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Article

Queueing Inventory System in Transport Problem

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Abstract: In this paper, we consider the batch arrival of customers to a transport station. Customers belonging to each category is considered as a single entity according to a BMMAP. An Erlang clock of order m starts ticking when the transport vessel reaches the station. When the Lth stage of the clock is reached, an order for the next vessel is placed. The lead time for arrival of the vessel follows exponential distribution. There are two types of rooms in this system: the waiting rooms and the service rooms for customers in the transport station and in the vessel, respectively. The waiting room capacity for customers of type 1 is infinite whereas those for customer of type $2, \ldots, k$ are of finite capacities. The service room capacity C_j for customer type $j, j = 1, 2, \ldots, k$ is finite. Upon arrival, customers of category j occupy seats designated for that category in the vessel, provided there is at least one vacancy belonging to that category. The total number of vessels with the operator is h^*. The service time of each vessel follows exponential distribution with parameter μ. Each group of customers belong to category j searches independently for customers of this category to mobilize passengers when the Erlang clock reaches L_1 where $L_1 < L$. The search time for customers of category j follows exponential distribution with parameter λ_j. The stability condition is derived. Some performance measures are estimated.

Keywords: batch marked markovian arrival process; batch arrival; Erlang clock; batch service; matrix analytic method

MSC: 60K25; 60K30

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1. Introduction

In the literature on queueing inventory, there are some studies that have focused on queueing inventory in transport problems. Shajin et al. [1] studied a queueing-inventory problem in passenger transport system. In their study, there were two types of customers: high priority and low priority. High priority customers have a finite buffer to wait but low priority customers to wait have an finite capacity queue. The arrival of customers forms a marked Poisson process. Service time follows exponential distribution for each customer. Melikov and Molchanov [2] studied stock optimization in transportation/storage systems. Sigman and Simchi [3] studied light traffic heuristic for an M/G/1 queue with limited inventory. Neuts [4] introduced the general bulk service (GBS) rule: customers can be served in batches between the minimum and maximum batch sizes. Krishnamoorthy et al. [5] analyzed a queueing system with k stages of services. Customers can join a service from the beginning of any of the stages. The customer arrives at the first node according to MAP. At all other nodes, the customer arrives according to a distinct Poisson process. The service at each stage relies on the stage and the number of customers served in a batch. In their study, they gave some real examples in our life, such as elevators and transport systems. A public transport system serves customers from different locations in the city.

Besides this, many studies have focused on transport systems based on queueing theory, such as [6–9]. In [8], the authors studied the optimization model of the buses number on the route based on queueing theory in a smart city.

In the case of classical queues, Neuts and Ramalhoto [10] introduced the concept of search by server in their study. In the retrial queueing set up, the concept of search was introduced by Artalejo et.al [11]. This study has been extended to more general cases by D'Apice and Manzo [12], Sekar et al. [13], and Ayyappan and Udayageetha [14]. Sekar et al. [13] studied a single server retrial queueing system with orbit search under Erlang-K service. D'Apice and Manzo [12] studied a single server queueing system with search for customers in a finite capacity.

In this paper, we consider the batch Marked Markovian arrival process (BMMAP) of customers to a transport station. Customers belonging to category j in each batch as a single entity and so even though there are j customers in the batch, we consider it as a single unit for our most of the purposes, where $j = 1, 2, \ldots, k$. Customer category j arrives to the transport station according to batch Marked Markovian arrival process (BMMAP) with representation $(D_0, D_1, \ldots, D_k)$ where the D_j s$'$ are each of order m_1.

There are two types of rooms in this system: the waiting rooms and the service rooms for customers type j are in the transport station and in the vessel, respectively. The waiting room capacity for category 1 customers is infinite, whereas those for category j customers are of finite capacities, which is denoted by W_j, $j = 2, \ldots, k$. The service room capacity C_j for category j customers $j = 1, 2, \ldots, k$ is finite. Category j customer batch, on arrival, occupies seats designated for that category in the vessel, provided that there is at least one of the required size vacancy belonging to that category; a vacant seat for category j means that there are j seats. We call such a "unit of seats" as one single entity.

An Erlang clock of order m starts when a transport vessel reaches the station. The Erlang clock moves from phase m$'$ to phase (m$'$ + 1) with parameter ϕ.

When the Erlang clock reaches the Lth phase, we place an order for the next vessel. The lead time for arrival of vessels follow exponential distribution with parameter θ. If the current vessel is in the transport station and a new vessel arrives to the station, the former leaves the station immediately. In the same time, an Erlang clock for the new vessel starts.

The total number of vessels in this queueing inventory system is denoted by h^*. The number of vessels, which are in operation to drop off passengers, is denoted by h. The service time of transport vessel follows exponential distribution with parameter μ. Each group of customers belonging to category j searches independently for customers of this category to mobilize passengers belonging to that category, provided there is vacancy. The search for customers starts only when the Erlang clock reaches L_1 where $L_1 < L$ and service room for type j has customer type j less than C_j. If the new arrival type j fills the last seat in this service room, then the search for this type of customers is terminated for the present vessel. The search time for category j customers follows exponential distribution with parameter λ_j. If category 1 customers find that seats are not available in service room for this category, they must wait in the waiting room for category 1 (infinite capacity). Customers of category $j (j = 2, 3, \ldots, k)$ leave the system if their waiting room is full at the time of arrival of this category.

The salient features of this paper are as follows:

- Transport problem with batch arrival of customers and reserved vessel of designated capacity for each category is presented and discussed for the first time.
- Search of customers to fill vacant seats belonging to each category, if any is done by customers belonging to that category only.
- An Erlang clock monitors the phase for starting search of customers for the current vehicle.
- The next vehicle is ordered when the Erlang clock reaches stage L, which requires an exponentially distributed amount of time to arrive at the station.
- The vehicle already present in the station leaves the moment given by min $\{$realization of the Erlang clock, arrival of next vehicle, all seats are filled$\}$.

This paper is arranged as follows. Section 2 discusses the mathematical modeling of the problem under investigation. In Section 3, the stability of this system is investigated and the stationary system state distribution is computed. Performance measures are elaborated in Section 4. In Section 5, numerical investigation of the model is elaborated. Cost analysis is shown in Section 6. Finally, we summarize with a concluding Section 7.

2. Mathematical Description of the Model

For the analysis of the model, we introduce the following notations:

$N_j(t)$: the number of category j customers in the waiting room for this category at time t where $j = 1, 2, \ldots, k$.

$Y(t)$: the phase of the arrival process.

$I_j(t)$: the number of category j customers in the service room for this category at time t where $j = 1, 2, \ldots, k$.

$B(t)$: the booking status for the transport vessel where

$$B(t) = \begin{cases} 1 & \text{If already order for a new vessel,} \\ 0 & \text{Not booking} \end{cases}$$

$H(t)$: the number of vessels which (in operation) leave the station on the way to drop off passengers at time t.

$G(t)$: the phase of an Erlang clock of order m.

The Erlang clock only starts when a transport vessel is available at station.

$$\gamma(t) = \begin{cases} 1 & \text{(On) If the transport vessel is available at a transport station,} \\ 0 & \text{(Off) If it is not available at a transport station.} \end{cases}$$

$X(t) = \{(N_1(t), \ldots, N_k(t), Y(t), B(t), H(t), \gamma(t), G(t), I_1(t), \ldots, I_k(t)); t \geq 0\}$ is a continuous time Markov Chain on the state space. Therefore, this model can be studied as a level independent quasi-birth-death (LIQBD) process with state space is given by

$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5 \cup \Omega_6 \cup \Omega_7$; where

$\Omega_1 = \{(n_1, \ldots, n_k, y, b = 1, h, \gamma = 0) : 0 \leq n_1; 0 \leq n_j \leq W_j; 1 \leq y \leq m_1; 0 \leq h \leq h^*\}$;

$\Omega_2 = \{(0, \ldots, 0, y, b = 0, h, \gamma = 1, g, i_1, \ldots, i_k) : 1 \leq y \leq m_1; 0 \leq h \leq (h^* - 1); 1 \leq g \leq (L-1); 0 \leq i_j \leq C_j \forall j \neq j'; i_{j'} \neq C_{j'}; j = 1, 2, \ldots, k\}$;

$\Omega_3 = \{(0, \ldots, 0, y, b = 1, h, \gamma = 1, g, i_1, \ldots, i_k) : 1 \leq y \leq m_1; 0 \leq h \leq (h^* - 1); L \leq g \leq m; 0 \leq i_j \leq C_j \forall j \neq j'; i_{j'} \neq C_{j'}; j = 1, 2, \ldots, k\}$;

$\Omega_4 = \{(0, n_2, \ldots, n_k, y, b = 0, h, \gamma = 1, g, i_1, \ldots, i_k) : 1 \leq y \leq m_1; 0 \leq n_j \leq W_j; j \neq 1; 0 \leq h \leq (h^* - 1); 1 \leq g \leq (L-1); 0 \leq i_j \leq C_j \forall j \neq j'; i_{j'} \neq C_{j'}; j = 1, 2, \ldots, k\}$;

$\Omega_5 = \{(0, n_2, \ldots, n_k, y, b = 1, h, \gamma = 1, g, i_1, \ldots, i_k) : 1 \leq y \leq m_1; 0 \leq n_j \leq W_j; j \neq 1; 0 \leq h \leq (h^* - 1); L \leq g \leq m; 0 \leq i_j \leq C_j \forall j \neq j'; i_{j'} \neq C_{j'}; j = 1, 2, \ldots, k\}$;

$\Omega_6 = \{(n_1, \ldots, n_k, y, b = 0, h, \gamma = 1, g, C_1, \ldots, i_k) : 1 \leq y \leq m_1; 1 \leq n_1; 0 \leq n_j \leq W_j; j \neq 1; 0 \leq h \leq (h^* - 1); 1 \leq g \leq (L-1); 0 \leq i_j \leq C_j \forall j \neq j'; i_{j'} \neq C_{j'}; j = 1, 2, \ldots, k\}$;

$\Omega_7 = \{(n_1, \ldots, n_k, y, b = 1, h, \gamma = 1, g, C_1, \ldots, i_k) : 1 \leq y \leq m_1; 1 \leq n_1; 0 \leq n_j \leq W_j; j \neq 1; 0 \leq h \leq (h^* - 1); L \leq g \leq m; 0 \leq i_j \leq C_j \forall j \neq j'; i_{j'} \neq C_{j'}; j = 1, 2, \ldots, k\}$;

With the following conditions (in case $\gamma = 1$):

- If $0 \leq i_j < C_j$ then $n_j = 0$ for $j = 1, 2, \ldots, k$.
- If $i_1 = C_1$ then $0 \leq n_1$.
- If $i_j = C_j$ then $0 \leq n_j \leq W_j$ for $j = 2, 3, \ldots, k$.
- If $i_j = C_j$ $\forall j$, then the Erlang clock expires.

For category j customers, C_j is the maximum capacity of their service room (in the vessel) where $j = 1, 2, \ldots, k$, whereas W_j is the maximum capacity of their waiting room (in the transport station) where $j = 2, 3, \ldots, k$. Regarding the service rooms, the total number of individual seats is visualized in Figure 1 below.

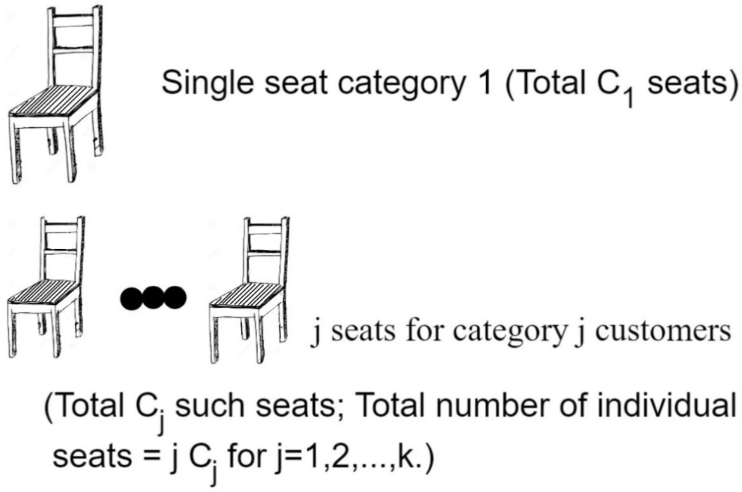

Figure 1. The total number of individual seats in a specific service room.

The terms of transitions of the states are given in Tables 1–5.

Table 1. Arrival rates for a new vessel.

From	$(n_1, \ldots, n_k, y, b = 1, h, \gamma = 0)$ $0 \leq h \leq h^*$
To	$(0, \ldots, 0, y, b = 0, h, \gamma = 1, g = 1, i_1, \ldots, i_k)$ $0 \leq h \leq (h^* - 1)$
Description	$n_j \leq C_j \forall j; i_j \leq C_j, j = 1, 2, \ldots, k.$
From	$(n_1, \ldots, n_k, y, b = 1, h, \gamma = 0)$
To	$(0, \ldots, 0, y, b = 0, (h + 1), \gamma = 0)$
Description	$C_j = n_j \forall j$ $0 \leq h \leq (h^* - 1)$
From	$(n_1, \ldots, n_k, y, 1, h, \gamma = 0)$
To	$(n'_1, \ldots, n'_k, y, 1, (h + 1), \gamma = 0)$
Description	$C_j < n_j \forall j$ $n'_j = n_j - C_j$
From	$(n_1, \ldots, n_k, y, b = 1, h^*, \gamma = 0)$
To	$(0, \ldots, 0, y, b = 0, (h^* - 1), \gamma = 1, g = 1, i_1, \ldots, i_k)$
Description	$n_j \leq C_j \ \forall j$ If $i_j = C_j \ \forall j \neq j'$, then $0 \leq i_{j'} \leq (C_{j'} - 1)$
From	$(n_1, \ldots, n_k, y, b = 1, h^*, \gamma = 0)$
To	$(0, \ldots, 0, y, b = 1, h^*, \gamma = 0)$
Description	$C_j = n_j \forall j$
From	$(n_1, \ldots, n_k, y, 1, h^*, \gamma = 0)$
To	$(n'_1, \ldots, n'_k, y, 1, h^*, \gamma = 0)$

Table 1. *Cont.*

Description	$C_j < n_j \forall j$ $n'_j = n_j - C_j$
From	$(n_1, \ldots, n_k, y, b = 1, (h^* - 1), \gamma = 1, g'', i_1, \ldots, i_k)$
To	$(n'_1, \ldots, n'_k, y, b, (h^* - 1), \gamma = 1, g = 1, i'_1, \ldots, i'_k)$
Description	$L \leq g'' \leq m$; If $i_j = C_j \forall j$, then $0 \leq n_j \leq W_j$; $j \neq 1$; $0 \leq n_1$; If $i_j < C_j \forall j$, then $n_j = 0 \forall j$;
From	$(n_1, \ldots, n_k, y, b = 1, h, \gamma = 1, g'', i_1, \ldots, i_k)$
To	$(0, \ldots, 0, y, b = 0, (h + 1), \gamma = 1, g = 1, i'_1 \ldots, i'_k)$
Description	$n_j < C_j, L \leq g'' \leq m, 0 \leq h \leq (h^* - 1)$
From	$(n_1, \ldots, n_k, y, b = 1, h, \gamma = 1, g'', i_1, \ldots, i_k)$
To	$(n'_1, \ldots, n'_k, y, b = 0, (h + 1), \gamma = 1, g = 1, i'_1, \ldots, i'_k)$
Description	$L \leq g'' \leq m$ $C_j \leq n_j \forall j, n'_j = n_j - C_j$
From	$(n_1, \ldots, n_k, y, b = 1, h, \gamma = 1, g'', i_1, \ldots, i_k)$
To	$(n'_1, \ldots, n'_k, y, b = 1, (h + 1), \gamma = 0)$
Description	$L \leq g'' \leq m$ $C_j < n_j \forall j, n'_j = n_j - (C_j - i_j)$
Arrival rate for a new vessel	θ

Table 2. Search rates.

From	$(n_1, \ldots, n_k, y, 1, h, \gamma = 1, g, i_1, \ldots, i_k)$
To	$(n_1, \ldots, n_k, y, 1, h, \gamma = 1, g, \ldots, C_j, \ldots)$
Description	$L_1 \leq g \leq m; L_1 \leq L, 0 \leq h < h^*$ $1 \leq i_j < C_j, j = 1, 2, \ldots, k.$ Only one among $i_1, \ldots, i_k$ can be change with positive probability.
From	$(n_1, \ldots, n_k, y, b = 1, h, \gamma = 1, g, C_1, C_1, \ldots, , i_j, \ldots, C_k)$
To	$(n_1, \ldots, n_k, y, b = 1, (h + 1), \gamma = 0)$
Description	$L_1 \leq g \leq m; L_1 \leq L, 0 \leq h < h^*$ $1 \leq i_j < C_j, j = 1, 2, \ldots, k.$
Search rate	$i_j \lambda_j, j = 1, 2, \ldots, k.$

Table 3. Departure rate for vessel.

From	$(n_1, \ldots, n_k, y, b, h, \gamma = 1, g, i_1, \ldots, i_k)$
To	$(n_1, \ldots, n_k, y, b, (h - 1), \gamma = 1, g, i_1, \ldots, i_k)$
Description	$1 \leq h < h^*, 1 \leq g \leq m$
From	$(n_1, \ldots, n_k, y, b = 1, h, \gamma = 0)$
To	$(n_1, \ldots, n_k, y, b = 1, (h - 1), \gamma = 0)$
Description	$1 \leq h \leq h^*$
rate	$h\mu$

Table 4. An Erlang distribution with m states and parameter ϕ.

From	$(n_1, \ldots, n_k, y, b, h, \gamma = 1, g, i_1, \ldots, i_k)$
To	$(n_1, \ldots, n_k, y, b, h, \gamma = 1, (g+1), i_1, \ldots, i_k)$
Description	$1 \leq h < h^*, 1 \leq g < m$
From	$(n_1, \ldots, n_k, y, b = 1, h, \gamma = 1, m, i_1, \ldots, i_k)$
To	$(n_1, \ldots, n_k, y, b = 1, (h+1), \gamma = 0)$
Description	$1 \leq h < h^*$
rate	ϕ

Table 5. Arrival rate.

From	$(n_1, \ldots, n_j, \ldots, n_k, y, b = 1, h, \gamma = 0)$
To	$(n_1, \ldots, (n_j+1), \ldots, n_k, y', b = 1, h, \gamma = 0)$
Description	$0 \leq h \leq h^*$ Only one among $n_1, \ldots, n_j, \ldots, n_k$ can be change with positive probability.
rate	$d^{(j)}_{yy'}$ where $j = 1, 2, \ldots, k$.
From	$(n_1, \ldots, n_j, \ldots, n_k, y, 1, h, \gamma = 0)$
To	$(n_1, \ldots, n_j, \ldots, n_k, y', 1, h, \gamma = 0)$
Description	$0 \leq h \leq h^*$
rate	$d^{(0)}_{yy'}$
From	$(n_1, \ldots, 0, \ldots, n_k, y, b, h, \gamma = 1, g, C_1, \ldots, i_j, \ldots, C_k)$
To	$(n_1, \ldots, 0, \ldots, n_k, y', b, h, \gamma = 1, g, C_1, \ldots, (i_j+1), \ldots, C_k)$
Description	$1 \leq g \leq m, 1 \leq h < h^*$ Only one among $i_1, i_2, \ldots, i_k$ can be change with positive probability.
rate	$d^{(j)}_{yy'}$ where $j = 1, 2, \ldots, k$.
From	$(n_1, \ldots, 0, \ldots, n_k, y, b, h, \gamma = 1, g, C_1, \ldots, i_j, \ldots, C_k)$
To	$(n_1, \ldots, 0, \ldots, n_k, y', b, h, \gamma = 1, g, C_1, \ldots, i_j, \ldots, C_k)$
Description	$1 \leq g \leq m, 1 \leq h < h^*$
rate	$d^{(0)}_{yy'}$
From	$(n_1, \ldots, 0, \ldots, n_k, y, b, h, \gamma = 1, g, C_1, \ldots, (C_j-1), \ldots, C_k)$
To	$(n_1, \ldots, 0, \ldots, n_k, y', b = 1, (h+1), \gamma = 0)$
Description	$1 \leq g < m, 1 \leq h < h^*$
rate	$d^{(j)}_{yy'}$ where $j = 1, 2, \ldots, k$.
From	$(n_1, \ldots, 0, \ldots, n_k, y, b, h, \gamma = 1, g, C_1, \ldots, (C_j-1), \ldots, C_k)$
To	$(n_1, \ldots, 0, \ldots, n_k, y', b, h, \gamma = 1, g, C_1, \ldots, (C_j-1), \ldots, C_k)$

Table 5. *Cont.*

Description	$1 \le g < m, 1 \le h < h^*$
rate	$d_{yy'}^{(0)}$
From	$(n_1, \ldots, 0, \ldots, n_k, y, b = 0, h, \gamma = 1, g, C_1, \ldots, (C_j - 1), \ldots, C_k)$
To	$(n_1, \ldots, 0, \ldots, n_k, y', b = 1, (h+1), \gamma = 0)$
Description	$1 \le g \le (L-1); 1 \le h < h^*$ Only one among $(C_1 - 1), (C_2 - 1), \ldots, (C_k - 1)$ can be change with positive probability.
rate	$d_{yy'}^{(j)} + \phi$, where $j = 1, 2, \ldots, k$.
From	$(n_1, \ldots, 0, \ldots, n_k, y, b = 0, h, \gamma = 1, m, C_1, \ldots, (C_j - 1), \ldots, C_k)$
To	$(n_1, \ldots, 0, \ldots, n_k, y', b = 0, h, \gamma = 1, m, C_1, \ldots, (C_j - 1), \ldots, C_k)$
Description	$1 \le h < h^*$
rate	$d_{yy'}^{(0)}$

The infinitesimal generator Q of the level independent quasi-birth-death (LIQBD) process with state space is of the form

$$Q = \begin{pmatrix} B_{00} & B_{01} & & & \\ B_{10} & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & A_0 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}; \text{where}$$

$$B_{00} = \begin{pmatrix} E_{00} & E_{01} & E_{02} & \ldots & E_{0(k-1)} \\ E_{10} & E_{11} & E_{12} & \ldots & E_{1(k-1)} \\ E_{20} & E_{21} & E_{22} & \ldots & E_{2(k-1)} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ E_{(k-1)0} & E_{(k-1)1} & E_{(k-1)2} & \ldots & E_{(k-1)(k-1)} \end{pmatrix}; B_{01} = \begin{pmatrix} E_{0(k,1)} & O & \ldots & O \\ E_{1(k,1)} & O & \ldots & O \\ E_{2(k,1)} & O & \ldots & O \\ \vdots & \vdots & \vdots & \vdots \\ E_{(k-1)(k,1)} & O & \ldots & O \end{pmatrix}$$

$$B_{10} = \begin{pmatrix} E_{(k,1)0} & O & \ldots & O \\ E_{(k,2)0} & O & \ldots & O \\ E_{(k,3)0} & O & \ldots & O \\ \vdots & \vdots & \vdots & \vdots \\ E_{(k,C_1)0} & O & \ldots & O \end{pmatrix} A_1 = \begin{pmatrix} E_1 & E_0 & O & \ldots & \ldots & \ldots & \ldots & O \\ O & E_1 & E_0 & O & \ldots & \ldots & \ldots & O \\ O & O & E_1 & E_0 & O & \ldots & \ldots & O \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & O \\ O & O & O & \ldots & \ldots & \ldots & O & E_1 \end{pmatrix};$$

$$A_2 = \begin{pmatrix} E_2 & O & O & \ldots & \ldots & O \\ O & E_2 & O & O & \ldots & O \\ O & O & E_2 & O & \ldots & O \\ \vdots & \vdots & \ddots & \ddots & \ddots & O \\ O & O & O & \ldots & \ldots & E_2 \end{pmatrix}; A_0 = \begin{pmatrix} O & O & O & \ldots & \ldots & O \\ O & O & O & \ldots & \ldots & O \\ O & O & O & \ldots & \ldots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots & O \\ E_0 & O & O & \ldots & \ldots & O \end{pmatrix};$$

where all sub-matrices (the zero matrices, E_0, E_1 and E_2) are matrices of same order.

For example, we fix $k = 3, C_1 = 3; L_1 = 2$ and $L = 3$. We get the following matrices as in Appendix A.

3. Steady-State Analysis

3.1. Stability Condition

Theorem 1. *The queueing inventory system generated by Q is stable if and only if*

$$\boldsymbol{\pi_{C_1}} E_0\, \boldsymbol{e_*} < \sum_{i=1}^{C_1} \boldsymbol{\pi_i} E_2\, \boldsymbol{e_*}$$

where $\boldsymbol{\pi_i}$ *is sub vector of order* $1 \times r_*$ *for* $i = 1, 2, \ldots, C_1$. $E_0 = (I_{(\frac{r_*}{m_1})} \otimes D_1)$ *where* $I_{(\frac{r_*}{m_1})}$ *is an identity matrix of order* $\frac{r_*}{m_1}$ *where* $\frac{r_*}{m_1}$ *is a positive integer number.* e_* *is column vector of 1's.*

Proof. Let $A = A_2 + A_1 + A_0$. We can notice that A is an irreducible matrix. Thus, the stationary vector $\boldsymbol{\pi}$ of A exists such that

$$\boldsymbol{\pi} A = 0$$

$$\boldsymbol{\pi}\, \boldsymbol{e} = 1.$$

The Markov chain with the infinitesimal generator Q of the level independent quasi-birth-death (LIQBD) is stable if and only if

$$\boldsymbol{\pi} A_0\, \boldsymbol{e} < \boldsymbol{\pi} A_2\, \boldsymbol{e}.$$

Recall, A_0 is a square matrix and $A_0 = \begin{pmatrix} O & O & O & \ldots & \ldots & O \\ O & O & O & \ldots & \ldots & O \\ O & O & O & \ldots & \ldots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots & O \\ E_0 & O & O & \ldots & \ldots & O \end{pmatrix}$. $E_0 = (I_{(\frac{r_*}{m_1})} \otimes D_1)$ where $I_{(\frac{r_*}{m_1})}$ is an identity matrix of order $\frac{r_*}{m_1}$ where $\frac{r_*}{m_1}$ is a positive integer number.

$\boldsymbol{\pi} A_0 e = (\boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \boldsymbol{\pi_2}, \ldots, \boldsymbol{\pi_{C_1}}) A_0\, \boldsymbol{e}$; where $\boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \ldots, \boldsymbol{\pi_{C_1}}$ are sub vectors of order $1 \times r_*$.

$$\begin{aligned} \boldsymbol{\pi} A_0\, \boldsymbol{e} &= (\boldsymbol{\pi_{C_1}} E_0, \boldsymbol{O}, \boldsymbol{O}, \ldots, \boldsymbol{O}) \begin{pmatrix} \boldsymbol{e_*} \\ \boldsymbol{e_*} \\ \vdots \\ \boldsymbol{e_*} \end{pmatrix}; \\ &= \boldsymbol{\pi_{C_1}} E_0\, \boldsymbol{e_*} \end{aligned}$$

Recall that A_2 is a square matrix and $A_2 = \begin{pmatrix} E_2 & O & O & \ldots & \ldots & O \\ O & E_2 & O & O & \ldots & O \\ O & O & E_2 & O & \ldots & O \\ \vdots & \vdots & \ddots & \ddots & \ddots & O \\ O & O & O & \ldots & \ldots & E_2 \end{pmatrix}$.

$\boldsymbol{\pi} A_2 e = (\boldsymbol{\pi_1},\ \boldsymbol{\pi_2},\ \boldsymbol{\pi_2}, \ldots,\ \boldsymbol{\pi_{C_1}}) A_2\ \boldsymbol{e}$; where $\boldsymbol{\pi_1},\ \boldsymbol{\pi_2}, \ldots,\ \boldsymbol{\pi_{C_1}}$ are sub vectors of order $1 \times r_*$.

$$\begin{aligned}
\boldsymbol{\pi} A_2\ \boldsymbol{e} &= (\ \boldsymbol{\pi_1} E_2,\ \boldsymbol{\pi_2} E_2,\ \boldsymbol{\pi_3} E_2, \ldots,\ \boldsymbol{\pi_{C_1}} E_2) \begin{pmatrix} \boldsymbol{e_*} \\ \boldsymbol{e_*} \\ \vdots \\ \boldsymbol{e_*} \end{pmatrix}; \\
&= \boldsymbol{\pi_1} E_2\ \boldsymbol{e_*} +\ \boldsymbol{\pi_2} E_2\ \boldsymbol{e_*} + \cdots +\ \boldsymbol{\pi_{C_1}} E_2\ \boldsymbol{e_*} \\
&= \sum_{i=1}^{C_1} \boldsymbol{\pi_i} E_2\ \boldsymbol{e_*}
\end{aligned}$$

Then

$$\boldsymbol{\pi} A_2\ \boldsymbol{e} = \sum_{i=1}^{C_1} \boldsymbol{\pi_i} E_2\ \boldsymbol{e_*};$$

Since $\boldsymbol{\pi} A_0\ \boldsymbol{e} =\ \boldsymbol{\pi_{C_1}} E_0\ \boldsymbol{e_*}$ and $\boldsymbol{\pi}\ A_2\ \boldsymbol{e} = \sum_{i=1}^{C_1} \boldsymbol{\pi_i} E_2\ \boldsymbol{e_*}$ then the queueing inventory system is stable if and only if

$$\boldsymbol{\pi_{C_1}} E_0\ \boldsymbol{e_*} < \sum_{i=1}^{C_1} \boldsymbol{\pi_i} E_2\ \boldsymbol{e_*}$$

□

3.2. Stationary Distribution

The stationary distribution of the Markov chain under consideration can obtained by solving the set of Equations (1) and (2).

$$\mathbf{X} Q = 0; \tag{1}$$

$$\mathbf{X} e = 1. \tag{2}$$

Let $\mathbf{X}$ be decomposed with Q as following:

$\mathbf{X} = (\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \ldots)$, where $\mathbf{X}_r = (\mathbf{Z}_{C_1 r - C_1 + 1}, \mathbf{Z}_{C_1 r - C_1 + 2}, \ldots, \mathbf{Z}_{C_1 r})$; $r = 1, 2, \ldots$ where

$\mathbf{X}_0 = (\mathbf{X}_{00}, \mathbf{X}_{0n_k}, \mathbf{X}_{0n_{(k-1)}}, \ldots, \mathbf{X}_{0n_2})$ where

$\mathbf{X}_{00} = (\mathbf{X}_{000}, \mathbf{X}_{0i_k}, \mathbf{X}_{0i_{(k-1)}}, \ldots, \mathbf{X}_{0i_2}, \mathbf{X}_{0i_1})$ where

$\mathbf{X}_{000} = (x_{0\ldots01(b=1)h(\gamma=0)}, x_{0\ldots02(b=1)h(\gamma=0)}, \ldots, x_{0\ldots0m_1(b=1)h(\gamma=0)})$; where $h = 0, 1, 2, \ldots, h^*$;

$\mathbf{X}_{0i_k} = (\mathbf{X}_{0\ldots0y(b=0)h(\gamma=1)g'0\ldots0i_k}, \mathbf{X}_{0\ldots0y(b=1)h(\gamma=1)g''0\ldots0i_k})$; where $i_k = 0, 1, 2, \ldots, C_k$; $y = 1, 2, \ldots, m_1$; $h = 0, 1, 2, \ldots, (h^* - 1)$; $g' = 1, 2, \ldots, (L-1)$; $g'' = L, (L+1), \ldots, m$;

$\mathbf{X}_{0i_{(k-1)}} = (\mathbf{X}_{0\ldots0y(b=0)h(\gamma=1)g'0\ldots0i_{(k-1)}i_k}, \mathbf{X}_{0\ldots0y(b=1)h(\gamma=1)g''0\ldots0i_{(k-1)}i_k})$; where $y = 1, 2, \ldots, m_1$; $h = 0, 1, 2, \ldots, (h^* - 1)$; $g' = 1, 2, \ldots, (L-1)$; $g'' = L, (L+1), \ldots, m$; $i_k = 0, 1, 2, \ldots, C_k$; $i_{(k-1)} = 1, 2, \ldots, C_{(k-1)}$;

$\vdots$

$\mathbf{X}_{0i_j} = (\mathbf{X}_{0\ldots0y(b=0)h(\gamma=1)g'0\ldots i_j \ldots i_k}, \mathbf{X}_{0\ldots0y(b=1)h(\gamma=1)g''0\ldots i_j \ldots i_k})$; where $y = 1, 2, \ldots, m_1$; $h = 0, 1, 2, \ldots, (h^* - 1)$; $g' = 1, 2, \ldots, (L-1)$; $g'' = L, (L+1), \ldots, m$; $i_k = 0, 1, 2, \ldots, C_k$; $i_j = 1, 2, \ldots, C_j$; $j = 1, 2, \ldots, k$;

$\vdots$

$\mathbf{X}_{0i_1} = (\mathbf{X}_{0\ldots0y(b=0)h(\gamma=1)g' i_1 i_2 \ldots i_k}, \mathbf{X}_{0\ldots0y(b=1)h(\gamma=1)g'' i_1 i_2 \ldots i_k})$; where $y = 1, 2, \ldots, m_1$; $h = 0, 1, 2, \ldots, (h^* - 1)$; $g' = 1, 2, \ldots, (L-1)$; $g'' = L, (L+1), \ldots, m$; $i_1 = 1, 2, \ldots, C_1$; $i_j = 0, 1, 2, \ldots, C_j$; $j = 2, 3, \ldots, k$;

$\mathbf{X}_{0n_k} = (\mathbf{X}_{00n_k}, \mathbf{X}_{00n_k C_k}, \mathbf{X}_{00n_k i_{(k-1)} C_k}, \ldots, \mathbf{X}_{00n_k i_2 C_k}, \mathbf{X}_{00n_k i_1 C_k})$ where $n_k = 1, 2, \ldots, W_k$;

$\mathbf{X}_{00n_k} = (\mathbf{X}_{0\ldots n_k 1(b=1)h(\gamma=0)}, \mathbf{X}_{0\ldots n_k 2(b=1)h(\gamma=0)}, \ldots, \mathbf{X}_{0\ldots n_k m_1(b=1)h(\gamma=0)})$; where $h = 0, 1, 2, \ldots, h^*$; $y = 1, 2, \ldots, m_1$;

$\mathbf{X}_{00n_k C_k} = (\mathbf{X}_{00n_k y(b=0)h(\gamma=1)g'0\ldots 0C_k}, \mathbf{X}_{00n_k y(b=1)h(\gamma=1)g''0\ldots 0C_k})$; where $n_k = 1, 2, \ldots, W_k$; $y = 1, 2, \ldots, m_1$; $h = 0, 1, 2, \ldots, (h^* - 1)$; $g' = 1, 2, \ldots, (L - 1)$; $g'' = L, (L+1), \ldots, m$;

$\mathbf{X}_{00n_k i_{(k-1)} C_k} = (\mathbf{X}_{00n_k y(b=0)h(\gamma=1)g'0\ldots 0i_{(k-1)}C_k}, \mathbf{X}_{00n_k y(b=1)h(\gamma=1)g''0\ldots 0i_{(k-1)}C_k})$; where $n_k = 1, 2, \ldots, W_k$; $y = 1, 2, \ldots, m_1$; $h = 0, 1, 2, \ldots, (h^* - 1)$; $g' = 1, 2, \ldots, (L - 1)$; $g'' = L, (L+1), \ldots, m$; $i_{(k-1)} = 1, 2, \ldots, C_{(k-1)}$;

$\vdots$

$\mathbf{X}_{00n_k i_2 C_k} = (\mathbf{X}_{00n_k y(b=0)h(\gamma=1)g'0i_2 i_3\ldots i_{(k-1)}C_k}, \mathbf{X}_{00n_k y(b=1)h(\gamma=1)g''0i_2 i_3\ldots i_{(k-1)}C_k})$; where $i_2 = 1, 2, \ldots, C_2$; $i_j = 0, 1, 2, \ldots, C_j$; $j = 3, 4, \ldots, k$; $h = 0, 1, 2, \ldots, (h^* - 1)$;

$\mathbf{X}_{00n_k i_1 C_k} = (\mathbf{X}_{00n_k y(b=0)h(\gamma=1)g'i_1 i_2 i_3\ldots i_{(k-1)}C_k}, \mathbf{X}_{00n_k y(b=1)h(\gamma=1)g''i_1 i_2 i_3\ldots i_{(k-1)}C_k})$; where $i_2 = 1, 2, \ldots, C_2$; $i_j = 0, 1, 2, \ldots, C_j$; $j = 3, 4, \ldots, k$;

$\mathbf{X}_{0n_{(k-1)}} = (\mathbf{X}_{00n_{(k-1)}}, \mathbf{X}_{00n_{(k-1)}C_{(k-1)}i_k}, \mathbf{X}_{00n_{(k-1)}i_{(k-2)}C_{(k-1)}}, \mathbf{X}_{00n_{(k-1)}i_{(k-3)}C_{(k-1)}}, \ldots, \mathbf{X}_{00n_{(k-1)}i_2 C_{(k-1)}}, \mathbf{X}_{00n_{(k-1)}i_1 C_{(k-1)}})$ where $n_{(k-1)} = 1, 2, \ldots, W_{(k-1)}$;

$\mathbf{X}_{00n_{(k-1)}} = (\mathbf{X}_{0\ldots n_{(k-1)}n_k 1(b=1)h(\gamma=0)}, \mathbf{X}_{0\ldots n_{(k-1)}n_k 2(b=1)h(\gamma=0)}, \ldots, \mathbf{X}_{0\ldots n_{(k-1)}n_k m_1(b=1)h(\gamma=0)})$; where $h = 0, 1, 2, \ldots, h^*$; $y = 1, 2, \ldots, m_1$; $n_{(k-1)} = 1, 2, \ldots, W_k$;

$\mathbf{X}_{00n_{(k-1)}C_{(k-1)}i_k} = (\mathbf{X}_{0\ldots 0n_{(k-1)}n_k y(b=0)h(\gamma=1)g'0\ldots 0C_{(k-1)}i_k}, \mathbf{X}_{0\ldots 0n_{(k-1)}n_k y(b=1)h(\gamma=1)g''0\ldots 0C_{(k-1)}} i_k)$; where $n_k = 0, 1, 2, \ldots, W_k$; $y = 1, 2, \ldots, m_1$; $h = 0, 1, 2, \ldots, (h^* - 1)$; $g' = 1, 2, \ldots, (L - 1)$; $g'' = L, (L+1), \ldots, m$; $n_{(k-1)} = 1, 2, \ldots, W_{(k-1)}$;

$\vdots$

$\mathbf{X}_{00n_{(k-1)}i_1 C_{(k-1)}} = (\mathbf{X}_{0\ldots 0n_{(k-1)}n_k y(b=0)h(\gamma=1)g'i_1 i_2\ldots C_{(k-1)}i_k}, \mathbf{X}_{0\ldots 0n_{(k-1)}n_k y(b=1)h(\gamma=1)g''i_1 i_2\ldots C_{(k-1)}i_k})$; where $n_k = 0, 1, 2, \ldots, W_k$; $ty = 1, 2, \ldots, m_1$; $h = 0, 1, 2, \ldots, (h^* - 1)$; $g' = 1, 2, \ldots, (L - 1)$; $g'' = L, (L+1), \ldots, m$; $n_{(k-1)} = 1, 2, \ldots, W_{(k-1)}$;

$\mathbf{X}_{0n_2} = (\mathbf{X}_{00n_2}, \mathbf{X}_{00n_2 0C_2 0\ldots 0i_k}, \mathbf{X}_{00n_2 0C_2 0\ldots 0i_{(k-1)}i_k}, \ldots, \mathbf{X}_{00n_2 0C_2 i_3\ldots i_{(k-1)}i_k})$; where $n_2 = 1, 2, \ldots, W_2$;

$\mathbf{X}_{00n_2} = (\mathbf{X}_{0n_2 n_3\ldots n_k 1(b=1)h(\gamma=0)}, \mathbf{X}_{0n_2 n_3\ldots n_k 2(b=1)h(\gamma=0)}, \ldots, \mathbf{X}_{0n_2 n_3\ldots n_k m_1(b=1)h(\gamma=0)})$; where $h = 0, 1, 2, \ldots, h^*$; $y = 1, 2, \ldots, m_1$; $n_2 = 1, 2, \ldots, W_2$;

$\mathbf{X}_{00n_2 i_1 C_2 00i_k} = (\mathbf{X}_{0n_2 n_3\ldots n_k y(b=0)h(\gamma=1)g'i_1 C_2 0\ldots 0i_k}, \mathbf{X}_{0n_2 n_3\ldots n_k y(b=1)h(\gamma=1)g''i_1 C_2 0\ldots 0i_k})$; where $n_2 = 1, 2, \ldots, W_2$; $y = 1, 2, \ldots, m_1$; $h = 0, 1, 2, \ldots, (h^* - 1)$; $g' = 1, 2, \ldots, (L - 1)$; $g'' = L, (L+1), \ldots, m$;

$\vdots$

$\mathbf{X}_{00n_2 0C_2 i_3\ldots i_k} = (\mathbf{X}_{0n_2 n_3\ldots n_k y(b=0)h(\gamma=1)g'0C_2 i_3\ldots i_k}, \mathbf{X}_{0n_2 n_3\ldots n_k y(b=1)h(\gamma=1)g''0C_2 i_3\ldots i_k})$; where $n_2 = 1, 2, \ldots, W_2$; $y = 1, 2, \ldots, m_1$; $h = 0, 1, 2, \ldots, (h^* - 1)$; $g' = 1, 2, \ldots, (L - 1)$; $g'' = L, (L+1), \ldots, m$;

$\mathbf{Z}_{n_1} = (\mathbf{Z}_{n_1 00}, \mathbf{Z}_{n_1 0}, \mathbf{Z}_{n_1 n_k}, \mathbf{Z}_{n_1 n_{(k-1)}}, \ldots, \mathbf{Z}_{n_1 n_2})$; where $0 < n_1$;

$\mathbf{Z}_{n_1 00} = (\mathbf{Z}_{n_1 n_2\ldots n_k 1(b=1)h(\gamma=0)}, \mathbf{Z}_{n_1 n_2\ldots n_k 2(b=1)h(\gamma=0)}, \ldots, \mathbf{Z}_{n_1 n_2\ldots n_k m_1(b=1)h(\gamma=0)})$; where $h = 0, 1, 2, \ldots, h^*$; $y = 1, 2, \ldots, m_1$; $0 < n_1$;

$\mathbf{Z}_{n_1 0} = (\mathbf{Z}_{n_1 0\ldots 0y(b=0)h(\gamma=1)g'C_1 i_2\ldots i_k}, \mathbf{Z}_{n_1 0\ldots 0y(b=1)h(\gamma=1)g''C_1 i_2\ldots i_k})$; where $h = 0, 1, 2, \ldots, (h^* - 1)$; $y = 1, 2, \ldots, m_1$;

$\mathbf{Z}_{n_1 n_k} = (\mathbf{Z}_{n_1 0\ldots n_k y(b=0)h(\gamma=1)g'C_1 i_2\ldots C_k}, \mathbf{Z}_{n_1 0\ldots n_k y(b=1)h(\gamma=1)g''C_1 i_2\ldots C_k})$; where $h = 0, 1, 2, \ldots, (h^* - 1)$; $y = 1, 2, \ldots, m_1$;

$\mathbf{Z}_{n_1 n_{(k-1)}} = (\mathbf{Z}_{n_1 0\ldots n_{(k-1)}n_k y(b=0)h(\gamma=1)g'C_1 i_2\ldots C_{(k-1)}i_k}, \mathbf{Z}_{n_1 0\ldots n_{(k-1)}n_k y(b=1)h(\gamma=1)g''C_1 i_2\ldots C_{(k-1)}i_k})$; where $h = 0, 1, 2, \ldots, (h^* - 1)$; $y = 1, 2, \ldots, m_1$;

$\vdots$

$\mathbf{Z}_{n_1 n_2} = (\mathbf{Z}_{n_1 n_2 \ldots n_k y(b=0)h(\gamma=1)g' C_1 C_2 i_3 \ldots i_k}, \mathbf{Z}_{n_1 n_2 \ldots n_k y(b=1)h(\gamma=1)g'' C_1 C_2 i_3 \ldots i_k})$; where $h = 0, 1, 2, \ldots,$
$(h^* - 1)$; $y = 1, 2, \ldots, m_1$; $n_2 = 1, 2, \ldots, W_k$;

Note: in case ($\gamma = 1$):

- If $i_j = C_j \forall j$, then Erlang clock expires.
- If $i_j = C_j$, then $0 \leq n_j \leq W_j$; $j = 2, 3, \ldots, k$.
- If $i_1 = C_1$, then $0 \leq n_1$.
- If $i_j < C_j$, then $n_j = 0$; $j = 1, 2, \ldots, k$.

From Equation (1), we obtain the following equations.

$$\mathbf{X}_0 B_{00} + \mathbf{X}_1 B_{10} = 0; \tag{3}$$

$$\mathbf{X}_0 B_{01} + \mathbf{X}_1 A_1 + \mathbf{X}_2 A_2 = 0; \tag{4}$$

$$\mathbf{X}_1 A_0 + \mathbf{X}_2 A_1 + \mathbf{X}_3 A_2 = 0; \tag{5}$$

$$\mathbf{X}_{j-1} A_0 + \mathbf{X}_j A_1 + \mathbf{X}_{j+1} A_2 = 0 \text{ for } j = 2, 3, \ldots \tag{6}$$

Then there exists a matrix R such that
$\mathbf{X}_j = \mathbf{X}_1 R^{(j-1)}$ for $j = 2, 3, \ldots$. For more details, we can see Stewart [15].

4. Performance Measures

We obtain some performance measures of the system under steady state as following:

1. Expected number of type 1 customers in the waiting room for type 1

$$E[N_1] = \sum_{n_1=1}^{\infty} n_1 \mathbf{Z}_{n_1} \mathbf{e}.$$

2. Expected number of type j customers in waiting room type j for $j = 2, \ldots, k$.

$$E[N_j] = \sum_{n_j=1}^{W_j} \sum_{n_{(j+1)}=0}^{W_{(j+1)}} \cdots \sum_{n_k=0}^{W_k} \sum_{y=1}^{m_1} \sum_{h=0}^{h^*} n_j \mathbf{X}_{0 n_j n_{(j+1)} \ldots n_k y(b=1)h(\gamma=0)} \mathbf{e}$$

$$+ \sum_{n_j=1}^{W_j} \sum_{n_{(j+1)}=0}^{W_{(j+1)}} \cdots \sum_{n_k=0}^{W_k} \sum_{y=1}^{m_1} \sum_{h=0}^{(h^*-1)} \sum_{g'=1}^{(L-1)} \sum_{i_k=0}^{C_k} n_j \mathbf{X}_{0 n_j n_{(j+1)} \ldots n_k y(b=0)h(\gamma=1)g' 0 C_j 0 \ldots 0 i_k} \mathbf{e}$$

$$+ \sum_{n_j=1}^{W_j} \sum_{n_{(j+1)}=0}^{W_{(j+1)}} \cdots \sum_{n_k=0}^{W_k} \sum_{y=1}^{m_1} \sum_{h=0}^{(h^*-1)} \sum_{g''=L}^{m} \sum_{i_k=0}^{C_k} n_j \mathbf{X}_{0 n_j n_{(j+1)} \ldots n_k y(b=1)h(\gamma=1)g'' 0 C_j 0 \ldots 0 i_k} \mathbf{e}$$

$$+ \ldots$$

$$+ \sum_{n_2=1}^{W_2} \sum_{n_3=0}^{W_3} \cdots \sum_{n_k=0}^{W_k} \sum_{y=1}^{m_1} \sum_{h=0}^{(h^*-1)} \sum_{g'=1}^{(L-1)} \sum_{i_{(j-1)}=0}^{C_{(j-1)}} \sum_{i_{(j+1)}=0}^{C_{(j+1)}} \cdots \sum_{i_k=0}^{C_k}$$

$$n_j \mathbf{X}_{0 n_2 n_3 \ldots n_j \ldots n_k y(b=0)h(\gamma=1)g' 0 C_2 \ldots i_{(j-1)} C_j i_{(j+1)} \ldots i_k} \mathbf{e}$$

$$+ \sum_{n_2=1}^{W_2} \sum_{n_3=0}^{W_3} \cdots \sum_{n_k=0}^{W_k} \sum_{y=1}^{m_1} \sum_{h=0}^{(h^*-1)} \sum_{g''=L}^{m} \sum_{i_{(j-1)}=0}^{C_{(j-1)}} \sum_{i_{(j+1)}=0}^{C_{(j+1)}} \cdots \sum_{i_k=0}^{C_k}$$

$$n_j \mathbf{X}_{0 n_2 n_3 \ldots n_k y(b=1)h(\gamma=1)g'' 0 C_2 \ldots i_{(j-1)} C_j i_{(j+1)} \ldots i_k} \mathbf{e}$$

$$+ \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{W_2} \cdots \sum_{n_j=1}^{W_j} \sum_{n_{(j+1)}=0}^{W_{(j+1)}} \cdots \sum_{n_k=0}^{W_k} \sum_{y=1}^{m_1} \sum_{h=0}^{h^*} n_j \mathbf{Z}_{n_1 n_2 \ldots n_k y(b=1)h(\gamma=0)} \mathbf{e}$$

$$+\sum_{n_1=1}^{\infty}\sum_{n_{2=1}}^{W_2}\cdots\sum_{n_{j=1}}^{W_j}\sum_{n_{(j+1)=0}}^{W_{(j+1)}}\cdots\sum_{n_{k=0}}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g'=1}^{(L-1)}\sum_{i_3=0}^{C_3}\cdots\sum_{i_k=0}^{C_k}$$

$$n_j\mathbf{Z}_{n_1n_2n_3\ldots n_j\ldots n_ky(b=0)h(\gamma=1)g'C_1C_2i_3\ldots C_j\ldots i_k}\mathbf{e}$$

$$+\sum_{n_1=1}^{\infty}\sum_{n_{2=1}}^{W_2}\cdots\sum_{n_{j=1}}^{W_j}\sum_{n_{(j+1)=0}}^{W_{(j+1)}}\cdots\sum_{n_{k=0}}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g''=L}^{m}\sum_{i_3=0}^{C_3}\cdots\sum_{i_k=0}^{C_k}$$

$$n_j\mathbf{Z}_{n_1n_2n_3\ldots n_j\ldots n_ky(b=1)h(\gamma=1)g''C_1C_2i_3\ldots C_j\ldots i_k}\mathbf{e}$$

3. Probability that the service room j is idle for $j = 1, 2, \ldots, k$.

$$b_1^0=\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g'=1}^{(L-1)}\sum_{i_1=0}^{C_1}\cdots\sum_{i_{(j-1)}=0}^{C_{(j-1)}}\sum_{i_{(j+1)}=0}^{C_{(j+1)}}\cdots\sum_{i_k=0}^{C_k}\mathbf{X}_{0\ldots0y(b=0)h(\gamma=1)g'i_1\ldots(i_j=0)\ldots i_k}\mathbf{e}$$

$$+\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g''=L}^{m}\sum_{i_1=0}^{C_1}\cdots\sum_{i_{(j-1)}=0}^{C_{(j-1)}}\sum_{i_{(j+1)}=0}^{C_{(j+1)}}\cdots\sum_{i_k=0}^{C_k}\mathbf{X}_{0\ldots0y(b=1)h(\gamma=1)g''i_1\ldots(i_j=0)\ldots i_k}\mathbf{e}$$

$$+\sum_{n_2=1}^{W_2}\cdots\sum_{n_{(j-1)}=0}^{W_{(j-1)}}\sum_{n_{(j+1)}=0}^{W_{(j+1)}}\cdots\sum_{n_k=0}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g'=1}^{(L-1)}\sum_{i_1=0}^{C_1}\cdots\sum_{i_{(j-1)}=0}^{C_{(j-1)}}\sum_{i_{(j+1)}=0}^{C_{(j+1)}}\cdots\sum_{i_k=0}^{C_k}$$

$$\mathbf{X}_{n_2\ldots n_{(j-1)}n_{(j+1)}\ldots n_ky(b=0)h(\gamma=1)g'i_1\ldots(i_j=0)\ldots i_k}\mathbf{e}$$

$$+\sum_{n_2=1}^{W_2}\cdots\sum_{n_{(j-1)}=0}^{W_{(j-1)}}\sum_{n_{(j+1)}=0}^{W_{(j+1)}}\cdots\sum_{n_k=0}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g''=L}^{m}\sum_{i_1=0}^{C_1}\cdots\sum_{i_{(j-1)}=0}^{C_{(j-1)}}\sum_{i_{(j+1)}=0}^{C_{(j+1)}}\cdots\sum_{i_k=0}^{C_k}$$

$$\mathbf{X}_{n_2\ldots n_{(j-1)}n_{(j+1)}\ldots n_ky(b=1)h(\gamma=1)g''i_1\ldots(i_j=0)\ldots i_k}\mathbf{e}$$

$$+\sum_{n_1=1}^{\infty}\cdots\sum_{n_{(j-1)}=0}^{W_{(j-1)}}\sum_{n_{(j+1)}=0}^{W_{(j+1)}}\cdots\sum_{n_k=0}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g'=1}^{(L-1)}\sum_{i_1=0}^{C_1}\cdots\sum_{i_{(j-1)}=0}^{C_{(j-1)}}\sum_{i_{(j+1)}=0}^{C_{(j+1)}}\cdots\sum_{i_k=0}^{C_k}$$

$$\mathbf{Z}_{n_1n_2\ldots n_{(j-1)}n_{(j+1)}\ldots n_ky(b=0)h(\gamma=1)g'i_1\ldots(i_j=0)\ldots i_k}\mathbf{e}$$

$$+\sum_{n_1=1}^{\infty}\cdots\sum_{n_{(j-1)}=0}^{W_{(j-1)}}\sum_{n_{(j+1)}=0}^{W_{(j+1)}}\cdots\sum_{n_k=0}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g''=L}^{m}\sum_{i_1=0}^{C_1}\cdots\sum_{i_{(j-1)}=0}^{C_{(j-1)}}\sum_{i_{(j+1)}=0}^{C_{(j+1)}}\cdots\sum_{i_k=0}^{C_k}$$

$$\mathbf{Z}_{n_1n_2\ldots n_{(j-1)}n_{(j+1)}\ldots n_ky(b=1)h(\gamma=1)g''i_1\ldots(i_j=0)\ldots i_k}\mathbf{e}$$

4. Probability that the service room j is busy for $j = 1, 2, \ldots, k$.

$$b_j^1 = 1 - b_j^0$$

5. Expected number of vessels, which are in operation to drop off passengers.

$$E[N_h]=\sum_{h=0}^{h^*}\sum_{y=1}^{m_1}hx_{0\ldots0y(b=1)h(\gamma=0)}$$

$$+\sum_{h=0}^{(h^*-1)}\sum_{y=1}^{m_1}\sum_{g'=1}^{(L-1)}\sum_{i_k=0}^{C_k}h\mathbf{X}_{0\ldots0y(b=0)h(\gamma=1)g'0\ldots0i_k}\mathbf{e}$$

$$+\sum_{h=0}^{(h^*-1)}\sum_{y=1}^{m_1}\sum_{g''=L}^{m}\sum_{i_k=0}^{C_k} h\mathbf{X}_{0\ldots0y(b=1)h(\gamma=1)g''0\ldots0i_k}\mathbf{e}$$

$$+\sum_{h=0}^{(h^*-1)}\sum_{y=1}^{m_1}\sum_{g'=1}^{(L-1)}\sum_{i_{(k-1)}=0}^{C_{(k-1)}}\sum_{i_k=0}^{C_k} h\mathbf{X}_{0\ldots0y(b=0)h(\gamma=0)g'0\ldots0i_{(k-1)}i_k}\mathbf{e}$$

$$+\sum_{h=0}^{(h^*-1)}\sum_{y=1}^{m_1}\sum_{g''=L}^{m}\sum_{i_{(k-1)}=0}^{C_{(k-1)}}\sum_{i_k=0}^{C_k} h\mathbf{X}_{0\ldots0y(b=1)h(\gamma=1)g''0\ldots0i_{(k-1)}i_k}\mathbf{e}$$

$$+\ldots$$

$$+\sum_{h=0}^{(h^*-1)}\sum_{y=1}^{m_1}\sum_{g'=1}^{(L-1)}\sum_{i_1=1}^{C_1}\sum_{i_2=1}^{C_2}\cdots\sum_{i_k=0}^{C_k} h\mathbf{X}_{0\ldots0y(b=0)h(\gamma=1)g'i_1i_2\ldots i_k}\mathbf{e}$$

$$+\sum_{h=0}^{(h^*-1)}\sum_{y=1}^{m_1}\sum_{g''=L}^{m}\sum_{i_1=1}^{C_1}\sum_{i_2=1}^{C_2}\cdots\sum_{i_k=0}^{C_k} h\mathbf{X}_{0\ldots0y(b=1)h(\gamma=1)g''i_1i_2\ldots i_k}\mathbf{e}$$

$$+\sum_{h=0}^{h^*}\sum_{y=1}^{m_1} h\mathbf{X}_{0\ldots n_ky(b=1)h(\gamma=0)}\mathbf{e}+\sum_{h=0}^{(h^*-1)}\sum_{y=1}^{m_1}\sum_{g'=1}^{(L-1)} h\mathbf{X}_{00n_ky(b=0)h(\gamma=1)g'0\ldots0C_k}\mathbf{e}$$

$$+\sum_{h=0}^{(h^*-1)}\sum_{y=1}^{m_1}\sum_{g''=L}^{m} h\mathbf{X}_{00n_ky(b=1)h(\gamma=1)g''0\ldots0C_k}\mathbf{e}$$

$$+\ldots$$

$$+\sum_{h=0}^{(h^*-1)}\sum_{y=1}^{m_1}\sum_{g'=1}^{(L-1)}\sum_{i_3=1}^{C_3}\sum_{i_4=1}^{C_4}\cdots\sum_{i_k=0}^{C_k} h\mathbf{X}_{0n_2n_3\ldots n_ky(b=0)h(\gamma=1)g'0C_2i_3\ldots i_k}\mathbf{e}$$

$$+\sum_{h=0}^{(h^*-1)}\sum_{y=1}^{m_1}\sum_{g''=L}^{m}\sum_{i_3=1}^{C_3}\sum_{i_4=1}^{C_4}\cdots\sum_{i_k=0}^{C_k} h\mathbf{X}_{0n_2n_3\ldots n_ky(b=1)h(\gamma=1)g''0C_2i_3\ldots i_k}\mathbf{e}$$

$$+\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{W_2}\cdots\sum_{n_k=0}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{h^*} h\mathbf{Z}_{n_1n_2\ldots n_ky(b=1)h(\gamma=0)}\mathbf{e}$$

$$+\sum_{n_1=1}^{\infty}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g'=1}^{(L-1)}\sum_{i_2=0}^{C_2}\sum_{i_3=0}^{C_3}\cdots\sum_{i_k=0}^{C_k} h\mathbf{Z}_{n_10\ldots0y(b=0)h(\gamma=1)g'C_1i_2\ldots i_k}\mathbf{e}$$

$$+\sum_{n_1=1}^{\infty}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g'=1}^{(L-1)}\sum_{i_2=0}^{C_2}\sum_{i_3=0}^{C_3}\cdots\sum_{i_k=0}^{C_k} h\mathbf{Z}_{n_10\ldots0y(b=0)h(\gamma=1)g'C_1i_2\ldots i_k}\mathbf{e}$$

$$+\sum_{n_1=1}^{\infty}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g''=L}^{m}\sum_{i_2=0}^{C_2}\sum_{i_3=0}^{C_3}\cdots\sum_{i_k=0}^{C_k} h\mathbf{Z}_{n_10\ldots0y(b=1)h(\gamma=1)g''C_1i_2\ldots i_k}\mathbf{e}$$

$$+\sum_{n_1=1}^{\infty}\sum_{n_k=1}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g'=1}^{(L-1)}\sum_{i_2=0}^{C_2}\sum_{i_3=0}^{C_3}\cdots\sum_{i_{(k-1)}=0}^{C_{(k-1)}} h\mathbf{Z}_{n_10\ldots n_ky(b=0)h(\gamma=1)g'C_1i_2\ldots C_k}\mathbf{e}$$

$$+\sum_{n_1=1}^{\infty}\sum_{n_k=1}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g''=L}^{m}\sum_{i_2=0}^{C_2}\sum_{i_3=0}^{C_3}\cdots\sum_{i_{(k-1)}=0}^{C_{(k-1)}} h\mathbf{Z}_{n_10\ldots n_ky(b=1)h(\gamma=1)g''C_1i_2\ldots C_k}\mathbf{e}$$

$$+\ldots$$

$$+\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{W_2}\cdots\sum_{n_k=1}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g'=1}^{(L-1)}\sum_{i_3=0}^{C_3}\cdots\sum_{i_k=0}^{C_k}h\mathbf{Z}_{n_1n_2\ldots n_ky(b=0)h(\gamma=1)g'C_1C_2i_3\ldots i_k}\mathbf{e}$$

$$+\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{W_2}\cdots\sum_{n_k=1}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g''=L}^{m}\sum_{i_3=0}^{C_3}\cdots\sum_{i_k=0}^{C_k}h\mathbf{Z}_{n_1n_2\ldots n_ky(b=1)h(\gamma=1)g''C_1C_2i_3\ldots i_k}\mathbf{e}$$

6. Expected number of type j customers in the service room type j for $j = 1, 2, \ldots, k$

$$E[I_j]=\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g'=1}^{(L-1)}\sum_{i_1=0}^{C_1}\cdots\sum_{i_j=0}^{C_j}\cdots\sum_{i_k=0}^{C_k}i_j\mathbf{X}_{0\ldots0y(b=0)h(\gamma=1)g'i_1\ldots i_j\ldots i_k}\mathbf{e}$$

$$+\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g''=L}^{m}\sum_{i_1=0}^{C_1}\cdots\sum_{i_j=0}^{C_j}\cdots\sum_{i_k=0}^{C_k}i_j\mathbf{X}_{0\ldots0y(b=1)h(\gamma=1)g''i_1\ldots i_j\ldots i_k}\mathbf{e}$$

$$+\sum_{n_k=1}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g'=1}^{(L-1)}\sum_{i_1=0}^{C_1}\cdots\sum_{i_j=0}^{C_j}\cdots\sum_{i_k=0}^{C_k}i_j\mathbf{X}_{0\ldots0n_ky(b=0)h(\gamma=1)g'i_1\ldots i_j\ldots C_k}\mathbf{e}$$

$$+\sum_{n_k=1}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g''=L}^{m}\sum_{i_1=0}^{C_1}\cdots\sum_{i_j=0}^{C_j}\cdots\sum_{i_k=0}^{C_k}i_j\mathbf{X}_{0\ldots0n_ky(b=1)h(\gamma=1)g''i_1\ldots i_j\ldots C_k}\mathbf{e}$$

$$+\ldots$$

$$+\sum_{n_2=1}^{W_2}\cdots\sum_{n_j=0}^{W_j}\cdots\sum_{n_k=1}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g'=1}^{(L-1)}\sum_{i_1=0}^{C_1}\cdots\sum_{i_j=0}^{C_j}\cdots\sum_{i_k=0}^{C_k}i_j\mathbf{X}_{0n_2\ldots n_j\ldots n_ky(b=0)h(\gamma=1)g'i_1\ldots i_j\ldots i_k}\mathbf{e}$$

$$+\sum_{n_2=1}^{W_2}\cdots\sum_{n_j=0}^{W_j}\cdots\sum_{n_k=1}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g''=L}^{m}\sum_{i_1=0}^{C_1}\cdots\sum_{i_j=0}^{C_j}\cdots\sum_{i_k=0}^{C_k}i_j\mathbf{X}_{0n_2\ldots n_j\ldots n_ky(b=1)h(\gamma=1)g''i_1\ldots i_j\ldots i_k}\mathbf{e}$$

$$+\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{W_2}\cdots\sum_{n_j=0}^{W_j}\cdots\sum_{n_k=1}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g'=1}^{(L-1)}\sum_{i_2=0}^{C_2}\cdots\sum_{i_j=0}^{C_j}\cdots\sum_{i_k=0}^{C_k}$$
$$i_j\mathbf{Z}_{n_1n_2\ldots n_j\ldots n_ky(b=0)h(\gamma=1)g'C_1i_2\ldots i_j\ldots i_k}\mathbf{e}$$

$$+\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{W_2}\cdots\sum_{n_j=0}^{W_j}\cdots\sum_{n_k=1}^{W_k}\sum_{y=1}^{m_1}\sum_{h=0}^{(h^*-1)}\sum_{g''=L}^{m}\sum_{i_2=0}^{C_2}\cdots\sum_{i_j=0}^{C_j}\cdots\sum_{i_k=0}^{C_k}$$
$$i_j\mathbf{Z}_{n_1n_2\ldots n_j\ldots n_ky(b=1)h(\gamma=1)g''C_1i_2\ldots i_j\ldots i_k}\mathbf{e}$$

7. Expected number of seats filled in the vessel that just left.

$$E[P]=\sum_{j=1}^{k}jE[I_j]$$

8. Expected number of unoccupied seats in the vessel that just left.

$$E[U]=\sum_{j=1}^{k}jC_j-\sum_{j=1}^{k}jE[I_j]$$

5. Numerical Example

In order to illustrate the performance measures of the system numerically, we fix $k = 3, C_1 = 3, C_2 = 3, C_3 = 3, W_1 = 3, W_2 = 3, \theta = 10, h = 4, \lambda_1 = 0.1, \lambda_2 = 0.2, \lambda_3 = 0.3, \mu = 5, m = 4, L_1 = 2, L = 3$ and $\phi = 3$.

For (BMMAP) with representation (D_0, D_1, D_2, D_3) of order m_1, we fix $m_1 = 3$,

$$D_0 = \begin{pmatrix} -14.8 & 1 & 2 \\ 0.3 & -21.3 & 4 \\ 0.5 & 0 & -21.5 \end{pmatrix}, D_1 = \begin{pmatrix} 0.3 & 5 & 0.5 \\ 1 & 2 & 3 \\ 1.2 & 1.5 & 2 \end{pmatrix}, D_2 = \begin{pmatrix} 1 & 0.2 & 2 \\ 0.4 & 1.1 & 2.5 \\ 7 & 0.8 & 1 \end{pmatrix} \text{ and}$$
$$D_3 = \begin{pmatrix} 0.3 & 0.5 & 2 \\ 3 & 1.5 & 2.5 \\ 1 & 3.5 & 3 \end{pmatrix}.$$

5.1. The Effect of Parameter ϕ on Performance Measures

To analyse the effect of parameter ϕ on various performance measures, we use Tables 6–9 and Figures 2–4 as following:

1. From Figures 2 and 3, we can see that the expected number of type j customers in waiting room j decreases as ϕ increases where $j = 1, 2, 3$.
2. Moreover, we can see from Figure 4 that the probability that service room j is idle increases as ϕ increases where $j = 1, 2, 3$.

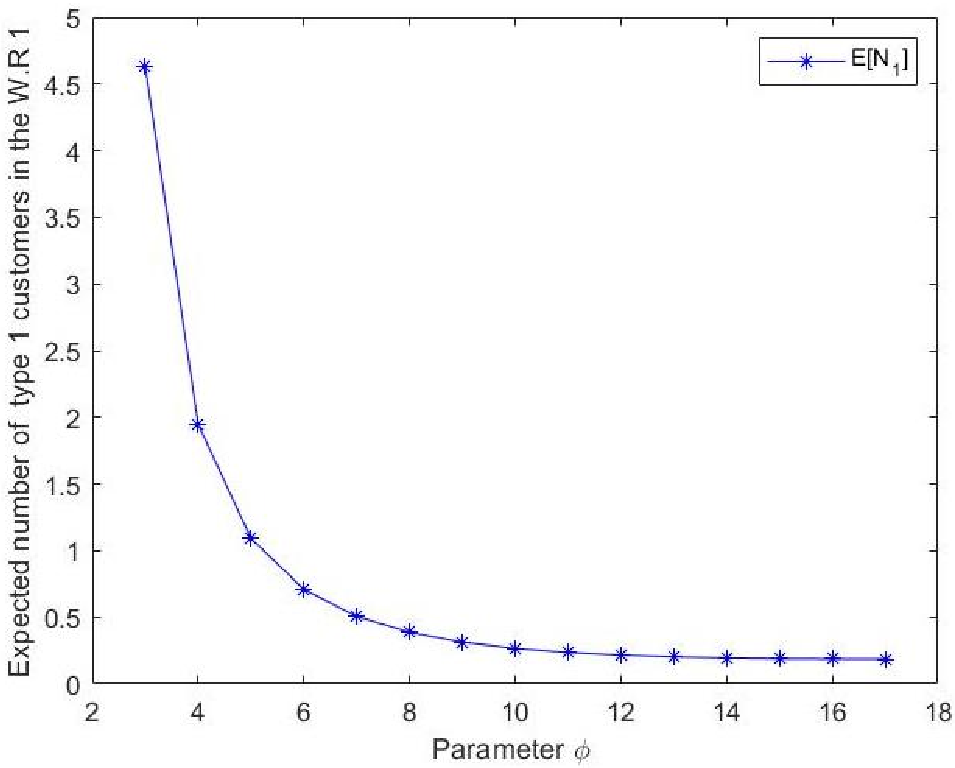

Figure 2. Effect of ϕ on expected number of type 1 customers in waiting room 1.

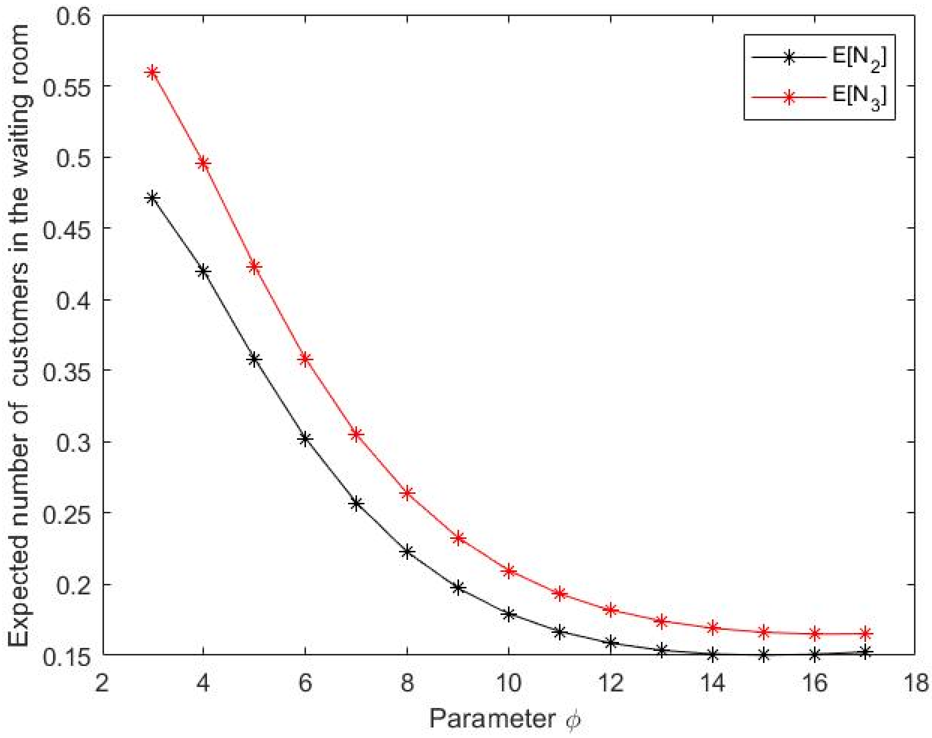

Figure 3. Effect of ϕ on expected number of customers in the waiting room.

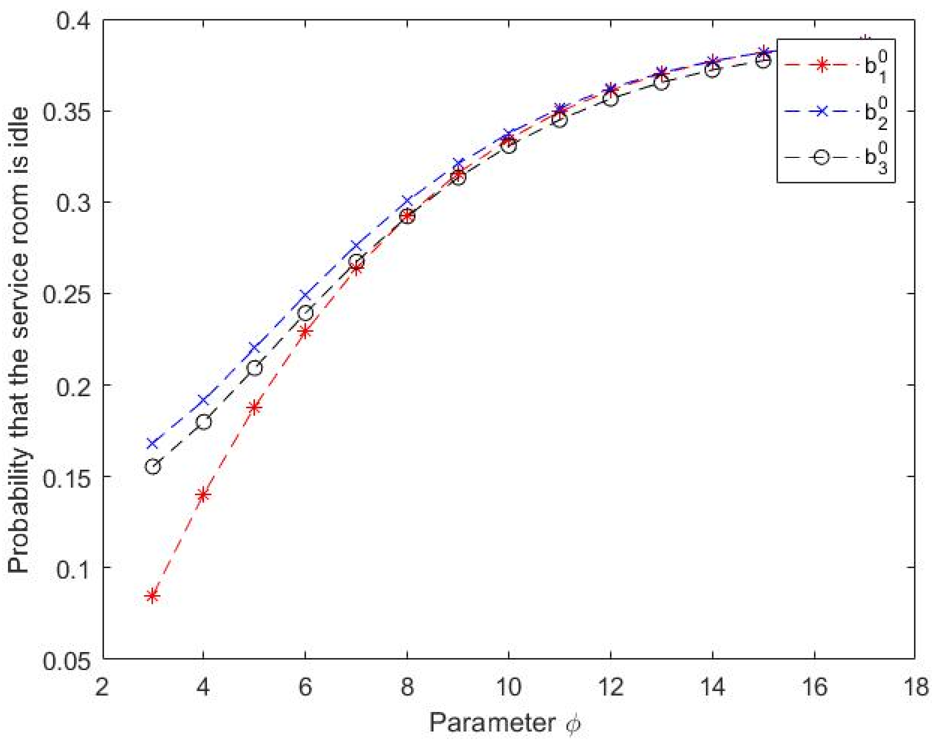

Figure 4. Effect of ϕ on probability that the service room is idle.

Table 6. Effect of ϕ on various performance measures ($L = 3$).

$L = 3$ ϕ	$E[N_1]$	$E[N_2]$	$E[N_3]$	b_1^0	b_1^1	b_2^0	b_2^1	b_3^0	b_3^1
3	4.6332	0.4708	0.5592	0.0851	0.9149	0.1684	0.8316	0.1555	0.8445
4	1.9472	0.4197	0.4954	0.1406	0.8594	0.1916	0.8084	0.1799	0.8201
5	1.0924	0.3581	0.4232	0.1882	0.8118	0.2201	0.7799	0.2093	0.7907
6	0.7093	0.3023	0.3581	0.2291	0.7709	0.2493	0.7507	0.2393	0.7607
7	0.5060	0.2572	0.3050	0.2637	0.7363	0.2765	0.7235	0.2673	0.7327
8	0.3873	0.2227	0.2637	0.2923	0.7077	0.3005	0.6995	0.2921	0.7079
9	0.3137	0.1974	0.2327	0.3157	0.6843	0.3209	0.6791	0.3132	0.6868
10	0.2665	0.1793	0.2099	0.3345	0.6655	0.3377	0.6623	0.3308	0.6692
11	0.2356	0.1668	0.1934	0.3494	0.6506	0.3514	0.6486	0.3450	0.6550
12	0.2154	0.1586	0.1819	0.3611	0.6389	0.3623	0.6377	0.3565	0.6435
13	0.2023	0.1536	0.1741	0.3701	0.6299	0.3707	0.6293	0.3654	0.6346
14	0.1941	0.1510	0.1691	0.3769	0.6231	0.3771	0.6229	0.3722	0.6278
15	0.1893	0.1502	0.1662	0.3818	0.6182	0.3818	0.6182	0.3773	0.6227
16	0.1871	0.1508	0.1650	0.3852	0.6148	0.3850	0.6150	0.3809	0.6191
17	0.1867	0.1524	0.1651	0.3873	0.6127	0.3871	0.6129	0.3832	0.6168

Table 7. Effect of ϕ on various performance measures and ETC per unit time ($L = 3$).

$L = 3$ ϕ	$E[N_h]$	$E[I_1]$	$E[I_2]$	$E[I_3]$	$E[P]$	$E[U]$	$ETC_{(L=3)}$
3	0.3907	2.0589	1.5158	1.6081	9.9149	8.0851	127.2037
4	0.4244	1.8016	1.4848	1.5689	9.4780	8.5220	102.6960
5	0.4639	1.6075	1.4144	1.4921	8.9127	9.0873	97.1599
6	0.5067	1.4511	1.3284	1.4003	8.3087	9.6913	96.9178
7	0.5507	1.3212	1.2397	1.3061	7.7191	10.2809	98.8133
8	0.5946	1.2117	1.1551	1.2164	7.1710	10.8290	101.6713
9	0.6374	1.1185	1.0775	1.1340	6.6753	11.3247	104.9368
10	0.6786	1.0387	1.0076	1.0597	6.2330	11.7670	108.3244
11	0.7180	0.9698	0.9453	0.9935	5.8410	12.1590	111.6826
12	0.7554	0.9101	0.8899	0.9347	5.4940	12.5060	114.9366
13	0.7908	0.8579	0.8407	0.8824	5.1865	12.8135	118.0396
14	0.8242	0.8119	0.7968	0.8358	4.9130	13.0870	120.9761
15	0.8557	0.7712	0.7576	0.7941	4.6688	13.3312	123.7429
16	0.8855	0.7348	0.7224	0.7567	4.4499	13.5501	126.3522
17	0.9136	0.7023	0.6906	0.7230	4.2525	13.7475	128.8058

Table 8. Effect of ϕ on various performance measures ($L = 4$).

$L = 4$ ϕ	$E[N_1]$	$E[N_2]$	$E[N_3]$	b_1^0	b_1^1	b_2^0	b_2^1	b_3^0	b_3^1
4	10.7258	0.4823	0.5823	0.0454	0.9546	0.1537	0.8463	0.1398	0.8602
5	3.9734	0.4703	0.5589	0.0866	0.9134	0.1607	0.8393	0.1482	0.8518
6	2.1304	0.4370	0.5159	0.1218	0.8782	0.1730	0.8270	0.1616	0.8384
7	1.3526	0.3971	0.4674	0.1522	0.8478	0.1881	0.8119	0.1776	0.8224
8	0.9521	0.3584	0.4209	0.1786	0.8214	0.2041	0.7959	0.1944	0.8056
9	0.7204	0.3244	0.3799	0.2014	0.7986	0.2197	0.7803	0.2108	0.7892
10	0.5759	0.2962	0.3452	0.2209	0.7791	0.2343	0.7657	0.2261	0.7739
11	0.4810	0.2735	0.3169	0.2374	0.7626	0.2473	0.7527	0.2399	0.7601
12	0.4164	0.2557	0.2942	0.2513	0.7487	0.2587	0.7413	0.2519	0.7481
13	0.3713	0.2422	0.2763	0.2629	0.7371	0.2685	0.7315	0.2623	0.7377
14	0.3392	0.2321	0.2624	0.2724	0.7276	0.2768	0.7232	0.2712	0.7288
15	0.3162	0.2247	0.2517	0.2802	0.7198	0.2837	0.7163	0.2785	0.7215
16	0.2996	0.2195	0.2437	0.2866	0.7134	0.2893	0.7107	0.2846	0.7154
17	0.2877	0.2162	0.2377	0.2916	0.7084	0.2938	0.7062	0.2895	0.7105
18	0.2792	0.2142	0.2335	0.2956	0.7044	0.2974	0.7026	0.2935	0.7065

Table 9. Effect of ϕ on various performance measures and ETC per unit time ($L = 4$).

$L = 4$ ϕ	$E[N_h]$	$E[I_1]$	$E[I_2]$	$E[I_3]$	$E[P]$	$E[U]$	$ETC_{(L=4)}$
4	0.3746	2.2301	1.4840	1.5856	9.9549	8.0451	187.4106
5	0.3933	1.9801	1.4898	1.5821	9.7058	8.2942	121.6603
6	0.4166	1.7851	1.4564	1.5415	9.3224	8.6776	105.2528
7	0.4431	1.6260	1.4003	1.4793	8.8644	9.1356	99.8471
8	0.4717	1.4921	1.3328	1.4063	8.3766	9.6234	98.5270
9	0.5016	1.3772	1.2615	1.3298	7.8897	10.1103	99.1191
10	0.5321	1.2774	1.1909	1.2544	7.4225	10.5775	100.7019
11	0.5626	1.1903	1.1235	1.1826	6.9852	11.0148	102.8096
12	0.5927	1.1136	1.0607	1.1157	6.5820	11.4180	105.1817
13	0.6223	1.0460	1.0028	1.0540	6.2137	11.7863	107.6789
14	0.6511	0.9859	0.9499	0.9977	5.8787	12.1213	110.2018
15	0.6791	0.9325	0.9016	0.9463	5.5746	12.4254	112.6978
16	0.7062	0.8846	0.8576	0.8995	5.2985	12.7015	115.1370
17	0.7323	0.8416	0.8176	0.8570	5.0476	12.9524	117.4941
18	0.7575	0.8027	0.7810	0.8181	4.8192	13.1808	119.7667

5.2. The Effect of λ_j on Performance Measures Where $j = 1, 2, 3$

To analyze the effect of parameter λ_1 on various performance measures, we use Table 10 and Figures 5–7 as follows:

1. From Figure 5, we can notice that the expected number of type 1 customers in waiting room 1 increases as λ_1 increases.
2. However, from Figure 6, we can see that the expected number of type j customers in waiting room j decreases as λ_1 increases where $j = 2, 3$.
3. Moreover, we can see from Figure 7 that the probability that service room 1 is busy increases as λ_1 increases but the probability that service room $j, j = 2, 3$ is busy decreases as λ_1 increases.

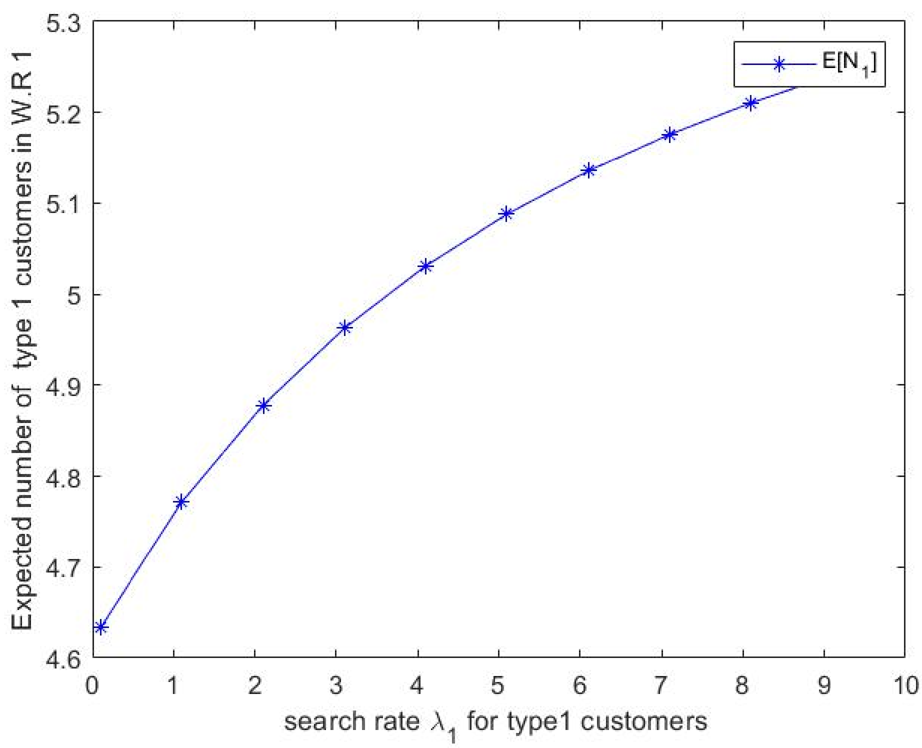

Figure 5. Effect of λ_1 on expected number of type 1 customers in waiting room 1.

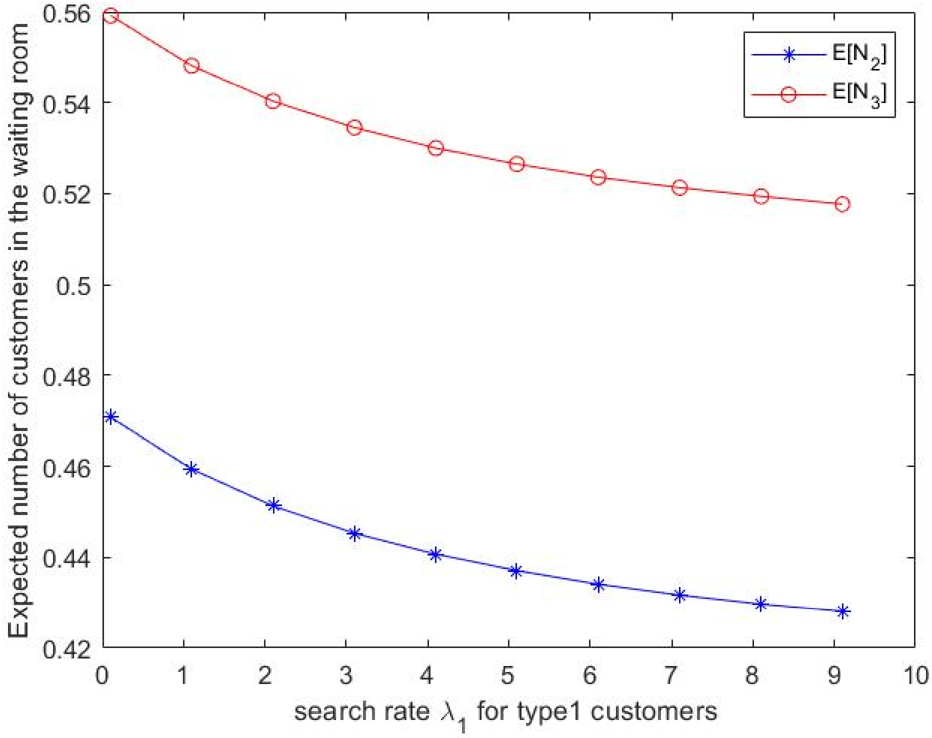

Figure 6. Effect of λ_1 on expected number of type 2, 3 customers in waiting room 2, 3.

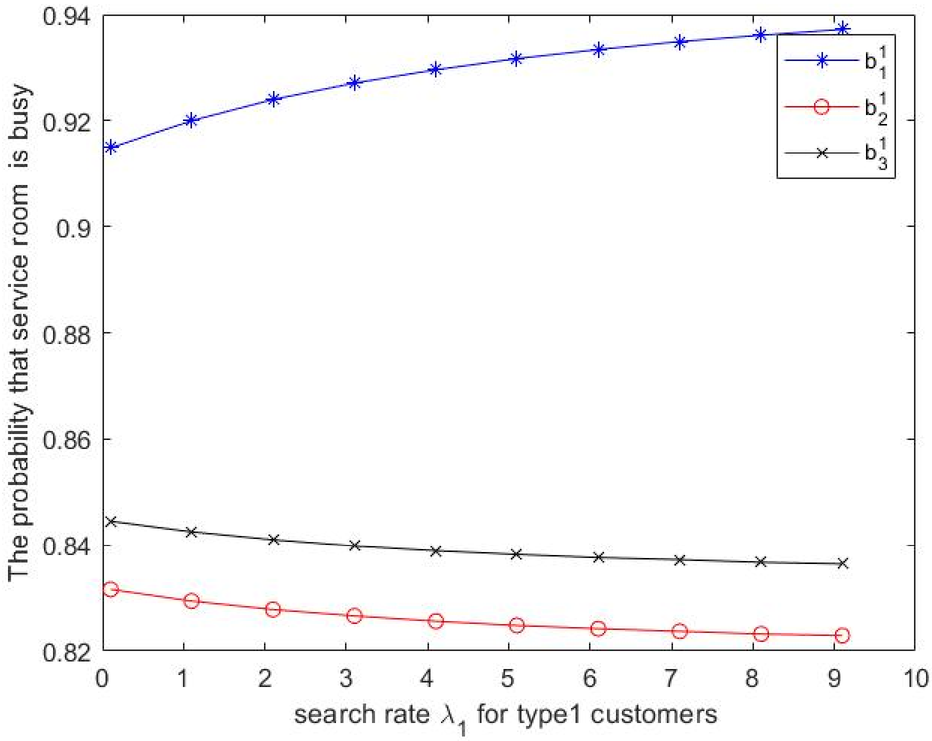

Figure 7. Effect of λ_1 on probability that the service room is idle.

Table 10. Effect of λ_1 on various performance measures.

λ_1	$E[N_1]$	$E[N_2]$	$E[N_3]$	b_1^0	b_1^1	b_2^0	b_2^1	b_3^0	b_3^1
0.1	4.6332	0.4708	0.5592	0.0851	0.9149	0.1684	0.8316	0.1555	0.8445
1.1	4.7715	0.4594	0.5482	0.0800	0.9200	0.1706	0.8294	0.1575	0.8425
2.1	4.8781	0.4513	0.5403	0.0760	0.9240	0.1722	0.8278	0.1590	0.8410
3.1	4.9627	0.4453	0.5345	0.0729	0.9271	0.1734	0.8266	0.1602	0.8398
4.1	5.0312	0.4407	0.5300	0.0704	0.9296	0.1744	0.8256	0.1611	0.8389
5.5	5.0878	0.4371	0.5265	0.0683	0.9317	0.1752	0.8248	0.1618	0.8382
6.1	5.1354	0.4341	0.5236	0.0666	0.9334	0.1758	0.8242	0.1624	0.8376
7.1	5.1759	0.4317	0.5213	0.0651	0.9349	0.1763	0.8237	0.1628	0.8372
8.1	5.2107	0.4297	0.5194	0.0639	0.9361	0.1768	0.8232	0.1633	0.8367
9.1	5.2411	0.4281	0.5177	0.0628	0.9372	0.1771	0.8229	0.1636	0.8364

To study the effect of parameter λ_2 on various performance measures, we use Table 11 and Figures 8–10 as following:

1. From Figure 8, we can notice that the expected number of type 1 customers in the waiting room 1 decreases as λ_2 increases.
2. We can notice from Figure 9 that the expected number of type 3 customers in the waiting room 3 decreases as λ_2 increases. However, the expected number of type 2 customers in the waiting room 2 increases when λ_2 increases.
3. Moreover, we can see from Figure 10 that the probability that service room 2 is busy increases as λ_2 increases but the probability that service room $j, j = 1, 3$ is busy decreases as λ_2 increases.

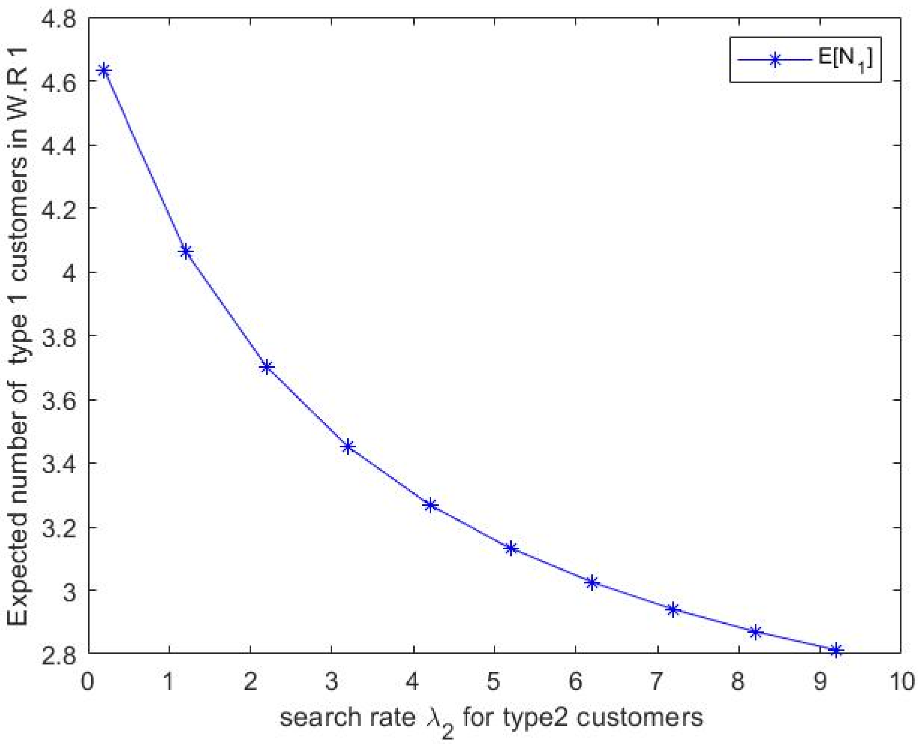

Figure 8. Effect of λ_2 on expected number of type 1 customers in the waiting room 1.

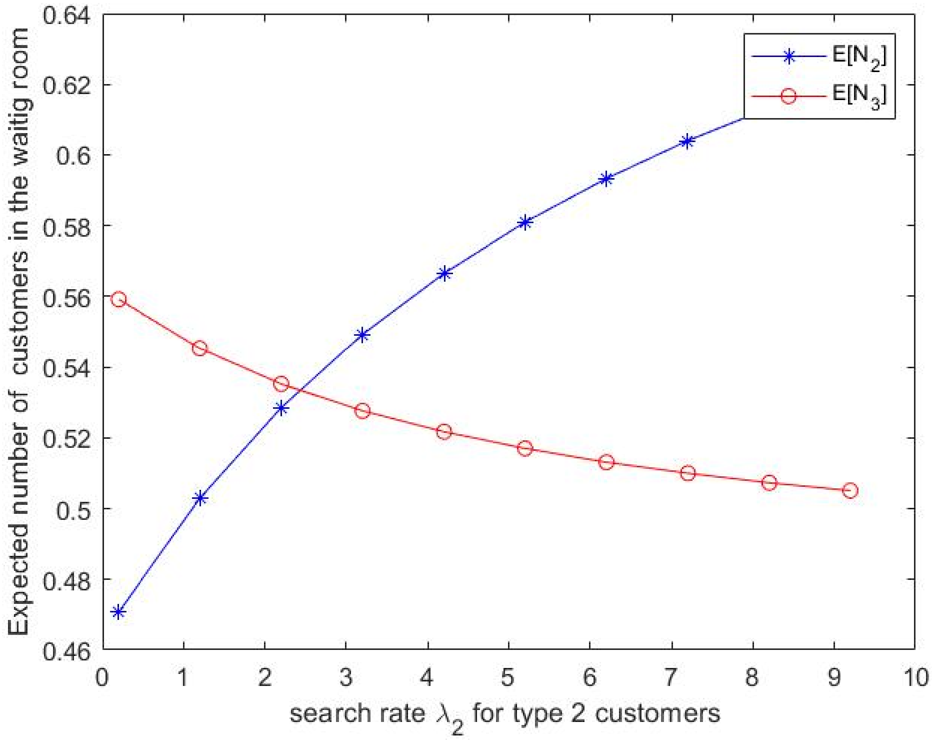

Figure 9. Effect of λ_2 on expected number of type 2, 3 customers in the waiting room 2, 3.

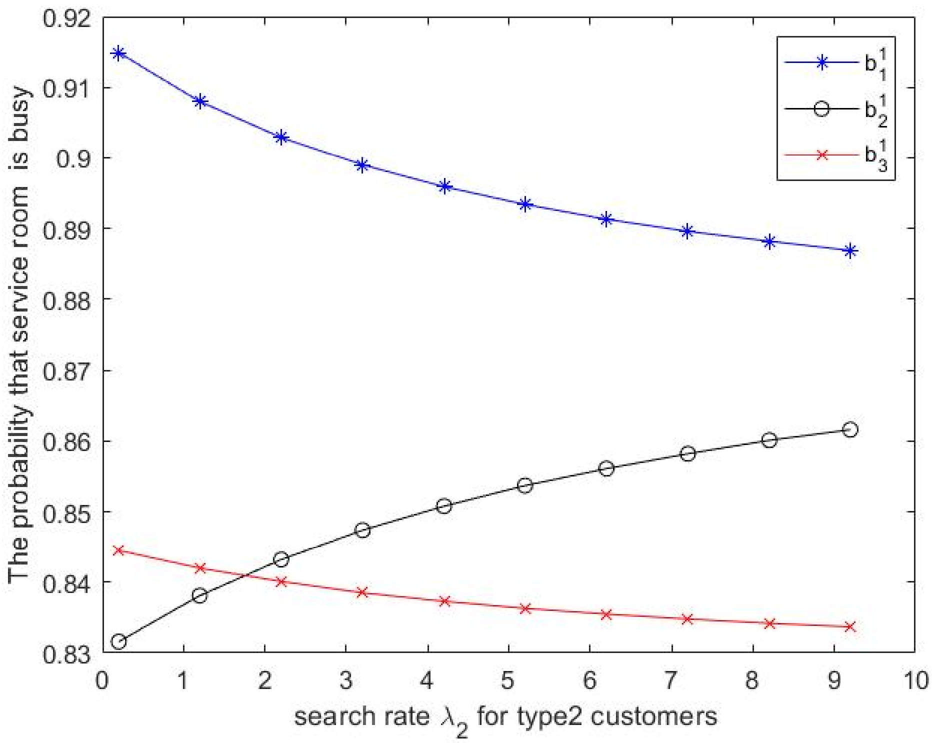

Figure 10. Effect of λ_2 on probability that the service room is idle.

Table 11. Effect of λ_2 on various performance measures.

λ_2	$E[N_1]$	$E[N_2]$	$E[N_3]$	b_1^0	b_1^1	b_2^0	b_2^1	b_3^0	b_3^1
0.2	4.6332	0.4708	0.5592	0.0851	0.9149	0.1684	0.8316	0.1555	0.8445
1.2	4.0646	0.5029	0.5453	0.0920	0.9080	0.1619	0.8381	0.1580	0.8420
2.2	3.7008	0.5285	0.5352	0.0971	0.9029	0.1568	0.8432	0.1599	0.8401
3.2	3.4504	0.5493	0.5276	0.1010	0.8990	0.1526	0.8474	0.1615	0.8385
4.2	3.2688	0.5665	0.5217	0.1041	0.8959	0.1492	0.8508	0.1627	0.8373
5.2	3.1317	0.5810	0.5170	0.1066	0.8934	0.1463	0.8537	0.1637	0.8363
6.2	3.0249	0.5933	0.5131	0.1087	0.8913	0.1439	0.8561	0.1645	0.8355
7.2	2.9396	0.6040	0.5100	0.1104	0.8896	0.1418	0.8582	0.1652	0.8348
8.2	2.8701	0.6132	0.5073	0.1118	0.8882	0.1399	0.8601	0.1658	0.8342
9.2	2.8123	0.6213	0.5051	0.1131	0.8869	0.1384	0.8616	0.1663	0.8337

From Table 12 and Figures 11–13, we can notice the following:

1. From Figure 11, we can notice that the expected number of type 1 customers in waiting room 3 decreases as λ_3 increases.
2. We can notice from Figure 12 that the expected number of type 2 customers in waiting room 2 decreases as λ_3 increases. However, the expected number of type 3 customers in waiting room 3 increases when λ_3 increases.
3. Moreover, we can see from Figure 13 that the probability that service room 3 is busy increases as λ_3 increases but the probability that service room $j, j = 1, 2$ is busy decreases as λ_3 increases.

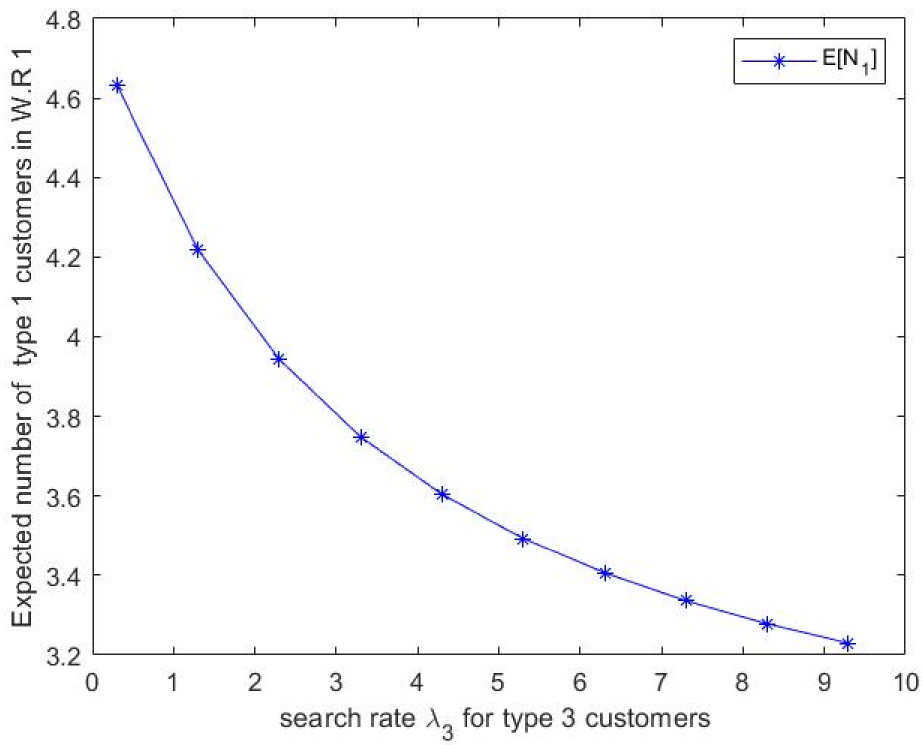

Figure 11. Effect of λ_3 on expected number of type 1 customers in waiting room 1.

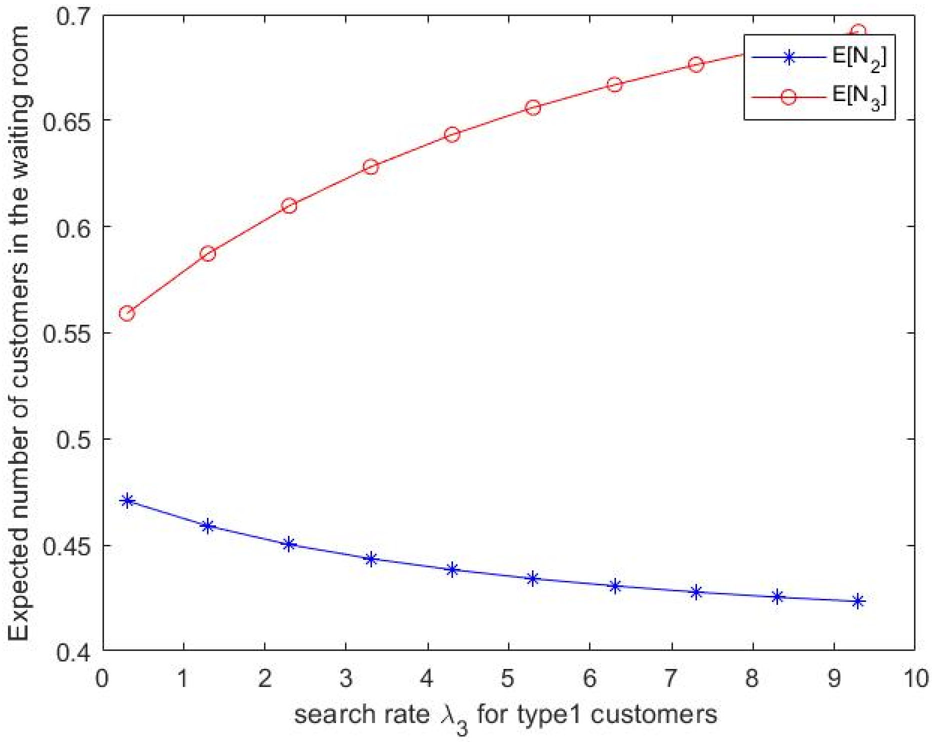

Figure 12. Effect of λ_3 on expected number of type $2,3$ customers in waiting room $2,3$.

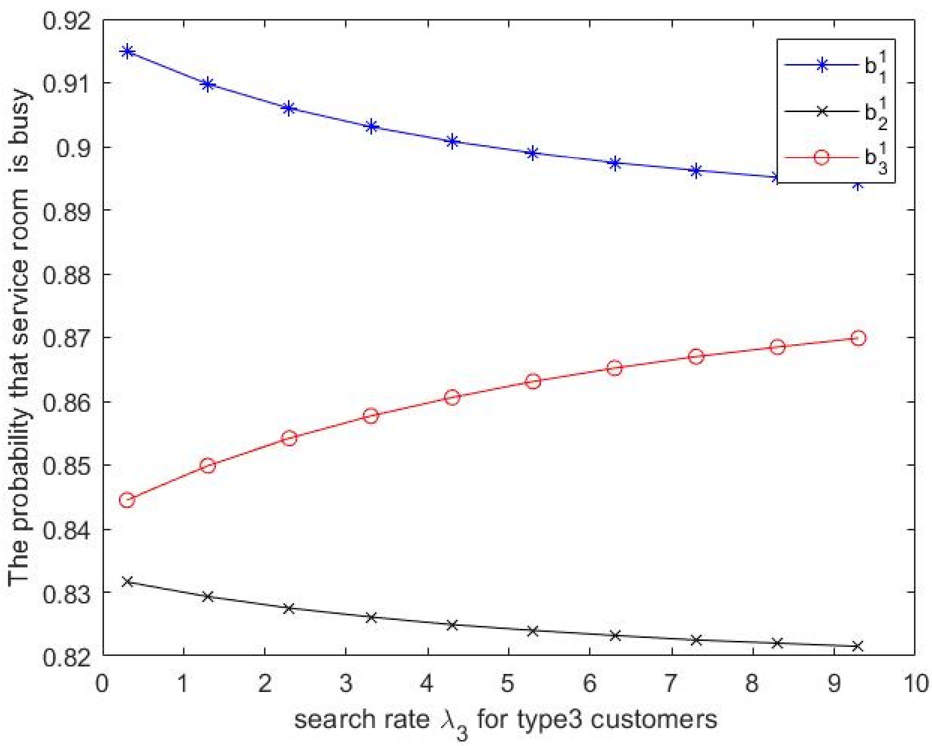

Figure 13. Effect of λ_3 on probability that the service room is idle.

Table 12. Effect of λ_3 on various performance measures.

λ_3	$E[N_1]$	$E[N_2]$	$E[N_3]$	b_1^0	b_1^1	b_2^0	b_2^1	b_3^0	b_3^1
0.3	4.6332	0.4708	0.5592	0.0851	0.9149	0.1684	0.8316	0.1555	0.8445
1.3	4.2186	0.4590	0.5873	0.0902	0.9098	0.1707	0.8293	0.1501	0.8499
2.3	3.9431	0.4502	0.6098	0.0940	0.9060	0.1725	0.8275	0.1458	0.8542
3.3	3.7483	0.4436	0.6281	0.0969	0.9031	0.1739	0.8261	0.1423	0.8577
4.3	3.6041	0.4384	0.6433	0.0992	0.9008	0.1751	0.8249	0.1394	0.8606
5.3	3.4936	0.4342	0.6560	0.1010	0.8990	0.1760	0.8240	0.1369	0.8631
6.3	3.4064	0.4308	0.6669	0.1026	0.8974	0.1768	0.8232	0.1348	0.8652
7.3	3.3359	0.4279	0.6764	0.1038	0.8962	0.1775	0.8225	0.1330	0.8670
8.3	3.2780	0.4255	0.6846	0.1049	0.8951	0.1780	0.8220	0.1315	0.8685
9.3	3.2296	0.4235	0.6918	0.1058	0.8942	0.1785	0.8215	0.1301	0.8699

6. Cost Analysis

Based on the above performance measures, we define the expected total cost (ETC) per unit time as

$$ETC = \sum_{j=1}^{k} V_{1j}E[N_j] + \sum_{j=1}^{k} V_{2j}b_j^{0*} + V_3E[N_h] + V_4E[U] \tag{7}$$

where

- $E[N_j]$ = Expected number of type j customers in waiting room $j; j = 1, 2, \ldots k$.
- $E[N_h]$ = Expected number of vessels, which are in operation to drop off passengers.
- $E[U]$ = Expected number of unoccupied seats in the vessel that just left.
- b_j^{0*} = Fraction of time seats for type j customers remain unoccupied, where $b_j^{0*} = \frac{b_j^0}{b_j^0+(1-b_j^1)}; j = 1, 2, \ldots k$.
- V_{1j} = Holding cost/customer of type j/unit time in waiting room $j; j = 1, 2, \ldots k$.
- V_{2j} = Idle time cost of service room $j; j = 1, 2, \ldots k$..
- V_3 = Operating cost per vessel.
- V_4 = Cost per unoccupied seat.

We fix $k = 3$, $V_{11} = \$10; V_{12} = \$15; V_{13} = \$20; V_{21} = \$3; V_{22} = \$6;$ $V_{23} = \$9; V_3 = \$50$ and $V_4 = \$5$.

Then, we compute $ETC_{L=3}$ and $ETC_{L=4}$ when $L = 3$ and $L = 4$, as mentioned in Table 7 and Table 9, respectively. From Tables 6–9 and Figure 14, we can realize that the optimal value of the expected total cost $(ETC_{L=3})$ per unit time is $ETC_{L=3} = \$96.9178$ when $\phi = 6$ and the optimal value of the expected total cost $(ETC_{L=4})$ per unit time is $ETC_{L=4} = \$98.5270$ when $\phi = 8$ for these two examples.

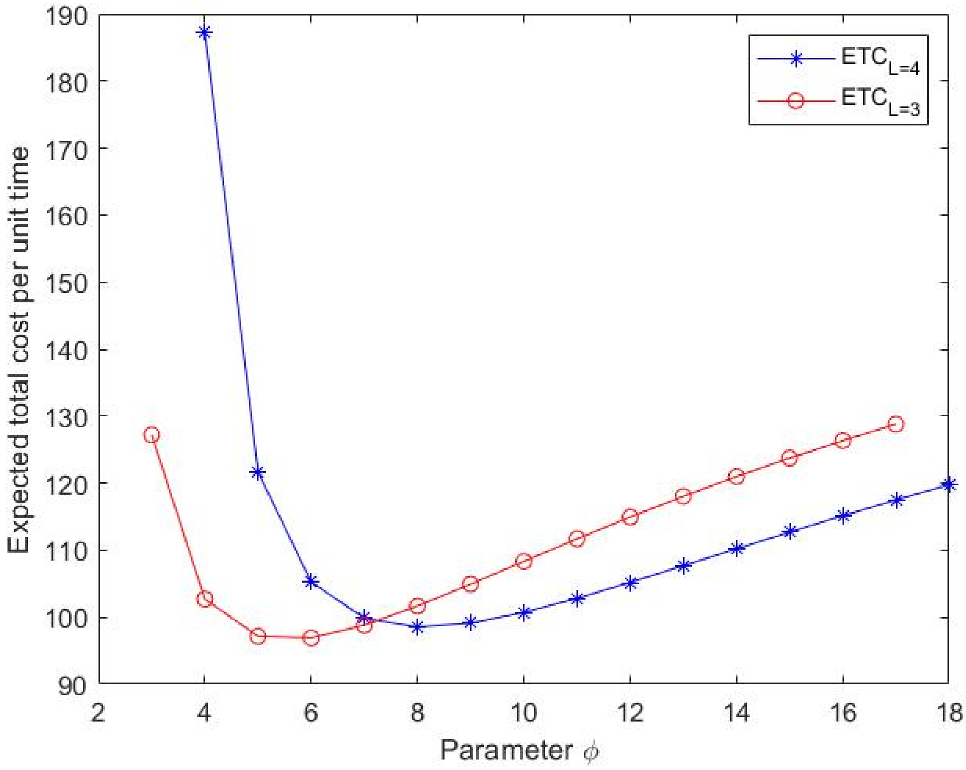

Figure 14. Effect of ϕ on the expected total cost (ETC).

7. Conclusions

In this paper we study the batch Marked Markovian arrival process (BMMAP) of customers to a transport station. Under steady state condition, various performance measures are estimated. We study the effect of parameter ϕ of Erlang distribution and search rate for customers on performance measures. Besides this, we compute the expected total cost per unit time.

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Appendix A

We fix $k = 3, C_1 = 3; L_1 = 2$ and $L = 3$ and we get the following matrices:

$$Q = \begin{pmatrix} B_{00} & B_{01} & & & & \\ B_{10} & A_1 & A_0 & & & \\ & A_2 & A_1 & A_0 & & \\ & & A_2 & A_1 & A_0 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}; \text{where}$$

$$B_{00} = \begin{pmatrix} E_{00} & E_{01} & E_{02} \\ E_{10} & E_{11} & E_{12} \\ E_{20} & O & E_{22} \end{pmatrix}; B_{01} = \begin{pmatrix} E_{03} & O & O \\ E_{13} & O & O \\ E_{23} & O & O \end{pmatrix}; B_{10} = \begin{pmatrix} E_{30} & O & O \\ E_{40} & O & O \\ E_{50} & O & O \end{pmatrix};$$

$$A_1 = \begin{pmatrix} E_{33} & E_{34} & O \\ O & E_{33} & E_{34} \\ O & O & E_{33} \end{pmatrix}; A_0 = \begin{pmatrix} O & O & O \\ O & O & O \\ E_{34} & O & O \end{pmatrix}; A_2 = \begin{pmatrix} E_{63} & O & O \\ O & E_{63} & O \\ O & O & E_{63} \end{pmatrix}; \text{where}$$

$$E_{00} = \begin{pmatrix} S_{00} & \Upsilon_{01} & O & O & \dots & O \\ \Psi_{01} & S_{01} & \Upsilon_{01} & O & \dots & O \\ O & \Psi_{02} & S_{02} & \Upsilon_{01} & & O \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ O & O & O & \Psi_{0(h^*-1)} & S_{0(h^*-1)} & \Upsilon_{02} \\ O & O & O & O & \Psi_{0h^*} & S_{0h^*} \end{pmatrix};$$

$$E_{01} = \begin{pmatrix} \Lambda_{01} & O & O & \dots & O \\ O & \Lambda_{01} & O & \dots & O \\ \vdots & & \ddots & & \vdots \\ O & O & \dots & O & \Lambda_{02} \end{pmatrix}; E_{02} = \begin{pmatrix} \Lambda_{03} & O & O & \dots & O \\ O & \Lambda_{03} & O & \dots & O \\ \vdots & & \ddots & & \vdots \\ O & O & \dots & O & \Lambda_{04} \end{pmatrix};$$

$$E_{03} = \begin{pmatrix} \Lambda_{05} & O & O & \dots & O \\ O & \Lambda_{05} & O & \dots & O \\ \vdots & & \ddots & & \vdots \\ O & O & \dots & O & \Lambda_{06} \end{pmatrix};$$

$$E_{11} = \begin{pmatrix} S_{10} & \Upsilon_{11} & O & O & \dots & O \\ \Psi_{11} & S_{11} & \Upsilon_{11} & O & \dots & O \\ O & \Psi_{12} & S_{12} & \Upsilon_{11} & & O \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ O & O & O & \Psi_{1(h^*-1)} & S_{1(h^*-1)} & \Upsilon_{12} \\ O & O & O & O & \Psi_{1h^*} & S_{1h^*} \end{pmatrix};$$

$$E_{12} = \begin{pmatrix} \Lambda_{11} & O & O & \dots & O \\ O & \Lambda_{11} & O & \dots & O \\ \vdots & & \ddots & & \vdots \\ O & O & \dots & O & \Lambda_{12} \end{pmatrix};$$

$$E_{13} = \begin{pmatrix} \Lambda_{13} & O & O & \dots & O \\ O & \Lambda_{13} & O & \dots & O \\ \vdots & & \ddots & & \vdots \\ O & O & \dots & O & \Lambda_{14} \end{pmatrix}; E_{10} = \begin{pmatrix} \Gamma_{10} & \delta_{10} & O & O & \dots & O \\ O & \Gamma_{10} & \delta_{10} & O & \dots & O \\ \vdots & & \ddots & & \ddots & \vdots \\ O & O & \dots & O & \Gamma_{11} & O \\ O & O & \dots & O & \Gamma_{12} & O \end{pmatrix};$$

$$E_{22} = \begin{pmatrix} S_{20} & \Upsilon_{21} & O & O & \dots & O \\ \Psi_{21} & S_{21} & \Upsilon_{21} & O & \dots & O \\ O & \Psi_{22} & S_{22} & \Upsilon_{21} & & O \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ O & O & O & \Psi_{2(h^*-1)} & S_{2(h^*-1)} & \Upsilon_{22} \\ O & O & O & O & \Psi_{2h^*} & S_{2h^*} \end{pmatrix};$$

$$E_{20}=\begin{pmatrix}\Gamma_{20}&\delta_{20}&O&O&\dots&O\\O&\Gamma_{20}&\delta_{20}&O&\dots&O\\\vdots&&\ddots&&\ddots&\vdots\\O&O&\dots&O&\Gamma_{21}&O\\O&O&\dots&O&\Gamma_{22}&O\end{pmatrix};E_{23}=\begin{pmatrix}\Lambda_{21}&O&O&\dots&O\\O&\Lambda_{21}&O&\dots&O\\\vdots&&\ddots&&\vdots\\O&O&\dots&O&\Lambda_{22}\end{pmatrix};$$

$$E_{33}=\begin{pmatrix}S_{30}&\Upsilon_{31}&O&O&\dots&O\\\Psi_{31}&S_{31}&\Upsilon_{31}&O&\dots&O\\O&\Psi_{32}&S_{32}&\Upsilon_{31}&&O\\\vdots&&\ddots&\ddots&\ddots&\vdots\\O&O&O&\Psi_{3(h^*-1)}&S_{3(h^*-1)}&\Upsilon_{32}\\O&O&O&O&\Psi_{3h^*}&S_{3h^*}\end{pmatrix};$$

$$E_{34}=\begin{pmatrix}\Lambda_{31}&O&O&\dots&O\\O&\Lambda_{31}&O&\dots&O\\\vdots&&\ddots&&\vdots\\O&O&\dots&O&\Lambda_{32}\end{pmatrix};E_{30}=\begin{pmatrix}\Gamma_{30}&\delta_{30}&O&O&\dots&O\\O&\Gamma_{30}&\delta_{30}&O&\dots&O\\\vdots&&\ddots&&\ddots&\vdots\\O&O&\dots&O&\Gamma_{31}&O\\O&O&\dots&O&\Gamma_{32}&O\end{pmatrix};$$

$$E_{40}=\begin{pmatrix}\Gamma_{40}&\delta_{40}&O&O&\dots&O\\O&\Gamma_{40}&\delta_{40}&O&\dots&O\\\vdots&&\ddots&&\ddots&\vdots\\O&O&\dots&O&\Gamma_{41}&O\\O&O&\dots&O&\Gamma_{42}&O\end{pmatrix};E_{50}=\begin{pmatrix}\Gamma_{50}&\delta_{50}&O&O&\dots&O\\O&\Gamma_{50}&\delta_{50}&O&\dots&O\\\vdots&&\ddots&&\ddots&\vdots\\O&O&\dots&O&\Gamma_{51}&O\\O&O&\dots&O&\Gamma_{52}&\Gamma_{53}\end{pmatrix};$$

$$E_{63}=\begin{pmatrix}\Gamma_{60}&\delta_{60}&O&O&\dots&O\\O&\Gamma_{60}&\delta_{60}&O&\dots&O\\\vdots&&\ddots&&\ddots&\vdots\\O&O&\dots&O&\Gamma_{61}&O\\O&O&\dots&O&\Gamma_{62}&\Gamma_{63}\end{pmatrix};S_{00}=\begin{pmatrix}\gamma_7&\gamma_8&O&O\\O&\gamma_{27}&\gamma_{28}&O\\O&O&\gamma_{29}&\gamma_{28}\\O&O&O&\gamma_{29}\end{pmatrix}\text{where}$$

$$\gamma_7=\begin{pmatrix}\gamma_0\\\gamma_1+\gamma_3+\gamma_5+\gamma_6\end{pmatrix}\text{where }\gamma_0=\begin{pmatrix}D_0-\theta I_{(m_1)}&\theta I_{(m_1)}&O\end{pmatrix};$$

$$\gamma_1=\begin{pmatrix}O&I_{((C_3+1)(C_2+1)(C_1+1)-1)}\otimes(D_0-\phi I_{(m_1)})\end{pmatrix};$$

$$\gamma_2=\begin{pmatrix}I_{(C_3)}\otimes D_3&O\\O&O\end{pmatrix};\gamma_3=\begin{pmatrix}O&I_{((C_2+1)(C_1+1)-1)}\otimes\gamma_2&O\\O&O&I_{(C_3-1)}\otimes D_3\\O&O&O\end{pmatrix};$$

$$\gamma_4=\begin{pmatrix}I_{((C_3+1)C_2)}\otimes D_2&O\\O&O\end{pmatrix};\gamma_5=\begin{pmatrix}O&I_{(C_1)}\otimes\gamma_4&O\\O&O&I_{((C_3+1)C_2-1)}\otimes D_2\\O&O&O\end{pmatrix};$$

$$\gamma_6=\begin{pmatrix}O&I_{((C_3+1)(C_2+1)C_1-1)}\otimes D_1\\O&O\end{pmatrix}\gamma_8=\begin{pmatrix}O\\\phi I_{((C_3+1)(C_2+1)(C_1+1)m_1-m_1)}\end{pmatrix};$$

$$\gamma_{27}=\gamma_{26}+\gamma_{11}+\gamma_{16}+\gamma_{18}+\gamma_{19}+\gamma_{20}+\gamma_{21}\text{ where}$$

$$\gamma_{26}=\gamma_{22}+\gamma_{23}+\gamma_{24}+\gamma_{25};$$

$$\gamma_{22}=\begin{pmatrix}I_{((C_3+1)(C_2+1)(C_1+1)-1)}\otimes(D_0-\phi I_{(m_1)})\end{pmatrix};$$

$$\gamma_{23}=\begin{pmatrix}O&I_{((C_2+1)(C_1+1)-1)}\otimes\gamma_2&O\\O&O&I_{(C_3-1)}\otimes D_3\\O&O&O\end{pmatrix};$$

$$\gamma_{24}=\begin{pmatrix}O&I_{(C_1)}\otimes\gamma_4&O\\O&O&I_{((C_3+1)C_2-1)}\otimes D_2\\O&O&O\end{pmatrix};$$

$$\gamma_{25}=\begin{pmatrix}O&I_{((C_3+1)(C_2+1)C_1-1)}\otimes D_1\\O&O\end{pmatrix};\gamma_{11}=\begin{pmatrix}I_{((C_2+1)(C_1+1)-1)}\otimes\gamma_{10}&O\\O&O\end{pmatrix};$$

$$\gamma_{10}=\begin{pmatrix}O&[0:(C_3-2)]^T\otimes\lambda_3 I_{(m_1)}\\O&O\end{pmatrix};$$

$$\gamma_{16}=\begin{pmatrix}\zeta_1\\\left([1:(C_2-2)]^T\otimes(I_{(C_3)}\otimes\lambda_2 I_{(m_1)})\right)\end{pmatrix};\zeta_1=\begin{pmatrix}O&I_{(C_1)}\otimes\gamma_{13}&O\end{pmatrix};$$

$\gamma_{13} = \begin{pmatrix} O & \gamma_{12} & O \\ O & O & O \end{pmatrix}$; $\gamma_{18} = \begin{pmatrix} O & O \\ O & [1:(C_1-2)]^T \otimes (I_{((C_3+1)(C_2+1)-1)} \otimes \lambda_1 I_{m_1}) \\ O & O \end{pmatrix}$;

$\gamma_{19} = diag((e_{1\times((C_2+1)(C_1+1)-1)} \otimes (O, -\lambda_3[1:(C_3-2)] \otimes e_{1\times m_1}, O), O, (-\lambda_3[1:(C_3-2)] \otimes e_{1\times m_1}), O)$;

$\gamma_{20} = diag((O, -\lambda_2[1:(C_2-2)] \otimes e_{1\times(C_3+1)m_1}, O), O, -\lambda_2[1:(C_2-2)] \otimes e_{1\times(C_3+1)m_1}, O)$;

$\gamma_{21} = diag(O, -\lambda_1[1:(C1-2)] \otimes e_{1\times(C_3+1)(C_2+1)m_1}, O)$;

$\gamma_{28} = \begin{pmatrix} \phi I_{((C3+1)(C_2+1)(C_1+1)m_1-m_1)} \end{pmatrix}$;

$\gamma_{29} = \gamma_{27} - diag\begin{pmatrix} \theta e_{1\times(C3+1)(C_2+1)(C_1+1)m_1-m_1} \end{pmatrix}$where

$S_{0h} = S_{00} - h\,\mu\, diag(e_{1\times(m_1+((C_3+1)(C_2+1)(C_1+1)m_1-m_1)m)})$ where $h = 1,2,\ldots,(h^*-2)$; $S_{0(h^*-1)} = S_{00} - h\,\mu\, diag(e_{1\times(m_1+((C_3+1)(C_2+1)(C_1+1)m_1-m_1)m)}) + \gamma_{48}$;

$S_{0h^*} = \begin{pmatrix} (D_0 - h^*\mu I_{(m_1)} - \theta I_{(m_1)}) \end{pmatrix}$;

$\Psi_{0h} = h\,\mu\, diag(e_{1\times(m_1+((C_3+1)(C_2+1)(C_1+1)m_1-m_1)m)})$ where $h = 1,2,\ldots,(h^*-1)$;

$\Psi_{0h^*} = \begin{pmatrix} h^*\mu I_{(m_1)} & \theta I_{(m_1)} & O \end{pmatrix}$;

$\gamma_{48} = \begin{pmatrix} O & O \\ \gamma_{35} & O \\ \gamma_{35} & O \end{pmatrix}$; $\Upsilon_{01} = \begin{pmatrix} \gamma_{31} & O \\ \gamma_{34} & O \\ \gamma_{36} & O \\ \gamma_{38} & O \end{pmatrix}$where; $\gamma_{31} = \begin{pmatrix} O & O \\ \gamma_{30} & O \end{pmatrix}$; $\gamma_{30} = \begin{pmatrix} O \\ D_1 \\ O \\ D_2 \\ O \\ D_3 \end{pmatrix}$; $\gamma_{34} = \gamma_{33} + \begin{pmatrix} \gamma_{30} & O \end{pmatrix}$where $\gamma_{33} = \begin{pmatrix} \gamma_{32} & O \end{pmatrix}$where

$\gamma_{32} = \begin{pmatrix} O \\ (1:(C_1-2))^T \otimes \zeta_2 \\ O \\ (1:(C_2-2))^T \otimes \zeta_3 \\ O \\ (1:(C_3-2))^T \otimes \lambda_3 I_{(m_1)} \\ O \end{pmatrix}$; $\zeta_2 = \begin{pmatrix} O \\ \lambda_1 I_{(m_1)} \end{pmatrix}$; $\zeta_3 = \begin{pmatrix} O \\ \lambda_2 I_{(m_1)} \end{pmatrix}$;

$\gamma_{36} = \gamma_{34} + \gamma_{35}$; $\gamma_{35} = \begin{pmatrix} O & e_{((C_3+1)(C_2+1)(C_1+1)-1)\times 1} \otimes \phi I_{(m_1)} & O \end{pmatrix}$;

$\gamma_{38} = \gamma_{36} + \gamma_{37}$; $\gamma_{37} = \begin{pmatrix} e_{((C_3+1)(C_2+1)(C_1+1)-1)\times 1} \otimes \phi I_{(m_1)} & O \end{pmatrix}$;

$\Upsilon_{02} = \begin{pmatrix} O \\ \gamma_{30} \\ \gamma_{32} + \gamma_{30} \\ \gamma_{32} + \gamma_{30} \\ \zeta_4 \end{pmatrix}$; $\zeta_4 = \gamma_{32} + \gamma_{30} + (e_{((C_3+1)(C_2+1)(C_1+1)-1)\times 1} \otimes \phi I_{(m_1)})$;

$\Lambda_{01} = \begin{pmatrix} D_3 & O & O \\ O & I_{(m)} \otimes \gamma_{40} & O \end{pmatrix}$where; $\gamma_{40} = \begin{pmatrix} I_{((C_2+1)(C_1+1)-1)} \otimes \zeta_5 \\ O \end{pmatrix}$;

$\zeta_5 = \begin{pmatrix} O \\ D_3 \end{pmatrix}$; $\Lambda_{02} = \begin{pmatrix} D_3 O \end{pmatrix}$; $\Lambda_{03} = \begin{pmatrix} D_2 & O & O \\ O & I_{(m)} \otimes \gamma_{43} & O \end{pmatrix}$;

$\gamma_{43} = \begin{pmatrix} I_{(C_1)} \otimes \zeta_6 & O \\ O & O \\ O & I_{(C_3)} \otimes D_2 \end{pmatrix}$; $\zeta_6 = \begin{pmatrix} O \\ (I_{(C_3+1)} \otimes D_2) \end{pmatrix}$;

$\Lambda_{04} = \begin{pmatrix} D_2 O \end{pmatrix}$; $\Lambda_{05} = \begin{pmatrix} D_1 & O & O & O \\ O & O & \gamma_{46} & O \end{pmatrix}$; $\gamma_{46} = \begin{pmatrix} I_{(m)} \otimes \zeta_7 \end{pmatrix}$;

$\zeta_7 = \begin{pmatrix} O \\ I_{((C_3+1)(C_2+1)-1)} \otimes D_1 \end{pmatrix}$; $\Lambda_{06} = \begin{pmatrix} D_1 O \end{pmatrix}$; $S_{10} = \begin{pmatrix} \gamma_{56} + \gamma_{57} & O & O \\ O & O & \gamma_{83} + \gamma_{85} \end{pmatrix}$;

$\gamma_{56} = \begin{pmatrix} I_{(W_3-1)} \otimes (-\theta I_{(m_1)} + D_0) & O \\ O & -\theta I_{(m_1)} + D_0 + diag(sum(D_3,2))^T \end{pmatrix}$;

$\gamma_{57} = \begin{pmatrix} O & I_{(W_3-1)} \otimes D_3 \\ O & O \end{pmatrix}$; $\gamma_{83} = \begin{pmatrix} \gamma_{81} & O \\ O & \gamma_{82} \end{pmatrix}$;

$\gamma_{82} = \gamma_{80} + (I_{(((C_2+1)(C_1+1)-1)m)} \otimes diag((sum(D_3,2))^T))$;

$\gamma_{81} = \begin{pmatrix} I_{(W_3-1)} \otimes \gamma_{80} \end{pmatrix}$; $\gamma_{80} = \gamma_{77} + \gamma_{79}$;

$$\gamma_{77} = \begin{pmatrix} \gamma_{63} & O & O \\ O & \gamma_{74} & O \\ O & O & I_{(m-2)} \otimes \gamma_{76} \end{pmatrix}; \gamma_{63} = \gamma_{60} + \gamma_{61} + \gamma_{62};$$
$$\gamma_{60} = \begin{pmatrix} O & I_{(C_1)} \otimes \gamma_{59} & O \\ O & O & I_{(C_2-1)} \otimes D_2 \\ O & O & O \end{pmatrix}; \gamma_{59} = \begin{pmatrix} I_{(C_2)} \otimes D_2 & O \\ O & O \end{pmatrix};$$
$$\gamma_{61} = \begin{pmatrix} O & I_{((C_1-1)(C_2+1))} \otimes D_1 & O \\ O & O & I_{(C_2)} \otimes D_1 \\ O & O & O \end{pmatrix};$$
$$\gamma_{62} = (I_{((C_2+1)(C_1+1)-1)} \otimes (D_0 - \phi I_{(m_1)})); \gamma_{74} = \gamma_{63} + \gamma_{66} + \gamma_{68} + \gamma_{71} + \gamma_{73};$$
$$\gamma_{66} = \begin{pmatrix} O & I_{(C_1)} \otimes \gamma_{65} & O \\ O & O & O \end{pmatrix}; \gamma_{65} = \begin{pmatrix} O & O & O \\ O & \gamma_{64} & O \\ O & O & O \end{pmatrix};$$
$$\gamma_{64} = ([1:(C_2-2)]^T \otimes \lambda_2 I_{(m_1)}); \gamma_{68} = \begin{pmatrix} O & O \\ O & \gamma_{67} \\ O & O \end{pmatrix} \text{where}$$
$$\gamma_{67} = diag([1:(C_1-2)] \otimes (e_{1\times C_2} \otimes \lambda_1 e_{1\times m_1}));$$
$$\gamma_{71} = diag((e_{1\times C_1} \otimes \gamma_{70}), O, \gamma_{69}, O);$$
$$\gamma_{70} = (O, \gamma_{69}, O);$$
$$\gamma_{69} = ([1:(C_2-2)] \otimes -\lambda_2 e_{1\times m_1});$$
$$\gamma_{73} = diag([O, \gamma_{72}, O]) + diag([O, ([1:(C_1-2)] \otimes (e_{1\times C_2} \otimes -\lambda_1 e_{1\times m_1})), O, O])$$
$$\gamma_{72} = ([1:(C_1-2)] \otimes [O_{1\times C_2 m_1}, -\lambda_1 e_{1\times m_1}]);$$
$$\gamma_{76} = \gamma_{74} + \gamma_{75}; \gamma_{75} = diag\begin{pmatrix} -\theta e_{1\times((C_2+1)(C_1+1)m_1-m_1)} \end{pmatrix};$$
$$\gamma_{79} = \begin{pmatrix} O & \gamma_{78} \\ O & O \end{pmatrix}; \gamma_{78} = I_{(m-1)} \otimes \begin{pmatrix} I_{((C_2+1)(C_1+1)-1)} \otimes \phi I_{(m_1)} \end{pmatrix};$$
$$\gamma_{85} = \begin{pmatrix} O & \gamma_{84} \\ O & O \end{pmatrix}; \gamma_{84} = \begin{pmatrix} I_{((C_2+1)(C_1+1)-1)(m)(W_3-1)} \otimes D_3 \end{pmatrix};$$
$$S_{1h} = S_{11} - h\,\mu\, diag(e_{1\times(W_3 m_1 + ((C_2+1)(C_1+1)m_1 - m_1)mW_3)}); \text{ where } h = 1, 2, \dots, (h^* - 1);$$
$$S_{1h^*} = \begin{pmatrix} I_{(W_3-1)} \otimes (D_0 - \theta I_{(m_1)} - h^* \mu I_{(m_1)}) & O \\ O & \gamma_* \end{pmatrix}$$
$$+ \begin{pmatrix} O & I_{(W_3-1)} \otimes D_3 \\ O & O \end{pmatrix} \text{where } \gamma_* = (D_0 + diag((sum(D_3,2))^T) - \theta I_{(m_1) - h^* \mu I_{(m_1)}});$$
$$\Psi_{1h} = h\,\mu\, diag(e_{1\times(W_3 m_1 + ((C_2+1)(C_1+1)m_1 - m_1)mW_3)}) \text{ where } h = 1, 2, \dots, (h^* - 1);$$
$$\Psi_{1h^*} = \begin{pmatrix} I_{(W_3)} \otimes h^* \mu I_{(m_1)} & O \end{pmatrix};$$
$$\Upsilon_{11} = \begin{pmatrix} O & O \\ \gamma_{98} & O \end{pmatrix} \text{where } \gamma_{98} = \begin{pmatrix} I_{(W_3)} \otimes \gamma_{97} \end{pmatrix} \text{where } \gamma_{97} = \gamma_{92} + \gamma_{94} + \gamma_{96} \text{ where}$$
$$\gamma_{96} = \begin{pmatrix} O \\ e_{(m-1)\times 1} \otimes \gamma_{95} \end{pmatrix} \text{where}$$
$$\gamma_{95} = \begin{pmatrix} O \\ [1:(C_1-2)]^T \otimes \zeta_8 \\ O \\ [1:(C_2-2)]^T \otimes \lambda_2 I_{(m_1)} \\ O \end{pmatrix}; \zeta_8 = \begin{pmatrix} O \\ \lambda_1 I_{(m_1)} \end{pmatrix};$$
$$\gamma_{94} = \begin{pmatrix} O \\ \gamma_{93} \end{pmatrix} \text{where } \gamma_{93} = \begin{pmatrix} e_{((C_2+1)(C_1+1)-1)\times 1} \otimes \phi I_{(m_1)} \end{pmatrix}; \gamma_{92} = \begin{pmatrix} e_{m\times 1} \otimes \gamma_{91} \end{pmatrix};$$
$$\gamma_{91} = \begin{pmatrix} O \\ D_1 \\ O \\ D_2 \end{pmatrix}; \Upsilon_{12} = \begin{pmatrix} O \\ \gamma_{98} \end{pmatrix}; \Lambda_{11} = \begin{pmatrix} O & I_{(W_3)} \otimes D_2 & O & O \\ O & O & \gamma_{102} & O \end{pmatrix} \text{where}$$
$$\gamma_{102} = \begin{pmatrix} I_{(W_3)} \otimes \begin{pmatrix} I_{(m)} \otimes \gamma_{101} \end{pmatrix} \end{pmatrix}; \gamma_{101} = \begin{pmatrix} I_{(C_1)} \otimes \zeta_9 \\ O \end{pmatrix}; \zeta_9 = \begin{pmatrix} O \\ D_2 \end{pmatrix};$$
$$\Lambda_{12} = \begin{pmatrix} O & I_{(W_3)} \otimes D_2 & O \end{pmatrix}; \Lambda_{13} = \begin{pmatrix} O & I_{(W_3)} \otimes D_1 & O & O & O \\ O & O & O & \gamma_{106} & O \end{pmatrix} \text{where}$$
$$\gamma_{106} = (I_{(W_3)} \otimes \gamma_{105}); \gamma_{105} = \begin{pmatrix} I_{(m)} \otimes \zeta_{10} \end{pmatrix}; \zeta_{10} = \begin{pmatrix} O \\ I_{(C_2)} \otimes D_1 \end{pmatrix};$$

$$\Lambda_{14} = \begin{pmatrix} O & (I_{(W_3)} \otimes D_1) & O \end{pmatrix}; \Gamma_{10} = \begin{pmatrix} O & I_{(W_3)} \otimes \theta I_{(m_1)} & O \\ O & O & O \end{pmatrix};$$
$$\delta_{10} = \begin{pmatrix} O & O & O \\ O & \gamma_{89} & O \end{pmatrix} \text{where } \gamma_{89} = \begin{pmatrix} I_{(W_3)} \otimes \zeta_{11} \end{pmatrix};$$
$$\zeta_{11} = \begin{pmatrix} O \\ e_{((C_2+1)(C_1+1)-1)(2)\times 1} \otimes \theta I_{(m_1)} \end{pmatrix};$$
$$\Gamma_{11} = \Gamma_{10} + \delta_{10}; \Gamma_{12} = \begin{pmatrix} O & I_{(W_3)} \otimes \theta I_{(m_1)} & O \end{pmatrix} \text{where } S_{20} = \gamma_{175} + \gamma_{125};$$
$$\gamma_{175} = \begin{pmatrix} \gamma_{127} & O & O & O & O & \dots & O \\ O & \gamma_{170} & \gamma_{173} & O & O & \dots & O \\ O & O & \gamma_{170} & \gamma_{173} & O & \dots & O \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ O & O & \dots & O & \gamma_{170} & \gamma_{173} & O \\ O & O & \dots & O & O & \gamma_{168} & \gamma_{169} \end{pmatrix} \text{where}$$
$$\gamma_{127} = \gamma_{120} + \gamma_{121} + \gamma_{122} + \gamma_{119}; \gamma_{120} = \begin{pmatrix} I_{((W_3+1)W_2)} \otimes (D_0 - \theta I_{(m_1)}) \end{pmatrix};$$
$$\gamma_{121} = diag(e_{1\times W_2} \otimes (O_{1\times C_3 m_1}, (sum(D_3, 2))^T));$$
$$\gamma_{122} = diag(O, (e_{1\times(C_3+1)} \otimes (sum(D_2, 2))^T));$$
$$\gamma_{119} = \gamma_{117} + \gamma_{118}; \gamma_{117} = \begin{pmatrix} I_{(W_2)} \otimes \gamma_{116} \end{pmatrix};$$
$$\gamma_{116} = \begin{pmatrix} O & I_{(W_3)} \otimes D_3 \\ O & O \end{pmatrix}; \gamma_{118} = \begin{pmatrix} O & I_{((W_3+1)(W_2-1))} \otimes D_2 \\ O & O \end{pmatrix};$$
$$\gamma_{170} = \begin{pmatrix} \gamma_{151} & O \\ O & \gamma_{165} \end{pmatrix} \text{where}$$
$$\gamma_{151} = \begin{pmatrix} \gamma_{134} & \gamma_{148} & O & O & & & \\ O & \gamma_{145} & \gamma_{148} & O & O & & \\ O & O & \gamma_{147} & \gamma_{148} & O & O & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ O & \dots & O & O & O & \gamma_{147} & \gamma_{148} \\ O & \dots & O & O & O & O & \gamma_{147} \end{pmatrix} \text{where}$$
$$\gamma_{134} = \gamma_{130} + \gamma_{131} + \gamma_{133};$$
$$\gamma_{130} = \begin{pmatrix} I_{(C_1)\otimes\gamma_{129}} & O & O \\ O & O & (I_{(C_3-1)} \otimes D_3) \\ O & O & O \end{pmatrix}; \gamma_{131} = \begin{pmatrix} O & I_{((C_3+1)C_1-1)} \otimes D_1 \\ O & O \end{pmatrix};$$
$$\gamma_{133} = \begin{pmatrix} I_{((C_3+1)(C_1+1)-1)} \otimes \gamma_{132} \end{pmatrix}; \gamma_{132} = \begin{pmatrix} D_0 - \phi I_{(m_1)} \end{pmatrix};$$
$$\gamma_{145} = \gamma_{134} + \gamma_{137} + \gamma_{140} + \gamma_{142} + \gamma_{144}; \gamma_{134} = \gamma_{130} + \gamma_{131} + \gamma_{133};$$
$$\gamma_{137} = \begin{pmatrix} I_{(C_1)} \otimes \gamma_{136} & O \\ O & O \end{pmatrix}; \gamma_{136} = \begin{pmatrix} O & \gamma_{135} \end{pmatrix};$$
$$\gamma_{135} = \begin{pmatrix} O \\ [1:(C_3-2)] \otimes \lambda_3 I_{(m_1)} \\ O \end{pmatrix};$$
$$\gamma_{140} = diag(e_{1\times C_1} \otimes \gamma_{138}, \gamma_{139}); \gamma_{139} = (O, [1:(C_3-2)] \otimes -\lambda_3 e_{1\times m_1}, O)$$
$$\gamma_{138} = (O, [1:(C_3-2)] \otimes -\lambda_3 e_{1\times m_1}, O)$$
$$\gamma_{142} = \begin{pmatrix} O & O \\ O & \gamma_{141} \\ O & O \end{pmatrix}; \gamma_{141} = \begin{pmatrix} [1:(C_1-2)]^T \otimes \zeta_{12} \end{pmatrix};$$
$$\zeta_{12} = \begin{pmatrix} I_{(C_3)} \otimes \lambda_1 I_{(m_1)} \\ O \end{pmatrix}; \gamma_{144} = diag\begin{pmatrix} O_{1\times(C_3+1)m_1} & \gamma_{143} & O \end{pmatrix};$$
$$\gamma_{143} = \begin{pmatrix} [1:(C_1-2)] \otimes (e_{1\times(C_3+1)} \otimes -\lambda_1 e_{1\times m_1}) \end{pmatrix};$$
$$\gamma_{147} = \gamma_{145} + \gamma_{146}; \gamma_{146} = diag\begin{pmatrix} -\theta e_{1\times((C_3+1)(C_1+1)m_1-m_1)} \end{pmatrix};$$
$$\gamma_{148} = \begin{pmatrix} \phi I_{((C_3+1)(C_1+1)m_1-m_1)} \end{pmatrix}; \gamma_{165} = \gamma_{163} + \gamma_{164};$$
$$\gamma_{163} = \begin{pmatrix} I_{(W_3-1)} \otimes \gamma_{160} & O \\ O & O \end{pmatrix}; \gamma_{160} = \gamma_{153} + \gamma_{154} + \gamma_{159};$$
$$\gamma_{153} = \begin{pmatrix} I_{(m)} \otimes \gamma_{152} \end{pmatrix}; \gamma_{152} = \begin{pmatrix} O & I_{(C_3-1)} \otimes D_1 \\ O & O \end{pmatrix};$$

$$\gamma_{154} = \begin{pmatrix} O & I_{(m-1)} \otimes (I_{(C_3)} \otimes \phi I_{(m_1)}) \\ O & O \end{pmatrix}; \gamma_{159} = \begin{pmatrix} \gamma_{155} & O & O \\ O & \gamma_{157} & O \\ O & O & I_{(m-2)} \otimes \gamma_{158} \end{pmatrix};$$

$\gamma_{155} = \begin{pmatrix} I_{(C_1)} \otimes (D_0 - \theta I_{(m_1)}) \end{pmatrix}; \gamma_{157} = \gamma_{155} + \gamma_{156};$

$\gamma_{156} = diag\begin{pmatrix} O & [1:(C_1-2)] \otimes -\lambda_1 e_{1\times m_1} & O \end{pmatrix};$

$\gamma_{158} = \gamma_{157} + \begin{pmatrix} I_{(C_1)} \otimes -\theta I_{(m_1)} \end{pmatrix};$

$\gamma_{162} = \gamma_{160} + \begin{pmatrix} I_{(C_1 m)} \otimes diag((sum(D_3, 2)^T)) \end{pmatrix};$

$$\gamma_{164} = \begin{pmatrix} O & I_{(W_3-1)} \otimes (I_{(C_1 m)} \otimes D_3) \\ O & O \end{pmatrix};$$

$$\gamma_{173} = \begin{pmatrix} \gamma_{166} & O \\ O & \gamma_{167} \end{pmatrix};$$

$\gamma_{166} = \begin{pmatrix} I_{(m)} \otimes (I_{((C_3+1)(C_1+1)-1)} \otimes D_2) \end{pmatrix};$

$\gamma_{167} = \begin{pmatrix} I_{(W_3)} \otimes (I_{(C_1 m)} \otimes D_2) \end{pmatrix};$

$\gamma_{168} = \gamma_{151} + \begin{pmatrix} I_{(((C_3+1)(C_1+1)-1)m)} \otimes diag((sum(D_2, 2)^T)) \end{pmatrix};$

$\gamma_{169} = \gamma_{165} + \begin{pmatrix} I_{(C_1 m W_3)} \otimes diag((sum(D_2, 2)^T)) \end{pmatrix};$

$$\gamma_{125} = \begin{pmatrix} O & O \\ O & \gamma_{124} \end{pmatrix};$$

$\gamma_{124} = \begin{pmatrix} I_{(W_2)} \otimes \begin{pmatrix} O & \gamma_{123} & O\gamma_{123} \end{pmatrix} \end{pmatrix}$ where

$$\gamma_{123} = \begin{pmatrix} I_{(m)} \otimes \zeta_{13} \\ O \end{pmatrix}; \zeta_{13} = \begin{pmatrix} I_{(C_1)} \otimes \zeta_{14} \\ O \end{pmatrix}; \zeta_{14} = \begin{pmatrix} O \\ D_3 \end{pmatrix};$$

$S_{2h} = S_{20} - h\,\mu\, diag(e_{1\times((W_3+1)W_2 m_1 + (((C_3+1)(C_1+1)m_1 - m_1)m + ((C_1+1)m_1 - m_1)mW_3)W_2)})$
where $h = 1, 2, \ldots, (h^* - 1)$; $S_{2h^*} = \gamma_{119} + \gamma_{206}$;

$$\gamma_{206} = \begin{pmatrix} \gamma_{204} & O \\ O & \gamma_{205} \end{pmatrix}; \gamma_{204} = \begin{pmatrix} I_{(W_2-1)} \otimes \gamma_{203} \end{pmatrix}; \gamma_{203} = \begin{pmatrix} I_{(W_3)} \otimes \gamma_{201} & O \\ O & \gamma_{202} \end{pmatrix};$$

$\gamma_{201} = D_0 - h^* \mu I_{(m_1)} - \theta I_{(m_1)}; \gamma_{202} = D_0 - h^* \mu I_{(m_1)} - \theta I_{(m_1)} + diag(sum(D_3, 2)^T);$

$\gamma_{205} = \gamma_{203} + (I_{(W_3+1)} \otimes diag(sum(D_2, 2)^T));$

$\Psi_{2h} = h\,\mu\, diag(e_{1\times((W_3+1)W_2 m_1 + (((C_3+1)(C_1+1)m_1 - m_1)m + ((C_1+1)m_1 - m_1)mW_3)W_2)})$
where $h = 1, 2, \ldots, (h^* - 1)$;

$\Psi_{2h^*} = \begin{pmatrix} I_{((W_3+1)W_2)} \otimes h^* \mu I_{(m_1)} & O \end{pmatrix};$

$$\Upsilon_{21} = \begin{pmatrix} O & O \\ \gamma_{196} & O \end{pmatrix} \text{where}$$

$$\gamma_{196} = I_{(W_2)} \otimes \begin{pmatrix} \gamma_{188} & O \\ O & \gamma_{195} \end{pmatrix}; \gamma_{195} = \begin{pmatrix} I_{(W_2)} \otimes \zeta_* \end{pmatrix}; \zeta_* = \begin{pmatrix} \gamma_{189} \\ \gamma_{192} \\ \gamma_{192} \\ e_{(m-3)\times 1} \otimes \gamma_{194} \end{pmatrix};$$

$\gamma_{194} = \gamma_{193} + \gamma_{192}; \gamma_{192} = \gamma_{191} + \gamma_{189};$

$$\gamma_{191} = \begin{pmatrix} O \\ \gamma_{190} \\ O \end{pmatrix}; \gamma_{189} = \begin{pmatrix} O \\ D_1 \end{pmatrix};$$

$\gamma_{193} = (e_{C_3 \times 1} \otimes \phi I_{(m_1)});$

$$\gamma_{188} = \begin{pmatrix} \gamma_{182} \\ \gamma_{186} \\ \gamma_{186} \\ \zeta_{15} \end{pmatrix}; \zeta_{15} = \begin{pmatrix} e_{(m-3)\times 1} \otimes (\gamma_{187} + \gamma_{186}) & O \end{pmatrix};$$

$\gamma_{187} = \begin{pmatrix} e_{((C_3+1)(C_1+1)-1)\times 1} \otimes \phi I_{(m_1)} \end{pmatrix}; \gamma_{186} = \gamma_{182} + \gamma_{185};$

$$\gamma_{185} = \begin{pmatrix} O \\ \gamma_{183} \\ O \\ \gamma_{184} \\ O \end{pmatrix}; \gamma_{182} = \begin{pmatrix} O \\ D_1 \\ O \\ D_3 \end{pmatrix}; \Upsilon_{22} = \begin{pmatrix} O \\ \gamma_{196} \end{pmatrix};$$

$$\Lambda_{21} = \begin{pmatrix} O & I_{(W_3+1)W_2} \otimes D_1 & O & O \\ O & O & O & I_{(W_{(2)})} \otimes \gamma_{199} \end{pmatrix} \text{where}$$

$$\gamma_{199}=\begin{pmatrix} I_{(m)}\otimes\zeta_{16} \\ O \end{pmatrix};\ \zeta_{16}=\begin{pmatrix} O \\ I_{(C_3)}\otimes D_1 \end{pmatrix};\ \Lambda_{22}=\begin{pmatrix} O & I_{(W_3+1)W_2}\otimes D_1 \end{pmatrix};$$
$$\Gamma_{20}=\begin{pmatrix} O & I_{(W_3+1)W_2)}\otimes I_{(m_1)} & O \\ O & O & O \end{pmatrix};$$
$$\Gamma_{21}=\gamma_{176}+\delta_{20};$$
$$\delta_{20}=\begin{pmatrix} O & O & O \\ O & I_{(W_2)}\otimes\zeta_{17} & O \end{pmatrix};\ \zeta_{17}=\begin{pmatrix} \gamma_{178} \\ \gamma_{180} \end{pmatrix};$$
$$\gamma_{178}=\begin{pmatrix} O & O \\ \gamma_{177} & O \end{pmatrix};\ \gamma_{177}=\begin{pmatrix} e_{(C_3+1)(C_1+1)(m-2)\times 1}\otimes\theta I_{(m_1)} \end{pmatrix};$$
$$\gamma_{180}=\begin{pmatrix} O & I_{(W_3)}\otimes\zeta_{18} \end{pmatrix};\ \zeta_{18}=\begin{pmatrix} O \\ \gamma_{179} \end{pmatrix};$$
$$\gamma_{179}=\begin{pmatrix} e_{C_1(m-2)\times 1}\otimes\theta I_{(m_1)} \end{pmatrix};\ \Gamma_{22}=\begin{pmatrix} O & I_{(W_2)}\otimes(I_{(W_3+1)}\otimes\theta I_{(m_1)}) & O \end{pmatrix};$$
$$S_{30}=\begin{pmatrix} \gamma_{226} & O & O & O \\ O & \gamma_{244} & \gamma_{270} & \gamma_{272} \\ O & O & \gamma_{257} & O \\ O & O & O & \gamma_{268} \end{pmatrix};\ \gamma_{226}=\gamma_{220}+\gamma_{222}+\gamma_{225};$$
$$\gamma_{225}=\begin{pmatrix} I_{(W_2)}\otimes\gamma_{223} & O \\ O & (\gamma_{223}+\gamma_{224}) \end{pmatrix};\ \gamma_{224}=\begin{pmatrix} I_{(W_3+1)}\otimes diag(sum(D_2,2)^T) \end{pmatrix};$$
$$\gamma_{223}=\begin{pmatrix} I_{(W_3)}\otimes(D_0-\theta I_{(m_1)}) & O \\ O & (D_0-\theta I_{(m_1)})\otimes diag(sum(D_3,2)^T)) \end{pmatrix};$$
$$\gamma_{222}=\begin{pmatrix} O & \gamma_{221} \\ O & O \end{pmatrix};\ \gamma_{221}=\begin{pmatrix} I_{((W_3+1)W_2)}\otimes D_2 \end{pmatrix};$$
$$\gamma_{220}=\begin{pmatrix} I_{((W_2+1))}\otimes\gamma_{219} \end{pmatrix};$$
$$\gamma_{219}=\begin{pmatrix} O & I_{(W_3)}\otimes D_3 \\ O & O \end{pmatrix};$$
$$\gamma_{244}=\gamma_{242}+\gamma_{243};$$
$$\gamma_{243}=\begin{pmatrix} O & I_{(m-1)}\otimes\phi I_{((C_3+1)(C_2+1)m_1-m_1)} \\ O & O \end{pmatrix};$$
$$\gamma_{242}=\begin{pmatrix} \gamma_{231} & O & O \\ O & \gamma_{240} & O \\ O & O & I_{(m-2)}\otimes\gamma_{241} \end{pmatrix};$$
$$\gamma_{241}=\gamma_{240}-\theta I_{((C_3+1)(C_2+1)m_1-m_1)};$$
$$\gamma_{240}=\gamma_{231}+\gamma_{234}+\gamma_{236}+\gamma_{238}+\gamma_{239};$$
$$\gamma_{239}=diag(O,[1:(C_2-2)]\otimes\Big[(e_{1\times(C_3+1)}\otimes-\lambda_2 e_{1\times m_1})\Big],O);$$
$$\gamma_{238}=diag((e_{1\times W_2}\otimes\gamma_{237}),O,([1:(C_3-2)]\otimes-\lambda_3 e_{1\times m_1}),O);$$
$$\gamma_{236}=\begin{pmatrix} O & O \\ O & \gamma_{235} \\ O & O \end{pmatrix};\ \gamma_{235}=\begin{pmatrix} [1:(C_2-2)]^T\otimes\zeta_{19} \end{pmatrix};\ \zeta_{19}=\begin{pmatrix} I_{(C_3)}\otimes\lambda_2 I_{(m_1)} \\ O \end{pmatrix};$$
$$\gamma_{234}=\begin{pmatrix} I_{(C_2)}\otimes\gamma_{233} & O \\ O & O \end{pmatrix};\ \gamma_{233}=\begin{pmatrix} O & \gamma_{232} \end{pmatrix};$$
$$\gamma_{232}=\begin{pmatrix} O \\ [1:(C_3-2)]^T\otimes\lambda_3 I_{(m_1)} \\ O \end{pmatrix};$$
$$\gamma_{231}=\gamma_{230}+\gamma_{229}+(I_{((C_3+1)(C_2+1)-1)}\otimes D_0)-\phi I_{((C_3+1)(C_2+1)m_1-m_1)};$$
$$\gamma_{230}=\begin{pmatrix} O & I_{((C_3+1)C_2-1)}\otimes D_2 \\ O & O \end{pmatrix};\ \gamma_{229}=\begin{pmatrix} I_{(C_2)}\otimes\gamma_{227} & O \\ O & \gamma_{228} \end{pmatrix};$$
$$\gamma_{228}=\begin{pmatrix} O & I_{(C_3-1)}\otimes D_3 \\ O & O \end{pmatrix};\ \gamma_{257}=\gamma_{255}+\gamma_{256};$$
$$\gamma_{256}=\begin{pmatrix} O & I_{(W_3-1)}\otimes(I_{(m)}\otimes(I_{(C_2)}\otimes D_3)) \\ O & O \end{pmatrix};\ \gamma_{255}=\begin{pmatrix} I_{(W_3-1)}\otimes\gamma_{253} & O \\ O & \gamma_{254} \end{pmatrix};$$
$$\gamma_{253}=\gamma_{251}+\gamma_{252};$$
$$\gamma_{252}=\begin{pmatrix} O & \phi I_{(C_2 m_1(m-1))} \\ O & O \end{pmatrix};\ \gamma_{251}=\begin{pmatrix} \gamma_{247} & O & O \\ O & \gamma_{249} & O \\ O & O & I_{(m-2)}\otimes\gamma_{250} \end{pmatrix};$$
$$\gamma_{250}=\gamma_{249}-\begin{pmatrix} I_{(C_2)}\otimes\theta I_{(m_1)} \end{pmatrix};\ \gamma_{249}=\gamma_{247}+\gamma_{248};$$

$\gamma_{248} = diag([O, ([1 : (C_2 - 2)] \otimes -\lambda_2 e_{1\times m_1}), O]);$
$\gamma_{247} = \gamma_{245} + \gamma_{246}; \gamma_{246} = \begin{pmatrix} I_{(C_2)} \otimes (D_0 - \phi I_{(m_1)}) \end{pmatrix};$
$\gamma_{245} = \begin{pmatrix} O & I_{(C_2-1)} \otimes D_2 \\ O & O \end{pmatrix}; \gamma_{254} = \gamma_{253} + \begin{pmatrix} I_{(C_2 m)} \otimes diag((sum(D_3, 2))^T) \end{pmatrix};$
$\gamma_{268} = \begin{pmatrix} I_{(W_2-1)} \otimes \gamma_{265} & O \\ O & (\gamma_{265} + \gamma_{266}) \end{pmatrix} + \gamma_{267}$ where
$\gamma_{267} = \begin{pmatrix} O & I_{(W_2-1)} \otimes (I_{(m)} \otimes (I_{(3)} \otimes D_2)) \\ O & O \end{pmatrix};$
$\gamma_{266} = \begin{pmatrix} I_{(m)} \otimes (I_{(C_3)} \otimes diag((sum(D_2, 2))^T)) \end{pmatrix};$
$\gamma_{265} = \gamma_{263} + \gamma_{264};$
$\gamma_{264} = \begin{pmatrix} O & I_{(m-1)} \otimes (I_{(C_3)} \otimes \phi I_{(m_1)}) \\ O & O \end{pmatrix};$
$\gamma_{263} = \begin{pmatrix} \gamma_{259} & O & O \\ O & \gamma_{261} & O \\ O & O & I_{(m-2)} \otimes \gamma_{262} \end{pmatrix};$
$\gamma_{259} = \gamma_{258} + \begin{pmatrix} I_{(C_3)} \otimes (D_0 - \phi I_{(m_1)}) \end{pmatrix};$
$\gamma_{258} = \begin{pmatrix} O & I_{(C_3-1)} \otimes D_3 \\ O & O \end{pmatrix};$
$\gamma_{262} = \gamma_{261} + \begin{pmatrix} I_{(C_3)} \otimes -\theta I_{(m_1)} \end{pmatrix};$
$\gamma_{261} = \gamma_{259} + \gamma_{260};$
$\gamma_{260} = diag([O, ([1 : (C_3 - 2)] \otimes -\lambda_3 e_{1\times m_1}), O]);$
$\gamma_{270} = \begin{pmatrix} \gamma_{269} & O \end{pmatrix};$
$\gamma_{269} = \begin{pmatrix} I_{(m)} \otimes \zeta_{20} \end{pmatrix}; \zeta_{20} = \begin{pmatrix} I_{(W_2)} \otimes \zeta_{21} \\ O \end{pmatrix}; \zeta_{21} = \begin{pmatrix} O \\ D_3 \end{pmatrix};$
$\gamma_{272} = \begin{pmatrix} \gamma_{271} & O \end{pmatrix};$
$\gamma_{271} = \begin{pmatrix} I_{(m)} \otimes \zeta_{22} \end{pmatrix}; \zeta_{22} = \begin{pmatrix} O \\ I_{(C_3)} \otimes D_2 \end{pmatrix};$
$S_{3h} = S_{30} - h\ \mu\ diag(e_{1\times((W_3+1)(W_2+1)m_1 + ((C_3+1)(C_2+1)m_1 - m_1)m + C_2 m_1 m W_3 + C_3 m_1 m W_2)})$ where $h = 1, 2, \ldots, (h^* - 1);$
$S_{3h^*} = \gamma_{306} + \gamma_{307} + \gamma_{313}; \gamma_{313} = \gamma_{311} + \gamma_{312};$
$\gamma_{312} = (I_{(W_3+1)(W_2+1)} \otimes (-h^* \mu I_{(m_1)} - \theta I_{(m_1)}));$
$\gamma_{311} = \begin{pmatrix} \gamma_{309} & O \\ O & \gamma_{310} \end{pmatrix}; \gamma_{310} = \gamma_{308} + (I_{(W_3+1)} \otimes diag(sum(D_2, 2)^T))$
$\gamma_{308} = \begin{pmatrix} I_{(C_3)} \otimes D_0 & O \\ O & D_0 + diag(sum(D_3, 2)^T) \end{pmatrix};$
$\gamma_{309} = \begin{pmatrix} I_{(W_2)} \otimes \gamma_{308} \end{pmatrix};$
$\Psi_{3h} = h\ \mu\ diag(e_{1\times((W_3+1)(W_2+1)m_1 + ((C_3+1)(C_2+1)m_1 - m_1)m + C_2 m_1 m W_3 + C_3 m_1 m W_2)})$ where $h = 1, 2, \ldots, (h^* - 1);$
$\Psi_{3h^*} = \begin{pmatrix} I_{((W_3+1)(W_2+1))} \otimes h^* \mu I_{(m_1)} & O \end{pmatrix};$
$\Upsilon_{31} = \begin{pmatrix} O & O \\ \gamma_{279} & O \\ \gamma_{286} & O \\ \gamma_{293} & O \end{pmatrix}; \gamma_{293} = \begin{pmatrix} O & I_{(W_2)} \otimes \gamma_{292} \end{pmatrix};$
$\gamma_{292} = \begin{pmatrix} \gamma_{287} & O \\ \gamma_{289} & O \\ \gamma_{289} & O \\ e_{(m-3)\times 1} \otimes \gamma_{291} & O \end{pmatrix};$
$\gamma_{291} = \gamma_{289} + \gamma_{290}; \gamma_{290} = \begin{pmatrix} e_{C_3\times 1} \otimes \phi I_{(m_1)} \end{pmatrix}; \gamma_{289} = \gamma_{287} + \gamma_{288};$
$\gamma_{288} = \begin{pmatrix} O \\ [1 : (C_2 - 2)]^T \otimes \lambda_3 I_{(m_1)} \\ O \end{pmatrix}; \gamma_{287} = \begin{pmatrix} O \\ D_3 \end{pmatrix};$
$\gamma_{286} = \begin{pmatrix} O & I_{(W_3)} \otimes \gamma_{285} & O \end{pmatrix};$

$$\gamma_{285} = \begin{pmatrix} \gamma_{280} \\ \gamma_{282} \\ \gamma_{282} \\ e_{(m-3)\times 1} \otimes \gamma_{284} \end{pmatrix};$$
$$\gamma_{284} = \gamma_{282} + \gamma_{283}; \gamma_{283} = \begin{pmatrix} e_{C_2\times 1} \otimes \phi I_{m_1} \end{pmatrix}; \gamma_{282} = \gamma_{280} + \gamma_{281};$$
$$\gamma_{281} = \begin{pmatrix} O \\ [1:(C_2-2)]^T \otimes \lambda_2 I_{(m_1)} \\ O \end{pmatrix}; \gamma_{280} = \begin{pmatrix} O \\ D_2 \end{pmatrix};$$
$$\Upsilon_{32} = \begin{pmatrix} O \\ \gamma_{279} \\ \gamma_{286} \\ \gamma_{293} \end{pmatrix}; \gamma_{279} = \begin{pmatrix} \gamma_{274} & O \\ \gamma_{276} & O \\ \gamma_{276} & O \\ e_{(m-3)\times 1} \otimes \gamma_{278} & O \end{pmatrix};$$
$$\gamma_{278} = \gamma_{276} + \gamma_{277}; \gamma_{277} = \begin{pmatrix} e_{((C_3+1)(C_2+1)-1)\times 1} \otimes \phi I_{m_1} \end{pmatrix};$$
$$\gamma_{276} = \gamma_{274} + \gamma_{275};$$
$$\gamma_{275} = \begin{pmatrix} O \\ [1:(C_2-2)]^T \otimes \zeta_{23} \\ O \\ [1:(C_3-2)]^T \otimes \lambda_3 I_{(m_1)} \\ O \end{pmatrix}; \zeta_{23} = \begin{pmatrix} O \\ \lambda_2 I_{(m_1)} \end{pmatrix}; \gamma_{274} = \begin{pmatrix} O \\ D_2 \\ O \\ D_3 \end{pmatrix};$$
$$\Gamma_{30} = \begin{pmatrix} O & I_{((W_3+1)(W_2+1))} \otimes \theta I_{(m_1)} & O \\ O & O & O \end{pmatrix};$$
$$\Gamma_{31} = \gamma_{296} + \delta_{30};$$
$$\delta_{30} = \begin{pmatrix} O & O & O & O & O \\ O & \gamma_{297} & O & O & O \\ O & O & \gamma_{299} & O & O \\ O & O & O & \gamma_{301} & O \end{pmatrix};$$
$$\gamma_{297} = \begin{pmatrix} e_{((C_3+1)(C_2+1)-1)(m-2)\times 1} \otimes \theta I_{(m_1)} \end{pmatrix};$$
$$\gamma_{299} = \begin{pmatrix} I_{(W_3)} \otimes \zeta_{24} \end{pmatrix}; \zeta_{24} = \begin{pmatrix} O \\ \gamma_{298} \end{pmatrix};$$
$$\gamma_{298} = \begin{pmatrix} e_{C_2(m-2)\times 1} \otimes \theta I_{(m_1)} \end{pmatrix};$$
$$\gamma_{301} = \begin{pmatrix} I_{(W_2)} \otimes \zeta_{25} \end{pmatrix}; \zeta_{25} = \begin{pmatrix} O \\ \gamma_{300} \end{pmatrix}; \gamma_{300} = \begin{pmatrix} e_{C_3(m-2)\times 1} \otimes \theta I_{(m_1)} & O \end{pmatrix};$$
$$\Gamma_{32} = \begin{pmatrix} O & I_{((W_3+1)(W_2+1))} \otimes \theta I_{(m_1)} & O \end{pmatrix};$$
$$\Lambda_{31} = \begin{pmatrix} D_1 I_{((W_3+1)(W_2+1)+((C_3+1)(C_2+1)-1)m+C_2 m W_3 + C_3 m W_2)} \end{pmatrix};$$
$$\Lambda_{32} = \begin{pmatrix} I_{((W_3+1)(W_2+1))} \otimes D_1 \end{pmatrix};$$
$$\Gamma_{40} = \begin{pmatrix} O & I_{((W_3+1)(W_2+1))} \otimes \theta I_{(m_1)} & O \\ O & O & O \end{pmatrix}; \delta_{40} = \begin{pmatrix} O & O & O \\ O & \gamma_{331} & O \end{pmatrix};$$
$$\gamma_{331} = \begin{pmatrix} \gamma_{326} \\ \gamma_{328} \\ \gamma_{330} \end{pmatrix}; \gamma_{330} = \begin{pmatrix} O & I_{(W_2)} \otimes \gamma_{329} \end{pmatrix};$$
$$\gamma_{329} = \begin{pmatrix} O & O \\ e_{C_3(m-2)\times 1} \otimes \theta I_{(m_1)} & O \end{pmatrix}; \gamma_{328} = \begin{pmatrix} O & \gamma_{327} & O \end{pmatrix};$$
$$\gamma_{327} = \begin{pmatrix} I_{(W_3)} \otimes \zeta_{26} \end{pmatrix}; \zeta_{26} = \begin{pmatrix} O \\ e_{C_2(m-2)\times 1} \otimes \theta I_{(m_1)} \end{pmatrix};$$
$$\gamma_{326} = \begin{pmatrix} O & O \\ e_{((C_3+1)(C_2+1)-1)(m-2))\times 1} \otimes \theta I_{(m_1)} & O \end{pmatrix};$$
$$\Gamma_{41} = \gamma_{325} + \gamma_{332}; \Gamma_{42} = \begin{pmatrix} O & I_{((W_3+1)(W_2+1))} \otimes \theta I_{(m_1)} & O \end{pmatrix};$$
$$\Gamma_{50} = \begin{pmatrix} O & \gamma_{339} & O \\ O & O & O \end{pmatrix}; \gamma_{339} = \begin{pmatrix} I_{((W_3+1)(W_2+1)-1)} \otimes \theta I_{(m_1)} \\ O \end{pmatrix};$$

$$\delta_{50} = \begin{pmatrix} O & O & O & O \\ \theta I_{(m_1)} & O & O & O \\ O & O & \gamma_{342} & O \\ O & O & \gamma_{343} & O \\ O & O & \gamma_{344} & O \\ O & O & \gamma_{345} & O \end{pmatrix}; \gamma_{345} = \begin{pmatrix} O & O & O \\ O & e_{C_3(m-2)\times 1} \otimes \theta I_{(m_1)} & O \end{pmatrix};$$

$$\gamma_{344} = \begin{pmatrix} O & I_{(W_2-1)} \otimes \gamma_{329} & O \end{pmatrix}; \gamma_{343} = \begin{pmatrix} O & \gamma_{327} & O \end{pmatrix};$$

$$\gamma_{342} = \begin{pmatrix} O & O \\ e_{((C_3+1)(C_2+1)-1)(m-2)\times 1} \otimes \theta I_{(m_1)} & O \end{pmatrix};$$

$$\Gamma_{52} = \begin{pmatrix} O & I_{((W_3+1)(W_2+1)-1)} \otimes \theta I_{(m_1)} & O \\ O & O & O \end{pmatrix};$$

$$\Gamma_{53} = \begin{pmatrix} O \\ \theta I_{(m_1)} \end{pmatrix}; \Gamma_{60} = \begin{pmatrix} \gamma_{353} \\ O \end{pmatrix};$$

$$\gamma_{353} = \begin{pmatrix} O & I_{((W_3+1)(W_2+1)-1)} \otimes \theta I_{(m_1)} & O \\ O & O & O \end{pmatrix};$$

$$\delta_{60} = \begin{pmatrix} O & O & O & O \\ \theta I_{(m_1)} & O & O & O \\ O & O & \gamma_{342} & O \\ O & O & \gamma_{343} & O \\ O & O & \gamma_{344} & O \\ O & O & \gamma_{345} & O \end{pmatrix}; \Gamma_{61} = \Gamma_{60} + \delta_{60};$$

$$\Gamma_{17} = \begin{pmatrix} O & I_{((W_3+1)(W_2+1)-1)} \otimes \theta I_{(m_1)} & O \\ O & O & O \end{pmatrix}; \Gamma_{63} = \begin{pmatrix} O & O \\ \theta I_{(m_1)} & O \end{pmatrix};$$

References

1. Shajin, D.; Jacob, J.; Vishnevskiy, V.M.; Krishnamoorthy, A. On a Queueing-Inventory Problem in Passenger Transport System. In *Distributed Computer and Communication Networks*; DCCN 2019, Communications in Computer and Information Science; Vishnevskiy, V., Samouylov, K., Kozyrev, D., Eds.; Springer: Cham, Switzerland, 2019; Volume 1141, pp. 215–229.
2. Melikov, A.Z.; Molchanov, A.A. Stock optimization in transportation/storage systems. *Cybern. Syst. Anal.* **1992**, *28*, 484–487. [CrossRef]
3. Sigman, K.; Simchi-Levi, D. Light traffic heuristic for an M/G/1 queue with limited inventory. *Ann. Oper. Res.* **1992**, *40*, 371–380. [CrossRef]
4. Neuts, M.F. A general class of bulk queues with Poisson input. *Ann. Math. Stat.* **1967**, *38*, 759–770. [CrossRef]
5. Krishnamoorthy, A.; Joshua, A.N.; Vishnevsky, V. Analysis of a k-Stage Bulk Service Queuing System with Accessible Batches for Service. *Mathematics* **2021**, *9*, 559. [CrossRef]
6. Jacyna, M.; Żak, J.; Gołębiowski, P. The Use of the Queueing Theory for the Analysis of Transport Processes. *Logist. Transp.* **2019**, *41*, 101–111. [CrossRef]
7. Radmilovic, Z.; Colic, V.; Hrle, Z. Some aspects of storage and bulk queueing systems in transport operations. *Transp. Plan. Technol.* **1996**, *20*, 67–81. [CrossRef]
8. Podlesna, L.; Bublyk, M.; Grybyk, I.; Matseliukh, Y.; Burov, Y.; Kravets, P.; Lozynska, O.; Karpov, I.; Peleshchak, I.; Peleshchak, R. Optimization Model of the Buses Number on the Route Based on Queueing Theory in a Smart City. In Proceedings of the MoMLeT+ DS, Lviv, Ukraine, 2–3 June 2020; pp. 502–515.
9. Corkindale, D.R. Queueing Theory in the Solution of a Transport Evaluation Problem. *J. Oper. Res. Soc.* **1975**, *26*, 259–271. [CrossRef]
10. Neuts, M.F.; Ramalhoto, M.F. A service model in which the server is required to search for customers. *J. Appl. Probab.* **1984**, *21*, 157–166. [CrossRef]
11. Artalejo, J.R.; Joshua, V.C.; Krishnamoorthy, A. An M/G/1 Retrial Queue with Orbital Search by Server. In *Advances in Stochastic Modelling*; Notable Publications Inc.: NJ, USA, 2002; pp. 41–54.
12. D'Apice, C.; Manzo, R. Search for customers in a finite capacity queueing system with phase-type distributions. *Inf. Process. Electron. Sci. J.* **2003**, *3*, 61–69.
13. Sekar, G.; Ayyappan, G.; Subdramanian, A.M.G. Single Server Retrial Queueing System with Orbital search under Erlang K Service. *Int. J. Appl. Math.* **2012**, *3*, 454–463.
14. Ayyappan, G.; Udayageetha, J. Transient analysis of M [X1], M [X2]/G1, G2/1 retrial queueing system with priority services, collisions, orbital search, working breakdown, start up/close down time, feedback, modified Bernoulli vacation and balking. *Int. J. Appl. Eng. Res.* **2018**, *13*, 8783–8795.

15. Stewart, W.J. *Probability, Markov Chains, Queues, and Simulation: The Mathematical Basis of Performance Modeling*; Princeton University Press: Princeton, NJ, USA, 2009.