mathematics

MDPI

Article

Liouville-Type Theorem for Nonlinear Elliptic Equations Involving Generalized Greiner Operator

Wei Shi

School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China; 3120205701@bit.edu.cn

Abstract: In this paper, we study the Liouville-type behaviors of the generalized Greiner operators with nonlinear boundary value conditions. We use the technique based upon the method of moving planes.

Keywords: Liouville-type theorem; moving plane; Greiner operator; nonlinear equations

MSC: 35J60; 35H20

1. Introduction

The purpose of this paper is to investigate Liouville properties for semi-linear elliptic equation with general nonlinearity

$$\begin{cases} \Delta_L u + f(u) = 0, & \text{in } \mathbb{R}^{2n+1}, \\ -\frac{\partial u}{\partial t} = g(u), & \text{on } \partial\mathbb{R}_+^{2n+1} \setminus \{0\}, \end{cases} \tag{1}$$

where Δ_L is the generalized Greiner operator, and f, g are nonnegative functions satisfying some appropriate conditions, which will be given later. The notation $\partial\mathbb{R}_+^{2n+1}$ denotes the boundary of set $\mathbb{R}_+^{2n+1} = \{(x, y, t) \in \mathbb{R}^{2n+1} | t > 0\}$. It is well known that the role played by the Liouville theorem is to establish a priori bounds for positive solutions of elliptic equations in bounded domains via the blow-up method.

Han and Zhao [1] studied a class of semi-linear elliptic equations with the principal part constructed by generalized Greiner vector fields, introducing the vector field method in their work. As an application, they studied the Liouville property of the following semi-linear equation:

$$\Delta_L u + h(\xi)u^p = 0, \tag{2}$$

on the generalized Greiner vector fields.

There are analogous results in the Euclidean case. In the splendid paper [2], Gidas and Spruck used the method of integral estimate to prove that, for $1 < p < \frac{n+2}{n-2}$, the following Equation (3) has no positive entire solution in the Euclidean space $\mathbb{R}^n$:

$$-\Delta u = u^q. \tag{3}$$

Similar results first appear in [3] using the main tools of the method of moving planes. Furthermore, the Liouville-type theorem for integral equation and system was established in paper [4]. Other results can be found in [5–9].

Recently, some Liouville-type theorems were obtained even for nonlinear elliptic equations with nonlinear boundary conditions in the Heisenberg group; see Theorem 1.1 in [10].

check for updates

Citation: Shi, W. Liouville-Type Theorem for Nonlinear Elliptic Equations Involving Generalized Greiner Operator. *Mathematics* **2023**, *11*, 61. https://doi.org/10.3390/math11010061

Academic Editor: Ravi P. Agarwal

Received: 29 November 2022
Revised: 17 December 2022
Accepted: 21 December 2022
Published: 24 December 2022

In addition to using the method of moving planes, the vector field method was also used to prove nonexistence results. Xu [11] obtained the nonexistence result on the Heisenberg group for the following equation:

$$\Delta_H u + h(x)u^p = 0, \ \text{ in } H^n, \tag{4}$$

and, supposing that weight function $h(x)$ satisfies some assumptions, then Equation (4) possesses no positive solutions providing $1 < p < 1 + \frac{8n+7}{(2n+1)^2}$. We note that the exponent $1 + \frac{8n+7}{(2n+1)^2}$ is smaller than $\frac{Q+2}{Q-2}$.

Yu [6] studied the following elliptic equation:

$$\begin{cases} -\Delta u = f(u), & \text{in } \mathbb{R}^N_+, \\ -\frac{\partial u}{\partial \nu} = g(u), & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{5}$$

He proved that this problem possess no positive solutions under some assumptions on nonlinear terms.

In recent years, the comparison principle and Liouville-type theorems for degenerate elliptic equations have been widely studied; see [12–16]. The Liouville-type theorem for cylindrical viscosity solutions of fully nonlinear CR invariant equations on the Heisenberg group were developed in [17]. As a by-product, the comparison principle with finite singularities for viscosity solutions to more general fully nonlinear operators on the Heisenberg group was obtained in [17].

The Hopf-type lemma and a CR type inversion for the generalized Greiner operator was first and extensively established in [18] .

In this paper, we study problem (1); it is very well known that both the equation and the boundary conditions are nonlinear. We are now ready to state the main result.

Theorem 1. *Let $u \in C(\mathbb{R}^{2n+1})$ be a nonnegative cylindrical solution for problem (1), and $f, g : [0, +\infty) \to [0, +\infty)$ are continuous functions satisfying*
(i) *$f(t), g(t)$ are nondecreasing in $(0, +\infty)$,*
(ii) *$h(t) = \frac{f(t)}{t^{\frac{Q+2}{Q-2}}}, l(t) = \frac{g(t)}{t^{\frac{4k}{Q-2}}}$ are nonincreasing in $(0, +\infty)$,*
(iii) *either h or l is not a constant,*
then $u \equiv c$ with $f(c) = g(c) = 0$ is the only solution of the problem (1).

The paper is organized as follows. In Section 2, we introduced some notations and facts that will be followed throughout the work. Theorem 1 is finally proved in Section 3. The disscussion is given in Section 4.

2. Preliminary Facts

The aim of this section is to introduce some notation and definitions about the generalized Greiner vector fields. We consider the Liouville property associated with generalized Greiner operators

$$\Delta_L = \sum_{j=1}^{n} (X_j^2 + Y_j^2),$$

where

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2}\frac{\partial}{\partial t}, \tag{6}$$

$$Y_j = \frac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2}\frac{\partial}{\partial t}, \tag{7}$$

$j = 1, \cdots, n, x, y \in \mathbb{R}^n, t \in \mathbb{R}, z = x + \sqrt{-1}y, |z| = \left[\sum_{j=1}^{n}(x_j^2 + y_j^2)\right]^{\frac{1}{2}}, k \geq 1.$

A function u is said to be cylindrical in $\mathbb{R}^{2n+1}$ with respect to the operator Δ_L if for any $(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, it has $u(x,y,t) = u(r,t), r = \sqrt{x^2+y^2}$.

If we denote by $\mathbf{A} = (a_{ij})$ the $(2n+1)\times(2n+1)$ symmetric matrix given by $a_{ij} = \delta_{ij}$ if $i,j = 1,\cdots,2n$, $a_{2n+1,j} = 2ky_j|z|^{2k-2}$ if $j = 1,\cdots,n$, $a_{2n+1,n+j} = -2kx_j|z|^{2k-2}$ if $j = 1,\cdots,n$, and $a_{2n+1,2n+1} = 4k^2|z|^{4k-2}$. We note that the matrix $\mathbf{A}$ is related to Δ_L by the formula

$$\Delta_L = \operatorname{div}(\mathbf{A}\nabla),$$

where ∇ and div denote the Euclidian gradient and Euclidian divergence operator of $\mathbb{R}^{2n+1}$, respectively.

Moreover, if we consider a $(2n)\times(2n+1)$ matrix whose rows are the coordinates of the vector field X_j, Y_j, that is

$$\sigma := \begin{pmatrix} I_n & 0 & 2ky|z|^{2k-2} \\ 0 & I_n & -2kx|z|^{2k-2} \end{pmatrix},$$

where I_n is the identity $n\times n$ matrix, then the generalized gradient ∇_L of a function $\psi : \mathbb{R}^{2n+1} \to \mathbb{R}$ is expressed by $\nabla_L\psi = (X_1\psi,\cdots,X_n\psi,Y_1\psi,\cdots,Y_n\psi) = \sigma\nabla\psi$, and

$$\Delta_L = \operatorname{div}(\sigma^T\sigma\nabla).$$

The dilation is defined as

$$\delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^{2k}t), \lambda > 0, \tag{8}$$

and the integer $Q = 2n+2k$ is called the homogeneous dimension with respect to dilation. Then, it is useful to consider the following homogeneous norm with respect to (8):

$$\rho(\xi) = \left[|z|^{4k} + t^2\right]^{\frac{1}{4k}}, \tag{9}$$

and the associated quasi distance between two point ξ,η in $\mathbb{R}^{2n+1}$ by setting

$$d(\xi,\eta) = \left[|z|^{4k} + |z'|^{4k} + (t-t')^2\right]^{\frac{1}{4k}}, \tag{10}$$

for $\xi = (z,t) \in \mathbb{R}^{2n+1}$, $\eta = (z',t') \in \mathbb{R}^{2n+1}$. We denote by $B_L(\xi,R)$ the quasi ball with center at ξ and radius R associated with the distance (10), that is

$$B_L(\xi,R) = \{\eta \in \mathbb{R}^{2n+1} | d(\xi,\eta) < R\}. \tag{11}$$

Note that for $R > 0$ sufficiently large, if $B(0,R)$ is the Euclidian ball of radius R centered at the origin, then

$$B(0,R) \subset B_L(0,R) \subset B(0,R^2).$$

Denote by

$$\nabla_L u = (X_1u,\cdots,X_nu,Y_1u,\cdots,Y_nu),$$

and

$$\begin{aligned}\Delta_L u &:= \sum_{j=1}^n X_j^2u + Y_j^2u \\ &= \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} + \frac{\partial^2 u}{\partial y_j^2} + 4ky_j|z|^{2k-2}\frac{\partial^2 u}{\partial x_j\partial t} - 4kx_j|z|^{2k-2}\frac{\partial^2 u}{\partial y_j \partial t} \\ &\quad + 4k^2|z|^{4k-2}\frac{\partial^2 u}{\partial t^2},\end{aligned} \tag{12}$$

the generalized gradient and the generalized Greiner operator, respectively.

Note that, when $k = 1$, the operator Δ_L becomes the well know sub-Laplacian Δ_{H^n} on the Heisenberg group H^n (see Folland [19]). If $k = 2, 3, \cdots, \Delta_L$ is the Greiner operator (see [20]). As is well known, the vector fields $X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n$ in (6) do not possess left translation invariance for $k > 1$ and, if $k = 1, 2, 3, \cdots$, they do not meet the Hörmander condition [21].

As in [18], we introduce the CR inversion of a function $u(x, y, t)$ in $\mathbb{R}^{2n+1}$ as

$$v(x, y, t) = \frac{1}{\rho^{Q-2}} u(\tilde{x}, \tilde{y}, \tilde{t}), \tag{13}$$

with $\tilde{x} = (\tilde{x_1}, \cdots, \tilde{x_n})$ and $\tilde{y} = (\tilde{y_1}, \cdots, \tilde{y_n})$, where

$$\tilde{x_i} = \frac{x_i t + |z|^{2k} y_i}{\rho^{2k+2}}, \ \tilde{y_i} = \frac{y_i t - |z|^{2k} x_i}{\rho^{2k+2}}, \ \tilde{t} = \frac{1}{\rho^{4k}}.$$

Lemma 2. *Suppose that $u \in C^2(\mathbb{R}^{2n+1}_+) \cap C(\overline{\mathbb{R}^{2n+1}_+})$ is a solution of (1), then v defined in (13) satisfies*

$$\begin{cases} \Delta_L v = \frac{1}{\rho^{Q+2}} f(u), & \text{in } \mathbb{R}^{2n+1}, \\ -\frac{\partial v}{\partial t} = \frac{1}{\rho^{Q+4k-2}} g(u), & \text{on } \partial\mathbb{R}^{2n+1}_+ \setminus \{0\}. \end{cases} \tag{14}$$

Proof. The first equation of this lemma has been proved in [18]. It remains to prove the second equation. In fact,

$$\begin{aligned} -\lim_{t\to 0} \frac{\partial v}{\partial t} &= -\lim_{\tilde{t}\to 0} \frac{1}{\rho^{Q-2}} \rho^{-4k} \frac{\partial u(\tilde{x}, \tilde{y}, \tilde{t})}{\partial \tilde{t}} \\ &= \frac{1}{\rho^{Q+4k-2}} g(u). \end{aligned} \tag{15}$$

□

3. Proof of Theorem 1

The proof of Theorem 1 uses the moving plane argument. Note that the function v might be singular at the origin and that

$$\lim_{\rho\to 0} \rho^{Q-2} v(r, t) = u(0). \tag{16}$$

We have previously seen that v satisfies the equation

$$\begin{cases} -\Delta_L v(x, y, t) = \frac{1}{\rho^{Q+2}} f(\rho^{Q-2} v(x, y, t)), & \text{in } \mathbb{R}^{2n+1} \setminus \{0\}, \\ -\frac{\partial v(x,y,t)}{\partial t} = \frac{1}{\rho^{Q+4k-2}} g(\rho^{Q-2} v(x, y, t)) & \text{on } \partial\mathbb{R}^{2n+1} \setminus \{0\}. \end{cases} \tag{17}$$

We define $h(t) = \frac{f(t)}{t^{\frac{Q+2}{Q-2}}}$, $l(t) = \frac{g(t)}{t^{\frac{Q+4k-2}{Q-2}}}$, and we obtain

$$f(\rho^{Q-2} v) = h(\rho^{Q-2} v) \rho^{Q+2} v^{\frac{Q+2}{Q-2}},$$

and

$$g(\rho^{Q-2} v) = l(\rho^{Q-2} v) \rho^{Q+4k-2} v^{\frac{Q+4k-2}{Q-2}},$$

then, the above equation can also be written as

$$\begin{cases} -\Delta_L v(x,y,t) = h(\rho^{Q-2} v(x,y,t)) v(x,y,t)^{\frac{Q+2}{Q-2}}, & \text{in } \mathbb{R}^{2n+1} \setminus \{0\}, \\ -\frac{\partial v(x,y,t)}{\partial t} = l(\rho^{Q-2} v(x,y,t)) v(x,y,t)^{\frac{Q+4k-2}{Q-2}}, & \text{on } \partial\mathbb{R}^{2n+1} \setminus \{0\}. \end{cases} \tag{18}$$

Let $\Sigma_\mu = \{(x,y,t) \in \mathbb{R}^{2n+1} | t \geq \mu\}, T_\mu = \partial\Sigma_\mu = \{(x,y,t) \in \mathbb{R}^{2n+1} | t = \mu\}$, and $p^\mu = (0,0,2\mu)$. We compare the values of the solution v on Σ_μ with those on its reflection. Let

$$v_\mu(x,y,t) = v_\mu(r_0,t) = v(r_0, 2\mu - t) = v(y, x, 2\mu - t),$$

for any (x,y) such that $(x^2+y^2)^{\frac{1}{2}} = r_0$. It is easy to see that v_μ satisfies the same equation as v does, that is

$$\begin{cases} -\Delta_L v_\mu(x,y,t) = h(\rho_\mu^{Q-2} v_\mu(x,y,t)) v_\mu(x,y,t)^{\frac{Q+2}{Q-2}}, & \text{in } \mathbb{R}^{2n+1} \setminus \{0\}, \\ -\frac{\partial v_\mu(x,y,t)}{\partial t} = l(\rho_\mu^{Q-2} v_\mu(x,y,t)) v_\mu(x,y,t)^{\frac{Q+4k-2}{Q-2}}, & \text{on } \partial\mathbb{R}^{2n+1} \setminus \{0\}. \end{cases} \tag{19}$$

If we define $w_\mu(\xi) = v_\mu(\xi) - v(\xi)$, then we can get the following key lemma.

Lemma 3. *For any fixed $\mu > 0$, the function $v_\mu, v \in L^{2^\sharp}(\Sigma_\mu) \cup L^\infty(\Sigma_\mu)$ with $2^\sharp = \frac{2Q}{Q-2}$. Furthermore, there exists $C_\mu > 0$, which is nonincreasing in μ, such that*

$$\begin{aligned} &\int_{\Sigma_\mu} |\nabla_L (v - v_\mu)^+|^2 d\xi \\ &\leq C_\mu \left[\left(\int_{A_\mu} \frac{1}{\rho^{2Q}} d\xi \right)^{\frac{2}{Q}} + \left(\int_{B_\mu} \frac{1}{\rho^{2kQ}} d\xi \right)^{\frac{2}{Q}} \right] \left(\int_{\Sigma_\mu} |\nabla_L (v - v_\mu)^+|^2 d\xi \right), \end{aligned} \tag{20}$$

where $A_\mu = \{(x,y,t) \in \Sigma_\mu | v > v_\mu\}$, $B_\mu = \{(x,y,t) \in \partial\Sigma_\mu | v > v_\mu, t = 0\}$.

Proof. If $\mu > 0$, then there exists $r > 0$, such that $\Sigma_\mu \subset \mathbb{R}^{2n+1} \setminus B_r(0)$; moreover, v is continuous and strictly positive in $\mathbb{R}^{2n+1} \setminus \{0\}$, with a possible singularity at the origin, and decays at infinity as $u(0)\rho^{2-Q}$, so that $v \in L^{2^\sharp} \cap L^\infty(\Sigma_\mu)$. Now, we give a cylinder cut-off function $0 \leq \eta_\epsilon \leq 1$, such that

$$\eta_\epsilon = \begin{cases} 1, & \text{if } 2\epsilon \leq |\xi - p^\mu| \leq \frac{1}{\epsilon}, \\ 0 & \text{if } |\xi - p^\mu| \leq \epsilon \ \text{ or } \ |\xi - p^\mu| \geq \frac{2}{\epsilon}, \end{cases} \tag{21}$$

$|\nabla_L \eta_\epsilon| \leq \frac{C}{\epsilon}$ for $\epsilon < |\xi - p^\mu| \leq 2\epsilon$ and $|\nabla_L \eta_\epsilon| \leq C\epsilon$ for $\frac{1}{\epsilon} \leq |\xi - p^\mu| \leq \frac{2}{\epsilon}$.

Next, $\psi = \eta_\epsilon (v - v_\mu)^+$ can be used as a test function, and we denote $\phi = \eta_\epsilon^2 (v - v_\mu)^+$; then we have

$$\begin{aligned} &\int_{\Sigma_\mu \cap \{2\epsilon \leq |\xi - p^\mu| \leq \frac{1}{\epsilon}\}} |\nabla_L (v - v_\mu)^+|^2 d\xi \leq \int_{\Sigma_\mu} |\nabla_L \psi|^2 d\xi \\ &= \int_{\Sigma_\mu} \nabla_L (v - v_\mu)^+ \nabla_L \phi d\xi + \int_{\Sigma_\mu} [(v - v_\mu)^+]^2 |\nabla_L \eta_\epsilon|^2 d\xi \\ &= \int_{A_\mu} -\Delta_L (v - v_\mu) \phi d\xi + \int_{B_\mu} \frac{\partial (v - v_\mu)}{\partial \nu} \phi d\xi' + I_\epsilon \\ &= \int_{A_\mu} \left[h(\rho^{Q-2} v) v^{\frac{Q+2}{Q-2}} - h(\rho_\mu^{Q-2} v_\mu) v_\mu^{\frac{Q+2}{Q-2}} \right] \phi d\xi \\ &\quad + \int_{B_\mu} \left[l(\rho^{Q-2} v) v^{\frac{Q+4k-2}{Q-2}} - l(\rho_\mu^{Q-2} v_\mu) v_\mu^{\frac{Q+4k-2}{Q-2}} \right] \phi d\xi' + I_\epsilon, \end{aligned} \tag{22}$$

where $I_\epsilon = \int_{\Sigma_\mu}[(v-v_\mu)^+]^2|\nabla_L \eta_\epsilon|^2 d\xi$.

Due to the monotonicity of h, l, $\rho > \rho_\mu$ and $v(\xi) > v_\mu(\xi)$ in A_μ and B_μ, we have

$$h(\rho^{Q-2}v) \le h(\rho_\mu^{Q-2}v_\mu), \tag{23}$$

and

$$l(\rho^{Q-2}v) \le l(\rho_\mu^{Q-2}v_\mu), \tag{24}$$

in A_μ and B_μ, respectively. Hence, we get

$$\begin{aligned}
&\int_{\Sigma_\mu \cap \{2\epsilon \le |\xi - p^\mu| \le \frac{1}{\epsilon}\}} |\nabla_L (v-v_\mu)^+|^2 d\xi \\
&\le \int_{A_\mu} h(\rho^{Q-2}v)[v^{\frac{Q+2}{Q-2}} - v_\mu^{\frac{Q+2}{Q-2}}]\phi d\xi \\
&\quad + \int_{B_\mu} l(\rho^{Q-2}v)[v^{\frac{Q+4k-2}{Q-2}} - v_\mu^{\frac{Q+4k-2}{Q-2}}]\phi d\xi' + I_\epsilon.
\end{aligned} \tag{25}$$

Moreover, since u is positive and bounded, there exists $0 < a = a_\mu < b = b_\mu < +\infty$ such that $a < \rho^{Q-2}v(\xi) = u(\tilde{\xi}) < b, \forall \xi, \tilde{\xi} \in \Sigma_\mu \setminus B_r(0)$, so that $h(\rho^{Q-2}v(\xi)) \le h(a_\mu) =: C_\mu^1$. Finally, if $0 \le v_\mu \le v$, we conclude that

$$\begin{aligned}
&\int_{A_\mu} h(\rho^{Q-2}v)[v^{\frac{Q+2}{Q-2}} - v_\mu^{\frac{Q+2}{Q-2}}]\phi d\xi \le C_\mu^1 \int_{A_\mu} v^{\frac{4}{Q-2}}(v-v_\mu)\phi d\xi \\
&\le C_\mu^1 \int_{A_\mu} \frac{1}{\rho^4}[(v-v_\mu)^+]^2 \eta_\epsilon^2 d\xi \\
&\le C_\mu^1 (\int_{A_\mu} \frac{1}{\rho^{2Q}} d\xi)^{\frac{2}{Q}} (\int_{\Sigma_\mu} [(v-v_\mu)^+]^{\frac{2Q}{Q-2}} d\xi)^{\frac{Q-2}{Q}},
\end{aligned} \tag{26}$$

in the last inequality, we have used the Hölder's inequality.

Similarly, by the decay estimate of v, there exists $C_\mu^2 > 0$, which is nonincreasing in μ, so that

$$\begin{aligned}
&\int_{B_\mu} l(\rho^{Q-2}v)[v^{\frac{Q+4k-2}{Q-2}} - v_\mu^{\frac{Q+4k-2}{Q-2}}]\phi d\xi' \le C_\mu^2 \int_{B_\mu} v^{\frac{4k}{Q-2}}(v-v_\mu)\phi d\xi' \\
&\le C_\mu^2 \int_{B_\mu} \frac{1}{\rho^{4k}}[(v-v_\mu)^+]^2 \eta_\epsilon^2 d\xi' \\
&\le C_\mu^2 (\int_{B_\mu} \frac{1}{\rho^{2kQ}} d\xi')^{\frac{2}{Q}} (\int_{\Sigma_\mu} [(v-v_\mu)^+]^{\frac{2Q}{Q-2}} d\xi')^{\frac{Q-2}{Q}}.
\end{aligned} \tag{27}$$

Therefore, it follows from the above inequalities that

$$\begin{aligned}
&\int_{\Sigma_\mu \cap \{2\epsilon \le |\xi - p_\mu| \le \frac{1}{\epsilon}\}} |\nabla_L (v-v_\mu)^+|^2 d\xi \\
&\le C_\mu^1 (\int_{A_\mu} \frac{1}{\rho^{2Q}} d\xi)^{\frac{2}{Q}} (\int_{\Sigma_\mu} [(v-v_\mu)^+]^{\frac{2Q}{Q-2}} d\xi)^{\frac{Q-2}{Q}} \\
&\quad + C_\mu^2 (\int_{B_\mu} \frac{1}{\rho^{2kQ}} d\xi)^{\frac{2}{Q}} (\int_{\partial\Sigma_\mu} [(v-v_\mu)^+]^{\frac{2Q}{Q-2}} d\xi')^{\frac{Q-2}{Q}} + I_\epsilon.
\end{aligned} \tag{28}$$

We claim now that $I_\epsilon \to 0$ as $\epsilon \to 0$. If we denote

$$B_\epsilon = \{\xi \in \Sigma_\mu | \epsilon < |\xi - p^u| < 2\epsilon \text{ or } \frac{1}{\epsilon} < |\xi - p^u| < \frac{2}{\epsilon}\},$$

then we get
$$\int_{B_\epsilon} |\nabla_L \eta_\epsilon| d\xi \leq C.$$

Hence, we infer from Hölder's inequality that

$$I_\epsilon \leq \left(\int_{B_\epsilon} [(v-v_\mu)^+]^{2^\sharp} d\xi\right)^{\frac{2}{2^\sharp}} \left(\int_{B_\epsilon} |\nabla_L \eta_\epsilon|^Q d\xi\right)^{\frac{2}{Q}} \leq C\left(\int_{B_\epsilon} [(v-v_\mu)^+]^{2^\sharp} d\xi\right)^{\frac{2}{2^\sharp}} \to 0,$$

as $\epsilon \to 0$ because $(v-v_\mu)^+ \in L^{2^\sharp}(\Sigma_\mu)$.

Finally, suppose for some $\epsilon \to 0$ in Equation (28), and set $C_\mu = C_\mu^1 S + C_\mu^2 S_T$ with S being the Sobolev constant and S_T being the Sobolev trace inequality constant, then we obtain

$$\begin{aligned} &\int_{\Sigma_\mu} |\nabla_L (v-v_\mu)^+|^2 d\xi \\ &\leq C_\mu \left[(\int_{A_\mu} \frac{1}{\rho^{2Q}} d\xi)^{\frac{2}{Q}} + (\int_{B_\mu} \frac{1}{\rho^{2kQ}} d\xi')^{\frac{2}{Q}} \right] \left(\int_{\Sigma_\mu} |\nabla_L (v-v_\mu)^+|^2 d\xi \right). \end{aligned} \tag{29}$$

□

We note that inequality (29) plays the same role as the maximum principle. If we can prove

$$C_\mu \left[(\int_{A_\mu} \frac{1}{\rho^{2Q}} d\xi)^{\frac{2}{Q}} + (\int_{B_\mu} \frac{1}{\rho^{2kQ}} d\xi')^{\frac{2}{Q}} \right] < 1,$$

then we get $v \leq v_\mu$ in Σ_μ, the same conclusion as the maximum principle implies.

The next lemma shows that

$$w_\mu(\xi) = v_\mu(\xi) - v(\xi) \geq 0, \forall \xi \in \Sigma_\mu. \tag{30}$$

Lemma 4. *Under the assumptions of Theorem 1, there exists some $\mu_0 > 0$, such that $v \leq v_\mu$ in Σ_μ for all $\mu > \mu_0$.*

Proof. By the decay behavior of v, see Equation (16), we can choose μ_0 large enough such that

$$C_\mu \left[(\int_{A_\mu} \frac{1}{\rho^{2Q}} d\xi)^{\frac{2}{Q}} + (\int_{B_\mu} \frac{1}{\rho^{2kQ}} d\xi)^{\frac{2}{Q}} \right] \leq \frac{1}{2},$$

then Equation (29) implies that

$$\int_{\Sigma_\mu} |\nabla_L (v-v_\mu)^+|^2 d\xi = 0,$$

the assertion follows. □

We now decrease the value of μ continuously, that is, we move the plane Σ_μ to the left as long as inequality (30) holds. We show that by moving in this way, the plane will not stop before hitting the origin. More precisely, let

$$\mu_1 = \inf\{\mu | w_\mu(\xi) \geq 0, \forall \xi \in \Sigma_{\bar{\mu}}, \bar{\mu} \geq \mu\}.$$

Lemma 5. *If $\mu_1 > 0$, then $w_{\mu_1}(\xi) \equiv 0$ for all $\xi \in \Sigma_{\mu_1}$.*

Proof. Arguing by contradiction, we claim that the plane Σ_{μ_1} can still be moved a small distance to the left. More precisely, there exists a $\delta_0 > 0$, such that, for all $0 < \delta < \delta_0$, we have

$$w_{\mu_1 - \delta}(\xi) \geq 0, \forall \xi \in \Sigma_{\mu_1 - \delta}. \tag{31}$$

This would contradict with the definition of μ_1, and hence (30) holds. Now, we prove our claim. Suppose that $w_{\mu_1}(\xi) \not\equiv 0$, then we infer from the continuity that $w_{\mu_1} \geq 0$. At the same time, from f being nondecreasing and (ii) in Theorem 1, we have

$$h(\rho^{Q-2}v)v^{\frac{Q+2}{Q-2}} = \frac{f(\rho^{Q-2}v)}{\rho^{Q+2}} \leq \frac{f(\rho^{Q-2}v_{\mu_1})}{\rho^{Q+2}} = \frac{f(\rho^{Q-2}v_{\mu_1})}{[\rho^{Q-2}v_{\mu_1}]^{\frac{Q+2}{Q-2}}} v_{\mu_1}^{\frac{Q+2}{Q-2}}$$
$$\leq \frac{f(\rho_{\mu_1}^{Q-2}v_{\mu_1})}{[\rho^{Q-2}v_{\mu_1}]^{\frac{Q+2}{Q-2}}} v_{\mu_1}^{\frac{Q+2}{Q-2}} = h(\rho_{\mu_1}^{Q-2}v_{\mu_1})v_{\mu_1}^{\frac{Q+2}{Q-2}}. \tag{32}$$

The above inequality implies

$$-\Delta_L v \leq -\Delta_L v_{\mu_1},$$

and, from the strong maximum principle in [18]

$$w_{\mu_1} > 0 \text{ in the interior of } \Sigma_{\mu_1}. \tag{33}$$

Moreover, since $\frac{1}{\rho^{2Q}}\chi_{A_\mu} \to 0$, $\frac{1}{\rho^{2Q}}\chi_{B_\mu} \to 0$, almost everywhere as $\mu \to \mu_1$, and $\frac{1}{\rho^{2Q}}\chi_{A_\mu} \leq \frac{1}{\rho^{2Q}}\chi_{\Sigma_{\mu_1-\delta}}$, $\frac{1}{\rho^{2Q}}\chi_{B_\mu} \leq \frac{1}{\rho^{2Q}}\chi_{\Sigma_{\mu_1-\delta}}$ for $\mu \in [\mu_1 - \delta, \mu_1]$ for some $\delta > 0$, then from the dominated convergence theorem, there holds

$$\int_{A_\mu} \frac{1}{\rho^{2Q}} d\xi \to 0,$$

and

$$\int_{B_\mu} \frac{1}{\rho^{2kQ}} d\xi' \to 0,$$

as $\mu \to \mu_1$. In particular, there exists $\delta > 0$, such that

$$C_\mu \left[\left(\int_{A_\mu} \frac{1}{\rho^{2Q}} d\xi \right)^{\frac{2}{Q}} + \left(\int_{B_\mu} \frac{1}{\rho^{2kQ}} d\xi' \right)^{\frac{2}{Q}} \right] \leq \frac{1}{2},$$

for all $\mu \in [\mu_1 - \delta, \mu_1]$, it follows from Lemma 4 that $v \leq v_\mu$ for all $\mu \in [\mu_1 - \delta, \mu_1]$; this contradicts the definition of μ_1. $\square$

Lemma 6. *Let f, g be as in Theorem 1 and assume also that u is positive. Let v be the CR inversion of u centered at a point $p = (0, 0, t_0)$; then v is symmetric with respect to T_{t_0}.*

Proof. We use the method of moving planes to prove this lemma. If $\mu_1 > t_0$, then we know from Lemma 5 that v is symmetric with respect to T_{μ_1}. On the other hand, the symmetry together with Equations (18) and (19) imply that $|\xi_\mu| = |\xi|$. By the assumption, either h or l is not a constant, which is impossible, hence we get $\mu_1 \leq t_0$. Similarly, we can also move the plane from the left and find a corresponding $\mu_1' \geq t_0$. Finally, we infer from $v_{\mu_1}(\xi) \geq v(\xi)$ and $v_{\mu_1'} \leq v(\xi)$ that $\mu_1 = \mu_1' = t_0$, that is, v hence u is symmetric with respect to T_{t_0}. $\square$

The following result from [6] plays a role in our proof.

Theorem 7. *Let $u \in C^0(\overline{\mathbb{R}^N_+})$ be a nonnegative solution of problem (5), where $f, g : [0, +\infty) \to [0, +\infty)$ are continuous functions with the properties*
(i) *$f(t), g(t)$ are nondecreasing in $(0, \infty)$.*
(ii) *$h(t) = \frac{f(t)}{t^{\frac{N+2}{N-2}}}, k(t) = \frac{g(t)}{t^{\frac{N}{N-2}}}$ are nonincreasing in $(0, \infty)$.*
Then $u \equiv c$ with $f(c) = g(c) = 0$.

Proof of Theorem 1. By Lemma 6, we have that, for any $t_0 \in \mathbb{R}$, the CR inversion function v of u at $p = (0, 0, t_0)$ is symmetric with respect to T_{t_0}. Since t_0 is arbitrary, then we have that u is independent of t, that is, u is a solution of

$$\begin{cases} -\Delta u = f(u), & \text{in } \mathbb{R}^{2n}_+, \\ -\frac{\partial u}{\partial t} = g(u) & \text{on } \partial\mathbb{R}^{2n}_+. \end{cases} \tag{34}$$

Since f, g is nondecreasing in $(0, \infty)$, and

$$\frac{f(t)}{t^{\frac{2n+2}{2n-2}}} = \frac{f(t)}{t^{\frac{Q+2}{Q-2}}} t^{\frac{Q+2}{Q-2} - \frac{2n+2}{2n-2}}, \tag{35}$$

and

$$\frac{g(t)}{t^{\frac{2n}{2n-2}}} = \frac{g(t)}{t^{\frac{4k}{Q-2}}} t^{\frac{4k}{Q-2} - \frac{2n}{2n-2}}, \tag{36}$$

is decreasing in t, then Theorem 7 implies that $u \equiv c$ with $f(c) = g(c) = 0$. □

4. Discussion

A useful tool for the study of symmetry for semi-linear equations with critical growth in $\mathbb{R}^n$ or in a ball is the moving planes method. This paper studied Liouville properties for semi-linear elliptic equations with general nonlinearity; the moving planes method based on integral inequalities was used to prove the Liouville theorem. As is known, when $k = 1$, Δ_L becomes the sub-Laplacian Δ_{H^n} on the Heisenberg group H^n, the nonexistence results for any positive of semi-linear or fully nonlinear equations in the Heisenberg group need further study.

Author Contributions: Writing—review and editing, W.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Beijing Natural Science Foundation (No. 1222017).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the anonymous referees for their thorough reading and insightful comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Han, Y.Z.; Zhao, Q.; Jin, Y.Y. Semi-linear Liouville theorems in the generalized Greiner vector fields. *Indian J. Pure Appl. Math.* **2013**, *44*, 311–342. [CrossRef]
2. Gidas, B.; Spruck, J. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.* **1981**, *34*, 525–598. [CrossRef]
3. Chen, W.; Li, C. Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.* **1991**, *63*, 615–622. [CrossRef]
4. Yu, X. Liouville type theorems for integral equations and integral systems. *Calc. Var. Partial Differ. Equ.* **2013**, *46*, 75–95. [CrossRef]
5. Yu, X. Liouville type theorem in the Heisenberg group with general nonlinearity. *J. Differ. Equ.* **2013**, *254*, 2173–2182. [CrossRef]
6. Yu, X. Liouville type theorem for nonlinear elliptic equation with general nonlinearity. *Discret. Contin. Dyn. Syst.* **2014**, *34*, 4947–4966. [CrossRef]
7. Jleli, M.; Ragusa, M.A.; Samet, B. Nonlinear Liouville-type theorems for generalized Baouendi-Grushin operator on Riemaniann manifolds. *Adv. Differ. Equ.* **2023**, *28*, 143–168. [CrossRef]
8. Sen, E. Transmission problem for the Sturm-Liouville equation involving a retarded argument. *Filomat* **2021**, *35*, 2071–2080.
9. Song, M.L.; Mei, S.Y. Existence of three solutions for nonlinear operator equations and applications to second-order differential equations. *J. Funct. Spaces* **2021**, *2021*, 6668037. [CrossRef]

10. Zhao, X. Liouville type theorem for nonlinear boundary value problem on Heisenberg group. *Ann. Acad. Sci. Fenn. Math. Vol.* **2017**, *42*, 723–733. [CrossRef]
11. Xu, L. Semi-linear Liouville theorems in the Heisenberg group via vector field methods. *J. Differ. Equ.* **2009**, *247*, 2799–2820.
12. Li, Y.Y.; Nguyen, L.; Wang, B. Towards a Liouville theorem for continuous viscosity solutions to fully nonlinear elliptic equations in conformal geometry. *Prog. Math.* **2019**, *333*, 199–222. [CrossRef]
13. Li, Y.Y.; Nguyen, L.; Wang, B. Comparison principles and Lipschitz regularity for some nonlinear degenerate elliptic equations. *Calc. Var. Partial Differ. Equ.* **2018**, *57*, 96.
14. Li, Y.Y.; Nguyen, L.; Wang, B. The axisymmetric σ_k-Nirenberg problem. *J. Funct. Anal.* **2021**, *281*, 109198. [CrossRef]
15. Li, Y.Y.; Wang, B. Strong comparison principles for some nonlinear degenerate elliptic equations. *Acta Math. Sci. Engl. Ser.* **2018**, 38, 1583–1590. [CrossRef]
16. Li, Y.Y.; Wang, B. Comparison principles for some fully nonlinear sub-elliptic equations on the Heisenberg group. *Anal. Theory Appl.* **2019**, *35*, 312–334.
17. Wang, B. A Liouville-type theorem for fully nonlinear CR invariant equations on the Heisenberg group. *Commun. Contemp. Math.* **2022**, 24, 2150060.
18. Niu, P.C.; Han, Y.W.; Han, J.Q. A Hopf type Lemma and a CR type inversion for the Generalized Greiner operator. *Can. Math. Bull.* **2004**, *47*, 417–430. [CrossRef]
19. Folland, G.B. A fundamental solution for a subelliptic operator. *Bull. Am. Math. Soc.* **1973**, *79*, 373–376.
20. Greiner, P.C. A fundamental solution for a nonelliptic partial differential operator. *Can. J. Math.* **1979**, *31*, 1107–1120. [CrossRef]
21. Hörmander, L. Hypoelliptic second order differential equations. *Acta Math.* **1967**, *119*, 147–171. [CrossRef]