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Comparing SSD-Efficient Portfolios with a Skewed Reference Distribution

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Abstract: Portfolio selection models based on second-order stochastic dominance (SSD) have the advantage of providing portfolios that reflect the behavior of risk-averse investors without the need to specify the utility function. Several scholars apply SSD conditions with respect to a reference distribution, typically that of the market index, to find its dominant SSD portfolio. However, since the reference distribution could strongly influence asset allocation, in this article, we compare two SSD-based portfolio selection strategies with a reshaping of the reference distribution in terms of its skewness and, consequently, its variance. Through an extensive empirical analysis based on multiasset investment universes, we empirically show that the SSD portfolios dominating the new skewed benchmark index generally perform better.

Keywords: stochastic dominance; portfolio optimization; reference distribution; skewness; multiasset investment

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JEL Classification: C02; C6; G1; G11

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1. Introduction

Stochastic dominance (SD) criteria have acquired significant relevance in portfolio selection problems thanks to their interesting theoretical properties and connection with expected utility theory (EUT). SD provides a partial order in the random returns space and considers the whole information regarding the distribution of the random portfolio return. Furthermore, portfolio selection strategies based on SD rules can overcome the issue of specifying a subjective utility function of investors (see, for example, Levy [1]). In a nutshell, an SD-based choice is able to satisfy all individuals represented by utility functions belonging to a class of functions with some general characteristics [2].

The concept of SD in financial applications dates back to Quirk and Saposnik [3], who originally introduced the first-order stochastic dominance (FSD) approach into the EUT framework based primarily on the monotony of investor preferences (i.e., the nonsatiety axiom). However, since FSD conditions do not express any investor's attitude to risk, they are very restrictive, namely, they have a poor ability to order random variables. Later on, Hadar and Russell [4], Hanoch and Levy [5], and Rothschild and Stiglitz [6] introduced the second-order stochastic dominance (SSD) criterion. SSD offers the investor a selection criterion also based on risk (e.g., volatility). Therefore, agents try to find the balance between reward and risk and, in this sense, since their utility functions share certain properties, the SD approach could ensure rules of unanimity [2].

Among those who contributed to SSD with relevant results, we can also mention the studies of Ogryczak and Ruszczynski [7,8]. First, they compare stochastic dominance versus mean-risk approaches where the "former is based on an axiomatic model of risk-averse preferences but does not provide a convenient computational recipe" [7]. Nevertheless, the SD approach considers the whole returns distributions, whereas a gain-risk model only uses a few statistics and, therefore, typically ignores many aspects of the return distribution. On the other hand, a gain-risk model is easy to use and simple to explain, and has the merit to describe the problem via a trade-off analysis. However, using variance as a risk measure, "the resulting mean-variance (Markowitz) model is, in general, not consistent with stochastic dominance rules" [7]. In [8], the authors provide some necessary conditions for stochastic dominance, which are connected to the location and dispersion parameters of a random variable. For an overview of SSD, see Fabian et al. [9] and Valle et al. [10].

Regarding portfolio selection, several contributions are provided in the literature about portfolio efficiency in terms of SSD, with respect to a set of feasible portfolios. For instance, Kuosmanen [11] developed practical tests for portfolio efficiency based on the SD criteria with a linear programming structure. Roman et al. [12] presented a portfolio selection model where the optimal solution dominates an (appropriate) reference distribution with respect to SSD. Fabian et al. [13] tackled the computational burden issues that typically affect the SD approach by solving the SSD-based models with cutting plane techniques. Using these findings, Roman et al. [14] then empirically investigated the effectiveness and computational improvement of the SSD-based models for addressing the enhanced indexation problem. More recently, Kopa and Post [15] developed a general test for SSD portfolio efficiency with a primal–dual representation that accounts for portfolio inefficiencies, using a linear programming formulation. The authors also provided a compact, reduced version of that test, which proved to be less computationally demanding.

Since SSD approaches select portfolios that are dominant with respect to a given benchmark index, the shape of its distribution could be a crucial issue. In this regard, Valle et al. [10] presented a novel portfolio optimization approach, where the reference distribution is reshaped, in order to select SSD-efficient portfolios that provide more appealing features [16].

This paper aims to compare the SSD models of Roman et al. [14] and Kopa and Post [15], where the distribution of the benchmark index is reshaped in terms of its skewness and, consequently, its variance (similarly to [10]). For this, we provide an empirical analysis based on four real-world datasets, consisting of multiasset investment universes: equity, bonds, ETFs, and commodities. We test the two SSD strategies with and without the skewed benchmark, along with the global minimum variance portfolio. Our empirical findings confirm that improving the benchmark index distribution leads to selecting SSD portfolios with superior out-of-sample performance results. Furthermore, among the set of reshaped benchmarks analyzed, we found that the approach of the model of Roman et al. [14] shows a greater ability to select SSD portfolios than that of Kopa and Post [15].

The rest of the paper is organized as follows. In Section 2, we introduce some preliminary concepts on the most commonly used exact stochastic dominance criteria. In Section 3, we first report the aforementioned SSD-based portfolio selection models of Roman et al. [14] and Kopa and Post [15]. Then, we discuss the procedure of reshaping the benchmark index used in both SSD models. In Section 4, we present and discuss the computational results of the empirical analysis based on real-world data, comparing the out-of-sample performance of the "improved" SSD-portfolios with respect to the original ones. Finally, Section 5 contains some concluding remarks and outlines possible future research such as third-order SD [17–19].

2. Theoretical Framework

In this section, we discuss some preliminary concepts on stochastic dominance rules, which have connections with expected utility theory (EUT). According to EUT, an investment is preferred over another if its expected utility is greater. This implies that all possible investments can be ranked and there is no situation where two bets cannot be compared (axiom of *complete ordering*). However, the EU criterion requires the specification of the investor's utility function $u(\cdot)$, which is a subjective matter. On the other hand, the stochastic dominance criteria do not require knowledge of the exact form of the utility function. Indeed, these criteria can satisfy all individuals represented by utility functions belonging to a class of functions with some general features.

For the sake of readability, we recall here the most commonly used exact stochastic dominance criteria. Let X and Y be two random variables, whose cumulative distribution functions (CDF) are $F_X(\alpha) = P(X \leq \alpha)$ and $F_Y(\alpha) = P(Y \leq \alpha)$, respectively, where $\alpha \in \mathbb{R}$.

Definition 1 (First-order stochastic dominance (FSD))**.** *X FSD-dominates Y if and only if*

$$F_X(\alpha) \leq F_Y(\alpha) \quad \forall \alpha \in \mathbb{R}, \tag{1}$$

or, in terms of expected utility,

$$E[u(X)] \geq E[u(Y)] \quad \forall u \in U_1\,, \tag{2}$$

where U_1 denotes the class of all nondecreasing utility functions u.

Note that both for relations (1) and (2), at least one inequality must be strict. Furthermore, the equivalence between (1) and (2) can be found, e.g., in [20].

Definition 2 (Second-order stochastic dominance (SSD))**.** *X FSD-dominates Y if and only if*

$$\int_{-\infty}^{\alpha} F_X(t)dt \leq \int_{-\infty}^{\alpha} F_Y(t)dt \quad \forall \alpha \in \mathbb{R}, \tag{3}$$

or, in terms of expected utility,

$$E[u(X)] \geq E[u(Y)] \quad \forall u \in U_2\,, \tag{4}$$

where U_2 indicates the class of all nondecreasing and concave utility functions u.
Again, for relations (3) and (4), it is required that at least one inequality must be strict. Note that the equivalence of (3) and (4) can be found, for example, in [20].

As shown by [21], the inequalities (3) are equivalent to the following relations:

$$E[\max(\alpha - X, 0)] \leq E[\max(\alpha - Y, 0)] \quad \forall \alpha \in \mathbb{R} \tag{5}$$

Furthermore, the inequalities (5) can also be expressed in terms of tails, as discussed in [22]. More precisely, X SSD-dominates Y if and only if

$$Tail_\beta(X) \geq Tail_\beta(Y) \quad \forall \beta \in (0,1]. \tag{6}$$

where $Tail_\beta(X)$ represents the unconditional expectation of the worst $\beta \cdot 100\%$ of the outcomes of X. In the case that X and Y represent random returns, this ensures that the tail of Y is fatter than the tail of X; therefore, Y is riskier than X.

The SSD rule is less demanding than that of FSD. More precisely, given an SD criterion of order v, as the order increases, the requirements of $v+1$-SD condition become less restrictive. Specifically, as described in [20], a stochastic dominance of the lower orders implies a stochastic dominance of the higher orders, while the opposite is not necessarily true.

3. Portfolio Selection via SSD Criteria

In this section, we describe two SSD models applied to portfolio selection that have received a large amount attention in the literature: the model developed by Roman–Mitra–Zverovich (RMZ) [14] and that of Kopa–Post (KP) [15].

The first approach aims to address the problem of enhanced indexation (EI). Given a benchmark index, the said approach consists of selecting a portfolio that is able not only to replicate it but at the same time also generate an additional return compared to the benchmark. This investment strategy is often referred to as *active*, differently from the index tracking problem that is defined as *passive*, and aims to replicate a given reference index. A comprehensive survey of these problems can be found in [23–28] for IT, and in [24,29–34] for EI. Furthermore, as shown by Agrrawal [35], a multiasset-class diversified portfolio, based on passive but liquid ETFs, can perform well "in mean–variance space and under varying market conditions, including the very adverse 2008 market crash".

The KP approach is originally developed as an LP test aimed at investigating whether a given portfolio dominates another with respect to SSD. It represents a generalization of the tests developed by Post in [36]. In [15], the authors also propose a dual formulation of their portfolio efficiency test, which has an interesting interpretation in terms of tails (see Section 3.2). As mentioned in the introduction, in this work, we compare the RMZ and KP models by appropriately reshaping the benchmark (see Section 3.3), and test them, in terms of performance, on multiasset investment universes.

In the remaining part of the section, we consider the asset returns defined on a discrete sample space with T states of the nature, where each state has a probability of occurrence equal to π_t with $t = 1, \ldots, T$. Furthermore, we use here a look-back approach, where the outcomes of the discrete random returns correspond to the historical scenarios, which are equally likely, i.e., $\pi_t = \frac{1}{T}\ \forall t$, as typically assumed in portfolio optimization (see, e.g., [37–39] and references therein). Thus, denoting by p_{it} the price of asset i at time t, the linear return of asset i at time t is $r_{it} = \frac{p_{it} - p_{i(t-1)}}{p_{i(t-1)}}$ with $i = 1, \ldots, n$ (n is the number of assets in the investment universe) and with $t = 1, \ldots, T$ (T is the length of an in-sample window of historical realizations). Hence, for a given vector of portfolio weights $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, where x_i denotes the fraction of capital invested in the i-th asset, adopting linear returns, we have that $R_t(\boldsymbol{x}) = \sum_{i=1}^{n} x_i r_{it}$ is the portfolio return at time t.

The RMZ and KP models examined in this work are then used to find portfolios $\boldsymbol{x}$ whose returns $R(\boldsymbol{x})$ SSD-dominate those of a given benchmark index R_I.

Note that if we denote by $R_{(1)}(\boldsymbol{x}) \leq \ldots \leq R_{(T)}(\boldsymbol{x})$ the ordered outcomes of the portfolio return, and by $R^I_{(1)} \leq \ldots \leq R^I_{(T)}$ the ordered outcomes of the benchmark index return, then

- $R(\boldsymbol{x})$ FSD-dominates R^I iff

$$R_{(j)}(\boldsymbol{x}) \geq R^I_{(j)} \quad \forall \quad j = 1, \ldots, T \tag{7}$$

- $R(\boldsymbol{x})$ SSD-dominates R^I iff

$$\sum_{t=1}^{j} R_{(t)}(\boldsymbol{x}) \geq \sum_{t=1}^{j} R^I_{(t)} \quad \forall \quad j = 1, \ldots, T \tag{8}$$

Since in the case of discrete random returns we have that $Tail_{\frac{j}{T}}(R(\boldsymbol{x})) = \frac{1}{T}\sum_{t=1}^{j} R_{(t)}(\boldsymbol{x})$, Conditions (8) are equivalent to (6). Recalling that $CVaR_{\frac{j}{T}}(R(\boldsymbol{x})) = -\frac{1}{j}\sum_{t=1}^{j} R_{(t)}(\boldsymbol{x})$, then $Tail_{\frac{j}{T}}(R(\boldsymbol{x})) = -\frac{j}{T} CVaR_{\frac{j}{T}}(R(\boldsymbol{x}))$. As a consequence, requiring that $Tail_{\frac{j}{T}}(R(\boldsymbol{x})) \geq Tail_{\frac{j}{T}}(R^I)$ is equivalent to imposing $CVaR_{\frac{j}{T}}(R(\boldsymbol{x})) \leq CVaR_{\frac{j}{T}}(R^I)$ for all $j = 1, \ldots, T$.

3.1. The Mitra–Roman–Zverovich Model

In this section, we describe the model proposed by Roman, Mitra, and Zverovich in [14]. The authors developed a multiobjective approach aimed at finding a portfolio x which dominates a given benchmark index with respect to SSD. Due to the aforementioned relationship between CVaR and tail, this approach can be expressed by both quantities. This then gives rise to two different formulations of the multiobjective problem, leading, in general, to different solutions (see Remark 1 in [14]). However, since the model with CVaR seems to show better out-of-sample performance with respect to that with tail (for more details, see [14]), in this paper, we consider only the most promising formulation, which we report below for the sake of exposition.

$$\begin{array}{ll} \min\limits_{x} & \left[(CVaR_{\frac{1}{T}}(R(x)) - CVaR_{\frac{1}{T}}(R_I)),\ldots,(CVaR_{\frac{T}{T}}(R(x)) - CVaR_{\frac{T}{T}}(R_I))\right] \\ \text{s.t.} & x \in C \end{array} \tag{9}$$

where C is a polyhedron. Among the infinite Pareto-optimal points of Problem (9) [14], consider the one obtained through the minimax scalarization as follows:

$$\begin{array}{ll} \min\limits_{x} & \max\limits_{1\le j\le T}(CVaR_{\frac{j}{T}}(R(x)) - CVaR_{\frac{j}{T}}(R_I)) \\ \text{s.t.} & x \in C\,. \end{array} \tag{10}$$

As shown in [13], a cutting plane representation of CVaR can be used. Thus, as proposed by [40], for all $j = 1,\ldots,T$, this approach leads to the following expressions:

$$\begin{array}{ll} CVaR_{\frac{j}{T}}(R(\boldsymbol{x})) = \max\limits_{\boldsymbol{x}} \dfrac{1}{j}\sum\limits_{j\in\mathcal{S}} -R_{(j)}(\boldsymbol{x}) & \\ \text{such that } \mathcal{S} \subset \{1,\ldots,T\}, & |\mathcal{S}| = j, \end{array} \tag{11}$$

where $|\mathcal{S}|$ represents the cardinality of $\mathcal{S}$. Hence, Problem (10) can be reformulated as follows:

$$\begin{cases} \min\limits_{x,\theta} & \theta \\ \text{s.t.} & \\ & \theta + CVaR_{\frac{j}{T}}(R^I) \ge \dfrac{1}{j}\sum\limits_{j\in\mathcal{S}_j} -R_{(j)}(\boldsymbol{x}) \quad \forall \mathcal{S}_j \subset \{1,\ldots,T\}, \quad |\mathcal{S}_j| = j \quad j = 1,\ldots,T \\ & \sum\limits_{i=1}^{n} x_i = 1 \\ & x_i \ge 0, \qquad\qquad i = 1,\ldots,n \\ & \theta \in \mathbb{R} \end{cases} \tag{12}$$

Although a large number of cuts may be required, in practice, this formulation yields the solution after a few cuts. Note that if the optimal value function $\theta^\star$ of Problem (12) is nonpositive, then the corresponding optimal portfolio $\boldsymbol{x}^\star$ is SSD-efficient. On the other hand, if $\theta^\star > 0$, then $\boldsymbol{x}^\star$ is almost SSD-efficient in the sense of [41] with $\varepsilon = \theta^\star$.

3.2. The Kopa–Post Model

As mentioned above, in [15] the authors proposed an LP test to evaluate whether a fixed portfolio is SSD-efficient with respect to all feasible portfolios represented by a polytope. On the one hand, their primal LP model shows an interesting interpretation in terms of utility functions; on the other hand, the dual formulation of this portfolio efficiency test has an appealing characterization in terms of tails and, as a further result, can select, if it exists, another portfolio (belonging to a polytope) that SSD-dominates the initially fixed portfolio. In this study, we then test the dual KP model which we report below for convenience.

$$
\begin{cases}
\max\limits_{x,d} & \sum\limits_{j=1}^{T} w_j d_j \\
\text{s.t.} & \\
& Tail_{\frac{j}{T}}(R(\boldsymbol{x})) - Tail_{\frac{j}{T}}(R_I) \geq d_j \quad j = 1,\dots,T \\
& \sum\limits_{i=1}^{n} x_i = 1 \\
& x_i \geq 0 \qquad i = 1,\dots,n \\
& d_j \geq 0 \qquad j = 1,\dots,T
\end{cases}
\tag{13}
$$

We implement Problem (13) in its LP formulation, namely, Model (10) of [15], where the positive weights w_j, with $j = 1, \dots, T$, are chosen as in [42]. More precisely, we use the following weighting scheme of [42] denoted by the acronym KP2011Power3:

$$
w_j = \begin{cases}
\dfrac{u'(1+R_j^I) - u'(1+R_{j+1}^I)}{u'(1+R_j^I)} & \text{for } j = 1,\dots,T-1 \\
\dfrac{u'(1+R_j^I)}{u'(1+R_1^I)} & \text{for } j = T
\end{cases}
$$

where $u(1+R_t^I) = \dfrac{(1+R_t^I)^{1-\alpha}}{1-\alpha}$, $u'(1++R_t^I) = (1++R_t^I)^{-\alpha}$, and $\alpha = 3$ is the risk aversion parameter.

3.3. Reshaping the Reference Distribution

In this section, we briefly provide some details on the reshaping procedure of the reference distribution applied for both the RMZ and KP approaches. As mentioned above, the RMZ and KP models examined in this work are used to find a portfolio $\boldsymbol{x}$ whose return $R(\boldsymbol{x})$ SSD-dominates that of a given benchmark index R_I. Note that the shape of the benchmark distribution could be a crucial issue for both models. In this regard, the aim is to reshape the original benchmark distribution, thus yielding a new (synthetic) distribution that should generate in the two models portfolios with better-performing returns [16,43], such as a higher expected return and skewness. Clearly, the reshaping of the original index has to be made so that the SSD-based optimization problems, in the case that (12) and (13), remain feasible.

Let μ_I, σ_I, and γ_I, respectively, denote the mean, standard deviation, and skewness of a fixed benchmark index R_I. The goal is to obtain a new improved benchmark index, $\widetilde{R_I}$, with the parameters $\widetilde{\mu}_I$, $\widetilde{\sigma}_I$, and $\widetilde{\gamma}_I$, which coincide with target values $\overline{\mu_I}$, $\overline{\sigma_I}$ and $\overline{\gamma_I}$, that are specified a priori. For this purpose, we follow the reshaping method provided by Valle et al. [10], who introduce an algorithmic procedure to construct a synthetic distribution with specified target values for its first three moments. This procedure is based on the quadratic curve equating method introduced by Wang and Kolen [44], which guarantees to preserve the first three moments of a distribution (see also [45]).

As suggested by Valle et al. [10], the target values $\overline{\mu_I}$, $\overline{\sigma_I}$, and $\overline{\gamma_I}$ have to be consistent or, at least, not independent, from the original parameters μ_I, σ_I, and γ_I, since it may make it more challenging to obtain efficient portfolios that SSD-dominate the improved benchmark index. Therefore, we consider two multiplicative parameters, Δ_γ and Δ_σ, by which we compute the deviation of the target skewness $\overline{\gamma_I}$ and standard deviation $\overline{\sigma_I}$ from the original values γ_I and σ_I, respectively. More precisely, we consider

$$
\begin{aligned}
\overline{\sigma_I} &= \sigma_I + \sigma_I \Delta_\sigma \\
\overline{\gamma_I} &= \gamma_I + |\gamma_I| \Delta_\gamma \,,
\end{aligned}
$$

where $\Delta_\sigma \geq -1$ and $\Delta_\gamma \geq 0$.

In the empirical analysis, we follow the same empirical setup of [10]. More precisely, we set the mean of the reshaped distribution $\overline{\mu_I} = \mu_I$, since the out-of-sample performance results seem to be poorly affected by large variations in the mean target value $\overline{\mu_I}$. Furthermore, we test several values of the two multiplicative parameters Δ_σ and Δ_γ, spanning the intervals $[-0.1, 0.5]$ and $[0, 5]$, respectively, with step size equal to 0.1. Clearly, when $\Delta_\sigma = 0$ and $\Delta_\gamma = 0$, $\overline{\sigma_I} = \sigma_I$ and $\overline{\gamma_I} = \gamma_I$, namely, we obtain the original benchmark distribution. Note that, for the sake of space, in the empirical analysis we provide only those results that are significant for discussion and for which the RMZ model and the KP model, with the new improved benchmark index $\widetilde{R_I}$, guarantee SSD portfolios (i.e., admit optimal solutions). Indeed, we observe that in the KP model, the corresponding optimal portfolio, if it exists, exactly satisfies conditions (6), namely, Problem (13) always provides an SSD-efficient portfolio. On the other hand, as discussed in Section 3.1, for the RMZ model (12), the corresponding optimal portfolio can be SSD-efficient or almost SSD-efficient. Therefore, it follows that the RMZ model is less demanding than the KP model, namely, the RMZ approach shows a greater ability to select SSD portfolios than the KP one. Thus, in the empirical analysis, we expect that when reshaping the reference index by varying Δ_σ and Δ_γ, more infeasible situations occur with the KP model than with the RMZ model.

4. Empirical Analysis and Discussion

This section reports the empirical analysis based on four real-world datasets, consisting of multiasset investment universes, in which we test and compare the RMZ model [14] and the KP model [15] with the new improved benchmark index $\widetilde{R_I}$ described in Section 3.3. More precisely, in Section 4.1 we describe the datasets considered and the experimental setup of the empirical analysis. Section 4.2 introduces a broad description of the performance measures used to evaluate the out-of-sample performances of the SSD-efficient portfolios. Then, in Section 4.3, we compare the out-of-sample performance of the "improved" SSD-portfolios with respect to the original ones.

All experiments were executed on a workstation with Intel(R) Xeon(R) CPU E5-2623 v4 (2.6 GHz, 64 gigabyte RAM) under Windows 10 Pro, using Matlab R2022a.

4.1. Datasets and Experimental Setup

The experiments were conducted on the following real-world datasets, which take into account both corporate actions and income generating events, as obtained from Refinitiv (Datastream). Regarding returns, we refer the reader to the paper of Agrrawal et al. [46] highlighting the fact that, at the time of the investigation, "most prominent finance websites excluded income-generating events such as dividends and interest" in comparative return graphics.

- Euro Stoxx 50 (ES50) is a stock index of the Eurozone, which includes the 50 largest-cap companies. It consists of the prices (in EUR) adjusted for dividends and stock splits, of 46 assets, with weekly frequency (from 2 January 2006 to 2 May 2022).
- Euro Bonds (EuroBonds) is a bond index of the Eurozone sovereign bonds of 11 countries. It contains the weekly total return indices of 72 assets, with maturities ranging from 1 to 30 years (from 1 January 2008 to 25 October 2022).
- ETF Emerging countries (ETF) is an ETF index of 22 emerging countries, and it consists of the total return indices (in USD) of 22 assets, with weekly frequency (from 1 January 2008 to 25 October 2022).
- Commodities and Italian Bonds (CIB) is a mixed index of commodities (agriculture, gold, energy, industrial metals) and Italian sovereign bonds, with maturities ranging from 1 to 30 years. It contains the weekly total return indices of 12 assets, with weekly frequency (from 1 January 2008 to 25 October 2022).

As described above, we aim to compare the RMZ and KP models, where the distribution of the benchmark index is reshaped in terms of its skewness and its variance. In our empirical analysis, the benchmark is represented by the ES50 market index for the equity universe, while we define an equally-weighted benchmark portfolio for the other datasets (for a similar approach, see [35]). We adopt a rolling time window scheme of evaluation, namely, we allow for the possibility of rebalancing the portfolio composition during the holding period, at fixed intervals. The length of the in-sample windows is set to 2 years (i.e., 104 weekly observations), while we consider one financial month (i.e., 4 weekly observations) for both the rebalancing and holding period (namely, the out-of-sample window). Then, we shift the in-sample window by 1 month, we again compute the optimal portfolio with respect to the new in-sample window, and therefore we evaluate the out-of-sample portfolio performance for the subsequent 1 month. This procedure is repeated up to the end of the time series of available returns. Finally, we collect all the computed out-of-sample portfolio returns, which are evaluated using several performance measures, described in the next section.

We point out that we choose to adopt a 2-year period for the in-sample window based on the empirical setup of [32,47], and of some preliminary tests. Furthermore, this choice also addresses the problem of the stability of the solutions for estimation errors of the covariance matrix, used in the GMinV model. Indeed, in the empirical setup we took into account some insights from the literature (see, e.g., [48,49]). Since the estimation error strongly depends on n and T, we set the length of the in-sample windows so that $n < T$, where n represents the number of assets available in the market and T is the number of observations considered. As a consequence, the perturbation of the input data becomes less significant, and we do not have any instability problems due to the singularity of the covariance matrix.

We then perform the following steps:

Step 1: Increase the skewness of the benchmark index R_I by increasing the multiplicative parameter Δ_γ, until Models (12) and (13) admit optimal solutions (see Section 3.3), and collect the resulting performances.

Step 2: For each performance measure, rank the computational result by assigning the value 1 to the most performing outcome. Score the iterations, by computing the median value of the overall performance rankings, and choose the iteration of Δ_γ that provides the lowest score.

Step 3: Given the best value of Δ_γ from Step 2, modify the volatility of the benchmark distribution by varying Δ_σ.

Step 4: Apply the scoring method as in Step 2, to choose the iteration of Δ_σ that provides the lowest score.

To evaluate the goodness of the reshaping procedure, we compare the performances of the portfolios that SSD-dominate the reshaped benchmark with those obtained by the portfolios that SSD-dominate the original benchmark. For comparison purposes, we also consider the global minimum variance portfolio (GMinV).

4.2. Performance Measures

As pointed out by Hodder et al. [42], while "Constructing portfolios based on second-order stochastic dominance (SSD) is theoretically attractive since all risk-averse investors would prefer a dominating portfolio. However, choosing among SSD efficient portfolios is a challenge without an obvious ranking metric". Thus, to evaluate the out-of-sample portfolio results, various performance measures widely used in the literature (e.g., [43,47,50–53]) are proposed. In the following, they are summarized for the reader's convenience.

- $\boldsymbol{\mu^{out}}$ is the average portfolio return, namelym $\mu^{out} = \mathbb{E}[R^{out}]$, where R^{out} represents the out-of-sample portfolio return. Clearly, the higher its value, the better the performance.
- $\boldsymbol{\sigma^{out}}$ is the standard deviation, i.e., $\sigma^{out} = \sqrt{\mathbb{E}[(R^{out} - \mu)^2]}$. As one may expect, lower values are associated with higher performances.
- The Sharpe ratio [54,55] is a measure of gain per unit of risk and is defined as

$$Sharpe = \frac{\mu^{out} - r_f}{\sigma^{out}}$$

where $r_f = 0$, μ^{out} is the sample mean of the out-of-sample portfolio return R^{out}, and σ^{out} is its standard deviation. The higher the Sharpe ratio, the better the portfolio performance.

- The Sortino ratio [56] is defined as

$$Sortino = \frac{\mu^{out} - r_f}{\sqrt{\mathbb{E}[((R^{out} - r_f)^{-})^2]}}$$

where $r_f = 0$ and $(R^{out} - r_f)^{-} = -\min\{R^{out} - r_f, 0\}$. A higher value of Sortino ratio is associated with better portfolio performance.

- The information ratio (InfoR) is defined as the expected value of the difference between the out-of-sample portfolio return and that of the benchmark index, divided by the standard deviation of such difference, namely,

$$InfoR = \frac{E[R^{out} - R_I^{out}]}{\sigma[R^{out} - R_I^{out}]}.$$

The larger its value, the better the portfolio performance.

- The turnover (see, e.g., DeMiguel et al. [57]) evaluates the amount of trading required to put into practice the portfolio strategy and is defined as

$$Turnover = \frac{1}{N_{reb}} \sum_{t=1}^{N_{reb}} \sum_{k=1}^{n} \mid x_{t,k} - x_{t-1,k} \mid$$

where N_{reb} indicates the number of rebalances, $x_{t,k}$ represents the portfolio weight of asset k after rebalancing, and $x_{t-1,k}$ is the portfolio weight before rebalancing at time t. This definition of portfolio turnover is a proxy of the effective one and evaluates the amount of trade generated by the models at each rebalancing time. Low portfolio turnover indicates better portfolio performance (thus confirming that trading is hazardous to wealth [58]).

- The Rachev ratio [59] is a measure of the relative gap between the mean of the best α% values of $R^{out} - r_f$ and that of the worst β% ones. It is computed as

$$Rachev = \frac{CVaR_{\alpha}(r_f - R^{out})}{CVaR_{\beta}(R^{out} - r_f)},$$

with $r_f = 0$ and $\alpha = \beta = 5\%$. A high Rachev ratio is preferred.

- The return on investment (ROI) is a performance measure used to analyze the profitability of investments and is defined as the time-by-time return generated by a given portfolio strategy [60], over a specified time horizon $\Delta\tau$. More formally, it is defined as

$$ROI_{P,\tau} = \frac{W_{P,\tau} - W_{P,\tau-\Delta\tau}}{W_{P,\tau-\Delta\tau}} \qquad \tau = \Delta\tau + 1, \ldots, T.$$

$W_{P,\tau-\Delta\tau}$ represents the amount of capital invested at the beginning of the time horizon, while the portfolio wealth is given by $W_{P,\tau} = W_{P,\tau-\Delta\tau} \prod_{t=\tau-\Delta\tau+1}^{\tau} (1 + R_{P,t}^{out})$ with T representing the number of historical observations. In our experiments, we fix $\Delta\tau$ equal to 3 years.

- The Jensen's alpha is defined as the intercept of the line given by the linear regression of R^{out} on R_I^{out}, namely,

$$\alpha = E[R^{out}] - r_f - \beta(E[R_I^{out}] - r_f)$$

where $\beta = Cov(R^{out}, R_I^{out}) / \sigma^2(R_I^{out})$.

- The Omega ratio [61] is defined as

$$\Omega_\eta(x) = \frac{\int_\eta^{+\infty}(1 - F_{R_P}(r))dr}{\int_{-\infty}^{\eta}(F_{R_P}(r))dr} = \frac{E[\max(0, R_P(x) - \eta)]}{E[\min(0, R_P(x) - \eta)]},$$

  where $F_{R_P^{out}}$ is the cumulative distribution function of the out-of-sample portfolio return and $\eta = 0$. In a nutshell, Omega is the ratio between the sum of positive deviations of R_P^{out} from η and the sum of its negative deviations. Higher values of the Omega ratio are always preferred.

4.3. Out-of-Sample Results

In this section, we report the out-of-sample performance results of the portfolios obtained by the RMZ and KP models, and by the GMinV portfolio, on the datasets described in Section 4.1. For reasons of clarity, we denote by RMZ^R and KP^R the efficient portfolios obtained by Models (12) and (13), respectively, with the reshaped benchmark index. We also report the performances of the (original) benchmark market's index, named INDEX.

4.3.1. ES50

In Tables 1–4, we report the results obtained by applying the reshaping procedure of the benchmark index on the ES50 dataset. As mentioned in Section 3.3, we provide only the results for which Problems (12) and (13), with the new improved benchmark index $\widetilde{R_I}$, are feasible. In this regard, note that the highest increment of the skewness parameter Δ_γ is equal to 2 for the RMZ^R model (Table 1), and it is 0.8 for the KP^R model (Table 3), since higher values of Δ_γ do not provide feasible solutions. As highlighted in Table 1, for the RMZ^R model, the best computational results are obtained when $\Delta_\gamma = 1.7$, in terms of almost all performance measures. Furthermore, note that the volatility of the out-of-sample returns (i.e., σ^{out}) tends to decrease as the target skewness of the improved benchmark increases [62], which is a desirable feature to risk-averse investors. For $\Delta_\gamma = 1.7$, Table 2 shows that the overall most desirable performance results are obtained with $\Delta_\sigma = 0.2$, which corresponds to a small increase in the reshaped benchmark volatility.

On the other hand, as reported in Table 3, for the KP^R model, the best performance (median) score is obtained when $\Delta_\gamma = 0.6$. Therefore, for $\Delta_\gamma = 0.6$, the best computational results are obtained with $\Delta_\sigma = -0.1$, namely, when decreasing the standard deviation of the benchmark distribution. In this case, the ensuing optimal portfolio's return distribution displays better risk characteristics in terms of lower standard deviation and higher tail values [10] (see Table 4). Indeed, we observe that the portfolio performance generally tends to worsen as Δ_σ increases.

Finally, in Table 5, we summarize the out-of-sample performance results obtained by the best RMZ^R and KP^R portfolios, by the RMZ and KP models with the original benchmark distribution, by the GMinV portfolio, and by the benchmark market's index. Note that the reshaping procedure generally enhances the performance of the SSD-based portfolio strategies. Indeed, the RMZ^R and KP^R portfolios show an improvement in terms of Sharpe, Sortino, information, and Omega ratios. Among these, the RMZ^R approach achieves the best overall performance score.

Table 1. ES50: out-of-sample performance results for the RMZ^R strategy by varying Δ_γ, where Δ_γ is the skewness parameter and score is the median value of the overall performance rankings. The best results are marked in bold.

Δ_γ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
0.0	0.001196	0.02934	0.0407	0.0554	0.5611	0.8303	0.04137	1.1190	0.0010032	0.33414	21.0
0.1	0.001254	0.02927	0.0428	0.0583	0.5614	0.8321	0.04389	1.1252	0.0010620	0.34010	20.0
0.2	0.001334	0.02928	0.0455	0.0622	0.5650	0.8373	0.04721	1.1333	0.0011431	0.35133	19.0
0.3	0.001358	0.02930	0.0463	0.0634	0.5622	0.8389	0.04800	1.1355	0.0011671	0.35104	18.0
0.4	0.001412	0.02924	0.0483	0.0662	0.5580	0.8416	0.05021	1.1416	0.0012223	0.35971	17.0
0.5	0.001469	0.02921	0.0503	0.0691	0.5581	0.8465	0.05257	1.1485	0.0012799	0.36956	16.0
0.6	0.001490	0.02952	0.0504	0.0690	0.5610	0.8409	0.05331	1.1507	0.0012992	0.36958	15.0
0.7	0.001559	0.02990	0.0521	0.0710	0.5520	0.8425	0.05610	1.1575	0.0013654	0.37597	14.0
0.8	0.001640	0.03007	0.0545	0.0746	0.5450	0.8519	0.05967	1.1657	0.0014446	0.38835	11.5
0.9	0.001656	0.03026	0.0547	0.0750	0.5456	0.8628	0.06023	1.1680	0.0014596	0.39784	10.5
1.0	0.001711	0.03033	0.0564	0.0777	0.5458	**0.8739**	0.06263	1.1743	0.0015144	0.40233	8.0
1.1	0.001737	0.02926	0.0593	0.0813	0.5430	0.8495	0.06466	1.1802	0.0015471	0.42404	7.0
1.2	0.001743	0.02889	0.0603	0.0830	0.5409	0.8405	0.06593	1.1820	0.0015536	0.42068	6.0
1.3	0.001783	0.02829	0.0630	0.0880	0.5425	0.8624	0.06861	1.1881	0.0015968	0.41258	4.0
1.4	0.001737	0.02824	0.0615	0.0861	0.5347	0.8667	0.06643	1.1830	0.0015520	0.40784	5.5
1.5	0.001808	0.02815	0.0642	0.0898	0.5228	0.8648	0.06964	1.1922	0.0016240	0.42698	3.0
1.6	0.001829	0.02826	0.0647	0.0909	0.5137	0.8672	0.06998	1.1941	0.0016445	0.43178	2.0
1.7	**0.001844**	0.02806	**0.0657**	**0.0923**	0.5164	0.8484	**0.07121**	**1.1979**	**0.0016603**	**0.43624**	**1.0**
1.8	0.001621	0.02776	0.0583	0.0818	0.4994	0.8428	0.06134	1.1734	0.0014389	0.38325	9.0
1.9	0.001579	0.02754	0.0573	0.0804	**0.4855**	0.8551	0.05969	1.1700	0.0013979	0.37681	9.5
2.0	0.001534	**0.02744**	0.0559	0.0783	0.4892	0.8525	0.05783	1.1652	0.0013540	0.36896	12.0

Table 2. ES50: out-of-sample performance results for the RMZ^R strategy by varying Δ_σ (with $\Delta_\gamma = 1.7$).

Δ_σ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
−0.1	0.00150	**0.0263**	0.0570	0.0801	**0.4659**	0.8567	0.0582	1.1699	0.00133	0.35998	5.0
0.0	0.00184	0.0281	0.0657	0.0923	0.5164	0.8484	0.0712	1.1979	0.00166	0.43624	2.5
0.1	0.00187	0.0298	0.0627	0.0881	0.5687	0.8761	0.0687	1.1879	0.00168	0.44082	4.0
0.2	**0.00217**	0.0311	**0.0696**	**0.0988**	0.5705	**0.8963**	**0.0776**	**1.2097**	**0.00197**	**0.48511**	**1.0**
0.3	**0.00217**	0.0331	0.0653	0.0919	0.5508	0.8763	0.0724	1.1952	0.00197	0.46788	2.5
0.4	0.00193	0.0358	0.0539	0.0745	0.5807	0.8453	0.0593	1.1611	0.00172	0.41589	6.0
0.5	0.00187	0.0382	0.0489	0.0673	0.6016	0.8581	0.0538	1.1481	0.00165	0.39221	7.0

Table 3. ES50: out-of-sample performance results for the KP^R strategy by varying Δ_γ.

Δ_γ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
0.0	**0.00198**	0.0354	0.0559	0.0761	**0.53829**	0.8349	0.0606	1.1730	**0.001772**	**0.47745**	6.0
0.1	0.00195	0.0351	0.0557	0.0758	0.54028	0.8316	0.0603	1.1720	0.001748	0.46986	7.0
0.2	0.00196	0.0347	0.0563	0.0768	0.54167	0.8302	0.0610	1.1735	0.001751	0.46820	6.0
0.3	0.00194	0.0344	0.0563	0.0769	0.54090	0.8290	0.0610	1.1731	0.001732	0.46103	6.0
0.4	0.00194	0.0340	0.0570	0.0779	0.54136	0.8306	0.0618	1.1750	0.001738	0.45808	5.0
0.5	0.00194	0.0338	0.0572	0.0783	0.54031	0.8320	0.0622	1.1760	0.001734	0.45169	3.5
0.6	0.00194	0.0336	**0.0578**	**0.0791**	0.54413	0.8362	**0.0632**	**1.1782**	0.001740	0.44870	**3.0**
0.7	0.00189	0.0333	0.0566	0.0776	0.54987	0.8382	0.0622	1.1745	0.001685	0.44557	4.0
0.8	0.00186	**0.0329**	0.0564	0.0773	0.55723	**0.8383**	0.0621	1.1731	0.001653	0.44314	6.0

Table 4. ES50: out-of-sample performance results for the KP^R strategy by varying Δ_σ (with $\Delta_\gamma = 0.6$).

Δ_σ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
−0.1	0.00184	**0.0312**	**0.0589**	**0.0809**	0.5334	0.8346	**0.0656**	**1.1800**	0.00164	0.45048	**1.5**
0.0	0.00194	0.0336	0.0578	0.0791	0.5441	**0.8362**	0.0632	1.1782	0.00174	0.44870	2.0
0.1	0.00195	0.0356	0.0546	0.0739	0.5333	0.8252	0.0590	1.1695	0.00174	**0.45202**	3.0
0.2	**0.00197**	0.0372	0.0529	0.0711	0.5155	0.8302	0.0571	1.1657	**0.00175**	0.43897	4.0
0.3	0.00196	0.0378	0.0519	0.0701	0.4926	0.8309	0.0563	1.1621	0.00174	0.41810	5.0
0.4	0.00177	0.0382	0.0463	0.0623	**0.4878**	0.8210	0.0498	1.1438	0.00155	0.36585	6.0
0.5	0.00168	0.0382	0.0440	0.0592	0.4908	0.8210	0.0469	1.1355	0.00146	0.34692	7.0

Table 5. Summary of results for ES50.

Approach	Δ_γ	$\Delta\sigma$	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
RMZ	–	–	0.00120	0.0293	0.0407	0.0554	0.5611	0.8303	0.0414	1.1190	0.00100	0.33414	5.0
KP	–	–	0.00198	0.0354	0.0559	0.0761	0.5383	0.8349	0.0606	1.1730	0.00177	0.47745	4.0
RMZ^R	1.7	0.2	**0.00217**	0.0311	0.0696	0.0988	0.5705	0.8963	0.0776	1.2097	**0.00197**	**0.48511**	2.0
KP^R	0.6	−0.1	0.00184	0.0312	0.0589	0.0809	0.5334	0.8346	0.0656	1.1800	0.00164	0.45048	3.0
GMinV	–	–	0.00173	**0.0225**	**0.0766**	**0.1084**	**0.2088**	**0.8987**	**0.0797**	**1.2362**	0.00156	0.37753	**1.0**
INDEX	–	–	0.00028	0.0311	0.0090	0.0122	–	0.8918	–	1.0256	–	0.07734	6.0

4.3.2. EuroBonds

In Tables 6–9, we report the computational results obtained by applying the reshaping procedure of the benchmark index on the EuroBonds dataset. Similarly to the equity case, $\Delta_\gamma = 1.4$ and $\Delta_\gamma = 0.8$ are the highest values of the skewness parameter, which ensure the feasibility of Models (12) and (13), respectively. For the RMZ^R model, the overall best performance is obtained with $\Delta_\gamma = 1.4$. Indeed, Table 6 shows that the lowest out-of-sample standard deviation (i.e., σ^{out}) and the highest Rachev ratio are obtained with this target skewness parameter. For $\Delta_\gamma = 1.4$, the most favorable statistics are obtained with the volatility parameter $\Delta_\sigma = -0.1$. Indeed, as shown in Table 7, note that the out-of-sample turnover, Sharpe, Sortino, Rachev, and Omega ratios tend to worsen as Δ_σ increases.

On the other hand, as reported in Table 8, for the KP^R model, the overall most desirable performance results are obtained with $\Delta_\gamma = 0.3$. Then, for $\Delta_\gamma = 0.3$, Table 9 shows that the best performance (median) score is obtained for two values of the volatility parameter, namely, $\Delta_\sigma = -0.1, 0.1$. Among these, we choose the one with the highest number of best performances out of 10, namely, $\Delta_\sigma = 0.1$. Finally, in Table 10, we report the summary results obtained by all the approaches considered. The reshaping procedure turns out to be a suitable choice, particularly for the KP^R approach, in terms of almost all performance measures. The RMZ^R portfolio also improved its statistics, obtaining the highest Sharpe, Sortino, and Omega ratios, and lower standard deviation and turnover.

Table 6. EuroBonds: out-of-sample performance results for the RMZ^R strategy by varying Δ_γ, where Δ_γ is the skewness parameter and score is the median value of the overall performance rankings. The best results are marked in bold.

Δ_γ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
0.0	**0.000748**	0.00587	0.1273	0.1785	0.8366	0.8732	0.03218	1.4525	0.0004206	**0.15521**	10.0
0.1	0.000736	0.00589	0.1248	0.1748	0.8149	0.8698	0.02983	1.4425	0.0004060	0.15398	13.5
0.2	0.000745	0.00585	0.1273	0.1795	0.8240	0.8843	0.03161	1.4528	0.0004193	0.15490	8.5
0.3	0.000752	0.00582	0.1292	0.1823	0.8020	0.8886	**0.03286**	1.4607	0.0004294	0.15545	7.0
0.4	0.000740	0.00583	0.1268	0.1785	0.7883	0.8861	0.03044	1.4518	0.0004173	0.15283	11.5
0.5	0.000739	0.00582	0.1269	0.1789	0.7861	0.8933	0.03022	1.4542	0.0004181	0.15176	9.0
0.6	0.000712	0.00579	0.1229	0.1725	0.8062	0.8891	0.02512	1.4391	0.0003938	0.14613	12.5
0.7	0.000733	0.00575	0.1273	0.1791	0.7980	0.8995	0.02883	1.4583	0.0004181	0.14939	9.0
0.8	0.000711	0.00569	0.1250	0.1759	0.7919	0.9059	0.02475	1.4498	0.0004027	0.14455	13.0
0.9	0.000728	0.00561	0.1297	0.1829	0.7831	0.9108	0.02770	1.4713	0.0004254	0.14783	6.0
1.0	0.000747	0.00558	0.1337	0.1904	**0.7373**	0.9388	0.03124	1.4893	0.0004459	0.15110	4.0
1.1	0.000723	0.00539	0.1340	0.1900	0.7630	0.9202	0.02672	1.4986	0.0004369	0.14449	5.0
1.2	0.000738	0.00529	**0.1394**	**0.1989**	0.7645	0.9476	0.02910	**1.5301**	**0.0004619**	0.14549	3.0
1.3	0.000713	0.00524	0.1360	0.1935	0.7576	0.9535	0.02444	1.5211	0.0004425	0.14007	3.0
1.4	0.000705	**0.00516**	0.1364	0.1944	0.7427	**0.9788**	0.02268	1.5285	0.0004417	0.13710	**2.0**

Table 7. EuroBonds: out-of-sample performance results for the RMZ^R strategy by varying Δ_σ (with $\Delta_\gamma = 1.4$).

Δ_σ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
−0.1	0.000643	**0.00468**	**0.1371**	0.1941	**0.7277**	**0.9818**	0.01179	**1.5351**	0.0004031	0.12339	**1.5**
0.0	0.000705	0.00516	0.1364	**0.1944**	0.7427	0.9788	0.02268	1.5285	0.0004417	0.13710	2.5
0.1	0.000758	0.00574	0.1320	0.1870	0.7505	0.9589	0.03160	1.5101	0.0004661	0.15042	4.5
0.2	0.000810	0.00622	0.1302	0.1856	0.7435	0.9790	0.03973	1.5016	0.0004941	0.16314	4.0
0.3	0.000830	0.00672	0.1234	0.1737	0.7670	0.9534	0.04173	1.4716	0.0004891	0.16798	5.0
0.4	0.000904	0.00726	0.1244	0.1740	0.7848	0.9548	0.05180	1.4860	0.0005367	0.18342	6.0
0.5	**0.000965**	0.00770	0.1252	0.1761	0.7765	0.9679	**0.05923**	1.4900	**0.0005768**	**0.19630**	4.5

Table 8. EuroBonds: out-of-sample performance results for the KP^R strategy by varying Δ_γ.

Δ_γ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
0.0	**0.000618**	0.006490	0.0951	0.1286	0.7561	0.8274	**0.00690**	1.3293	0.0002767	**0.13722**	6.0
0.1	0.000613	0.006433	0.0952	0.1288	0.7508	0.8321	0.00604	1.3293	0.0002737	0.13576	5.0
0.2	0.000604	0.006387	0.0945	0.1279	0.7598	0.8345	0.00450	1.3261	0.0002663	0.13387	6.0
0.3	0.000608	0.006125	**0.0992**	**0.1351**	0.7249	**0.8465**	0.00533	**1.3479**	**0.0002819**	0.12859	**2.0**
0.4	0.000573	0.006133	0.0933	0.1265	0.7311	0.8328	−0.00093	1.3241	0.0002449	0.12468	8.5
0.5	0.000587	0.006088	0.0964	0.1313	0.7195	0.8408	0.00172	1.3358	0.0002554	0.12727	3.5
0.6	0.000583	0.006091	0.0956	0.1298	0.7188	0.8340	0.00089	1.3324	0.0002507	0.12910	5.0
0.7	0.000587	0.006062	0.0968	0.1324	0.7195	0.8445	0.00170	1.3330	0.0002550	0.13066	3.5
0.8	0.000576	**0.006061**	0.0949	0.1299	**0.7057**	0.8447	−0.00044	1.3240	0.0002430	0.12742	7.0

Table 9. EuroBonds: out-of-sample performance results for the KP^R strategy by varying Δ_σ (with $\Delta_\gamma = 0.3$).

Δ_σ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
−0.1	0.000563	**0.005563**	0.1011	**0.1373**	0.7396	0.8221	−0.00280	1.3590	0.0002622	0.12132	**2.5**
0.0	0.000608	0.006125	0.0992	0.1351	**0.7249**	**0.8465**	0.00533	1.3479	0.0002819	0.12859	3.0
0.1	0.000683	0.006744	**0.1012**	0.1372	0.7857	0.8311	0.01808	**1.3646**	0.0003238	0.14800	**2.5**
0.2	0.000688	0.007298	0.0942	0.1266	0.7988	0.8091	0.01816	1.3343	0.0003012	0.15598	4.0
0.3	0.000726	0.007646	0.0949	0.1264	0.8111	0.8002	0.02384	1.3392	0.0003204	0.17005	4.0
0.4	0.000722	0.008303	0.0869	0.1141	0.8256	0.7613	0.02153	1.3077	0.0002891	0.17395	6.5
0.5	**0.000824**	0.008827	0.0933	0.1232	0.7868	0.7848	**0.03496**	1.3365	**0.0003670**	**0.19012**	4.5

Table 10. Summary of results for EuroBonds.

Approach	Δ_γ	$\Delta\sigma$	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
RMZ	–	–	**0.00075**	0.0059	0.1273	0.1785	0.8366	0.8732	**0.0322**	1.4525	**0.00042**	0.15521	**2.0**
KP	–	–	0.00062	0.0065	0.0951	0.1286	0.7561	0.8274	0.0069	1.3293	0.00028	0.13722	4.0
RMZ^R	1.4	−0.1	0.00064	0.0047	**0.1371**	**0.1941**	0.7277	0.9818	0.0118	**1.5351**	0.00040	0.12339	**2.0**
KP^R	0.3	0.1	0.00068	0.0067	0.1012	0.1372	0.7857	0.8311	0.0181	1.3646	0.00032	0.14800	3.0
GMinV	–	–	0.00008	**0.0011**	0.0756	0.1140	**0.1546**	**1.0851**	−0.0782	1.2870	0.00002	0.01125	5.0
INDEX	–	–	0.00058	0.0070	0.0827	0.1162	–	0.8812	–	1.2667	–	**0.16166**	5.0

4.3.3. ETF

In Tables 11–14, we report the results obtained by applying the reshaping procedure of the benchmark index on the ETF dataset (see Section 4.1). In this case, $\Delta_\gamma = 2$ and $\Delta_\gamma = 0.8$ are the highest values of the skewness parameter, which ensure the feasibility of Models (12) and (13), respectively. As demonstrated in Table 11, for the RMZ^R model, the best performance is achieved with $\Delta_\gamma = 0.3$, which obtained the highest out-of-sample returns (i.e., μ^{out}), Jensen's alpha, Sharpe, Sortino, Omega, and information ratios. For $\Delta_\gamma = 0.3$, the overall most promising results are obtained with $\Delta_\sigma = 0$, namely, when the target volatility is equal to the volatility of the original benchmark index (see Table 12). Indeed, we observe that the portfolio performance generally tends to worsen as Δ_σ increases.

Regarding the KP^R model, Table 13 shows that $\Delta_\gamma = 0.3$ obtains the highest (median) score in terms of almost all performance measures. Therefore, for $\Delta_\gamma = 0.3$, the overall most desirable results are obtained when decreasing the standard deviation of the benchmark distribution, namely, when $\Delta_\sigma = -0.1$ (see Table 14). Indeed, further increments in the target standard deviation generally do not improve the statistics. Then, in Table 15, we summarize the computational results obtained by all the approaches considered. Clearly, the reshaping procedure improved both SSD-based models. Among these, the RMZ^R portfolio achieved the highest scores on eight out of 10 performance measures. The KP^R portfolio also enhanced its statistics, showing better out-of-sample standard deviation, turnover, Jensen's alpha, Sharpe, Sortino, information, and Omega ratios, with respect to the (original) KP portfolio.

Table 11. ETF: out-of-sample performance results for the RMZ^R strategy by varying Δ_γ, where Δ_γ is the skewness parameter and score is the median value of the overall performance rankings. The best results are marked in bold.

Δ_γ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
0.0	0.002749	0.022780	0.1206	0.1714	0.4071	0.8387	0.10454	1.3824	0.0017476	0.40711	7.0
0.1	0.002811	0.022865	0.1229	0.1752	0.4068	0.8453	0.10741	1.3895	0.0018127	0.42281	5.0
0.2	0.002821	0.022879	0.1232	0.1758	0.4086	0.8457	0.10747	1.3902	0.0018242	0.42933	4.0
0.3	**0.002865**	0.022973	**0.1246**	**0.1774**	0.4057	0.8437	**0.10967**	**1.3959**	**0.0018673**	0.44226	**1.0**
0.4	0.002856	0.023021	0.1240	0.1765	0.4030	0.8424	0.10846	1.3927	0.0018582	**0.44414**	2.0
0.5	0.002846	0.023038	0.1234	0.1760	0.4032	0.8504	0.10768	1.3899	0.0018480	0.44035	3.0
0.6	0.002781	0.022926	0.1212	0.1728	0.4136	**0.8591**	0.10372	1.3826	0.0017881	0.42314	6.0
0.7	0.002653	0.022758	0.1165	0.1649	0.4250	0.8450	0.09528	1.3670	0.0016682	0.38775	8.0
0.8	0.002562	0.022420	0.1142	0.1617	0.4377	0.8465	0.08890	1.3602	0.0015983	0.36324	9.0
0.9	0.002464	**0.022418**	0.1098	0.1549	0.4307	0.8408	0.08174	1.3419	0.0015049	0.34031	10.0
1.0	0.001817	0.022669	0.0801	0.1099	0.3984	0.7945	0.03922	1.2385	0.0008404	0.21234	11.0
1.1	0.001552	0.022828	0.0679	0.0921	0.3884	0.7732	0.02172	1.1997	0.0005636	0.17491	12.0
1.2	0.001313	0.023282	0.0563	0.0758	0.3956	0.7740	0.00561	1.1642	0.0003120	0.14099	13.0
1.3	0.001113	0.023674	0.0470	0.0625	0.3720	0.7583	−0.00722	1.1358	0.0001055	0.12021	14.0
1.4	0.001068	0.023750	0.0449	0.0594	0.3781	0.7439	−0.01009	1.1299	0.0000557	0.11245	15.0
1.5	0.000972	0.023821	0.0408	0.0535	0.3861	0.7284	−0.01620	1.1178	−0.0000450	0.09374	16.0
1.6	0.000889	0.023923	0.0371	0.0487	0.3786	0.7277	−0.02122	1.1069	−0.0001273	0.07658	17.0
1.7	0.000848	0.024030	0.0353	0.0462	0.3767	0.7318	−0.02357	1.1012	−0.0001681	0.06730	18.0
1.8	0.000839	0.024116	0.0348	0.0456	**0.3634**	0.7333	−0.02392	1.1001	−0.0001771	0.06564	19.0
1.9	0.000820	0.024092	0.0340	0.0446	0.3636	0.7380	−0.02491	1.0978	−0.0001894	0.06211	20.0
2.0	0.000794	0.024113	0.0329	0.0432	0.3647	0.7375	−0.02629	1.0948	−0.0002117	0.05751	21.0

Table 12. ETF: out-of-sample performance results for the RMZ^R strategy by varying Δ_σ (with $\Delta_\gamma = 0.3$).

Δ_σ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
−0.1	0.00263	**0.0211**	**0.12466**	0.1760	**0.3759**	0.8451	0.1053	**1.4027**	0.00168	0.38138	3.0
0.0	0.00287	0.0230	0.12463	**0.1774**	0.4057	0.8437	**0.1097**	1.3959	0.00187	0.44226	**2.0**
0.1	0.00293	0.0243	0.12048	0.1738	0.4293	0.8620	0.1062	1.3738	0.00189	0.45709	3.0
0.2	**0.00309**	0.0256	0.12086	0.1751	0.4302	**0.8763**	0.1085	1.3732	**0.00203**	0.50000	3.0
0.3	0.00303	0.0271	0.11160	0.1601	0.4412	0.8530	0.0974	1.3403	0.00192	0.51049	5.0
0.4	0.00299	0.0285	0.10493	0.1498	0.4709	0.8470	0.0890	1.3178	0.00185	**0.51563**	6.0
0.5	0.00280	0.0296	0.09455	0.1341	0.4669	0.8521	0.0754	1.2830	0.00163	0.47891	7.0

Table 13. ETF: out-of-sample performance results for the KP^R strategy by varying Δ_γ.

Δ_γ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
0.0	0.002264	0.025354	0.0892	0.12480	0.4257	0.8207	0.06075	1.2661	0.001206	0.31986	6.0
0.1	0.002273	0.025251	0.0899	0.12592	0.4228	0.8212	0.06148	1.2687	0.001220	0.32082	4.0
0.2	0.002282	0.025136	0.0907	0.12710	0.4169	0.8214	0.06216	1.2716	0.001235	0.32128	3.0
0.3	**0.002286**	0.025033	**0.0912**	**0.12789**	0.4117	0.8230	**0.06258**	1.2738	**0.001242**	**0.32215**	**1.0**
0.4	0.002271	0.024901	0.0911	0.12779	**0.4092**	**0.8248**	0.06200	**1.2739**	0.001233	0.31871	2.5
0.5	0.002207	0.024698	0.0893	0.12496	0.4149	0.8227	0.05881	1.2680	0.001174	0.30244	5.0
0.6	0.002101	0.024450	0.0858	0.1198	0.4223	0.8164	0.05297	1.2565	0.001076	0.27831	7.0
0.7	0.001977	0.024158	0.0818	0.1137	0.4343	0.8059	0.04573	1.2428	0.000966	0.24934	8.0
0.8	0.001858	**0.023563**	0.0788	0.1090	0.4793	0.8062	0.03990	1.2350	0.000860	0.22631	9.0

Table 14. ETF: out-of-sample performance results for the KP^R strategy by varying Δ_σ (with $\Delta_\gamma = 0.3$).

Δ_σ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
−0.1	0.00222	**0.0234**	**0.09480**	**0.1328**	**0.4082**	0.8180	**0.0634**	**1.2867**	0.00123	0.29862	**1.0**
0.0	**0.00229**	0.0250	0.09124	0.1279	0.4117	0.8230	0.0626	1.2738	**0.00124**	0.32215	2.0
0.1	0.00226	0.0261	0.08662	0.1217	0.4355	**0.8382**	0.0584	1.2578	0.00118	**0.33112**	3.0
0.2	0.00216	0.0270	0.07999	0.1115	0.4332	0.8242	0.0512	1.2371	0.00105	0.32119	4.0
0.3	0.00208	0.0276	0.07524	0.1043	0.4426	0.8192	0.0455	1.2222	0.00095	0.30526	5.0
0.4	0.00196	0.0279	0.07010	0.0967	0.4469	0.8150	0.0383	1.2063	0.00082	0.27488	6.0
0.5	0.00191	0.0280	0.06826	0.0941	0.4581	0.8123	0.0355	1.2007	0.00078	0.26149	7.0

Table 15. Summary of results for ETF.

Approach	Δ_γ	$\Delta\sigma$	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
RMZ	–	–	0.002749	0.0228	0.1206	0.1714	0.4071	0.8387	0.1045	1.3824	0.001748	0.40711	2.0
KP	–	–	0.002264	0.0254	0.0892	0.1248	0.4257	0.8207	0.0607	1.2661	0.001206	0.31986	4.0
RMZ^R	0.3	0.0	**0.002865**	0.0230	**0.1246**	**0.1774**	0.4057	**0.8437**	**0.1097**	**1.3959**	**0.001867**	**0.44226**	**1.0**
KP^R	0.3	−0.1	0.002224	0.0234	0.0948	0.1328	0.4082	0.8180	0.0634	1.2867	0.001228	0.29862	3.5
GMinV	–	–	0.000893	**0.0185**	0.0484	0.0641	**0.1467**	0.7859	−0.0287	1.1453	0.000021	0.09889	5.5
INDEX	–	–	0.001227	0.0220	0.0556	0.0742	–	0.7685	–	1.1688	–	0.12658	5.0

4.3.4. CIB (Commodities and Italian Bonds)

In Tables 16–19, we report the results obtained by applying the reshaping procedure of the benchmark index on the CIB dataset. Note that the highest increment of the skewness parameter Δ_γ is equal to 2 for the RMZ^R model (Table 16), and it is 0.8 for the KP^R model (Table 18), since higher values of Δ_γ do not ensure the feasibility of Models (12) and (13). As highlighted in Table 16 for the RMZ^R model, the best computational results are achieved with $\Delta_\gamma = 1.2$ in terms of almost all performance measures. For $\Delta_\gamma = 1.2$, Table 17 shows that the overall highest score is obtained with $\Delta_\sigma = -0.1$, namely, when decreasing the standard deviation of the benchmark distribution. In this regard, we found that the portfolio performances generally tend to worsen as Δ_σ increases.

On the other hand, for the KP^R model, Table 18 indicates that the most promising statistics are achieved with $\Delta_\gamma = 0.8$, which is the highest increment of the skewness parameter. Then, for $\Delta_\gamma = 0.8$, Table 19 shows that the best performance (median) score is obtained for two values of the volatility parameter, namely, $\Delta_\sigma = 0, 0.2$. Among these, we choose the one with the highest number of best performances out of 10, namely, $\Delta_\sigma = 0.2$.

Finally, Table 20 summarizes the out-of-sample results obtained by all the strategies considered. The reshaping procedure generally improved the statistics of the SSD-based approaches. In particular, the RMZ^R portfolio exhibits the most favorable results in terms of almost all performance measures.

Table 16. CIB: out-of-sample performance results for the RMZ^R strategy by varying Δ_γ, where Δ_γ is the skewness parameter and score is the median value of the overall performance rankings. The best results are marked in bold.

Δ_γ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
0.0	0.001059	0.01046	0.1012	0.1513	0.5196	1.0449	0.03238	1.3318	0.0004956	0.15278	19.0
0.1	0.001055	0.01049	0.1005	0.1505	0.5468	1.0453	0.03177	1.3292	0.0004904	0.15163	19.5
0.2	0.001049	0.01042	0.1005	0.1501	0.5171	1.0403	0.03109	1.3278	0.0004872	0.15045	19.0
0.3	0.001074	0.01046	0.1026	0.1541	0.5105	1.0488	0.03398	1.3363	0.0005123	0.15496	17.0
0.4	0.001099	0.01045	0.1051	0.1588	0.4957	1.0610	0.03717	1.3462	0.0005380	0.15877	14.0
0.5	0.001090	0.01041	0.1047	0.1583	0.4950	1.0642	0.03614	1.3449	0.0005318	0.15729	14.0
0.6	0.001103	0.01037	0.1062	0.1607	0.4917	1.0717	0.03786	1.3508	0.0005446	0.15963	11.5
0.7	0.001096	0.01033	0.1060	0.1607	0.5027	1.0793	0.03715	1.3507	0.0005390	0.15815	12.5
0.8	0.001118	0.01030	0.1085	0.1647	0.5182	1.0929	0.04000	1.3606	0.0005635	0.15977	11.0
0.9	0.001133	0.01024	0.1106	0.1683	0.5131	1.0936	0.04195	1.3686	0.0005815	0.16253	9.0
1.0	0.001161	0.01020	0.1137	0.1738	0.5044	1.0982	0.04515	1.3809	0.0006145	0.16652	7.0
1.1	0.001173	0.01016	0.1153	0.1768	0.5014	1.1123	0.04651	1.3884	0.0006311	0.16713	4.5
1.2	**0.001208**	0.01015	**0.1190**	**0.1831**	0.5054	**1.1316**	**0.05053**	**1.4053**	**0.0006712**	**0.17103**	**1.0**
1.3	0.001193	0.01007	0.1184	0.1823	0.4978	1.1315	0.04891	1.4044	0.0006589	0.16765	2.0
1.4	0.001141	0.00986	0.1157	0.1771	0.4951	1.1084	0.04341	1.3945	0.0006132	0.16016	5.0
1.5	0.001121	0.00966	0.1160	0.1778	0.4843	1.1029	0.04123	1.3966	0.0006038	0.15686	6.0
1.6	0.001110	0.00948	0.1170	0.1791	0.4637	1.0871	0.04012	1.3999	0.0006007	0.15576	6.0
1.7	0.001061	0.00922	0.1150	0.1758	0.4465	1.0833	0.03396	1.3928	0.0005679	0.14912	8.0
1.8	0.001008	0.00904	0.1115	0.1702	0.4133	1.0711	0.02731	1.3807	0.0005263	0.14319	11.0
1.9	0.000960	0.00874	0.1097	0.1669	0.4145	1.0565	0.02075	1.3767	0.0005019	0.13728	14.0
2.0	0.000888	**0.00846**	0.1049	0.1596	**0.4032**	1.0568	0.01165	1.3671	0.0004408	0.12828	16.0

Table 17. CIB: out-of-sample performance results for the RMZ^R strategy by varying Δ_σ (with $\Delta_\gamma = 1.2$).

Δ_σ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
−0.1	0.001135	**0.00920**	**0.1232**	**0.1908**	**0.4451**	**1.1482**	0.04390	**1.4241**	0.0006392	0.16165	**1.0**
0.0	0.001208	0.01015	0.1190	0.1831	0.5054	1.1316	0.05053	1.4053	0.0006712	0.17103	3.5
0.1	0.001297	0.01114	0.1163	0.1791	0.5387	1.1288	0.05759	1.3931	0.0007163	0.18288	4.0
0.2	0.001396	0.01209	0.1154	0.1789	0.5743	1.1354	0.06405	1.3881	0.0007782	0.19601	4.0
0.3	0.001489	0.01296	0.1148	0.1786	0.5683	1.1309	0.06931	1.3840	0.0008362	0.20975	5.0
0.4	0.001564	0.01374	0.1137	0.1780	0.5557	1.1465	**0.07175**	1.3791	**0.0008867**	**0.21710**	3.0
0.5	**0.001579**	0.01463	0.1079	0.1688	0.5575	1.1455	0.06828	1.3567	0.0008723	0.21418	4.0

Table 18. CIB: out-of-sample performance results for the KP^R strategy by varying Δ_γ.

Δ_γ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
0.0	0.000981	0.011789	0.0832	0.12415	0.5066	1.0381	0.02016	1.2685	0.000373	0.12541	6.5
0.1	0.000974	0.011693	0.0832	0.12423	0.4953	1.0387	0.01951	1.2690	0.000369	0.12424	6.0
0.2	0.000971	0.011585	0.0837	0.12500	0.4909	1.0408	0.01928	1.2709	0.000371	0.12324	5.0
0.3	0.000976	0.011449	0.0852	0.12729	**0.4868**	1.0417	0.02011	1.2766	0.000382	0.12349	3.5
0.4	0.000895	0.011070	0.0808	0.11965	0.4870	1.0231	0.01113	1.2627	0.000319	0.10851	8.0
0.5	0.000944	0.010695	0.0882	0.13118	0.4893	1.0358	0.01736	1.2969	0.000382	0.11614	4.0
0.6	0.000840	0.010536	0.0796	0.1173	0.5092	1.0257	0.00501	1.2627	0.000284	0.09765	9.0
0.7	0.000957	0.010540	0.0908	0.1352	0.4951	1.0528	0.01911	1.3033	0.000403	0.11932	2.5
0.8	**0.000993**	**0.010276**	**0.0965**	**0.1482**	0.5007	**1.1015**	**0.02422**	**1.3210**	**0.000441**	**0.12933**	**1.0**

Table 19. CIB: out-of-sample performance results for the KP^R strategy by varying Δ_σ (with $\Delta_\gamma = 0.8$).

Δ_σ	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
−0.1	0.000778	**0.00924**	0.0842	0.1274	**0.4700**	1.0758	−0.00265	1.2761	0.0002787	0.09576	6.0
0.0	0.000993	0.01028	0.0965	0.1482	0.5007	1.1015	0.02422	**1.3210**	0.0004409	0.12933	**2.0**
0.1	0.001048	0.01128	0.0929	0.1400	0.5118	1.0795	0.02856	1.3133	0.0004615	0.13328	4.0
0.2	0.001162	0.01201	**0.0967**	**0.1496**	0.5514	1.0990	0.03934	1.3199	0.0005459	0.15526	**2.0**
0.3	0.001178	0.01281	0.0919	0.1416	0.5598	1.0883	0.03857	1.3003	0.0005300	0.15399	3.5
0.4	**0.001240**	0.01378	0.0899	0.1396	0.5956	**1.1196**	**0.04128**	1.2920	**0.0005601**	**0.16652**	3.0
0.5	0.001103	0.01486	0.0742	0.1131	0.5618	1.0747	0.02599	1.2334	0.0003947	0.12415	6.5

Table 20. Summary of results for CIB.

Approach	Δ_γ	$\Delta\sigma$	μ^{out}	σ^{out}	Sharpe	Sortino	Turnover	Rachev	InfoR	Omega	Jensen	Avg. ROI	Score
RMZ	–	–	0.001059	0.0105	0.1012	0.1513	0.5196	1.0449	0.0324	1.3318	0.000496	0.15278	3.0
KP	–	–	0.000981	0.0118	0.0832	0.1242	0.5066	1.0381	0.0202	1.2685	0.000373	0.12541	4.0
RMZ^R	1.2	−0.1	0.001135	0.0092	**0.1232**	**0.1908**	0.4451	**1.1482**	**0.0439**	**1.4241**	**0.000639**	**0.16165**	**1.0**
KP^R	0.8	0.2	**0.001162**	0.0120	0.0967	0.1496	0.5514	1.0990	0.0393	1.3199	0.000546	0.15526	2.5
GMinV	–	–	0.000291	**0.0044**	0.0667	0.0951	**0.0193**	1.0322	−0.0633	1.2778	0.000060	0.04965	5.0
INDEX	–	–	0.000798	0.0104	0.0769	0.1098	–	0.9042	–	1.2411	–	0.12483	5.0

4.4. Discussion

Thus far, we provided an empirical analysis of a wide range of assets (from equities to bonds, from ETFs to commodities) of two SSD strategies with and without the skewed benchmark, along with the global minimum variance portfolio. Our empirical findings revealed that improving the benchmark index distribution leads to selecting SSD portfolios with superior out-of-sample performance results. This confirms earlier results by [14,36] who found that "the Fama and French market portfolio is significantly inefficient relative to benchmark portfolios formed on market capitalization and book-to-market equity ratio". In this sense, see Faff et al. [63] and Jurdi [64] for Australia, Kubota et al. [65] for Japan, Alrabadi et al. [66] for Jordan, Eyvazloo et al. [67] for Iran, and Griffin et al. [68] at a global level. As a methodological note, in terms of misallocation and correctness of economic financial decisions, a proper performance measure is key. Section 4.2 lists some of them, including the Omega ratio, which some refer to as a "universal performance measure" [61,69]. However, Bernard et al. [70] introduced compatibility conditions (i.e., the so-called non-strict dominance compatibility and the strict dominance compatibility conditions). Thus, the Omega ratio is not compatible with the second-order stochastic dominance criterion when using the strict dominance compatibility condition. Further, a critical meta-analysis points out that "the use of Omega in asset selection and optimal asset allocation may entail real computational economics issues and may lead to unreasonable financial decisions" [70]. This is mainly due to the incompatibility with the second-order stochastic dominance criterion under the strict dominance compatibility condition. However, due to the different ranges of performance measures adopted, we can reasonably assume that this study was unaffected by the issues mentioned above.

In terms of testing, apart from the aforementioned variety of metrics that extend the usual results based on the Sharpe ratio and information ratio [42], this study identified those portfolios that dominate the respective benchmarks. However, as pointed out by [42], the main limitation of some tests [11,15,36,71] on SSD dominance is that they only analyze in-sample performance. Instead, "for practical portfolio allocation problems, it is important to establish the out-of-sample properties of SSD-efficient portfolios" [42]. To this end, following [42,72–74], we implemented an out-of-sample assessment to properly judge the performance of the considered portfolios.

5. Conclusions

In this paper, we empirically compared the SSD-based portfolio selection models of Roman et al. [14] and Kopa and Post [15], with a reshaping of the skewness and variance of the benchmark distribution. To this end, following Valle et al. [10], the reference distribution was reshaped to select SSD-efficient portfolios that provide more appealing features for investors. Through an extensive empirical analysis based on multiasset investment universes, we showed that the SSD portfolios that dominate the new skewed benchmark generally perform better. In fact, comparing the out-of-sample performance of the “improved” SSD portfolios with respect to the original ones and the global minimum variance portfolio, we demonstrated that the improvement in benchmark distribution leads to select SSD portfolios that deliver superior out-of-sample performance results.

In terms of policy implications, failures in understanding the risk are often related to SSD dominance. Therefore, analysis may help in discovering the neglected downside risk, as opposed to overstated upside opportunities [75]. The application of this analysis ranges from the relationship between bank capital and risk [75], which has regulatory and allocation effects, to the evaluation of the Chinese IPO market [76], from portfolio diversification and its links between renewable energy, commodities, and financial stock markets [77] to evaluation of multidimensional measures of poverty [78], etc.

Future research could be directed to extend the analysis to exact higher-order stochastic dominance (see, e.g., [17–19]) and approximate stochastic dominance rules (see, e.g., [41,52,79]), disclosing the policy implication by testing different techniques for remodeling the reference index in accordance with real-world distributions [16,43]. Forming CAPM β ranked portfolios would be another dimension to control for systemic risk [80].

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