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Large Time Decay Rates of the 2D Micropolar Equations with Linear Velocity Damping

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Abstract: This paper studies the large time behavior of solutions to the 2D micropolar equations with linear damping velocity. It is proven that the global solutions have rapid time decay rates $\|\nabla\omega\|_{L^2} + \|\nabla u\|_{L^2} \leq C(1+t)^{-\frac{3}{2}}$ and $\|u\|_{L^2} \leq C(1+t)^{-\frac{3}{2}}, \|\omega\|_{L^2} \leq C(1+t)^{-1}$. The findings are mainly based on the new observation that linear damping actually improves the low-frequency effect of the system. The methods here are also available for complex fluid models with linear damping structures.

Keywords: 2D micropolar fluid; angular viscosity dissipation; large time decay

MSC: 35B40; 35D35; 76B03

1. Introduction and Main Results

The micropolar equations were first introduced in the 1960s by C.A. Eringen to model micropolar fluids (see [1–6]). The 2D micropolar equations are written as follows:

$$\begin{cases} \partial_t u - (\nu + \kappa)\Delta u - 2\kappa\nabla \times \omega + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0, \\ \partial_t \omega - \gamma\Delta\omega + 4\kappa\omega - 2\kappa\nabla \times u + u \cdot \nabla\omega = 0, \end{cases} \tag{1}$$

where $u = (u_1, u_2)$ is the velocity vector field and p and ω denote the scalar pressure and the microrotation angular velocity. $\nu \geq 0$ is the Newtonian kinetic viscosity, $\kappa > 0$ is the dynamic microrotation viscosity, and $\gamma \geq 0$ is the angular viscosities and

$$\nabla \times u = \partial_{x_1} u_2 - \partial_{x_2} u_1, \quad \nabla \times \omega = (\partial_{x_2}\omega, -\partial_{x_1}\omega).$$

When $\omega = 0$, the micropolar Equation (1) reduces to the classic Navier–Stokes equations [7–9]. Because these equations are mathematically significant, the well-posedness and large time behavior of the micropolar equations attract considerable attention. Ye-Wang-Jia [10] recently investigated the global small solutions of 3D micropolar equations. Song [11] proved the Gevrey analyticity and decay for the micropolar system in the critical Besov space. Wang-Wu-Ye [12] investigated the global regularity of the three-dimensional fractional micropolar equations. Liu [13] considers the global regularity and time decay for a 2D magneto-micropolar system with fractional dissipation and examines the large time behavior of the solution. For the large time behavior, Dong and Chen [14] derived large time L^2 decay rates of solutions to the 2D micropolar equations. When (1) possess linear velocity damping, Dong-Li-Wu [15] (see also [16]) recently examined the global regularity and large time behavior for solutions. They proved the solutions' decay in L^2 as $(1+t)^{-1/2}$ based on a complex diagonalization process to eliminate the linear terms. One may also refer to some important progress in this direction (see [17–25] and references therein).

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Motivated by the decay results [15], this paper is focused on the improved time decay rates for the following 2D micropolar equations with linear velocity damping:

$$\begin{cases} \partial_t u + \kappa u - 2\kappa \nabla \times \omega + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0, \\ \partial_t \omega - \gamma \Delta \omega - 2\kappa \nabla \times u + u \cdot \nabla \omega = 0. \\ u(x,0) = u_0(x), \quad \omega(x,0) = \omega_0(x). \end{cases} \tag{2}$$

On one hand, the complex micropolar fluid flow is usually anisotropic [6], and the linear or nonlinear damping structure shows that the stress momentum is linear or nonlinear in rotation of the particles. This sort micropolar fluid, such as some polymeric fluids and fluids containing certain additives, is less prone to instability than a classical fluid (see [1,4]). On the other hand, physical experiments with the fluids, for example, containing additional additives show that the skin friction near a rigid body are lower, and the dissipation effect here is actually linear [26].

As for the large time decay issue of this model, it should be mentioned that the linear velocity damping implies an exponential decay in the linear equation, which may improve the low-frequency effect of the system (2). Moreover, the Laplace dissipation in w is enough to derive an auxiliary estimate for ∇u and $\nabla \omega$,

$$\|(\nabla u(t),\, \nabla w(t))\|_{L^2} \le C(1+t)^{-1}. \tag{3}$$

Here, and in sequence, C is a constant which may be different from line to line. The above observation allows us to derive the improved decay rates for $\|\nabla u\|_{L^2}$, $\|\nabla \omega\|_{L^2}$ and $\|(u,\, \omega)\|_{L^2}$ using generalized Fourier splitting methods. More precisely, our results read as follows.

Theorem 1. *Assume $u_0 \in H^s(\mathbb{R}^2)$, $\omega_0 \in H^s(\mathbb{R}^2)$ with $s > 2$, div $u_0 = 0$, and*

$$|\hat{u}_0(\xi)| < C|\xi|, \quad |\hat{\omega}_0(\xi)| < C|\xi|. \tag{4}$$

Let $(u,\, \omega)$ be the global solution of the system (2). If

$$\gamma > 4\kappa, \tag{5}$$

then we have the following decay estimates for global solutions:

$$\begin{aligned} &\|u(t)\|_{L^2} \le C(1+t)^{-\frac{3}{2}}, \quad \|\nabla u(t)\|_{L^2} \le C(1+t)^{-\frac{3}{2}}, \\ &\|\omega(t)\|_{L^2} \le C(1+t)^{-1}, \quad \|\nabla \omega(t)\|_{L^2} \le C(1+t)^{-\frac{3}{2}}. \end{aligned}$$

Remark 1. *The results here are obviously improve those of Dong-Li-Wu [15]. The argument here is more direct and can be applied to the other complex fluid models on the large time behavior issues. It should be mentioned that we have no idea whether or not the results hold true in the bounded domain. We will focus this issue in the future.*

2. The Proof of the Theorem 1

This section proves Theorem 1. We first recall some basic facts on linear equations.

Lemma 1. *Let $\alpha > 0$, $\mu > 0$, $1 \le p \le q \le \infty$, and $m \ge 0$. The following $L^p - L^q$ estimates on the semigroup $e^{-\kappa(-\Delta)^\alpha t}$ is valid for any $t > 0$:*

$$\left\| \nabla^m e^{-\mu(-\Delta)^\alpha t} f \right\|_{L^q(\mathbb{R}^2)} \le\ C t^{-\frac{m}{2\alpha} - \frac{1}{\alpha}\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^2)}.$$

where $C = C(p,\, q)$ are constants.

Proof of Theorem 1. We first recall an auxiliary decay estimate of $\|\nabla u(t)\|_{L^2}$ and $\|\nabla\omega(t)\|_{L^2}$ which was proven by [15] (more details give in Appendix A):

$$\|\nabla u(t)\|_{L^2} + \|\nabla\omega(t)\|_{L^2} \le C(1+t)^{-1} \tag{6}$$

Step 1. In this step, we prove the improved decay rate of $\|u\|_{L^2}$. Taking the L^2 inner product of $(2)_1$ with u gives

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 + \kappa\|u\|_{L^2}^2 &= 2\kappa\int_{\mathbb{R}^2}\nabla\times\omega\cdot u\,dx\\ &\le 2\kappa\|u\|_{L^2}\|\nabla\omega\|_{L^2}\end{aligned}$$

Integrating time yields

$$\begin{aligned}\|u(t)\|_{L^2}^2 &\le e^{-2\kappa t}\|u_0\|_{L^2}^2 + 4\kappa\int_0^t e^{-2\kappa(t-s)}\|u(s)\|_{L^2}\|\nabla w(s)\|_{L^2}\,ds\\ &= e^{-2\kappa t}\|u_0\|_{L^2}^2 + 4\kappa\int_0^{\frac{t}{2}} e^{-2\kappa(t-s)}\|u(s)\|_{L^2}\|\nabla w(s)\|_{L^2}\,ds\\ &\quad + 4\kappa\int_{\frac{t}{2}}^t e^{-2\kappa(t-s)}\|u(s)\|_{L^2}\|\nabla w(s)\|_{L^2}\,ds.\end{aligned} \tag{7}$$

Taking the L^2 inner product on (2) with $(u,\,\omega)$, we can obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2) + \kappa\|u\|_{L^2}^2 + \gamma\|\nabla\omega\|_{L^2}^2\\ &= 4\kappa\int_{\mathbb{R}^2}\nabla\times\omega\cdot u\,dx\\ &\le \frac{\gamma+4\kappa}{2}\|\nabla\omega\|_{L^2}^2 + \frac{8\kappa^2}{\gamma+4\kappa}\|u\|_{L^2}^2.\end{aligned}$$

Namely,

$$\frac{d}{dt}(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2) + \frac{2\kappa(\gamma-4\kappa)}{(\gamma+4\kappa)}\|u\|_{L^2}^2 + (\gamma-4\kappa)\|\nabla\omega\|_{L^2}^2 \le 0. \tag{8}$$

Integrating (8) in time from s to t yields

$$\begin{aligned}\int_s^t \|u(\tau)\|_{L^2}^2\,d\tau &\le \frac{\gamma+4\kappa}{2\kappa(\gamma-4\kappa)}\Big(\|u(s)\|_{L^2}^2 + \|w(s)\|_{L^2}^2\Big),\ \ 0\le s\le t\le\infty,\\ \int_s^t \|\nabla w(\tau)\|_{L^2}^2\,d\tau &\le \frac{1}{\gamma-4\kappa}\Big(\|u(s)\|_{L^2}^2 + \|w(s)\|_{L^2}^2\Big),\ \ 0\le s\le t\le\infty.\end{aligned} \tag{9}$$

Using (9), we obtain

$$\begin{aligned}&4\kappa\int_0^{\frac{t}{2}} e^{-2\kappa(t-s)}\|u(s)\|_{L^2}\|\nabla w(s)\|_{L^2}\,ds\\ &\le 4\kappa e^{-\kappa t}\int_0^{\frac{t}{2}}\|u(s)\|_{L^2}\|\nabla w(s)\|_{L^2}\,ds\\ &\le 4\kappa e^{-\kappa t}\left(\int_0^{\frac{t}{2}}\|\nabla w(s)\|_{L^2}^2\,ds\right)^{\frac{1}{2}}\left(\int_0^{\frac{t}{2}}\|u(s)\|_{L^2}^2\,ds\right)^{\frac{1}{2}}\\ &\le C\sqrt{\frac{\gamma+4\kappa}{2\kappa(\gamma-4\kappa)^2}}\Big(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2\Big)e^{-\kappa t}\\ &\le Ce^{-\kappa t},\end{aligned}$$

Then, (7) obeys

$$\|u(t)\|_{L^2}^2 \le Ce^{-\kappa t} + 4\kappa \int_{\frac{t}{2}}^{t} e^{-2\kappa(t-s)} \|u(s)\|_{L^2} \|\nabla w(s)\|_{L^2}\, ds. \tag{10}$$

Multiplying (10) by $(1+t)^2$ yields

$$\begin{aligned}(1+t)^2\|u(t)\|_{L^2}^2 \le{}& C(1+t)^2 e^{-\kappa t} \\ &+ 4\kappa(1+t)^2 \int_{\frac{t}{2}}^{t} e^{-2\kappa(t-s)} (1+s)^{-2} ((1+s)\|u(s)\|_{L^2} (1+s)\|\nabla w(s)\|_{L^2})\, ds.\end{aligned}$$

Next, we shall prove that

$$(1+t)\|u(t)\|_{L^2} \to 0 \ \text{ as } \ t \to \infty. \tag{11}$$

Denote

$$\mathcal{M}(t) = \sup_{0\le s\le t} \{(1+s)\|u(s)\|_{L^2}\}, \tag{12}$$

since

$$\begin{aligned}\mathcal{M}^2(t) &\le C + 4\kappa(1+t)^2 \mathcal{M}(t) \int_{\frac{t}{2}}^{t} e^{-2\kappa(t-s)} (1+s)^{-2}\, ds \\ &\le C + C\mathcal{M}(t) \le \frac{1}{2}\mathcal{M}^2(t) + C.\end{aligned}$$

Thus, we have

$$\mathcal{M}(t) \le C,$$

and

$$\begin{aligned}(1+t)^2\|u(t)\|_{L^2}^2 &\le C(1+t)^2 e^{-\kappa t} + 4\kappa(1+t)^2 \mathcal{M}(t)\mathcal{P}(t) \int_{\frac{t}{2}}^{t} e^{-2\kappa(t-s)} (1+s)^{-2}\, ds \\ &\le C(1+t)^2 e^{-\kappa t} + C\mathcal{P}(t) \to 0 \ \text{ as } \ t \to \infty,\end{aligned} \tag{13}$$

if

$$\mathcal{P}(t) \equiv \sup_{\frac{t}{2}\le s\le t} \{(1+t)\|\nabla w(t)\|_{L^2}\} \to 0 \ \text{ as } \ t \to \infty.$$

Step 2. In this step, we will give the improved uniform decay rates of $\|\nabla u\|_{L^2}$.

The basic energy estimates yield

$$\begin{aligned}&\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + \int_0^t \left(\frac{2\kappa(\gamma - 4\kappa)}{(\gamma + 4\kappa)} \|u\|_{L^2}^2 + (\gamma - 4\kappa)\|\nabla\omega\|_{L^2}^2 \right) d\tau \\ &\le \|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2.\end{aligned} \tag{14}$$

Applying Fourier transformation on $(2)_1$, we can obtain

$$\begin{aligned}\frac{d\hat{u}}{dt} + \kappa\hat{u} &= 2\kappa\widehat{\nabla\times\omega} - \widehat{u\cdot\nabla u} - \widehat{\nabla p}, \\ \frac{d\bar{\hat{u}}}{dt} + \kappa\bar{\hat{u}} &= 2\kappa\overline{\widehat{\nabla\times\omega}} - \overline{\widehat{u\cdot\nabla u}} - \overline{\widehat{\nabla p}}.\end{aligned} \tag{15}$$

Multiplying the above two equations by $\bar{\hat{u}}$ and $\hat{u}$, respectively, and adding the resulting equations together, we obtain

$$\frac{1}{2}\frac{d}{dt}|\hat{u}|^2 + \kappa|\hat{u}|^2 = \mathrm{Re}\{2\kappa\widehat{\nabla\times\omega}\bar{\hat{u}} - \widehat{u\cdot\nabla u}\bar{\hat{u}} - \widehat{\nabla p}\bar{\hat{u}}\}. \tag{16}$$

Performing the same actions on $(2)_3$ gives

$$\frac{1}{2}\frac{d}{dt}|\hat{\omega}|^2+\gamma|\xi|^2|\hat{\omega}|^2=\mathrm{Re}\{2\kappa\widehat{\nabla\times u}\bar{\tilde{\omega}}-\widehat{u\cdot\nabla\omega}\bar{\tilde{\omega}}\}. \tag{17}$$

Combining (16) with (17), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|\hat{u}|^2+|\hat{\omega}|^2)+\kappa|\hat{u}|^2+\gamma|\xi|^2|\hat{\omega}|^2\\&=\mathrm{Re}\{2\kappa\widehat{\nabla\times\omega}\bar{\tilde{u}}+2\kappa\widehat{\nabla\times u}\bar{\tilde{\omega}}\}+\mathrm{Re}\{-\widehat{u\cdot\nabla u}\bar{\tilde{u}}-\widehat{u\cdot\nabla\omega}\bar{\tilde{\omega}}\}+\mathrm{Re}\{-\widehat{\nabla p}\bar{\tilde{u}}\}\\&=I+II+III.\end{aligned} \tag{18}$$

The three terms on the right hand side are bounded by

$$\begin{aligned}I&=\mathrm{Re}\{2\kappa\widehat{\nabla\times\omega}\bar{\tilde{u}}+2\kappa\widehat{\nabla\times u}\bar{\tilde{\omega}}\}\\&\le 4\kappa|\xi||\hat{u}||\hat{\omega}|\\&\le\frac{\gamma+4\kappa}{2}|\xi|^2|\hat{\omega}|^2+\frac{8\kappa^2}{\gamma+4\kappa}|\hat{u}|^2,\\II&=\mathrm{Re}\{-\widehat{u\cdot\nabla u}\bar{\tilde{u}}-\widehat{u\cdot\nabla\omega}\bar{\tilde{\omega}}\}\\&\le|\xi||\widehat{u\otimes u}|\hat{u}|+|\xi||\widehat{u\otimes\omega}|\hat{\omega}|\\&\le C|\xi|\|u\|_{L^2}^2|\hat{u}|+C|\xi|\|u\|_{L^2}\|\omega\|_{L^2}|\hat{\omega}|\\&\le C|\xi|(\|u\|_{L^2}^2+\|\omega\|_{L^2}^2)(|\hat{u}|+|\hat{\omega}|)\\III&=\mathrm{Re}\{-\widehat{\nabla p}\bar{\tilde{u}}\}\\&=\mathrm{Re}\{-i\xi_j\frac{\xi_k\xi_l}{|\xi|^2}\widehat{u_ku_l}\overline{\widehat{u_j}}\}\\&\le|\xi||\widehat{u\otimes u}||\hat{u}|\\&\le C|\xi|\|u\|_{L^2}^2|\hat{u}|.\end{aligned}$$

Then, (18) reduces to

$$\frac{d}{dt}(|\hat{u}|^2+|\hat{\omega}|^2)+\frac{2\kappa(\gamma-4\kappa)}{\gamma+4\kappa}|\hat{u}|^2+(\gamma-4\kappa)|\xi|^2|\hat{\omega}|^2\le\ C|\xi|(\|u\|_{L^2}^2+\|\omega\|_{L^2}^2)(|\hat{u}|+|\hat{\omega}|).$$

If $|\xi|\le 1$, then we have

$$\begin{aligned}&\frac{d}{dt}(|\hat{u}|^2+|\hat{\omega}|^2)+2\min\left\{\frac{\kappa(\gamma-4\kappa)}{\gamma+4\kappa},\ \frac{\gamma-4\kappa}{2}\right\}|\xi|^2(|\hat{u}|^2+|\hat{\omega}|^2)\\&\le C|\xi|(\|u\|_{L^2}^2+\|\omega\|_{L^2}^2)(|\hat{u}|^2+|\hat{\omega}|^2)^{\frac{1}{2}}.\end{aligned} \tag{19}$$

Denote

$$\eta=\min\left\{\frac{2\kappa(\gamma-4\kappa)}{\gamma+4\kappa},\ \frac{\gamma-4\kappa}{2}\right\}.$$

Dividing both sides of (19) by $(|\hat{u}|^2+|\hat{\omega}|^2)^{\frac{1}{2}}$, we can obtain

$$\frac{d}{dt}(|\hat{u}|^2+|\hat{\omega}|^2)^{\frac{1}{2}}+\eta|\xi|^2(|\hat{u}|^2+|\hat{\omega}|^2)^{\frac{1}{2}}\le\ C|\xi|(\|u\|_{L^2}^2+\|\omega\|_{L^2}^2).$$

Using Duhamel's Principle and Lemma 1, we obtain

$$\begin{aligned}
&(|\hat{u}|^2+|\hat{\omega}|^2)^{\frac{1}{2}}\\
&\leq e^{-\eta|\xi|^2 t}(|\hat{u}_0|^2+|\hat{\omega}_0|^2)^{\frac{1}{2}}+C\int_0^t e^{-\eta|\xi|^2(t-s)}|\xi|(\|u\|_{L^2}^2+\|\omega\|_{L^2}^2)\,ds\\
&\leq e^{-\eta|\xi|^2 t}|\hat{u}_0|+e^{-\eta|\xi|^2 t}|\hat{\omega}_0|+C\int_0^t e^{-\eta|\xi|^2(t-s)}|\xi|(\|u\|_{L^2}^2+\|\omega\|_{L^2}^2)\,ds.
\end{aligned} \tag{20}$$

Taking the L^2-inner product of (2) with $(-\Delta u,\ -\Delta\omega)$, we can obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2)+\frac{2\kappa(\gamma-4\kappa)}{\gamma+12\kappa}\|\nabla u\|_{L^2}^2+(\gamma-4\kappa)\|\nabla^2\omega\|_{L^2}^2\\
&\leq\frac{2}{\gamma-4\kappa}(\|\nabla u\|_{L^2}^2\|\nabla\omega\|_{L^2}^2).
\end{aligned}$$

Since $\frac{2\kappa(\gamma-4\kappa)}{\gamma+12\kappa}<(\gamma-4\kappa)$,

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2)+\frac{2\kappa(\gamma-4\kappa)}{\gamma+12\kappa}(\|\nabla u\|_{L^2}^2+\|\nabla^2\omega\|_{L^2}^2)\\
&\leq\frac{2}{\gamma-4\kappa}(\|\nabla u\|_{L^2}^2\|\nabla\omega\|_{L^2}^2).
\end{aligned}$$

Let $\mu=\frac{2\kappa(\gamma-4\kappa)}{\gamma+12\kappa}$, and applying Plancherel's theorem, we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\widehat{\nabla u}\|_{L^2}^2+\|\widehat{\nabla\omega}\|_{L^2}^2)+\mu(\|\widehat{\nabla u}\|_{L^2}^2+\|\widehat{\nabla^2\omega}\|_{L^2}^2)\\
&\leq\frac{2}{\gamma-4\kappa}(\|\nabla u\|_{L^2}^2\|\nabla\omega\|_{L^2}^2).
\end{aligned}$$

Denote

$$B(t)=\left\{\xi\in\mathbb{R}^2;|\xi|^2\leq\frac{n}{\mu(1+t)}\right\},$$

since

$$\begin{aligned}
\|\widehat{\nabla^2\omega}\|_{L^2}^2&=\int_{\mathbb{R}^2}|\xi|^2|\widehat{\nabla\omega}|^2\mathrm{d}\xi\\
&\geq\int_{\mathcal{B}(t)^c}|\xi|^2|\widehat{\nabla\omega}|^2\mathrm{d}\xi\\
&\geq\frac{n}{\mu(1+t)}\int_{\mathcal{B}(t)^c}|\widehat{\nabla\omega}|^2\mathrm{d}\xi\\
&=\frac{n}{\mu(1+t)}\left(\int_{\mathbb{R}^2}|\widehat{\nabla\omega}|^2\mathrm{d}\xi-\int_{\mathcal{B}(t)}|\widehat{\nabla\omega}|^2\mathrm{d}\xi\right),
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\widehat{\nabla u}\|_{L^2}^2+\|\widehat{\nabla\omega}\|_{L^2}^2)+\frac{n}{1+t}(\|\widehat{\nabla u}\|_{L^2}^2+\|\widehat{\nabla\omega}\|_{L^2}^2)\\
&\leq\frac{n}{1+t}\int_{\mathcal{B}(t)}|\widehat{\nabla\omega}|^2d\xi+\frac{2}{\gamma-4\kappa}(\|\nabla u\|_{L^2}^2\|\nabla\omega\|_{L^2}^2).
\end{aligned}$$

Multiplying both sides by $(1+t)^n$ and integrating the resulting inequality in time from 0 to t gives

$$\begin{aligned}
&(1+t)^n(\|\widehat{\nabla u}\|_{L^2}^2+\|\widehat{\nabla\omega}\|_{L^2}^2)\\
&\le(\|\nabla u_0\|_{L^2}^2+\|\nabla\omega_0\|_{L^2}^2)+C\int_0^t(1+s)^n\|\nabla u\|_{L^2}^2\|\nabla\omega\|_{L^2}^2ds\\
&\quad+C\int_0^t(1+s)^{n-1}\int_{\mathcal{B}(s)}|\xi|^2|\hat{\omega}|^2d\xi ds.
\end{aligned}\tag{21}$$

Since for any $\frac{1}{p}+\frac{1}{q}=1, 2<p<\infty, 1<q<2$,

$$\begin{aligned}
\int_0^s\|\omega(\tau)\|_{L^2}^2\,d\tau&\le(s)^{\frac{1}{p}}\left(\int_0^s\|\omega(\tau)\|_{L^2}^{2q}\,d\tau\right)^{\frac{1}{q}}\\
&\le(s)^{\frac{1}{p}}\left(\int_0^s\|\omega(\tau)\|_{L^2}^{2q}\,d\tau\right)^{\frac{1}{q}}\\
&\le(1+s)^{\frac{1}{p}}\left(\int_0^s(1+\tau)^{-q}\,d\tau\right)^{\frac{1}{q}}\\
&\le(1+s)^{\frac{1}{p}}\left((1-q)(1+s)^{1-q}-(1-q)\right)^{\frac{1}{q}}\\
&\le C(1+s)^{\frac{1}{p}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathcal{B}(s)}|e^{-\eta|\xi|^2t}\hat{u}_0|^2+|e^{-\eta|\xi|^2t}\hat{\omega}_0|^2\,d\xi\\
&\le\|e^{-\eta|\xi|^2t}\hat{u}_0\|_{L^2(\mathbb{R}^2_\xi)}^2+\|e^{-\eta|\xi|^2t}\hat{\omega}_0\|_{L^2(\mathbb{R}^2_\xi)}^2\\
&\le C\|e^{-\eta|\xi|^2t}|\xi|\|_{L^2(\mathbb{R}^2_\xi)}^2\le C(1+t)^{-2},
\end{aligned}$$

then applying (13) and (20) to the third term of the right hand side of (21), we obtain

$$\begin{aligned}
&\int_0^t(1+s)^{n-1}\int_{\mathcal{B}(s)}|\xi|^2|\hat{\omega}|^2d\xi ds\\
&\le\int_0^t(1+s)^{n-2}\int_{\mathcal{B}(s)}|\hat{\omega}|^2d\xi ds\\
&\le C\int_0^t(1+s)^{n-2}\int_{\mathcal{B}(s)}|e^{-\eta|\xi|^2t}\hat{u}_0|^2+|e^{-\eta|\xi|^2t}\hat{\omega}_0|^2\,d\xi ds\\
&\quad+C\int_0^t(1+s)^{n-2}\int_{\mathcal{B}(s)}\left(|\xi|\int_0^s(\|u(\tau)\|_{L^2}^2+\|\omega(\tau)\|_{L^2}^2)\,d\tau\right)^2d\xi ds\\
&\le C\int_0^t(1+s)^{n-4}ds+C\int_0^t(1+s)^{n-3}\int_{\mathcal{B}(s)}\left(\int_0^s\|u(\tau)\|_{L^2}^2\,d\tau\right)^2d\xi ds\\
&\quad+C\int_0^t(1+s)^{n-3}\int_{\mathcal{B}(s)}\left(\int_0^s\|\omega(\tau)\|_{L^2}^2\,d\tau\right)^2d\xi ds\\
&\le C\int_0^t(1+s)^{n-4}ds+C\int_0^t(1+s)^{n-5}ds+C\int_0^t(1+s)^{n-4}ds\\
&\quad+C\int_0^t(1+s)^{n-3}\int_{\mathcal{B}(s)}\left(\int_0^s\|\omega(\tau)\|_{L^2}^2\,d\tau\right)^2d\xi ds\\
&\le C\int_0^t(1+s)^{n-4}ds+C\int_0^t(1+s)^{n-5}ds+C\int_0^t(1+s)^{n-4}ds\\
&\quad+C\int_0^t(1+s)^{n-4+\frac{2}{p}}ds\\
&\le C\int_0^t(1+s)^{n-4+\frac{2}{p}}ds\\
&\le C(1+t)^{n-3+\frac{2}{p}}
\end{aligned}\tag{22}$$

for any $2 < p < \infty$. Combining this with (21) and applying (6), we obtain

$$\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 \leq C(1+t)^{-3+\frac{1}{r}} \tag{23}$$

for any $1 < r < \infty$.

Next, we shall prove that

$$\|u(t)\|_{L^2}^2 \leq C(1+t)^{-3+\frac{1}{r}}, \ 1 < r < \infty$$

Multiplying (10) by $(1+t)^{3-\frac{1}{r}}$ $(1 < r < \infty)$ yields

$$\begin{aligned}(1+t)^{3-\frac{1}{r}}\|u(t)\|_{L^2}^2 \leq\ & C(1+t)^{3-\frac{1}{r}}e^{-\kappa t} \\ & + 4\kappa(1+t)^{3-\frac{1}{r}}\int_{\frac{t}{2}}^{t} e^{-2\kappa(t-s)}(1+s)^{-3+\frac{1}{r}}\left((1+s)^{\frac{3-\frac{1}{r}}{2}}\|u(s)\|_{L^2}(1+s)^{\frac{3-\frac{1}{r}}{2}}\|\nabla w(s)\|_{L^2}\right)ds.\end{aligned}$$

Denoting

$$\mathcal{N}(t) = \sup_{0\leq s\leq t}\left\{(1+s)^{\frac{3-\frac{1}{r}}{2}}\|u(s)\|_{L^2}\right\}$$

and using the uniform bounds

$$\begin{aligned}(1+t)^{3-\frac{1}{r}}e^{-\kappa t} &\leq C,\\ (1+t)^{\frac{3-\frac{1}{r}}{2}}\|\nabla w(t)\|_{L^2} &\leq C,\end{aligned}$$

we have

$$\begin{aligned}\mathcal{N}^2(t) &\leq C + 4\kappa(1+t)^{3-\frac{1}{r}}\mathcal{M}(t)\int_{\frac{t}{2}}^{t} e^{-2\kappa(t-s)}(1+s)^{-3+\frac{1}{r}}ds\\ &\leq C + C\mathcal{N}(t)\\ &\leq \frac{1}{2}\mathcal{N}^2(t) + C,\end{aligned}$$

which implies the uniform bound $\mathcal{N}(t) \leq C$ for all $t \geq 0$. So, we have

$$\|u(t)\|_{L^2}^2 \leq C(1+t)^{-3+\frac{1}{r}}, \ 1 < r < \infty. \tag{24}$$

Since

$$\frac{d}{dt}\|\omega\|_{L^2}^2 + 2\gamma\|\nabla\omega\|_{L^2}^2 = 4\kappa\int_{\mathbb{R}^2}(\nabla\times u)\cdot\omega\,dx \leq 4\kappa\|\nabla\omega\|_{L^2}\|u\|_{L^2}.$$

Applying Plancherel's theorem yields

$$\frac{d}{dt}\|\hat{\omega}\|_{L^2}^2 + 2\gamma\int_{\mathbb{R}^2}|\xi|^2|\hat{\omega}|^2\,d\xi \leq 4\kappa\|\nabla\omega\|_{L^2}\|u\|_{L^2}.$$

Denote

$$T(t) = \left\{\xi\in\mathbb{R}^2; |\xi|^2 \leq \frac{n}{2\gamma(1+t)}\right\}.$$

Since

$$\begin{aligned}\int_{\mathbb{R}^2}|\xi|^2|\hat{\omega}|^2\,d\xi &\geq \int_{\mathcal{T}(t)^c}|\xi|^2||\hat{\omega}|^2\,d\xi\\ &\geq \frac{n}{2\gamma(1+t)}\left(\int_{\mathbb{R}^2}|\hat{\omega}|^2\,d\xi - \int_{\mathcal{T}(t)}|\hat{\omega}|^2\,d\xi\right),\end{aligned}$$

we have

$$\frac{d}{dt}\|\hat{\omega}\|_{L^2}^2 + \frac{n}{1+t}\|\hat{\omega}\|_{L^2}^2 \leq \frac{n}{1+t}\int_{\mathcal{T}(t)} |\hat{\omega}|^2\, d\xi + 4\kappa\|\nabla\omega\|_{L^2}\|u\|_{L^2}.$$

Multiplying both sides by $(1+t)^n$ and integrating the resulting inequality in time from 0 to t gives

$$\begin{aligned} &(1+t)^n\|\omega\|_{L^2}^2 \\ &\leq \|\omega_0\|_{L^2}^2 + n\int_0^t (1+s)^{n-1}\int_{\mathcal{T}(s)} |\hat{\omega}|^2\, d\xi ds + 2\kappa\int_0^t (1+s)^n \|\nabla\omega\|_{L^2}\|u\|_{L^2}\, ds \end{aligned} \tag{25}$$

Applying (24) for the second term on the right hand side of (25), we can obtain

$$\begin{aligned} &\int_0^t (1+s)^{n-1}\int_{\mathcal{T}(s)} |\hat{\omega}|^2 d\xi ds \\ &\leq C\int_0^t (1+s)^{n-1}\int_{\mathcal{T}(s)} |e^{-\eta|\xi|^2 t}\hat{u}_0|^2 + |e^{-\eta|\xi|^2 t}\hat{\omega}_0|^2\, d\xi ds \\ &\quad + C\int_0^t (1+s)^{n-1}\int_{\mathcal{T}(s)} \left(|\xi|\int_0^s (\|u(\tau)\|_{L^2}^2 + \|\omega(\tau)\|_{L^2}^2)\, d\tau\right)^2 d\xi ds \\ &\leq C\int_0^t (1+s)^{n-3} ds + C\int_0^t (1+s)^{n-2}\int_{\mathcal{T}(s)} \left(\int_0^s \|u(\tau)\|_{L^2}^2\, d\tau\right)^2 d\xi ds \\ &\quad + C\int_0^t (1+s)^{n-2}\int_{\mathcal{T}(s)} \left(\int_0^s \|\omega(\tau)\|_{L^2}^2\, d\tau\right)^2 d\xi ds \\ &\leq C\int_0^t (1+s)^{n-3} ds + C\int_0^t (1+s)^{n-4} ds + C\int_0^t (1+s)^{n-3} ds \\ &\quad + C\int_0^t (1+s)^{n-2}\int_{\mathcal{T}(s)} \left(\int_0^s \|\omega(\tau)\|_{L^2}^2\, d\tau\right)^2 d\xi ds \\ &\leq C\int_0^t (1+s)^{n-3} ds + C\int_0^t (1+s)^{n-4} ds + C\int_0^t (1+s)^{n-3} ds \\ &\quad + C\int_0^t (1+s)^{n-3+\frac{2}{p}} ds \\ &\leq C\int_0^t (1+s)^{n-3+\frac{2}{p}} ds \\ &\leq C(1+t)^{n-2+\frac{2}{p}} \end{aligned} \tag{26}$$

for any $2 < p < \infty$. Plugging this estimate into (25), and combining it with (23) and (24), we obtain

$$\|\omega\|_{L^2}^2 \leq C(1+t)^{-2+\frac{1}{r}} \tag{27}$$

for any $1 < r < \infty$. Plugging this estimate into (22), we can obtain

$$\begin{aligned} &\int_0^t (1+s)^{n-1}\int_{\mathcal{B}(s)} |\xi|^2|\hat{\omega}|^2 d\xi ds \\ &\leq C\int_0^t (1+s)^{n-4} ds + C\int_0^t (1+s)^{n-5} ds + C\int_0^t (1+s)^{n-4} ds \\ &\quad + C\int_0^t (1+s)^{n-3}\int_{\mathcal{B}(s)} \left(\int_0^s \|\omega(\tau)\|_{L^2}^2\, d\tau\right)^2 d\xi ds \\ &\leq C\int_0^t (1+s)^{n-4} ds + C\int_0^t (1+s)^{n-5} ds + C\int_0^t (1+s)^{n-4} ds \\ &\quad + C\int_0^t (1+s)^{n-4} ds \\ &\leq C\int_0^t (1+s)^{n-4} ds \\ &\leq C(1+t)^{n-3}. \end{aligned}$$

Combining this with (21) and applying (23), we obtain

$$\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 \le C(1+t)^{-3}. \tag{28}$$

Multiplying (10) by $(1+t)^3$ yields

$$\begin{aligned}(1+t)^3\|u(t)\|_{L^2}^2 \le{}& C(1+t)^3 e^{-\kappa t}\\ &+ 4\kappa(1+t)^3\int_{\frac{t}{2}}^{t} e^{-2\kappa(t-s)}(1+s)^{-3}\Big((1+s)^{\frac{3}{2}}\|u(s)\|_{L^2}(1+s)^{\frac{3}{2}}\|\nabla w(s)\|_{L^2}\Big)ds.\end{aligned}$$

Similarly, we have

$$\|u(t)\|_{L^2}^2 \le C(1+t)^{-3}. \tag{29}$$

Applying (27) and (29) for the second term on the right hand side of (25), we can obtain

$$\begin{aligned}&\int_0^t (1+s)^{n-1}\int_{\mathcal{T}(s)} |\hat{\omega}|^2 d\xi ds\\ &\le C\int_0^t (1+s)^{n-3}ds + C\int_0^t (1+s)^{n-4}ds + C\int_0^t (1+s)^{n-3}ds\\ &\quad + C\int_0^t (1+s)^{n-2}\int_{\mathcal{T}(s)}\left(\int_0^s \|\omega(\tau)\|_{L^2}^2\, d\tau\right)^2 d\xi ds\\ &\le C\int_0^t (1+s)^{n-3}ds + C\int_0^t (1+s)^{n-4}ds + C\int_0^t (1+s)^{n-3}ds\\ &\quad + C\int_0^t (1+s)^{n-3}ds\\ &\le C\int_0^t (1+s)^{n-3}ds\\ &\le C(1+t)^{n-2}.\end{aligned}$$

Plugging this estimate into (25), and combining it with (28) and (29), we obtain

$$\|\omega\|_{L^2}^2 \le C(1+t)^{-2}. \tag{30}$$

To sum up, we have the improved decay rates

$$\begin{aligned}&\|u(t)\|_{L^2} \le C(1+t)^{-\frac{3}{2}}, \quad \|\nabla u(t)\|_{L^2} \le C(1+t)^{-\frac{3}{2}},\\ &\|\omega(t)\|_{L^2} \le C(1+t)^{-1}, \quad \|\nabla \omega(t)\|_{L^2} \le C(1+t)^{-\frac{3}{2}}.\end{aligned}$$

Thus, we complete the proof of Theorem 1. □

3. Conclusions

The 2D microplar Equation (2) with linear velocity damping implies faster time decay rates. In fact, in comparison with the 2D classic Navier–Stokes equations (see [7]), where the decay rates of velocity are

$$\|u\|_{L^2} \le C(1+t)^{-\frac{1}{2}}$$

and the gradient of velocity decays as

$$\|\nabla u\|_{L^2} \le C(1+t)^{-1},$$

the system (2) shows the velocity field and rotational velocity fields decay as

$$\|u\|_{L^2} \le C(1+t)^{-\frac{3}{2}}, \quad \|\omega\|_{L^2} \le C(1+t)^{-1}$$

and

$$\|\nabla\omega\|_{L^2} + \|\nabla u\|_{L^2} \leq C(1+t)^{-\frac{3}{2}}.$$

The second conclusion is that the methods present in this study can be applied to the other complex fluid flows where the stress is linear; that is to say, if the fluid system has a linear damping structure, we can examine the large time decay rates of the solutions.

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Appendix A. Decay Estimates of Global Weak Solutions

This appendix provides the proof of the basic L^2 decay estimates of global solutions using a direct method. The same results have been proven in [15] by a complex diagonalization process. Here, we only use the generalized Fourier splitting methods.

Proposition A1. *Under the same condition in Theorem 1 with* $u_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, $\omega_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, *then*

$$\|u\|_{L^2} + \|\omega\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}.$$

Proof of Proposition A1. We apply the generalized Fourier splitting methods to the partially dissipative micropolar equations. Denote

$$S(t) = \left\{\xi \in \mathbb{R}^2; |\xi|^2 \leq \frac{f'(t)}{2\eta f(t)}\right\}. \tag{A1}$$

Assume $f(t)$ to be a continuously differentiable function with respect to $t \geq 0$ and satisfying $f(0) = 1$, $f^{'}(t) > 0$, and

$$\frac{f^{'}(t)}{2\eta f(t)} \leq 1, \text{ for any } t > 0.$$

Multiplying both sides of inequality (8) by $f(t)$ yields

$$\begin{aligned}&\frac{d}{dt}\left(f(t)\left(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2\right)\right) + 2\eta f(t)\left(\|u\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2\right)\\ \leq\ & f^{'}(t)\left(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2\right).\end{aligned}$$

Using Plancherel's theorem, we have

$$\begin{aligned}&\frac{d}{dt}\left(f(t)\left(\|\hat{u}\|_{L^2}^2 + \|\hat{\omega}\|_{L^2}^2\right)\right) + 2\eta f(t)\int_{\mathbb{R}^2}\left(|\hat{u}(\xi)|^2 + |\xi|^2|\hat{\omega}(\xi)|^2\right)d\xi\\ \leq\ & f^{'}(t)\int_{\mathbb{R}^2}\left(|\hat{u}(\xi)|^2 + |\hat{\omega}(\xi)|^2\right)d\xi.\end{aligned}$$

Since

$$\begin{aligned}
&2\eta f(t)\int_{\mathbb{R}^2}\Big(|\hat{u}(\xi)|^2+|\xi|^2|\hat{\omega}(\xi)|^2\Big)d\xi\\
&\geq 2\eta f(t)\int_{\mathcal{S}(t)^c}|\xi|^2\Big(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2\Big)d\xi+2\eta f(t)\int_{\mathcal{S}(t)}|\xi|^2\Big(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2\Big)d\xi\\
&\geq 2\eta f(t)\int_{\mathcal{S}(t)^c}|\xi|^2\Big(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2\Big)d\xi\\
&\geq f'(t)\int_{\mathcal{S}(t)^c}\Big(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2\Big)d\xi+f'(t)\int_{\mathcal{S}(t)}\Big(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2\Big)d\xi\\
&\quad-f'(t)\int_{\mathcal{S}(t)}\Big(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2\Big)d\xi\\
&=f'(t)\int_{\mathbb{R}^2}\Big(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2\Big)d\xi-f'(t)\int_{\mathcal{S}(t)}\Big(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2\Big)d\xi;
\end{aligned}$$

therefore, we obtain

$$\frac{d}{dt}\big(f(t)\Big(\|\hat{u}\|_{L^2}^2+\|\hat{\omega}\|_{L^2}^2\Big)\big)\leq f'(t)\int_{\mathcal{S}(t)}\Big(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2\Big)d\xi.$$

Applying (20), it follows that

$$\begin{aligned}
\frac{d}{dt}\big(f(t)\Big(\|\hat{u}\|_{L^2}^2+\|\hat{\omega}\|_{L^2}^2\Big)\big)&\leq f'(t)\int_{\mathcal{S}(t)}\Big(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2\Big)d\xi\\
&\leq Cf'(t)\int_{\mathcal{S}(t)}\left(1+|\xi|\int_0^t\Big(\|u\|_{L^2}^2+\|\omega\|_{L^2}^2\Big)d\tau\right)^2d\xi\\
&\leq Cf'(t)\left\{\frac{f'(t)}{f(t)}+C\frac{(f'(t))^2}{f^2(t)}\left(\int_0^t\Big(\|u\|_{L^2}^2+\|\omega\|_{L^2}^2\Big)d\tau\right)^2\right\}.
\end{aligned}$$

Integrating in time from 0 to t, we can obtain

$$\begin{aligned}
&f(t)\big(\|\hat{u}\|_{L^2}^2+\|\hat{\omega}\|_{L^2}^2\big)\\
&\leq\big(\|u_0\|_{L^2}^2+\|\omega_0\|_{L^2}^2\big)+C\int_0^t\frac{(f'(s))^2}{f(s)}\,ds\\
&\quad+C\int_0^t\frac{(f'(s))^3}{f^2(s)}\left(\int_0^s\Big(\|u\|_{L^2}^2+\|\omega\|_{L^2}^2\Big)d\tau\right)^2ds.
\end{aligned}\tag{A2}$$

Taking $f(t)=[\ln(e+t)]^3$, we have the fact that

$$\frac{\Big(f'(t)\Big)^2}{f(t)}=\frac{C\ln(e+t)}{(e+t)^2},\quad\frac{\Big(f'(t)\Big)^3}{f^2(t)}=\frac{C}{(e+t)^3}.\tag{A3}$$

Inserting (A3) into (A2) and then applying (21), we obtain

$$\begin{aligned}
&(\ln(e+t))^3\Big(\|\hat{u}\|_{L^2}^2+\|\hat{\omega}\|_{L^2}^2\Big)\\
&\leq C+C\int_0^t\frac{\ln(e+s)}{(e+s)^2}\,ds+C\int_0^t\frac{1}{(e+s)^3}\left(\int_0^s\Big(\|u\|_{L^2}^2+\|\omega\|_{L^2}^2\Big)d\tau\right)^2ds\\
&\leq C+C\ln(e+t).
\end{aligned}\tag{A4}$$

Applying Plancherel's Theorem again, we have

$$\begin{aligned}
\|u\|_{L^2}^2+\|\omega\|_{L^2}^2&\leq C(\ln(e+t))^{-3}+C(\ln(e+t))^{-2}\\
&\leq C(\ln(e+t))^{-2}.
\end{aligned}\tag{A5}$$

Choosing $f(t) = (1+t)^2$ in (A2) again, one shows that

$$\begin{aligned}
&(1+t)^2\Big(\|\hat{u}\|_{L^2}^2 + \|\hat{\omega}\|_{L^2}^2\Big)\\
&\le \Big(\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2\Big) + C\int_0^t 1\,ds + C\int_0^t \frac{1}{1+s}\left(\int_0^s \Big(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2\Big)d\tau\right)^2 ds\\
&\le C + C(1+t) + C\int_0^t \frac{s}{1+s}\int_0^s \Big(\|u\|_{L^2}^4 + \|\omega\|_{L^2}^4\Big)d\tau ds\\
&\le C(1+t) + C(1+t)\int_0^t \Big(\|u\|_{L^2}^4 + \|\omega\|_{L^2}^4\Big)ds\\
&\le C(1+t) + C(1+t)\int_0^t (\ln(e+s))^{-2}\Big(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2\Big)ds,
\end{aligned}$$

and

$$\begin{aligned}
&(1+t)\Big(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2\Big)\\
&\le C + C\int_0^t (1+s)^{-1}(\ln(e+s))^{-2}(1+s)\Big(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2\Big)ds.
\end{aligned} \tag{A6}$$

Let

$$\begin{aligned}
g(t) &= (1+t)(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2),\\
h(t) &= (1+t)^{-1}[\ln(e+t)]^{-2}.
\end{aligned}$$

Then, (A6) becomes

$$g(t) \le C + C\int_0^t h(s)g(s)\,ds.$$

Applying the Gronwall's inequality to this inequality yields

$$g(t) \le g(0)e^{C\int_0^t h(s)\,ds},$$

thanks to

$$\int_0^t h(s)\,ds = \int_0^t (1+s)^{-1}(\ln(e+s))^{-2}\,ds < \infty.$$

So,

$$g(t) \le C,$$

which implies the decay

$$\|u\|_{L^2} + \|\omega\|_{L^2} \le C(1+t)^{-\frac{1}{2}} \tag{A7}$$

Thus, the proof of Proposition A1 is completed. □

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