + \Sigma\rho_r = (n-1)\big[e^{2\rho(r)}\bar{k} - k\big], \end{cases} \tag{1}$$

where λ, μ, ν are given by (3)–(5), *Δ, Θ, Σ are given by* (11)–(13), *and $k = k(r), \bar{k} = \bar{k}(r)$ are the Einstein scalars of F and $\bar{F}$, respectively.*

2. Preliminaries

Let M be an n-dimensional smooth manifold and TM be the tangent bundle. If the function $F = F(x, y) : TM \to [0, \infty)$ satisfies the following properties:

(i) F is C^∞ on $TM\backslash\{0\}$:=$\{(x, y)|x \in M, y \in T_xM, y \neq 0\}$,

(ii) $F(x, \lambda y) = \lambda F(x, y)$ for any $\lambda > 0$,

(iii) the Hessian matrix $\big(g_{ij}\big) := \dfrac{1}{2}\big(F^2_{y^i y^j}\big)$ is positive-definite at any point of $TM\backslash\{0\}$,

then $F = F(x, y)$ is called a Finsler metric on M, and the tensor $g = g_{ij}(x, y)dx^i \otimes dx^j$ is called the fundamental tensor of Finsler metric F. A smooth manifold M endowed with a Finsler structure F is called a Finsler manifold, which is denoted by (M, F).

Let F be a Finsler metric on an n-dimensional manifold M. G^i are the spray coefficients of F, which are defined by

$$G^i = \frac{1}{4}g^{il}\left\{\Big[F^2\Big]_{x^k y^l} y^k - \Big[F^2\Big]_{x^l}\right\},$$

where $\big(g^{ij}\big) := \big(g_{ij}\big)^{-1}$.

The Cartan tensor is defined by $\mathbf{C} := C_{ijk}dx^i \otimes dx^j \otimes dx^k$, where

$$C_{ijk} := \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

The Landsberg curvature $L = L_{ijk}(x, y)dx^i \otimes dx^j \otimes dx^k$ is a horizontal tensor on $TM\backslash\{0\}$, defined by

$$L_{ijk} := -\frac{1}{2}FF_{y^m}\frac{\partial^3 G^m}{\partial y^i \partial y^j \partial y^k}.$$

The Riemann curvature $\mathbf{R}_y = R^i{}_k dx^k \otimes \frac{\partial}{\partial x^i}\Big|_p : T_pM \to T_pM$ is a family of linear transformations on tangent spaces, which is defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The trace of the Riemann curvature

$$\mathrm{Ric}(y) := (n-1)R(y) = R^m{}_m(y)$$

is called the Ricci curvature.

The Finsler metric F is called an Einstein metric if there is a scalar function $k = k(x)$ on the manifold M such that

$$Ric = (n-1)kF^2.$$

We denote k as the Einstein scalar of F. If $k(x) = 0$, F is said to be Ricci-flat.

Let F and $\bar{F}$ be two Finsler metrics on an n-dimensional manifold M. If there exists a smooth function $\rho(x)$ on the manifold such that

$$\bar{F}(x,y) = e^{\rho(x)}F(x,y),$$

then F and $\bar{F}$ are said to be locally conformally related, where the smooth function $\rho(x)$ is called the conformal factor. If the conformal factor is a constant, then the conformal transformation is called a homothetic transformation. In this article, we do not consider the case of homothetic transformations.

For conformally related Finsler metrics, Bácsó and Cheng gave the transformation formula for their Ricci curvature, which is as follows.

Lemma 1 ([1]). *Let F and $\bar{F}$ be two Finsler metrics on an n-dimensional manifold M. If they are conformally related, that is, if there exists a scalar function $\rho = \rho(x)$ such that $\bar{F} = e^{\rho(x)}F$, then Ric and $\overline{Ric}$ satisfy*

$$\begin{aligned}\overline{Ric} = {} & Ric + (n-2)\left[(\rho_i v^i)^2 - \rho_{ij}v^i v^j + G^i\rho_i\right] + F^2\left[-(n-2)|\nabla\rho|_F^2 - 2\rho^i g^{jk}L_{ijk}\right.\\ & - g^{ij}\rho_{i;j} - (\rho^k g^{ij}C_{ijk})_{;0} - 2\rho_0(\rho^k g^{ij}C_{jik}) + 2F^2 g^{ij}C_{ijk}\rho^l\rho^m C^k{}_{lm}\\ & \left. - F^2\rho^j\rho^k(g^{il}C_{ijl})_{.k} - F^2\rho^j\rho^k C^i{}_{jl}C^l{}_{ik}\right],\end{aligned} \tag{2}$$

where $\rho_i := \dfrac{\partial\rho}{\partial x^i}$, $\rho_0 := \rho_i y^i$, $|\nabla\rho|_F^2 = g^{ij}\rho_i\rho_j$, $\rho^i := g^{ij}\rho_j$, $\rho_{ij} := \frac{\partial^2\rho}{\partial x^i\partial x^j}$, ";" denotes the horizontal covariant derivatives with respect to the Berwald connection of F, and "." denotes the vertical covariant derivatives.

In what follows, we always let the index conventions be as follows:

$$\begin{aligned}1 \leqslant \mathrm{A, B, C}\cdots \leqslant \mathrm{n},\\ 2 \leqslant \mathrm{i, j, k}\cdots \leqslant \mathrm{n}.\end{aligned}$$

The definition and some conclusions about the Finsler warped product metrics are as follows.

Consider an n-dimensional product manifold $M := I \times \check{M}$, where I is an interval of $\mathbb{R}$, and $\check{M}$ is an $(n-1)$-dimensional manifold equipped with a Riemannian metric $\check{\alpha}$. A Finsler warped product metric on M is a Finsler metric of the form

$$F(u,v) = \check{\alpha}(\check{u},\check{v})\sqrt{\omega(r,s)},$$

where $u = \left(u^1, \check{u}\right) \in M$, $v = v^1\dfrac{\partial}{\partial u^1} + \check{v} \in T_{\check{u}}\check{M}$, $r = u^1$, $s = \dfrac{v^1}{\check{\alpha}(\check{u},\check{v})}$, and $\omega(r,s)$ is a positive function defined on a domain of $\mathbb{R}^2$. (M,F) is called a Finsler warped product manifold.

Lemma 2 ([6]). *The matrix of the Finsler warped product metric g has the following form:*

$$\begin{pmatrix} g_{11} & g_{1j} \\ g_{i1} & g_{ij} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\omega_{ss} & \frac{1}{2}\Xi_s\check{l}_j \\ \frac{1}{2}\Xi_s\check{l}_i & \frac{1}{2}\Xi\check{a}_{ij} - \frac{1}{2}s\Xi_s\check{l}_i\check{l}_j \end{pmatrix}, \tag{3}$$

where $\check{l}_i := \check{\alpha}_{v^i}$, $\check{l}^i := \check{a}^{ij}\check{l}_j = \dfrac{v^i}{\check{\alpha}}$, $\omega_s := \dfrac{\partial}{\partial s}\omega$, $\Xi := 2\omega - s\omega_s$, *and* $\Xi_s := \dfrac{\partial}{\partial s}\Xi = \omega_s - s\omega_{ss}$. *Then the inverse matrix of* (g_{AB}) *is*

$$\left(g^{AC}\right) = \begin{pmatrix} g^{11} & g^{1k} \\ g^{i1} & g^{ik} \end{pmatrix} = \begin{pmatrix} U & V\check{l}^k \\ Vl^i & W\check{a}^{ik} + X\check{l}^i\check{l}^k \end{pmatrix},$$

where $\Lambda := 2\omega\omega_{ss} - \omega_s^2$, $U := \dfrac{2(\Xi - s\Xi_s)}{\Lambda}$, $V := -\dfrac{2\Xi_s}{\Lambda}$, $W := \dfrac{2}{\Xi}$, $andX := \dfrac{2\omega_s\Xi_s}{\Xi\Lambda}$.

Lemma 3 ([6]). *The Finsler warped product metric* $F = \check{\alpha}\sqrt{\omega}$ *is strongly convex if and only if* $\Xi > 0$ *and* $\Lambda > 0$.

The expression of the Riemann curvature tensor for the Finsler warped product metric is as follows [6]:

$$R^1{}_1 = \lambda\check{\alpha}^2,\ R^1{}_k = -s\lambda\check{\alpha}^2\check{l}_k,\ R^i{}_1 = \tau\check{\alpha}^2\check{l}^i,\ R^i{}_k = \check{R}^i{}_k + \left(\mu\delta^i{}_k - \nu\check{l}^i\check{l}_k\right)\check{\alpha}^2,$$

where $\check{R}^i{}_k$ are coefficients of the Riemann curvature tensor of the Riemannian metric $\check{\alpha}$, and

$$\begin{aligned}
\Phi &:= \frac{s^2(\omega_r\omega_{ss} - \omega_s\omega_{rs}) - 2\omega(\omega_r - s\omega_{rs})}{2(2\omega\omega_{ss} - \omega_s^2)}, \\
\Psi &:= \frac{s(\omega_r\omega_{ss} - \omega_s\omega_{rs}) + \omega_r\omega_s}{2(2\omega\omega_{ss} - \omega_s^2)}, \\
\lambda &:= (2\Phi_r - s\Phi_{sr}) + \left(2\Phi\Phi_{ss} - \Phi_s^2\right) + 2(\Phi_s - s\Phi_{ss})\Psi - (2\Phi - s\Phi_s)\Psi_s, && (4) \\
\mu &:= \Psi^2 - 2s\Psi\Psi_s - s\Psi_r + 2\Phi\Psi_s, && (5) \\
\nu &:= s(\Psi_r - s\Psi_{sr}) + (\Psi - s\Psi_s)^2 - 2s^2\Psi\Psi_{ss} + 2s\Psi_{ss}\Phi + (2\Phi - s\Phi_s)\Psi_s, && (6) \\
\tau &:= \frac{(\nu - \mu)}{s} = 2\Psi_r - s\Psi_{rs} + s\left(\Psi_s^2 - 2\Psi\Psi_{ss}\right) + 2\Psi_{ss}\Phi - \Psi_s\Phi_s.
\end{aligned}$$

Therefore, the Ricci curvature of the Finsler warped product metric can be expressed as

$$Ric = \check{Ric} + \left[\lambda + (n-1)\mu - \nu\right]\check{\alpha}^2, \tag{7}$$

where $\check{Ric}$ is the Ricci curvature of the Riemannian metric $\check{\alpha}$.

Chen, Shen, and Zhao characterized the Einstein metric on Finsler warped product manifolds, and their conclusion is as follows.

Lemma 4 ([6]). *The Finsler warped product metric* $F = \check{\alpha}\sqrt{w}$ *is an Einstein metric if and only if the Riemannian metric* $\check{\alpha}$ *has isotropic Ricci curvature* χ, *and the following equation holds:*

$$(n-1)(Kw - \mu) + \nu - \lambda = \chi(\theta), \tag{8}$$

where $\chi = \chi(\theta)$ *is a function on* $\check{M}$. *If* $\dim \check{M} \geq 3$, *then* $\chi = const.$

Based on Lemma 4, we have optimized the equivalent characterization of the Einstein metric on Finsler warped product manifolds and obtained that the Einstein scalar of F is only a function with respect to the variable r. The conclusion is as follows.

Proposition 1. *The Finsler warped product metric* $F = \check{\alpha}\sqrt{w}$ *is an Einstein metric if and only if the Riemannian metric* $\check{\alpha}$ *has isotropic Ricci curvature* χ, *and the following equation holds:*

$$\left[\lambda_s + (n-1)\mu_s - \nu_s\right]\omega = \left[\chi + \lambda + (n-1)\mu - \nu\right]\omega_s, \tag{9}$$

where $\chi = \chi(\theta)$ is a function on $\check{M}$. Furthermore,

$$\frac{\chi + \lambda + (n-1)\mu - \nu}{(n-1)\omega}$$

is the Einstein scalar of F, and it is a function only with respect to the variable r.

Proof. Firstly, we prove the necessity. If F is an Einstein metric, then by Lemma 4, we know that

$$\check{Ric} = \chi\check{\alpha}.$$

From (7), we can obtain

$$\begin{aligned} Ric &= \check{Ric} + \left[\lambda + (n-1)\mu - \nu\right]\check{\alpha}^2 \\ &= \chi\check{\alpha}^2 + \left[\lambda + (n-1)\mu - \nu\right]\check{\alpha}^2 \\ &= \frac{\chi + \lambda + (n-1)\mu - \nu}{\omega}\check{\alpha}^2\omega \\ &= \frac{\chi + \lambda + (n-1)\mu - \nu}{\omega}F^2. \end{aligned}$$

Also, by the definition of the Einstein metric, we have

$$Ric = (n-1)k(r, \check{u})F^2.$$

Therefore,

$$(n-1)k(r, \check{u}) = \frac{\chi + \lambda + (n-1)\mu - \nu}{\omega}.$$

Taking partial derivatives with respect to v^A on both sides of the above equation yields (9). It follows that $k = k(r)$.

To prove the sufficiency, from (9) we have

$$\left(\frac{\chi + \lambda + (n-1)\mu - \nu}{\omega}\right)_s = 0.$$

This implies that

$$\frac{\chi + \lambda + (n-1)\mu - \nu}{\omega}$$

is independent of both $\check{u}$ and v^A. Therefore, it is a function only with respect to the variable r on M. Furthermore, we can see from (7) that

$$\begin{aligned} Ric &= \check{Ric} + \left[\lambda + (n-1)\mu - \nu\right]\check{\alpha}^2 \\ &= \chi\check{\alpha}^2 + \left[\lambda + (n-1)\mu - \nu\right]\check{\alpha}^2 \\ &= \frac{\chi + \lambda + (n-1)\mu - \nu}{\omega}F^2. \end{aligned}$$

Since

$$\frac{\chi + \lambda + (n-1)\mu - \nu}{\omega}$$

is a function only with respect to the variable r on M, it follows that F is an Einstein metric, and

$$k = k(r) = \frac{\chi + \lambda + (n-1)\mu - \nu}{(n-1)\omega}$$

is its Einstein scalar. □

3. Conformal Transformations Preserving Einstein Metrics

In this section, we characterize conformal transformations, which preserve Einstein metrics on Finsler warped product manifolds. Firstly, we have the transformation formula for their Ricci curvature, which is as follows.

Proposition 2. *Let F be a Finsler warped product metric on a manifold M of dimension $n(\geq 3)$. If $\bar{F} = e^{\rho(r,\check{u})}F$, then Ric and $\overline{Ric}$ satisfy*

$$\overline{Ric} = Ric + F^2 \cdot \Pi, \tag{10}$$

where, in what follows, "$|$" denotes the horizontal covariant derivative with respect to the Levi-Civita connection of the Riemannian metric $\check{\alpha}$, and

$$\begin{aligned}
\Pi :=& \frac{n-2}{\omega}\big[(s\rho_r+\rho_i\check{l}^i)^2 - s^2\rho_{rr} - 2s\rho_{ri}\check{l}^i - \rho_{ij}\check{l}^i\check{l}^j + 2\Phi\rho_r + 2\check{\alpha}^{-2}\check{G}^i\rho_i + 2\Psi\rho_i\check{l}^i\big]\\
&-(n-2)\big[U\rho_r^2 + 2V\rho_r\rho_i\check{l}^i + W|\nabla\rho|_{\check{\alpha}}^2 + X(\rho_i\check{l}^i)^2\big]\\
&+(U-2sV)(\rho_r+V\rho_i\check{l}^i)\big[\frac{1}{2}\omega_s\Phi_{sss} + (\omega-\frac{1}{2}s\omega_s)\Psi_{sss}\big]\\
&+\frac{1}{2}s(sV-2U)\big[V\rho_r+(W+X)\rho_i\check{l}^i\big]\big[\frac{1}{2}\omega_s\Phi_{sss}+(\omega-\frac{1}{2}s\omega_s)\Psi_{sss}\big]\\
&+(U\rho_r+V\rho_i\check{l}^i)\Big\{s^2(W+X)\big[\frac{1}{2}\omega_s\Phi_{sss}+(\omega-\frac{1}{2}s\omega_s)\Psi_{sss}\big]\\
&+(n-2)W\big[\frac{1}{2}\omega_s(\Phi_s-\Phi_{ss})-s(\omega-\frac{1}{2}s\omega_s)\Psi_{ss}\big]\Big\}\\
&+s\big[V\rho_r+(W+X)\rho_i\check{l}^i\big]\Big\{-s^2(W+X)\big[\frac{1}{2}\omega_s\Phi_{sss}+(\omega-\frac{1}{2}s\omega_s)\Psi_{sss}\big]\\
&-(n-2)W\big[\frac{1}{2}\omega_s(\Phi_s-\Phi_{ss})-s(\omega-\frac{1}{2}s\omega_s)\Psi_{ss}\big]\Big\}\\
&+U(-\rho_{rr}+\rho_r\Phi_{ss})-2V\rho_{ri}\check{l}^i+2V\rho_r(\Phi_s-s\Phi_{ss})+(U-2sV)\Psi_{ss}\rho_i\check{l}^i+2V\Psi_s\rho_i\check{l}^i\\
&+W\big[-\rho^k{}_{|k}+\rho_r(2\Phi-2s\Phi_s+s^2\Phi_{ss})+(n-2)\rho_r(2\Phi-s\Phi_s)\\
&+s^2\Psi_{ss}\rho_r\rho_i\check{l}^i+(n-1)(\Psi-s^2\Psi_s)\rho_r\rho_i\check{l}^i\big]\\
&+X\big[\rho_{i|j}\check{l}^i\check{l}^j+\rho_r(2\Phi-2s\Phi_s+s^2\Phi_{ss})+s^2\Psi_{ss}\rho_r\rho_k\check{l}^k+2(\Psi-s\Psi_s)\rho_r\rho_i\check{l}^i\big]\\
&-s\check{\alpha}\Big\{\big[U_r-2sV_r+s^2(W_r+X_r)\big]C_{111}+\big[U-2sV+s^2(W+X)\big]C_{111r}\\
&+\frac{n-2}{4}\alpha^{-1}\Xi_{rs}W+\frac{n-2}{4}\alpha^{-1}\Xi_sW_r\Big\}\big[U\rho_r+V\rho_i\check{l}^i-sV\rho_r-s(W+X)\rho_i\check{l}^i\big]\\
&-s\check{\alpha}\Big\{\big[U-2sV+s^2(W+X)\big]C_{111}+\frac{n-2}{4}\alpha^{-1}\Xi_sW\Big\}\\
&\Big\{(U_r-sV_r)\rho_r+(U-sV)\rho_{rr}+\big[V_r-s(W_r+X_r)\big]\rho_i\check{l}^i+\big[V-s(W+X)\big]\rho_{ri}\check{l}^i\Big\}\\
&+\frac{1}{2}(\Phi-s\Psi)\Big\{\omega_{sss}\big[U_s-2V+2s(W+X-V_s)+s^2(W_s+X_s)\big]+\omega_{ssss}\big[U-2sV\\
&+s^2(W+X)\big]+(n-2)(\Xi_{ss}W+\Xi_sW_s)\Big\}\big[U\rho_r+V\rho_i\check{l}^i-sV\rho_r-s(W+X)\rho_i\check{l}^i\big]\\
&+\frac{1}{2}\Big\{\omega_{sss}\big[U-2sV+s^2(W+X)\big]+(n-2)\Xi_sW\Big\}\\
&\Big\{(U_s-V-sV_s)\rho_r+\big[V_s-(W+X)-s(W_s+X_s)\big]\rho_i\check{l}^i\Big\}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}(\Psi+s\rho_r+\rho_i\check{l}^i)\Big\{\omega_{sss}\big[U-2sV+s^2(W+X)\big]+(n-2)\Xi_sW\Big\}\\
&\Big\{(U-sV)\rho_r-\big[s(W+X)-V\big]\rho_i\check{l}^i\Big\}\\
&-\frac{1}{8}\omega\Big\{\omega_{sss}\big[U-2sV+s^2(W+X)\big]+(n-2)\Xi_sW\Big\}\\
&\Big\{\omega_{sss}\big[(U-sV)\rho_r+(V-sW-sX)\rho_i\check{l}^i\big]^2+\Xi_sW^2\big[|\nabla\rho|^2_{\check{\alpha}}-(\rho_i\check{l}^i)^2\big]\Big\}\\
&-\check{\alpha}\omega\big(g^{jk}C_{1jk}\big)_s\Big\{(U-sV)\rho_r+(V-sW-sX)\rho_i\check{l}^i\Big\}^2\\
&+\check{\alpha}\omega\big(g^{jk}C_{1jk}\big)\Big\{V(U-2sV)\rho_r^2+\big[U(W+X)+V^2-4sV(W+X)\big]\rho_r\rho_i\check{l}^i\\
&+\big[V(W+X)-s(3W-4WX-2X^2)\big](\rho_i\check{l}^i)^2+s^2W^2|\nabla\rho|^2_{\check{\alpha}}\Big\}\\
&-\frac{1}{16}\omega\big[(U-sV)\rho_r+(V-sW-sX)\rho_i\check{l}^i\big]^2\Big\{\omega_{sss}\big[(U-2sV+s^2W+s^2X)\big]^2\\
&+(n-2)\omega^2\Xi_s^2\Big\}-\frac{1}{8}\check{\alpha}^{-2}\omega\Xi_s^2W^3\big[U-2sV+s^2(W+X)\big]\big[|\nabla\rho|^2_{\check{\alpha}}-(\rho_i\check{l}^i)^2\big].
\end{aligned}
$$

Proof. Assume that F and $\bar{F}$ are conformally related. According to Lemma 1, (2) holds. In [8], Yang and Zhang showed that the quantities L_{ijk} of the warped product metric F are

$$
\begin{aligned}
L_{111}=&-\frac{1}{2}\check{\alpha}^{-1}F\left[\frac{\omega_s}{2\sqrt{\omega}}\Phi_{sss}+\left(\sqrt{\omega}-\frac{s}{2\sqrt{\omega}}\omega_s\right)\Psi_{sss}\right],\\
L_{11i}=&\frac{1}{2}s\check{\alpha}^{-1}F\left[\frac{\omega_s}{2\sqrt{\omega}}\Phi_{sss}+\left(\sqrt{\omega}-\frac{s}{2\sqrt{\omega}}\omega_s\right)\Psi_{sss}\right]\check{l}_i,\\
L_{1ij}=&-\frac{1}{2}\check{\alpha}^{-1}F\Big\{s^2\left[\frac{\omega_s}{2\sqrt{\omega}}\Phi_{sss}+\left(\sqrt{\omega}-\frac{s}{2\sqrt{\omega}}\omega_s\right)\Psi_{sss}\right]\check{l}_i\check{l}_j\\
&+\left[\frac{\omega_s}{2\sqrt{\omega}}(\Phi_s-s\Phi_{ss})-s\left(\sqrt{\omega}-\frac{s}{2\sqrt{\omega}}\omega_s\right)\Psi_{ss}\right]\check{h}_{ij}\Big\},\\
L_{ijk}=&\frac{1}{2}s\check{\alpha}^{-1}F\Big\{\left[\frac{s^2\omega_s}{2\sqrt{\omega}}\Phi_{sss}+s^2\left(\sqrt{\omega}-\frac{s}{2\sqrt{\omega}}\omega_s\right)\Psi_{sss}-\frac{3\omega_s}{2\sqrt{\omega}}(\Phi_s-s\Phi_{ss}),\right.\\
&+s\left(\sqrt{\omega}-\frac{s}{2\sqrt{\omega}}\omega_s\right)\Psi_{ss}\Big]\check{l}_i\check{l}_j\check{l}_k+\Big[-\frac{3\omega_s}{2\sqrt{\omega}}(\Phi_s-s\Phi_{ss})\\
&+s\left(\sqrt{\omega}-\frac{s}{2\sqrt{\omega}}\omega_s\right)\Psi_{ss}\Big](\check{a}_{ij}\check{l}_k+\check{a}_{jk}\check{l}_i+\check{a}_{ki}\check{l}_j)\Big\}.
\end{aligned}
$$

Substituting the above equations and (3) into (2) yields (10). □

A metric conformally related to a Finsler warped product metric may not necessarily be a Finsler warped product metric. Specially, when the conformal factor ρ depends only on r, the conformally related metric must also be a Finsler warped product metric. The transformation formula for their Ricci curvature is as follows.

Proposition 3. *Let F and $\bar{F}$ be two Finsler warped product metrics on a manifold M of dimension $n(\geq 3)$. If $\bar{F}=e^{\rho(r)}F$, then Ric and $\overline{Ric}$ satisfy*

$$
\overline{Ric}=Ric+F^2\cdot\left(\Delta\rho_{rr}+\Theta\rho_r^2+\Sigma\rho_r\right), \tag{11}
$$

where

$$\begin{aligned}
\Delta :=& -\frac{n-2}{\omega}s^2 - 2s\omega\omega_{sss}\frac{\Xi}{\Lambda^2} - (n-4)\frac{s\Xi_s}{\Lambda} - \frac{2\Xi}{\Lambda},\\
\Theta :=& 2\omega\omega_{sss}\left[(n-4)\frac{s\Xi}{\Lambda^2} + \omega\omega_{sss}\frac{\Xi^2}{\Lambda^3} + 2\omega^2\omega_{sss}\frac{\Xi^2}{\Lambda^4} - 2\omega_s\frac{\Xi^2}{\Lambda^3} - 2\omega(2\omega - s^2\omega_{ss})\frac{\Xi_s}{\Lambda^3}\right]\\
&+ (n-2)\left[-\frac{2\Xi}{\Lambda} + \omega\omega_{sss}\frac{\Xi_s\Xi}{\Lambda^2} + \omega\frac{\Xi_s^2}{\Lambda^2} - 2\omega(\omega - s^2\omega_{ss})\frac{\Xi_s^2}{\Lambda^2\Xi} + \frac{s^2}{\omega}\right]\\
&- 4\omega^2\omega_{ssss}\frac{\Xi_s^2}{\Lambda^3},\\
\Sigma :=& 2(n-2)\left(\frac{2}{\Xi} + \frac{1}{\omega}\right)\Phi + \frac{4\omega_{ss}}{\Lambda}\Phi - 2\left[\frac{(n-2)s}{\Xi} + \frac{2\omega_s}{\Lambda}\right]\Phi_s + \frac{4\omega}{\Lambda}\Phi_{ss}\\
&+ 2\left[\frac{2\omega\omega_{sss}}{\Lambda} + (n-2)\frac{\Xi_s}{\Xi}\right]\left[\left(\frac{\Xi}{\Lambda}\right)_s - \Psi\frac{\Xi}{\Lambda}\right] - s\left[\frac{2\omega\omega_{sss}\Xi}{\Lambda^2} + (n-2)\frac{\Xi_s}{\Lambda}\right]_r\\
&+ 2\left[\frac{\Xi\big(-2\omega\omega_r - s\omega_r\omega_s + 2s\omega\omega_{rs}\big)}{\Lambda^2}\right]\left[\frac{2\omega\omega_{sss}}{\Lambda} + (n-2)\frac{\Xi_s}{\Xi}\right]_s\\
&+ 2\big[\omega_s\Phi_{sss} + (2\omega - s\omega_s)\Psi_{sss}\big]\left[\frac{2\omega\Xi}{\Lambda^2} + \frac{n-2}{\Lambda}\right].
\end{aligned}$$

Proof. Assuming that the conformal factor is $\rho(r)$, then by Proposition 2, (10) can be simplified as

$$\overline{Ric} = Ric + F^2\cdot\big(\Delta\rho_{rr} + \Theta\rho_r^2 + \Sigma\rho_r\big),$$

where

$$\Delta = -\frac{n-2}{\omega}s^2 - \frac{1}{4}s(U - sV)\big[(U - 2sV + s^2W + s^2X)\omega_{sss} + (n-2)\Xi_sW\big] - U, \tag{12}$$

$$\begin{aligned}
\Theta =& \frac{1}{16}\omega_{sss}\Big\{\Big[-8s(U-sV) + 2\omega\omega_{sss}(U-sV)^2 + 4\omega V(U-2sV)\Big]\Big[U - 2sV\\
&+ s^2(W+X)\Big] - \omega\omega_{sss}\big[U - 2sV + s^2(W+X)\big]^2(U-sV)^2 - 4\omega(U-sV)^2\Big[U_s\\
&- 2V + 2s(W + X - V_s) + s^2(W_s + X_s)\Big]\Big\} + \frac{(n-2)}{16}\Big[-8s(U-sV)\\
&+ 2\omega\omega_{sss}(U-sV)^2 - \omega\Xi_sW(U-sV)^2 + 4\omega V(U-2sV)\Big]\Xi_sW + (n-2)\left(\frac{s^2}{\omega} - U\right)\\
&- \frac{1}{4}\omega\omega_{ssss}(U-sV)^2(U - 2sV + s^2W + s^2X) + (n-2)(\Xi_sW_s + \Xi_{ss}W),
\end{aligned} \tag{13}$$

$$\begin{aligned}
\Sigma =& \frac{2(n-2)}{\omega}\Phi + \Phi_{ss}U + 2(\Phi_sV - s\Phi_{ss}) + (2\Phi - 2s\Phi_s + s^2\Phi_{ss})(W+X)\\
&+ (n-2)(2\Phi - s\Phi_s)W + \left[\frac{1}{2}\omega_s\Phi_{sss} + (\omega - \frac{1}{2}s\omega_s)\Psi_{sss}\right]\Big\{U^2 + (n-2)UW\\
&- 3sUV - (n-2)sVW + s^2\big[(W+X)U + V^2\big] - s^3(W+X)V\Big\}\\
&+ \frac{1}{4}\big[U - 2sV + s^2\omega_{sss}(W+X) + (n-2)\Xi_sW\big]\big[-s(U_r - sV_r) + 2(U_s - V - sV_s)\\
&- 2\Psi(U-sV)\big] - \frac{1}{4}s\omega_{sss}(U-sV)\big[U_r - 2sV_r + s^2(X_r + W_r)\\
&+ \omega_{rsss}(U - 2sV + s^2W + s^2X) + (n-2)\Xi_{rs}W + (n-2)\Xi_sW_r\big]\\
&+ \frac{1}{2}(\Phi - s\Phi)(U-sV)\Big\{\omega_{sss}\big[U_s - 2V + 2s(W + X - V_s) + s^2(W_s + X_s)\big]\\
&+ \omega_{ssss}\big[U - 2sV + s^2(W+X)\big] + (n-2)(\Xi_{ss}W + \Xi_sW_s)\Big\}.
\end{aligned} \tag{14}$$

Furthermore, it is observed that

$$U - sV = 2\frac{\Xi}{\Lambda},\ U - 2sV = 2\frac{2\omega - s^2\omega_{ss}}{\Lambda},\ W + X = 2\frac{\omega_{ss}}{\Lambda},$$
$$V - s(W + X) = -2\frac{\omega_s}{\Lambda},\quad U - 2sV + s^2(W + X) = 4\frac{\omega}{\Lambda},$$
$$\Phi - s\Psi = \frac{-2\omega\omega_r - s\omega_r\omega_s + 2s\omega\omega_{rs}}{\Lambda}.$$

Substituting the above equations into (12), (13), and (14), we can obtain (11). The proposition is proved. □

Based on Proposition 3, for the conformal transformation preserving Einstein metrics on a warped product manifold, we obtain Theorem 1. The proof is as follows.

Proof of Theorem 1. Firstly, we prove the necessity.

Suppose $F = \breve{\alpha}\sqrt{\omega}$ is an Einstein metric with Einstein scalar k. According to Proposition 1, it is known that the Riemannian metric $\breve{\alpha}$ has isotropic Ricci curvature χ and

$$\big[\lambda_s + (n-1)\mu_s - \nu_s\big]\omega = \big[\chi + \lambda + (n-1)\mu - \nu\big]\omega_s$$

holds. Furthermore, since F and $\bar{F}$ are conformally related, according to Proposition 3, it follows that

$$\begin{aligned}\overline{Ric} &= Ric + \big(\Delta\rho_{rr} + \Theta\rho_r^2 + \Sigma\rho_r\big)F^2\\ &= (n-1)kF^2 + \big(\Delta\rho_{rr} + \Theta\rho_r^2 + \Sigma\rho_r\big)F^2.\end{aligned}$$

Because $\bar{F}$ is an Einstein metric with Einstein scalar $\bar{k}$, we have

$$\overline{Ric} = (n-1)\bar{k}\,\bar{F}^2 = (n-1)e^{2\rho}\bar{k}F^2.$$

Thus,

$$(n-1)e^{2\rho(r)}\bar{k}F^2 = (n-1)kF^2 + \big(\Delta\rho_{rr} + \Theta\rho_r^2 + \Sigma\rho_r\big)F^2.$$

Therefore, we can obtain

$$\Delta\rho_{rr} + \Theta\rho_r^2 + \Sigma\rho_r = (n-1)\big[e^{2\rho(r)}\bar{k} - k\big].$$

Next, we prove the sufficiency.

Assume that the Riemannian metric $\breve{\alpha}$ has isotropic Ricci curvature χ and

$$\big[\lambda_s + (n-1)\mu_s - \nu_s\big]\omega = \big[\chi + \lambda + (n-1)\mu - \nu\big]\omega_s.$$

According to Proposition 1, it is known that F is an Einstein metric with Einstein scalar k. Furthermore, since F and $\bar{F}$ are conformally related, according to Proposition 3, it follows that

$$\begin{aligned}\overline{Ric} &= Ric + \big(\Delta\rho_{rr} + \Theta\rho_r^2 + \Sigma\rho_r\big)F^2\\ &= (n-1)kF^2 + \big(\Delta\rho_{rr} + \Theta\rho_r^2 + \Sigma\rho_r\big)F^2.\end{aligned}$$

Since

$$\Delta\rho_{rr} + \Theta\rho_r^2 + \Sigma\rho_r = (n-1)\big[e^{2\rho(r)}\bar{k}(r) - k\big],$$

we have

$$\begin{aligned}\overline{Ric} &= (n-1)kF^2 + \big[\Delta\rho_{rr} + \Theta\rho_r^2 + \Sigma\rho_r\big]F^2\\ &= (n-1)\bar{k}(r)e^{2\rho}F^2\\ &= (n-1)\bar{k}(r)\bar{F}^2.\end{aligned}$$

That is to say, $\bar{F}$ is the Einstein metric, and $\bar{k} = \bar{k}(r)$ is the Einstein scalar of $\bar{F}$. □

Based on Theorem 1, we provide an example of a conformal transformation that preserves Einstein metrics on a Finsler warped product manifold, where the conformal factor $\rho(r)$ can be any function.

Example 1. *Let $\omega(r,s)$ be a function defined by*

$$\omega = s^{-2n},$$

where n is any positive integer. Then it satisfies

$$\Xi = 2\omega - s\omega s > 0,\ \Lambda = 2\omega\omega ss - \omega_s^2 > 0.$$

We have that the Finsler warped product metric

$$F = \breve{\alpha}\sqrt{\omega} = \breve{\alpha}s^{-n}$$

is a positive non-Riemannian metric by Lemma 3. By Proposition 1, the Finsler warped product metric F is an Einstein metric if and only if the Ricci curvature of the Riemannian metric $\breve{\alpha}$ is zero. In this case, F is Ricci-flat.

Furthermore, the conformal related Finsler warped product metric $\bar{F} = e^{\rho(r)}F$ is given by

$$\bar{F} = \breve{\alpha}e^{\rho(r)}s^{-n},$$

where $\rho = \rho(r)$ is any positive function. According to Proposition 1, it is known that $\bar{F}$ is also Ricci-flat.

4. Conformal Transformations Preserving Einstein Metrics on Riemannian Warped Product Manifolds

In this section, we study conformal transformations on Riemannian warped product manifolds and obtain the complete classification of conformal transformations preserving Einstein metrics. First, we need to find equations that characterize Riemannian metrics of the warped product metric type.

Proposition 4. *$F = \breve{\alpha}\sqrt{\omega}$ is a Riemannian warped product metric on an $n(\geq 3)$-dimensional manifold if and only if*

$$\omega = f(r) + g(r)s^2,$$

where $f(r)$ and $g(r)$ are any differentiable positive functions.

Proof. The Finsler warped product metric F is a Riemannian metric if and only if its metric tensor coefficients satisfy $\frac{\partial g_{AB}}{\partial v^C} = 0$. By Lemma 2, we can obtain the following PDE:

$$\begin{cases} \frac{\partial g_{11}}{\partial v^1} = \frac{1}{2}\breve{\alpha}^{-1}\omega_{sss}, \\ \frac{\partial g_{1j}}{\partial v^1} = \frac{1}{2}\breve{\alpha}^{-1}\Xi_{ss}\breve{l}_j, \\ \frac{\partial g_{ij}}{\partial v^1} = \frac{1}{2}\breve{\alpha}^{-1}(\Xi_s\breve{a}_{ij} - \Xi_s\breve{l}_i\breve{l}_j - s\Xi_{ss}\breve{l}_i\breve{l}_j), \\ \frac{\partial g_{11}}{\partial v^k} = -\frac{1}{2}\breve{\alpha}^{-1}s\omega_{sss}\breve{l}_k, \\ \frac{\partial g_{i1}}{\partial v^k} = -\frac{1}{2}\breve{\alpha}^{-1}\left[s\Xi_{ss}\breve{l}_j\breve{l}_k + \Xi_s(\breve{a}_{jk} - \breve{l}_j\breve{l}_k)\right], \\ \frac{\partial g_{ij}}{\partial v^k} = \frac{1}{2}\breve{\alpha}^{-1}s\left[-\Xi_s\breve{a}_{ij}\breve{l}_k + \Xi_s\breve{l}_i\breve{l}_j\breve{l}_k + s\Xi_{ss}\breve{l}_i\breve{l}_j\breve{l}_k - \Xi_s\breve{l}_j(\breve{a}_{ik} - \breve{l}_i\breve{l}_k)\right]. \end{cases}$$

Since $\frac{\partial g_{11}}{\partial v^1} = 0$, we can obtain $\omega_{sss} = 0$. Thus,

$$\Xi_{ss} = -s\omega_{sss} = 0.$$

Furthermore, by $\frac{\partial g_{i1}}{\partial v^k} = 0$, one can obtain $\Xi_s = 0$. Conversely, it is obvious that $\frac{\partial g_{AB}}{\partial v^C} = 0$ if $\Xi_s = 0$. That means F is a Riemannian metric.

In summary, $\Xi_s = 0$ is a necessary and sufficient condition for a Finsler warped product metric F to be a Riemannian metric. Solving this equation yields

$$\omega = f(r) + g(r)s^2,$$

where $f(r)$ and $g(r)$ are arbitrary differentiable positive functions. □

Proposition 5. *The Riemannian warped product metric $F = \breve{\alpha}\sqrt{f(r) + g(r)s^2}$ on a manifold M of dimension $n(\geq 3)$ is an Einstein metric if and only if the Riemannian metric $\breve{\alpha}$ has isotropic Ricci curvature χ and F is one of the following cases:*

$$F = \breve{\alpha}\sqrt{(e^{c_1 r} + c_2 e^{-c_1 r})^2 + gs^2}$$

or

$$F = \breve{\alpha}\sqrt{f(r) + \frac{c_3\left[\frac{df(r)}{dr}\right]^2}{f(r)\left[f(r) + c_4\right]}s^2},$$

where $g(> 0)$, c_1, c_3 are any constants, $c_2 = -\frac{\chi g}{4(n-2)c_1^2}$, $c_4 = \frac{4\chi c_3}{n-2}$, $f(r)$ is any differentiable positive function, and $c_3\left[f(r) + c_4\right] > 0$. Furthermore, the Einstein scalar of F is $-\frac{c_1^2}{g}$ or $-\frac{4}{c_3}$, respectively.

Proof. Since the Riemannian warped product metric F is the Einstein metric, then by Proposition 1 we get the following ODE:

$$\frac{1}{f}(\frac{df}{dr})^2 - \frac{d^2 f}{dr^2} + \frac{1}{2g}\frac{df}{dr}\frac{dg}{dr} = \frac{2\chi g}{(n-2)}. \tag{15}$$

We divide the problem into two cases.

Case 1: g is a constant.

Equation (15) is equivalent to

$$\frac{1}{f}(\frac{df}{dr})^2 - \frac{d^2 f}{dr^2} = \frac{2\chi g}{(n-2)}.$$

One has

$$f = (e^{c_1 r} + c_2 e^{-c_1 r})^2,$$

where c_1 is any constant, $c_2 = -\frac{\chi g}{4(n-2)c_1^2}$. We get

$$\omega = (e^{c_1 r} + c_2 e^{-c_1 r})^2 + gs^2.$$

In this case,

$$F = \breve{\alpha}\sqrt{\omega} = \breve{\alpha}\sqrt{(e^{c_1 r} + c_2 e^{-c_1 r})^2 + gs^2}$$

is a positive Einstein Riemannian metric of the warped product type where $g > 0$.

Case 2: g is not a constant.

Fixing the function f, (15) yields

$$g = \frac{c_3\left[\frac{df(r)}{dr}\right]^2}{f(r)[f(r)+c_4]},$$

where c_3 is any constant and $c_4 = \frac{4\chi c_3}{n-2}$. In this case,

$$F = \breve{\alpha}\sqrt{\omega} = \breve{\alpha}\sqrt{f(r) + \frac{c_3\left[\frac{df(r)}{dr}\right]^2}{f(r)\left[f(r)+c_4\right]}s^2}$$

is a positive Einstein Riemannian metric of the warped product type where $f(r) > 0$, $c_3\left[f(r)+c_4\right] > 0$.

According to Proposition 1, the Einstein scalar of the Riemannian warped product metric F is $-\frac{c_1^2}{g}$ or $-\frac{4}{c_3}$, respectively. □

Based on Proposition 2, the classification of conformal transformations, preserving Einstein metrics on a Riemannian warped product manifold, is obtained.

Theorem 2. *Let F and $\bar{F}$ be conformally related Riemannian warped product metrics on a manifold M of dimension $n(\geq 3)$, i.e., $\bar{F} = e^{\rho(r)}F$. Conformal transformations preserve Einstein metrics if and only if the Riemannian metric $\breve{\alpha}$ has isotropic Ricci curvature χ and F is one of the following cases:*

$$F = \breve{\alpha}\sqrt{(e^{c_1 r} + c_2 e^{-c_1 r})^2 + gs^2}$$

or

$$F = \breve{\alpha}\sqrt{f(r) + \frac{c_3\left[\frac{df(r)}{dr}\right]^2}{f(r)\left[f(r)+c_4\right]}s^2},$$

the corresponding conformal factor is

$$\rho = -\ln\left|-\int e^{c_1 r} + c_2 e^{-c_1 r}dr\right|$$

or

$$\rho = -\ln\left|-\int\sqrt{\frac{c_3\left[\frac{df(r)}{dr}\right]^2}{f(r)+c_4}}dr\right|,$$

where $g(>0)$, c_1, c_3 are any constants, $c_2 = -\frac{\chi g}{4(n-2)c_1^2}$, $c_4 = \frac{4\chi c_3}{n-2}$, $f(r)$ is any differentiable positive function, and $c_3\left[f(r)+c_4\right] > 0$. Furthermore, the Einstein scalar of F is $-\frac{c_1^2}{g}$ or $-\frac{4}{c_3}$, and the Einstein scalar of $\bar{F}$ is $-\frac{(c_4^2+4c_2)c_1^2}{g}$ or $\frac{4c_4}{c_3}$.

Proof. We only need to prove the necessity.

Since the conformal transformation preserves Einstein metrics, by Theorem 1, the second equation of (1) holds. Meanwhile, the Riemannian warped product metric F must be one of two cases stated in Proposition 5. Substituting the expression of the Riemannian warped product metric F into the second equation of (1) yields

$$\rho = -\ln\left|-\int e^{c_1 r} + c_2 e^{-c_1 r}dr\right|$$

or

$$\rho = -\ln\left|-\int\sqrt{\frac{c_3\left[\frac{df(r)}{dr}\right]^2}{f(r)+c_4}}dr\right|.$$

Thus, we have

$$e^{2\rho} = (e^{c_1 r} - c_2 e^{-c_1 r} + c_5)^{-2}$$

or

$$e^{2\rho} = \left[f(r)+c_4\right]^{-1},$$

where c_5 is any positive constant.

Let k be the Einstein scalar of F. According to Proposition 1, the Einstein scalar of $\bar{F}$ is $(c_4^2+4c_2)k$ or $-c_4 k$. That is to say, the Einstein scalar of $\bar{F}$ is $-\frac{(c_4^2+4c_2)c_1^2}{g}$ or $\frac{4c_4}{c_3}$. □

5. Conclusions

The Einstein metric is of great importance in both mathematics and physics. As a representative class of Finsler metrics, the Finsler warped product metric has attracted the attention of many scholars in exploring their geometric properties. This paper presented an equivalent characterization of conformal transformations preserving Einstein metrics on Finsler warped product manifolds. In particular, we classified the conformal transformations that preserve Einstein metrics on Riemannian warped product manifolds.

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