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On Queues with Working Vacation and Interdependence in Arrival and Service Processes

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Abstract: In this paper, we consider two queuing models. Model 1 considers a single-server working vacation queuing system with interdependent arrival and service processes. The arrival and service processes evolve by transitions on the product space of two Markovian chains. The transitions in the two Markov chains in the product space are governed by a semi-Markov rule, with sojourn times in states governed by the exponential distribution. In contrast, in the second model, we consider independent arrival and service processes following phase-type distributions with representation $(\boldsymbol{\alpha}, T)$ of order m and $(\boldsymbol{\beta}, S)$ of order n, respectively. The service time during normal working is the above indicated phase-type distribution whereas that during working vacation is a phase-type distribution with representation $(\boldsymbol{\beta}, \theta S)$, $0 < \theta < 1$. The duration of the latter is exponentially distributed. The latter model is already present in the literature and will be briefly described. The main objective is to make a theoretical comparison between the two. Numerical illustrations for the first model are provided.

Keywords: working vacation; interdependence; phase-type distribution; semi-Markov process

MSC: 60K25

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1. Introduction

This paper considers two vacation queuing models. Model 1 considers a single-server working vacation queuing system with interdependent arrival and service processes, whereas Model 2 considers a vacation queueing system with independent arrival and service processes. Before delving into the topic, the following are some important definitions of terms we used.

A stochastic process refers to a group of random variables $(X_t, t \in T)$ indexed by a parameter, often time. When the parameter set T is countable, the process is called a discrete time stochastic process. On the other hand, if the parameter set T is uncountable, the process is referred to as a continuous time process.

A continuous time Markov chain is a stochastic process $\{X(t) : t \geq 0\}$ that has the Markov property, which means that the probability of the system transitioning from one state to another depends only on the current state and not on the past of the process. Specifically, a continuous time Markov chain satisfies the following condition:

For any $r, v, t \geq 0$ and non-negative integers i, j, k,

$$P[X(t+r) = j/X(r) = i, X(v) = k, 0 \leq v \leq r] = P[X(t+r) = j/X(r) = i]$$

If the probability $P[X(t+r) = j/X(r) = i]$ is independent of r for all $t \geq 0$, then the process $X(t) : t \geq 0$ is said to be time homogeneous. This means that the transition probabilities of the process do not change over time and that the process has a stationary behavior.

A continuous random variable X is said to have an exponential distribution with parameter λ if X has a probability distribution function given by $F(t) = P[X \leq t] = 1 - e^{-\lambda t}$ for $t \geq 0$, where λ is a positive real number. Exponential distribution is widely used in queueing models due to its memoryless property. The memoryless property of the exponential distribution implies that the probability of the random variable exceeding a given time interval is independent of the time already elapsed. That is, $Pr\{X > t + u/X > u\} = Pr\{X > t\}$ holds for $t \geq 0$ and $u \geq 0$.

Phase-type distribution is introduced by Neuts as a generalization of the exponential distribution. Consider a finite state Markov chain $\bar{Y} = \{Y(t) : t \geq 0\}$ with m transient states and one absorbing state. Let the states be $\{1, 2, 3, \ldots, m, m+1\}$, where $1, 2, 3, \ldots, m$ are transient states and $m+1$ is the absorbing state. The infinitesimal generator matrix of this Markov chain be partitioned as $\tilde{Q} = \begin{bmatrix} S & S^0 \\ \mathbf{0} & 0 \end{bmatrix}$, where S is a square matrix of order m, S^0 is a column vector and $S\boldsymbol{e} + S^0 = \mathbf{0}$. Let $\mathcal{Z} = inf\{t \geq 0 : Y(t) = m+1\}$ be a random variable of the time to reach the absorbing state. The initial probability distribution is denoted by $(\boldsymbol{\beta}, \beta_{m+1})$ where $\boldsymbol{\beta}$ is a row vector of dimension m and $\beta_{m+1} = 1 - \boldsymbol{\beta}\boldsymbol{e}$. The distribution of $\mathcal{Z}$ is called a continuous phase-type distribution (PH distribution) of order m with parameter $(\boldsymbol{\beta}, S)$. The distribution function of $\mathcal{Z}$ is given by $F(t) = P(\mathcal{Z} \leq t) = 1 - \boldsymbol{\beta} e^{St}\boldsymbol{e}$ for $t \geq 0$ and probability density function of $\mathcal{Z}$ is $f(t) = \boldsymbol{\beta} e^{St} S^0$ for $t \geq 0$. The Laplace Stieltjes transform of $PH(\boldsymbol{\beta}, S)$ is given by $\psi(s) = \beta_{m+1} + \boldsymbol{\beta}(sI - S)^{-1} S^0$ for all $s \in C$ with $Re(s) \geq 0$.

A semi-Markov process is a stochastic process that changes states according to a Markov chain but with a random amount of time spent in each state. The process can be in any N state, denoted as $1, 2, 3, \ldots, N$. When it enters state i, it remains there for a random amount of time before transitioning to state j with the probability Pij, where Fij is the distribution of the time until the transition from i to j. The state of the process at time t is denoted by $Z(t)$. Then $\{Z(t) : t \geq 0\}$ is called a semi-Markov process. Unlike a Markov process, a semi-Markov process does not have the Markovian property since, to predict the future state, we need to know the present state and the time spent in that state.

A QBD process is a type of Markov chain where transitions are allowed only between states in the same level or two adjacent levels. In other words,

$$(i-1, j') \rightleftharpoons (i, j) \rightleftharpoons (i+1, j'') \text{ for } i \geq 1$$

If the transition rates in a QBD process are level-independent, i.e., the same for all levels, it is called an LIQBD process. On the other hand, if the transition rates vary by level, it is called an LDQBD process.

If $A = (a_{ij})$is an $n_1 \times m_1$ matrix and $B = (b_{ij})$ is an $n_2 \times m_2$ matrix, then their Kronecker product $A \otimes B$ is defined as the $n_1 n_2 \times m_1 m_2$ block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & & a_{1m_1}B \\ a_{11}B & a_{12}B & & a_{1m_1}B \\ \vdots & \vdots & & \vdots \\ a_{n_1 1}B & a_{n_1 2}B & & a_{n_1 m_1}B \end{bmatrix}.$$

The Kronecker sum of two square matrices A and B of orders n_1 and n_2, respectively, is defined as $A \oplus B = A \otimes I_{n_2} + I_{n_1} \otimes B$ where I_{n_1} and I_{n_2} are identity matrices of orders n_1 and n_2 respectively.

Kendall notation is used to represent a queueing system. For example, the notation $A/B/C/D/E$ was introduced by Kendall (1951). A describes the arrival pattern: Poisson process (M), or a renewal process (GI); B represents service time distribution: M stands for exponentially distributed service time, G stands for general; C represents the number of servers; D represents the system capacity (finite or countably infinite); E represents the queue discipline. However, our manuscript cannot be represented in Kendall notation because the arrival and service processes evolve through transitions in the product space of two Markov chains.

Considering these definitions, we will review the current literature on interdependent and vacation queueing models. Distinctive methods are utilized broadly in the literature to analyze the interdependency between random variables and processes. To begin with, Adan et al. [1] emphasize a single-server queue with Markov-dependent inter-arrival and service times. In this, the sequence of inter-arrival times is $\{A_n : n \geq 0, A_0 = 0\}$ and that of service times is$\{S_n : n \geq 1\}$, and the authors assume that there is a correlation between A_n and S_n. A study is carried out by Mitchell et al. [2] on an M/M/1 queueing system with the assumption that a customer service time and the inter-arrival time are correlated random variables having a bivariate exponential distribution via simulation. Yoneyama et al. [3] consider an M/M/c queueing system with dependencies between arrival and service patterns. Fendick et al. [4] dissect the impact of various dependencies between the arrival and service processes in packet communication network queues. The fluid production and inventory models with dependence between service and subsequent inter-arrival time are examined by Boxma et al. [5]. Sengupta [6] analyzes a semi-Markovian queue with correlated inter-arrival and service times using the technique developed by Sengupta [7]. Badila et al. [8] analyze a G/G/1 queue with dependence between inter-arrival and service times.

A vacation queuing system is one in which a server may become unavailable from a primary service center for random periods. The time away from the primary service center is called vacation and can be attributed to several factors. For a comprehensive and complete review of vacation queuing systems, we refer the readers to Doshi [9], Takagi [10], Tian and Zhang [11], Ke et al. [12] and Chandrasekaran et al. [13], who conducted a survey describing the latest research on the working vacation queueing system. Recently, Panta et al. [14] surveyed vacation queuing models. Servi and Finn [15] introduced a working vacation model with the idea of offering services at a low rate when the server is on vacation. While Wu and Takagi [16] generalized the model in [15] to an M/G/1 queue with general working vacations, Baba [17] studied a G1/M/1 queue with working vacations using the matrix analytic method.

Pankaj Sharma [18] looks at a loss and delay queueing model under the N-policy constraint for situations where the arrival and service of customers are correlated and follow a bivariate Poisson process. Subhapriya et al. [19] examine an M/M/1/K interdependent queuing model with vacation and controllable arrival rates. William J. Gray [20] considers an M/G1 vacation queuing model exceptional service for specific customers. The service counter is opened only when a certain number of customers, R + 1, are present in each busy period. The first R customers who have to wait additionally are given exceptional service divided into two phases. After that, the remaining customers might receive a different kind of service. Anilkumar et al. [21] consider an N-policy interdependent finite capacity queuing model with controllable arrival rates, and derive steady solutions and system characteristics for this model.

Various analyses have been conducted in the literature to study the interdependencies between random variables and processes using different techniques. One such approach, introduced and popularized by Achyutha Krishnamoorthy [22], involves interdependence analysis through the semi-Markov approach. According to their hypothesis, the evolution of the system follows a multi-dimensional semi-Markov process, where the dimension depends on the number of processes involved. Some processes can remain independent and neutral from all other related processes, while others may be interdependent in groups, but there will be no interdependence between distinct groups. In paper [23], Achyutha Krishnamoorthy introduces a new direction of analysis for studying system reliability, such as the k-out-of-n: G system, and in particular serial and parallel systems with interdependence between the components.

The k-out-of-n models have a wide range of practical applications in many spheres of human activity, including engineering, telecommunication, industry, biology, etc., and are effectively used to study the reliability of complex real-world redundant systems, such as aircraft multi-engine systems, multi-pump hydraulic control systems, unmanned underwater vehicles, unmanned multi-rotor flight modules of high-altitude telecommunications platforms, etc. In a series of works by Rykov, Kozyrev, et al. [24,25] a range of k-out-of-n-type models have been proposed for the reliability study of systems taking into account the dependence of system failure not only on the number of failed components but also on their location in the system, on their proximity to each other, on the state of the random environment and other dependence aspects. For example, in [26] Rykov et al. performed the reliability study of a k-out-of-n system, whose components' residual lifetimes depend on the increase in load after the failure of any component.

Though the interdependence of processes has already been analyzed in the literature, we introduce a recent type of analysis in this paper, as mentioned in Achyutha Krishnamoorthy [22,23].

In this paper, we consider two queueing models. In Model 1, we consider a single-server working vacation queueing system in which arrival and service processes are interdependent. The arrival and service processes evolve by transitions on the product space of two Markov chains. The transitions in two Markov chains on the product space are governed by a semi-Markov rule, with sojourn times in states governed by the exponential distribution. In contrast, in the second model, we consider independent arrival and service processes following phase-type distributions with representation $(\boldsymbol{\alpha}, T)$ of order m and $(\boldsymbol{\beta}, S)$ of order n, respectively. The service time during normal working is the above indicated phase-type distribution whereas that during working vacation is the phase-type distribution with representation $(\boldsymbol{\beta}, \theta S), 0 < \theta < 1$. The duration of the latter is exponentially distributed. The latter model is briefly described as it is already in the literature.

The salient features of the model discussed in this paper are

- It introduces a new approach to analyzing working vacation queuing systems with interdependent arrival and service processes.
- Theoretical comparisons with the independent systems are provided.

Notations and abbreviations used in this paper are

- $CTMC$: Continuous time Markov chain.
- I_a: Identity matrix of order a.
- $LIQBD$: Level-independent quasi-birth and death.
- $\boldsymbol{e}$: Column vector of $1'$s of appropriate order.
- QBD: Quasi-birth and death.
- PH: Phase type

Sections 2–7 of the paper are organized as follows. In Section 2, the mathematical formulation of Model 1 is presented. Section 3 deals with the steady-state analysis of queuing Model 1, followed by the computation and presentation of specific performance measures in Section 4. The mathematical formulation of Model 2 is discussed in Section 5, while Section 6 focuses on the steady-state analysis of Model 2. Finally, numerical results are presented and discussed in Section 7.

2. Mathematical Formulation of Model 1

Model 1 considers a single-server working vacation queueing system with interdependent arrival and service processes. The server will take a working vacation when the system is empty during service completion. A customer arriving during working vacation will be served at a low rate. The duration of vacation is assumed to be exponentially distributed with the parameter η. The server switches to normal mode when the vacation expires. Changes in the first coordinate due to transitions indicate service phase changes and those in the second indicate arrival phase changes. Two Markov chains govern arrival and service processes. Consider two Markov chains $Y = \{Y_n\}$ and $X = \{X_n\}$ with finite state spaces $\{1, 2, 3......, m, m+1\}$ and {1,2,3,.....,n,n+1}, respectively. Absorbing states in arrival and service processes are $m+1$ and $n+1$, respectively. Consider the Markov chain on the product space $Z = X \times Y$ with state space $\{(1,1), (1,2), ..., (1,m), (2,1), (2,2), ..., (2,m), ..., (n,1), (n,2), ...(n,m), (1,m+1), (2,m+1), ...(n,m+1), (n+1,1), (n+1,2), ..., (n+1,m), (n+1,m+1)\}$. Changes in the first coordinate due to transitions indicate service phase changes and those in the second indicate arrival phase changes. The transitions are interdependent in the sense that the sojourn time in any stage (i,j) depends on this as well as the state (i', j'), to be visited next. This sojourn time distribution is exponential with parameter $\delta(i,j)(i', j')$. Since the transitions are interdependent, none, one or both coordinates can change with a positive probability in a short time. However, one can very well assume that at most one change takes place with a positive possibility. This assumption leads to an infinitesimal generator which is highly sparse. On the other hand, if we proceed with the assumption of both coordinate changes with positive probability, then the infinitesimal generator will not be sparse. In this paper, we assume that at most one coordinate change in a transition has a positive probability. Let the initial probability vectors of arrival and service processes be $\boldsymbol{\alpha} = (\bar{\boldsymbol{\alpha}}, \alpha_{m+1})$ and $\boldsymbol{\beta} = (\bar{\boldsymbol{\beta}}, \beta_{n+1})$, respectively, where $\bar{\boldsymbol{\alpha}} = (\alpha_1, \alpha_2, \alpha_3, ...\alpha_m)$ and $\bar{\boldsymbol{\beta}} = (\beta_1, \beta_2, \beta_3, ...\beta_n)$. Absorbing states of $Z = \{Z_n\}$ are $\{(n+1, j) : 1 \leq j \leq m\}$ and $\{(i, m+1) : 1 \leq i \leq n\}$. In the absence of customers, no service can be provided; this is indicated by * in the position of service coordinate (third coordinate in the 4-tuples). Since the service is slow during working vacation, we multiply the corresponding service rate by θ, $0 < \theta < 1$.

The QBD Process

The model described above can be studied as an LIQBD process. First, we define the following notations: At time t, let

$N(t)$: number of customers in the system.

$$J(t) = \begin{cases} 0, if\ the\ server\ is\ in\ vacation\ mode. \\ 1, if\ the\ server\ is\ in\ normal\ mode. \end{cases}$$

$S(t)$: the phase of service.
$A(t)$: the phase of arrival.

$\{(N(t), J(t), S(t), A(t)) : t \geq 0\}$ is an LIQBD with state space $\bar{\Omega}$= $\{\{(0,0,*,j) : 1 \leq j \leq m\} \bigcup \{(q,0,i,j) : q \geq 1, 1 \leq i \leq n, 1 \leq j \leq m,\} \bigcup \{(q,1,i,j) : q \geq 1, 1 \leq i \leq n, 1 \leq j \leq m\}\}$.
The transitions are described in Table 1.

Table 1. Transitions and corresponding rates.

From	To	Rate		Remarks
$(0,*,j)$	$(0,*,j')$	$\delta(*,j)(*,j')$	$1 \leq j, j' \leq m$	arrival phase change
$(0,*,j)$	$(1,0,i,j')$	$\alpha'_j \beta_i \delta(*,j)(i,m+1)$	$1 \leq j, j' \leq m; 1 \leq i \leq n$	arrival occurs
$(1,0,i,j)$	(0,*,j)	$\theta\delta(i,j)(n+1,j)$	$1 \leq j \leq m, 1 \leq i \leq n$	service completion
$(1,1,i,j)$	$(0,*,j)$	$\delta(i,j)(n+1,j)$	$1 \leq j \leq m, 1 \leq i \leq n$	service completion
$(h,0,i,j)$	$(h,0,i,j')$	$\delta(i,j)(i,j')$	$1 \leq j, j' \leq m; 1 \leq i \leq n; h \geq 1$	arrival phase change
$(h,0,i,j)$	$(h,0,i',j)$	$\theta\delta(i,j)(i',j)$	$1 \leq j \leq m; 1 \leq i, i' \leq n; h \geq 1$	service phase change
$(h,0,i,j)$	$(h,1,i,j)$	η	$1 \leq j \leq m; 1 \leq i \leq n; h \geq 1$	vacation realization
$(h,0,i,j)$	$(h-1,0,i',j)$	$\theta\delta(i,j)(n+1,j)\beta'_i$	$1 \leq j, j' \leq m; 1 \leq i \leq n; h \geq 2$	service completion
$(h,0,i,j)$	$(h+1,0,i,j')$	$\delta(i,j)(i,m+1)\alpha'_j$	$1 \leq j, j' \leq m; 1 \leq i \leq n; h \geq 1$	arrival occurs

Table 1. *Cont.*

From	To	Rate		Remarks
$(h,1,i,j)$	$(h,1,i,j')$	$\delta(i,j)(i,j')$	$1 \le j,j' \le m; 1 \le i \le n; h \ge 1$	arrival phase change
$(h,1,i,j)$	$(h,1,i',j)$	$\delta(i,j)(i',j)$	$1 \le j \le m; 1 \le i,i' \le n; h \ge 1$	service phase change
$(h,1,i,j)$	$(h-1,1,i',j)$	$\delta(i,j)(n+1,j)\beta_i'$	$1 \le j,j' \le m; 1 \le i \le n; h \ge 2$	service completion
$(h,1,i,j)$	$(h+1,1,i,j')$	$\delta(i,j)(i,m+1)\alpha_j'$	$1 \le j,j' \le m; 1 \le i \le n; h \ge 1$	arrival phase change

Let

$$\mathcal{B}_1 = \begin{bmatrix} D_{*1} & \delta(*,1)(*,2) & \delta(*,1)(*,3) & \dots & \delta(*,1)(*,m) \\ \delta(*,2)(*,1) & D_{*2} & \delta(*,2)(*,3) & \dots & \delta(*,2)(*,m) \\ \delta(*,3)(*,1) & \delta(*,3)(*,2) & D_{*3} & \dots & \delta(*,3)(*,m) \\ \delta(*,4)(*,1) & \delta(*,4)(*,2) & \delta(*,4)(*,3) & \dots & \delta(*,4)(*,m) \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ \delta(*,m)(*,1) & \delta(*,m)(*,2) & \delta(*,m)(*,3) & \dots & D_{*m} \end{bmatrix}$$

where $D_{*i} = -[\sum_{j=1,j\neq i}^{j=m} \delta(*,i)(*,j) + \sum_{k=1}^{n} \beta_k \delta(*,i)(k,m+1)]$ for $i = 1,2,...m$.
Let

$$F_i = \begin{bmatrix} \alpha_1\beta_i\delta(*,1)(i,m+1) & \alpha_2\beta_i\delta(*,1)(i,m+1) & \alpha_3\beta_i\delta(*,1)(i,m+1) & \dots\dots & \alpha_m\beta_i\delta(*,1)(i,m+1) \\ \alpha_1\beta_i\delta(*,2)(i,m+1) & \alpha_2\beta_i\delta(*,2)(i,m+1) & \alpha_3\beta_i\delta(*,2)(i,m+) & \dots\dots & \alpha_m\beta_i\delta(*,2)(i,m+1) \\ \alpha_1\beta_i\delta(*,3)(i,m+1) & \alpha_2\beta_i\delta(*,3)(i,m+1) & \alpha_3\beta_i\delta(*,3)(i,m+1) & \dots\dots & \alpha_m\beta_i\delta(*,3)(i,m+1) \\ \alpha_1\beta_i\delta(*,4)(i,m+1) & \alpha_2\beta_i\delta(*,4)(i,m+1) & \alpha_3\beta_i\delta(*,4)(i,m+1) & \dots\dots & \alpha_m\beta_i\delta(*,4)(i,m+1) \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ \alpha_1\beta_i\delta(*,m)(i,m+1) & \alpha_2\beta_i\delta(*,m)(i,m+1) & \alpha_3\beta_i\delta(*,m)(i,m+1) & \dots\dots & \alpha_m\beta_i\delta(*,m)(i,m+1) \end{bmatrix}.$$

F_i is a square matrix of order m for $i = 1,2,3,\dots,n$.
$\mathcal{B}_0 = \begin{bmatrix} F_1 & F_2 & F_3 & F_4 & \dots\dots & F_n \end{bmatrix}$ is a matrix of order $m \times mn$.
Let

$$E_i = \begin{bmatrix} \delta(i,1)(n+1,1) & & & & \\ & \delta(i,2)(n+1,2) & & & \\ & & \delta(i,3)(n+1,3) & & \\ & & & \ddots & \\ & & & & \delta(i,m)(n+1,m) \end{bmatrix}$$

E_i is an $m \times m$ matrix for $i = 1,2,3,....n$.

Furthermore, let $P = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ \vdots \\ \vdots \\ E_n \end{bmatrix}$ be an $mn \times m$ matrix. $\mathcal{B}_2 = \begin{bmatrix} \theta P \\ P \end{bmatrix}$ is a $2mn \times m$ matrix.

Let

$$U_i = \begin{bmatrix} \alpha_1\delta(i,1)(i,m+1) & \alpha_2\delta(i,1)(i,m+1) & & \ldots\ldots & \alpha_m\delta(i,1)(i,m+1) \\ \alpha_1\delta(i,2)(i,m+1) & \alpha_2\delta(i,2)(i,m+1) & & \ldots\ldots & \alpha_m\delta(i,2)(i,m+1) \\ \alpha_1\delta(i,3)(i,m+1) & \alpha_2\delta(i,3)(i,m+1) & & \ldots\ldots & \alpha_m\delta(i,m)(i,m+1) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \alpha_1\delta(i,m)(i,m+1) & \alpha_2\delta(i,m)(i,m+1) & & \ldots\ldots & \alpha_m\delta(i,m)(i,m+1) \end{bmatrix}.$$

U_i is a square matrix of order m for $i = 1,2,3,\ldots,n$.

Let $V = \begin{bmatrix} U_1 & & & & \\ & U_2 & & & \\ & & U_3 & & \\ & & & \ddots & \\ & & & & U_n \end{bmatrix}$ be a square matrix of order mn.

$\mathcal{A}_0 = \begin{bmatrix} V & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix}$ is a square matrix of order 2mn.

Let $G = \begin{bmatrix} \beta_1E_1 & \beta_2E_1 & \beta_3E_1 & \ldots\ldots & \beta_nE_1 \\ \beta_1E_2 & \beta_2E_2 & \beta_3E_2 & \ldots\ldots & \beta_nE_2 \\ \beta_1E_3 & \beta_2E_3 & \beta_3E_3 & \ldots\ldots & \beta_nE_3 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \beta_1E_n & \beta_2E_n & \beta_3E_n & \ldots\ldots & \beta_nE_n \end{bmatrix}$ be a square matrix of order mn.

$\mathcal{A}_2 = \begin{bmatrix} G\theta & \mathbf{0} \\ \mathbf{0} & G \end{bmatrix}$ is a square matrix of order 2mn.

Let $J_{ih} = \begin{bmatrix} \delta(i,1)(i+h,1) & & & & \\ & \delta(i,2)(i+h,2) & & & \\ & & \delta(i,3)(i+h,3) & & \\ & & & \ddots & \\ & & & & \delta(i,m)(i+h,m) \end{bmatrix}$

be a square matrix of order m. i varies from 1 to n-1, h varies from 1 to n-1 and $i + h \leq n$.

Let $K_{ih} = \begin{bmatrix} \delta(i,1)(i-h,1) & & & & \\ & \delta(i,2)(i-h,2) & & & \\ & & \delta(i,3)(i-h,3) & & \\ & & & \ddots & \\ & & & & \delta(i,m)(i-h,m) \end{bmatrix}$

be a square matrix of order m. i varies from 2 to n, h from 1 to n-1 and $i > h$.

Let $H_i = \begin{bmatrix} C_{i1} & \delta(i,1)(i,2) & \delta(i,1)(i,3) & \ldots\ldots & \delta(i,1)(i,m) \\ \delta(i,2)(i,1) & C_{i2} & \delta(i,2)(i,3) & \ldots\ldots & \delta(i,2)(i,m) \\ \delta(i,3)(i,1) & \delta(i,3)(i,2) & C_{i3} & \ldots\ldots & \delta(i,3)(i,m) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \delta(i,m)(i,1) & \delta(i,m)(i,2) & \delta(i,m)(i,3) & \ldots\ldots & C_{im} \end{bmatrix}.$

H_i is a square matrix of order m for $i = 1,2,3,...,n$.

Here, $C_{ih} = -[\sum_{j=1,j\neq h}^{m+1} \delta(i,h)(i,j) + \sum_{k=1,i\neq k}^{n+1} \delta(i,h)(k,h)\theta + \eta]; 1 \leq i \leq n; 1 \leq h \leq m$.

Let $N_i = \begin{bmatrix} D_{i1} & \delta(i,1)(i,2) & \delta(i,1)(i,3) & \ldots\ldots & \delta(i,1)(i,m) \\ \delta(i,2)(i,1) & D_{i2} & \delta(i,2)(i,3) & \ldots\ldots & \delta(i,2)(i,m) \\ \delta(i,3)(i,1) & \delta(i,3)(i,2) & D_{i3} & \ldots\ldots & \delta(i,3)(i,m) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \delta(i,m)(i,1) & \delta(i,m)(i,2) & \delta(i,m)(i,3) & \ldots\ldots & D_{im} \end{bmatrix}.$

N_i is a square matrix of order m for $i = 1,2,3,...,n$.

Here, $D_{ih} = -[\sum_{j=1,j\neq h}^{m+1} \delta(i,h)(i,j) + \sum_{k=1,i\neq k}^{n+1} \delta(i,h)(k,h)]; 1 \leq i \leq n; 1 \leq h \leq m$.

Let

$$L = \begin{bmatrix} H_1 & J_{11}\theta & J_{12}\theta & J_{13}\theta & \cdots\cdots & J_{1(n-3)}\theta & J_{1(n-2)}\theta & J_{1(n-1)\theta} \\ K_{21}\theta & H_2 & J_{21}\theta & J_{22}\theta & \cdots\cdots & J_{2(n-4)}\theta & J_{2(n-3)}\theta & J_{2(n-2)\theta} \\ K_{32}\theta & K_{31}\theta & H_3 & J_{31}\theta & \cdots\cdots & J_{3(n-5)}\theta & J_{3(n-4)}\theta & J_{3(n-3)\theta} \\ K_{43}\theta & K_{42}\theta & K_{41}\theta & H_4 & \cdots\cdots & J_{4(n-6)}\theta & J_{4(n-5)}\theta & J_{4(n-4)\theta} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \\ \\ K_{(n-2)(n-3)}\theta & K_{(n-2)(n-4)}\theta & K_{(n-2)(n-5)}\theta & K_{(n-2)(n-6)}\theta & \cdots\cdots & H_{(n-2)} & J_{(n-2)1}\theta & J_{(n-2)2}\theta \\ K_{(n-1)(n-2)}\theta & K_{(n-1)(n-3)}\theta & K_{(n-1)(n-4)}\theta & K_{(n-1)(n-5)}\theta & \cdots\cdots & K_{(n-1)1}\theta & H(n-1) & J_{(n-1)1}\theta \\ K_{n(n-1)}\theta & K_{n(n-2)}\theta & K_{n(n-3)}\theta & K_{n(n-4)}\theta & \cdots\cdots & K_{n2}\theta & K_{n1}\theta & H_n \end{bmatrix}.$$

$$M = \begin{bmatrix} N_1 & J_{11} & J_{12} & J_{13} & \cdots\cdots & J_{1(n-3)} & J_{1(n-2)} & J_{1(n-1)} \\ K_{21} & N_2 & J_{21} & J_{22}\theta & \cdots\cdots & J_{2(n-4)} & J_{2(n-3)} & J_{2(n-2)} \\ K_{32} & K_{31} & N_3 & J_{31} & \cdots\cdots & J_{3(n-5)} & J_{3(n-4)} & J_{3(n-3)} \\ K_{43} & K_{42} & K_{41} & N_4 & \cdots\cdots & J_{4(n-6)} & J_{4(n-5)} & J_{4(n-4)} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ K_{(n-2)(n-3)} & K_{(n-2)(n-4)} & K_{(n-2)(n-5)} & K_{(n-2)(n-6)} & \cdots\cdots & N_{(n-2)} & J_{(n-2)1} & J_{(n-2)2} \\ K_{(n-1)(n-2)} & K_{(n-1)(n-3)} & K_{(n-1)(n-4)} & K_{(n-1)(n-5)} & \cdots\cdots & K_{(n-1)1} & N_{(n-1)} & J_{(n-1)1} \\ K_{n(n-1)} & K_{n(n-2)} & K_{n(n-3)} & K_{n(n-4)} & \cdots\cdots & K_{n2} & K_{n1} & N_n \end{bmatrix}.$$

Then $\mathcal{A}_1 = \begin{bmatrix} L & I_{mn}\eta \\ \mathbf{0} & M \end{bmatrix}$.

Thus, we have the following theorem.

Theorem 1. *The infinitesimal generator of continuous time Markov chain $\bar{\Omega}$ is*

$$\overline{Q} = \begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_0 & & & \\ \mathcal{B}_2 & \mathcal{A}_1 & \mathcal{A}_0 & & \\ & \mathcal{A}_2 & \mathcal{A}_1 & \mathcal{A}_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

The matrix representations $\mathcal{B}_1$, $\mathcal{B}_0$ and $\mathcal{B}_2$ describe transitions within level 0, transitions from level 0 to level 1 and transitions from level 1 to level 0, respectively. These matrices are of sizes $m \times m$, $m \times mn$, and $2\,mn \times m$, respectively. On the other hand, the matrices $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$ represent transitions between levels for $q \geq 1$. Specifically, $\mathcal{A}_0$ describes transitions from level q to $q+1$, $\mathcal{A}_1$ describes transitions within level q and $\mathcal{A}_2$ describes transitions from level q to $q-1$. All these matrices are of size $2\,mn \times 2\,mn$.

$$\textit{Let } \mathcal{B}_1 = \begin{bmatrix} D_{*1} & \delta(*,1)(*,2) & \delta(*,1)(*,3) & \ldots & \delta(*,1)(*,m) \\ \delta(*,2)(*,1) & D_{*2} & \delta(*,2)(*,3) & \ldots & \delta(*,2)(*,m) \\ \delta(*,3)(*,1) & \delta(*,3)(*,2) & D_{*3} & \ldots & \delta(*,3)(*,m) \\ \delta(*,4)(*,1) & \delta(*,4)(*,2) & \delta(*,4)(*,3) & \ldots & \delta(*,4)(*,m) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \delta(*,m)(*,1) & \delta(*,m)(*,2) & \delta(*,m)(*,3) & \ldots & D_{*m} \end{bmatrix}$$

$$\mathcal{B}_0 = \begin{bmatrix} F_1 & F_2 & F_3 & F_4 & \cdots\cdots & F_n \end{bmatrix}$$

$$\mathcal{B}_2 = \begin{bmatrix} \theta P \\ P \end{bmatrix}, \mathcal{A}_0 = \begin{bmatrix} V & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix}, \mathcal{A}_1 = \begin{bmatrix} L & I_{mn}\eta \\ \mathbf{0} & M \end{bmatrix}, \mathcal{A}_2 = \begin{bmatrix} G\theta & \mathbf{0} \\ \mathbf{0} & G \end{bmatrix}.$$

3. Steady-State Analysis

In this section, we assess the steady-state analysis of the queueing model by first determining the queueing system's stability condition.

3.1. Stability Condition

The generator matrix $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2$

$$\mathcal{A} = \begin{bmatrix} V + L + G\theta & I_{mn}\eta \\ \mathbf{0} & V + M + G \end{bmatrix}. \tag{1}$$

Let $\boldsymbol{\pi} = (\boldsymbol{\pi_1}, \boldsymbol{\pi_2})$ denote the steady-state probability vector of the generator matrix $\mathcal{A}$. Here $\boldsymbol{\pi}$ is of order $1 \times 2mn$ and $\boldsymbol{\pi_r}$ is of order $1 \times nm$ for $r = 1, 2$.
Steady-state probability vector $\boldsymbol{\pi}$ satisfying the equations

$$\boldsymbol{\pi}\mathcal{A} = 0, \boldsymbol{\pi}\mathbf{e} = 1. \tag{2}$$

Using Equation (2), we get

$$\boldsymbol{\pi_1}[V + L + G\theta] = \mathbf{0} \tag{3}$$

$$\boldsymbol{\pi_1} I_{mn}\eta + \boldsymbol{\pi_2}[V + M + G] = \mathbf{0} \tag{4}$$

From Equation (4);

$$\boldsymbol{\pi_1} = -\boldsymbol{\pi_2}[V + M + G]\frac{1}{\eta} I_{mn} \tag{5}$$

By using Equations (2) and (5)

$$-\boldsymbol{\pi_2}[V + M + G]\frac{1}{\eta} I_{mn}\mathbf{e} + \boldsymbol{\pi_2}\mathbf{e} = 1 \tag{6}$$

$$\boldsymbol{\pi_2}[I_{mn} - (V + M + G)\frac{1}{\eta} I_{mn}]\mathbf{e} = 1 \tag{7}$$

Using Equations (5) and (7) we can find $\boldsymbol{\pi_1}, \boldsymbol{\pi_2}$.

The LIQBD description of the model indicates that the queueing system is stable if and only if the left drift exceeds that of the right drift. That is,

$$\boldsymbol{\pi}\mathcal{A}_0\mathbf{e} < \boldsymbol{\pi}\mathcal{A}_2\mathbf{e}. \tag{8}$$

$$\boldsymbol{\pi}\mathcal{A}_0\mathbf{e} = \pi_1 V\mathbf{e} + \pi_2 V\mathbf{e} = \sum_{r=1}^{2} \pi_r V\mathbf{e} \tag{9}$$

$$\boldsymbol{\pi}\mathcal{A}_2\mathbf{e} = \boldsymbol{\pi_1} G\theta\mathbf{e} + \boldsymbol{\pi_2} G\mathbf{e} \tag{10}$$

Thus, we have the following theorem.

Theorem 2. *The given system is stable if and only if*

$$\sum_{r=1}^{2} \pi_r Ve < \boldsymbol{\pi_1} G\theta e + \boldsymbol{\pi_2} Ge \tag{11}$$

3.2. The Steady-State Probability Vector of $\overline{\mathcal{Q}}$

Let $\boldsymbol{x}$ be the steady-state probability vector of $\overline{\mathcal{Q}}$. $\boldsymbol{x} = (\boldsymbol{x_0}, \boldsymbol{x_1}, \boldsymbol{x_2} \ldots)$, where $\boldsymbol{x_0}$ is of dimension $1 \times m$ and $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots$ are of dimension $1 \times 2mn$. Under the stability condition, we have

$$\boldsymbol{x_i} = \boldsymbol{x}_1 R^{i-1}, i \geq 2.$$

The matrix R is the minimal non-negative solution to the matrix quadratic equation.

$$R^2\mathcal{A}_2 + R\mathcal{A}_1 + \mathcal{A}_0 = 0$$

and the vectors $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$are obtained by solving the equations

$$\boldsymbol{x_0}\mathcal{B}_1 + \boldsymbol{x_1}\mathcal{B}_2 = 0 \tag{12}$$

$$\boldsymbol{x_0}\mathcal{B}_0 + \boldsymbol{x_1}(\mathcal{A}_1 + R\mathcal{A}_2) = 0 \tag{13}$$

subject to the normalizing condition

$$\boldsymbol{x_0 e} + \boldsymbol{x_1}(I - R)^{-1}\boldsymbol{e} = 1 \tag{14}$$

Solving Equations (12)–(14), we get $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$. Hence we can find all $\boldsymbol{x_i}$'s.

4. Some Performance Measures

Theorem 3. *Let* $\boldsymbol{x_0} = \sum_{j=1}^{m} \boldsymbol{x_{0*j}}$, $\boldsymbol{x_q} = \sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{q0ij}} + \sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{q1ij}}$ *for* $q = 1, 2, 3, \ldots, n$. $\sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{q0ij}}$ *is the probability that the system is in a state with q customers and is in working vacation mode and* $\sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{q1ij}}$ *is the probability that the system is in a state with q customers and is in normal mode. Then*

- *Probability that the system is empty:*

$$p_{empty} = \mathbf{x_0 e}.$$

- *Probability that the server is working in vacation mode:*

$$P_{vac} = \sum_{q=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{q0ij}}$$

- *Probability that server is working in normal mode:*

$$P_{nor} = \sum_{q=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{q1ij}}$$

- *Probability that the server is busy:*

$$P_{busy} = \sum_{q=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{q0ij}} + \sum_{q=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{q1ij}}$$

- *Probability that q customers are in the system:*

$$P_q = \boldsymbol{x_q e}.$$

- *Mean number of customers in the system:*

$$ECS = \sum_{q=1}^{\infty} q\boldsymbol{x_q}\mathbf{e}$$

- *Mean number of customers in the queue:*

$$ECQ = \sum_{q=1}^{\infty} (q-1)\boldsymbol{x_q}\mathbf{e}$$

- *Rate of switching to the normal mode:*

$$RSN = \sum_{q=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{q0ij}}\eta$$

Cost Function

To find the expected cost, we constructed a cost function as follows.

CV- Cost per unit time when the server is in vacation mode.
CN- Cost per unit time when the server is in normal mode.
CSN- Cost per unit time for switching to normal mode.
HCQ- Holding cost per customer in the queue.

$$EC = CV * P_{vac} + CN * P_{nor} + CSN * RSN + ECQ * HCQ.$$

Next, we consider a queueing system in which arrival and service processes are independent.

5. Mathematical Formulation of Model 2

We consider a single-server queueing system in which the arrival process follows phase-type distribution $(\boldsymbol{\alpha}, T)$ of order m and the service process follows phase-type distribution $(\boldsymbol{\beta}, S)$ of order n. Once the server completes a service and the system becomes empty, the server goes on a working vacation. During a working vacation, a customer who arrives will be served at a low rate, with the service time following a phase-type distribution represented by $(\boldsymbol{\beta}, \theta S)$, where $0 < \theta < 1$. Furthermore, the duration of the working vacation is assumed to follow an exponential distribution with parameter η. Once the vacation ends, the server switches back to normal mode.

The QBD Process

The model described in Section 5 can be studied as an LIQBD process. First, we define the following notations. At time t, let

$N(t)$: number of customers in the system.

$$J(t) = \begin{cases} 0, if\ the\ server\ is\ in\ vacation\ mode. \\ 1, if\ the\ server\ is\ in\ normal\ mode. \end{cases}$$

$S(t)$: the phase of service.
$A(t)$: the phase of arrival.

$\{(N(t), J(t), S(t), A(t)\} : t \geq 0\}$ is a LIQBD with state space
$\tilde{\Omega}= \{\{(0,j) : 1 \leq j \leq m\} \bigcup \{(q,0,i,j) : q \geq 1, 1 \leq i \leq n, 1 \leq j \leq m\} \bigcup \{(q,1,i,j)/q \geq 1, 1 \leq i \leq n, 1 \leq j \leq m\}\}$.

The infinitesimal generator of this CTMC is

$$\mathcal{Q}^* = \begin{bmatrix} B_1 & B_0 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

The matrix B_1 is a square matrix of size m that contains the transitions within level 0. The matrix B_0 is an $m \times 2mn$ matrix that contains the transitions from level 0 to level 1. The matrix B_2 is a $2mn \times m$ matrix that contains the transitions from level 1 to level 0. The matrix A_0 represents transitions from level n to level $n+1$ for $n \geq 1$, A_1 represents transitions within level n for $n \geq 1$ and A_2 represents transitions from level n to level $n-1$ for $n \geq 2$. All of these matrices are square matrices of order $2mn$.

$B_1 = T$; $B_0 = \begin{bmatrix} T^0\boldsymbol{\beta} \otimes \boldsymbol{\alpha} & \mathbf{0} \end{bmatrix}$.

$$B_2 = \begin{bmatrix} \theta S^0 \otimes I_m \\ S^0 \otimes I_m \end{bmatrix}.$$
$$A_1 = \begin{bmatrix} \theta S \otimes I_m + I_n \otimes T - I_{mn}\eta & I_{mn}\eta \\ \mathbf{0} & S \otimes I_m + I_n \otimes T \end{bmatrix}.$$
$$A_2 = \begin{bmatrix} \theta S^0 \boldsymbol{\beta} \otimes I_m & \mathbf{0} \\ \mathbf{0} & S^0 \boldsymbol{\beta} \otimes I_m \end{bmatrix}.$$
$$A_0 = \begin{bmatrix} I_n \otimes T^0 \alpha & \mathbf{0} \\ \mathbf{0} & I_n \otimes T^0 \alpha \end{bmatrix}.$$

6. Steady-State Analysis

In this section, we perform the steady-state analysis of the queueing model under study by first establishing the stability condition of the queueing system.

6.1. Stability Condition

The generator matrix $A = A_0 + A_1 + A_2$

$$A = \begin{bmatrix} I_n \otimes T^0 \alpha + \theta S \otimes I_m + I_n \otimes T + \theta S^0 \boldsymbol{\beta} \otimes I_m & I_{mn}\eta \\ \mathbf{0} & I_n \otimes T^0 \alpha + S \otimes I_m + I_n \otimes T + S^0 \boldsymbol{\beta} \otimes I_m \end{bmatrix}. \tag{15}$$

Let $\boldsymbol{\pi} = (\boldsymbol{\pi_1}, \boldsymbol{\pi_2})$ denote the steady-state probability vector of the generator matrix A.

Here $\boldsymbol{\pi}$ is of order $1 \times 2mn$ and the $\boldsymbol{\pi_r}$ is of order $1 \times nm$ for $r = 1, 2$.
Steady-state probability vector π satisfying the equations

$$\boldsymbol{\pi A} = 0, \boldsymbol{\pi}\mathbf{e} = 1. \tag{16}$$

Using Equation (16), we get

$$\boldsymbol{\pi_1}[I_n \otimes T^0 \alpha + \theta S \otimes I_m + I_n \otimes T + \theta S^0 \boldsymbol{\beta} \otimes I_m] = \mathbf{0} \tag{17}$$
$$\boldsymbol{\pi_1} I_{mn}\eta + \boldsymbol{\pi_2}[I_n \otimes T^0 \alpha + S \otimes I_m + I_n \otimes T + S^0 \boldsymbol{\beta} \otimes I_m] = \mathbf{0} \tag{18}$$
$$\boldsymbol{\pi_1} \times \mathbf{e} + \boldsymbol{\pi_2} \times \mathbf{e} = 1 \tag{19}$$

Using Equations (17)–(19), we can find $\boldsymbol{\pi_1}, \boldsymbol{\pi_2}$. The *LIQBD* description of the model indicates that the queueing system is stable if and only if the left drift exceeds that of the right drift. That is,

$$\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e}. \tag{20}$$

Thus, we have the following theorem.

Theorem 4. *The system is stable if and only if*

$$\sum_{r=1}^{2} \pi_r (I_n \otimes T^0 \alpha) \boldsymbol{e} < \boldsymbol{\pi_1}[\theta S^0 \boldsymbol{\beta} \otimes I_m] \boldsymbol{e} + \boldsymbol{\pi_2}[S^0 \boldsymbol{\beta} \otimes I_m] \boldsymbol{e} \tag{21}$$

6.2. The Steady-State Probability Vector of Q^*

Let $\boldsymbol{x}$ be the steady-state probability vector of Q^*.

$\boldsymbol{x} = (\boldsymbol{x_0}, \boldsymbol{x_1}, \boldsymbol{x_2} \ldots)$, where $\boldsymbol{x_0}$ is of dimension $1 \times m$ and $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots$ are each of dimension $1 \times 2mn$. Under the stability condition, we have $\boldsymbol{x_i} = \boldsymbol{x}_1 R^{i-1}, i \geq 2$, where the matrix R is the minimal non-negative solution to the matrix quadratic equation

$$R^2 A_2 + R A_1 + A_0 = 0,$$

and the vectors $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$ are obtained by solving the equations

$$\boldsymbol{x_0} B_1 + \boldsymbol{x_1} B_2 = 0 \tag{22}$$
$$\boldsymbol{x_0} B_0 + \boldsymbol{x_1}(A_1 + R A_2) = 0 \tag{23}$$

subject to the normalizing condition

$$\boldsymbol{x_0 e} + \boldsymbol{x_1}(I - R)^{-1}\boldsymbol{e} = 1 \tag{24}$$

Solving Equations (22)–(24), we get $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$. Hence we can find all $\boldsymbol{x_i}$'s.

With a comparison between Equations (1) and (15) on the one hand and between (11) and (21) on the other, we see that there is a significant difference between the interdependent cases of arrival and service. However, a numerical comparison between the two models is not possible because it is impossible to identify an infinitesimal generator for the first model, which gives the exact arrival and service rates chosen for Model 2. It is important to note that the analysis of Model 1 does not involve Kronecker sum and/or product, unlike Model 2.

7. Numerical Results

We take $n = 3$ and $m = 2$, $\alpha_{m+1} = \beta_{n+1} = 0$. The state space of the arrival process is $1, 2, 3$ where the transient states are 1,2 and the absorbing state is 3. The state space of the service process is $1, 2, 3, 4$ where the transient states are 1,2,3 and the absorbing state is 4. Then the state space of the Markov chain

$Z = \{Z_n\}$ is {(1,1),(1,2),(2,1),(2,2),(3,1),(3,2),(1,3),(2,3),(3,3),(4,1),(4,2)(4,3)}. Absorbing states of Z are $(1,3)$, $(2,3)$, $(3,3)$, $(4,1)$, $(4,2)$ and $(4,3)$. We assume that, at most, one coordinate change in a transition has a positive probability. In the absence of customers, no service can be provided; this is indicated by * in the position of service coordinate (third coordinate in the 4-tuples). $\bar{\boldsymbol{\alpha}} = (0.6, 0.4)$ and $\bar{\boldsymbol{\beta}} = (0.3, 0.4, 0.3)$. To calculate expected cost, we take CV = 10; CN = 15; CSN = 100; HCQ = 5.

Transition rates are as follows (Table 2).

Table 2. Transition rates.

	$(*,1)$	$(*,2)$	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$	$(3,1)$	$(3,2)$	$(1,3)$	$(2,3)$	$(3,3)$	$(4,1)$	$(4,2)$	$(3,4)$
$(*,1)$	−6.5	1.7	0	0	0	0	0	0	0.8	0.8	0.8	0	0	0
$(*,2)$	2.1	−8.7	0	0	0	0	0	0	2.2	2.2	2.2	0	0	0
$(1,1)$	0	0	−8.8	1.7	1.5	0	1.8	0	0.8	0	0	3	0	0
$(1,2)$	0	0	2.1	−9.9	0	1.5	0	1.9	2.2	0	0	0	2.2	0
$(2,1)$	0	0	1.2	0	−15.5	3.5	3.1	0	0	3.9	0	3.8	0	0
$(2,2)$	0	0	0	2.3	1.7	−10.6	0	2.8	0	1.3	0	0	2.5	0
$(3,1)$	0	0	1.3	0	2.5	0	−10.2	1.1	0	0	2.8	2.5	0	0
$(3,2)$	0	0	0	2.1	0	3.4	1.2	−12	0	0	2.4	0	2.9	0
$(1,3)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$(2,3)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$(3,3)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$(4,1)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$(4,2)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$(3,4)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0

7.1. Effect of η on Performance Measures

When the η value increases, the duration of vacation decreases. Therefore P_{normal}, P_{empty} and RSN increase. However, ECS and P_{vac} decrease. When η increases, the expected cost (EC) increases and reaches the maximum at $\eta = 0.5$ and, after that, EC decreases (Table 3).

Table 3. Effect of η: fix $n = 3$, $m = 2$, $\theta = 0.6$.

η	ECS	ECQ	P_{empty}	P_{vac}	P_{normal}	RSN	EC
0.3	3.9654	3.1822	0.1854	0.3614	0.4219	0.1084	36.6945
0.4	3.4572	2.7152	0.2006	0.3149	0.4271	0.1260	35.7274
0.5	3.1453	2.4333	0.2123	0.2818	0.4302	0.1409	**35.5274**
0.6	2.9335	2.2446	0.2217	0.2566	0.4322	0.1540	35.6703
0.7	2.7795	2.1093	0.2296	0.2366	0.4335	0.1656	35.9804
0.8	2.6621	2.0074	0.2363	0.2202	0.4345	0.1762	36.3743
0.9	2.5695	1.9279	0.2421	0.2064	0.4352	0.1858	36.8088
1	2.4943	1.8641	0.2472	0.1946	0.4356	0.1946	37.2602
1.1	2.4321	1.8118	0.2517	0.1843	0.4360	0.2027	37.7151
1.2	2.3796	1.7680	0.2557	0.1752	0.4363	0.2103	38.1656
1.3	2.3347	1.7310	0.2594	0.1672	0.4365	0.2173	38.6071
1.4	2.2958	1.6991	0.2627	0.1599	0.4367	0.2239	39.0370

The bold is to show that the minimum cost is attained at this point

7.2. Effect of θ on Performance Measures

When the θ value increases, the expected cost decreases. As the service rate in the vacation mode increases, ECS, ECQ, P_{normal} and P_{vac} decrease. However, P_{idle} increases when θ increases (Table 4 and Figure 1).

Table 4. Effect of θ: fix $n = 3$, $m = 2$, $\eta = 0.9$.

θ	ECS	ECQ	P_{empty}	P_{vac}	P_{normal}	RSN	EC
0.4	2.9154	2.2434	0.1953	0.2059	0.4661	0.1853	38.7982
0.45	2.8259	2.1609	0.2065	0.2063	0.4586	0.1856	38.3120
0.5	2.7383	2.0808	0.2180	0.2065	0.4510	0.1858	37.8183
0.55	2.6528	2.0031	0.2299	0.2065	0.4431	0.1859	37.3172
0.6	2.5695	1.9279	0.2421	0.2064	0.4352	0.1858	36.8088
0.65	2.4885	1.8553	0.2545	0.2061	0.4271	0.1855	36.2936
0.7	2.4099	1.7854	0.2671	0.2056	0.4189	0.1851	35.7724
0.75	2.3338	1.7182	0.2798	0.2049	0.4107	0.1845	35.2460
0.8	2.2602	1.6536	0.2927	0.2041	0.4025	0.1837	34.7154
0.85	2.1892	1.5918	0.3055	0.2031	0.3943	0.1828	34.1819
0.9	2.1207	1.5326	0.3184	0.2019	0.3861	0.1817	33.6467
0.95	2.0547	1.4761	0.3313	0.2006	0.3781	0.1805	33.1109

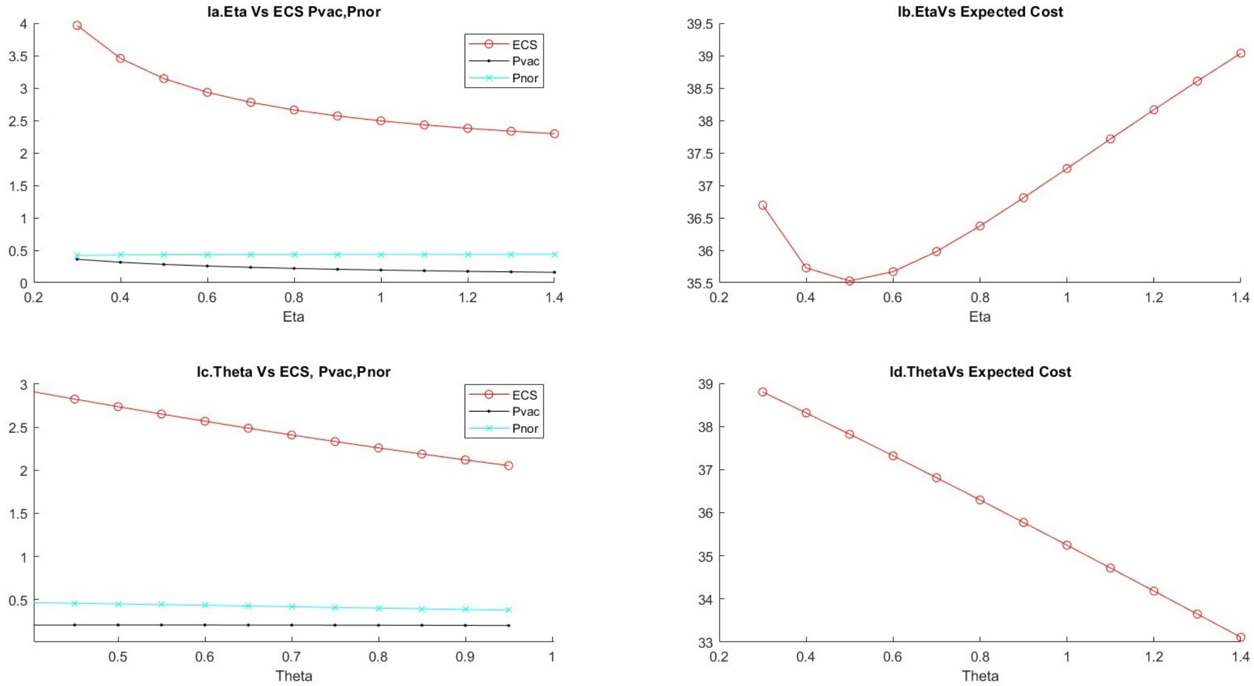

Figure 1. Effect of θ and η on ECS, Pvac, Pnor and cost function.

8. Conclusions

In this paper, we theoretically compared two single-server working vacation queueing models. In Model 1, arrival and service processes are interdependent. The arrival and service processes evolve by transitions on the product space of two Markov chains governed by a semi-Markov rule, with sojourn times in states governed by the exponential distribution. In the second model, we consider independent arrival and service processes following phase-type distributions with representation $(\boldsymbol{\alpha}, T)$ of order m and $(\boldsymbol{\beta}, S)$ of order n, respectively. We analyzed these models by using the matrix-analytic method. We also performed some numerical experiments to evaluate performance measures for Model 1.

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