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Higher-Order Associativity in Field Algebras

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Abstract: Field algebras were defined by Bakalov and Kac as an associative analogue of vertex algebras. We define the notion of higher-order associativity for field algebras and construct examples to show that higher-order associativity imposes a strictly stronger condition on field algebras than lower-order associativity.

Keywords: field algebra; general associative law; binary tree; formal calculus

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1. Introduction

Vertex algebras were defined by Borcherds in the study of the monstrous moonshine [1]. Vertex algebras, while the product map is allowed to have a meromorphic singularity, are analogous to unital commutative associative algebras with a group action [2,3]. Such an algebra of quantum fields, known as the operator product expansion, was first studied in physics [4,5], and the notion of vertex algebras gives a mathematical formulation of the chiral algebras in two-dimensional conformal field theories [6,7].

Field algebras were introduced by Bakalov and Kac as an associative analogue of vertex algebras [8], and similar notions were studied [9,10]. Examples of field algebras include quantum vertex operator algebras [11] and the smash product of a vertex algebra and its finite automorphism group [8]; they give some noncommutative generalizations of vertex algebras.

We first introduce some notation and give a definition of a field algebra. Let V be a vector space over $\mathbb{C}$. Let $V[[x]]$ and $V((x))$ be the vector spaces of formal power series and the formal Laurent series in x with coefficients in V, respectively. Consider a bilinear map $V \times V \to V((x))$, denoted by $(a, b) \mapsto a^x b \in V((x))$. In other words, for $a, b \in V$,

$$a^x b = \sum_{n\in\mathbb{Z}} x^{-1-n} a_{(n)} b, \text{ with } a_{(n)} b \in V, \tag{1}$$

where each map $\cdot_{(n)}\cdot : V \times V \to V$ is bilinear, and for each $a, b \in V$, $a_{(n)} b = 0$ eventually $n \to \infty$. By abuse of notation, a linear map is understood to act on a formal series by acting on the coefficients. For example,

$$a^x(b^y c) = \sum_{m,n\in Z} x^{-1-m} y^{-1-n} a_{(m)}(b_{(n)} c) \in V((x))((y)),$$
$$(a^{x-y} b)^y c = \sum_{m,n\in Z} (x-y)^{-1-m} y^{-1-n} (a_{(m)} b)_{(n)} c \in V((y))((x-y)).$$

Note $V((x))((y))$ and $V((y))((x-y))$ are both modules over $\mathbb{C}[[x, y]]$ and contain a common submodule $V[[x, y]]$. If $a^x b \in V[[x]]$ for some $a, b \in V$; then $a^x b|_{x=0} \in V$ denotes the evaluation at $x = 0$, namely, the constant term $a_{(-1)} b$.

Definition 1. *A* field algebra *is a vector space* V *over* $\mathbb{C}$ *with an identity* $1 \in V$ *and a bilinear map* $V \times V \to V((x))$, $(a, b) \mapsto a^x b \in V((x))$, *satisfying the following properties:*

(i) *(identity) For every* $a \in V$, $1^x a = a$, *and* $a^x 1 \in V[[x]]$ *with* $a^x 1|_{x=0} = a$.
(ii) *(associativity) For every* $a, b, c \in V$, *there exists* $N \in \mathbb{N}$ *(depending on* a, b, c*), such that the equality*

$$(xy(x-y))^N a^x (b^y c) = (xy(x-y))^N (a^{x-y} b)^y c$$

holds in $V[[x, y]]$.

The notation is chosen so that it expresses the analogy between field algebras and the unital associative algebras with a group action $a \mapsto a^x$, as first suggested by Borcherds for vertex algebras. The fact that Definition 1 is equivalent to the definition in [8] is explained in [12]. We use Definition 1 in order to make the present article self contained.

We now give one aspect of the associativity of field algebras that differs from the usual associative algebras: it is the general associative law. For example, consider replacing condition (ii) in Definition 1 by condition

(ii′) *(4-associativity) For every* $a, b, c, d \in V$, *the following elements*

$$\begin{gathered} a^x (b^y (c^z d)) \in V((x))((y))((z)) \\ a^x ((b^{y-z} c)^z d) \in V((x))((z))((y-z)) \\ (a^{x-y} b)^y (c^z d) \in V((y))((x-y, z)) \\ (a^{x-z} (b^{y-z} c))^z d \in V((z))((x-z))((y-z)) \\ ((a^{x-y} b)^{y-z} c)^z d \in V((z))((y-z))((x-y)), \end{gathered}$$

if multiplied by $\big(xyz(x-y)(y-z)(x-z)\big)^N$ *for sufficiently large* $N \in \mathbb{N}$, *belong to the subspace* $V[[x, y, z]]$ *and are equal.*

More generally, we can define

(ii″) *(n-associativity) For every* $a_1, \cdots, a_n \in V$, *the formal expressions corresponding to all binary bracketings of* $a_1, \cdots, a_n$ *satisfy the analogous condition.*

We can define ∞-associativity" to mean that n-associativity holds for all $n \geq 3$. If we let $\mathcal{F}_n$ denote the class of field algebras that are n-associative, then we have

$$\mathcal{F}_3 \supseteq \mathcal{F}_4 \supseteq \mathcal{F}_5 \supseteq \cdots \text{ with } \mathcal{F}_\infty = \bigcap_{n \geq 3} \mathcal{F}_n. \tag{2}$$

For the usual associative algebras as well as vertex algebras, 3-associativity automatically implies ∞-associativity. A natural question that arises in the case of field algebras is whether the higher-order associativity conditions are *strictly* stronger for higher n. With the restrictions imposed by the formal calculus, one may wonder whether requiring n-associativity for high enough n implies ∞-associativity. Although $\mathcal{F}_{n+1}$ is heuristically stronger than $\mathcal{F}_n$, no concrete example of field algebras showing their difference has been rigorously written down.

The main result of this study is the formulation of n-associativity (Section 3) and the construction of examples (Theorem 2) to show that unlike the ordinary associative algebras or vertex algebras, we indeed have proper inclusions

$$\mathcal{F}_n \supsetneq \mathcal{F}_{n+1}$$

for all $n \geq 3$.

2. Prefield Algebras

In this section, we fix notations and introduce some terminology.

Lemma 1. *Let* V *be a field algebra. Then, there exists a linear map* $T : V \to V$, *called the* translation operator, *with the following properties for every* $a, b \in V$.

(i) $a^x 1 = \mathrm{e}^{xT} a = \sum_{k \geq 0} x^k T^{(k)} a$, *with the notation* $T^{(k)} = T^k / k!$.

(ii) $(Ta)^x b = \partial_x(a^x b)$.
(iii) $T(a^x b) = (Ta)^x b + a^x(Tb)$.

Proof. We define the map $T : V \to V$ by

$$Ta = \partial_x(a^x 1)|_{x=0} = a_{(-2)}1$$

when $a_{(n)}b$ is given by (1). In particular, $T1 = 0$. If (ii) and (iii) hold, then $a^x 1 \in V[[x]]$ should satisfy $T(a^x 1) = \partial_x(a^x 1)$ with $a^x 1|_{x=0} = a$, and thus (i) follows. Hence, it suffices to show (ii) and (iii). Let $a, b \in V$. By Definition 1, there exists $N \in \mathbb{N}$, such that

$$(tx(t+x))^N (a^t 1)^x b = (tx(t+x))^N a^{t+x}(1^x b) = (tx(t+x))^N a^{t+x} b \tag{3}$$

in $V[[t,x]]$ with $(t+x)^N a^{t+x} b \in V[[t+x]]$. Hence, both sides of (3) belong to $(tx)^N V[[t,x]]$, and we must have

$$(t+x)^N (a^t 1)^x b = (t+x)^N a^{t+x} b \tag{4}$$

in $V[[t,x]]$. By applying $(t+x)\partial_t$ to (4), we obtain

$$N(t+x)^N (a^t 1)^x b + (t+x)^{N+1} \partial_t (a^t 1)^x b = N(t+x)^N a^{t+x} b + (t+x)^{N+1} \partial_t (a^{t+x} b),$$

where the first terms cancel by (4). Hence,

$$(t+x)^{N+1} \partial_t (a^t 1)^x b = (t+x)^{N+1} \partial_t (a^{t+x} b),$$

and setting $t = 0$, we obtain $x^{N+1}(Ta)^x b = x^{N+1} \partial_t (a^{t+x} b)|_{t=0} = x^{N+1} \partial_x (a^x b)$, and, therefore, $(Ta)^x b = \partial_x (a^x b)$. Similarly, we have $M \in \mathbb{N}$, such that

$$(tx(t+x))^M (a^x b)^t 1 = (tx(t+x))^M a^{t+x} (b^t 1)$$

in $V[[t,x]]$ with $x^M a^x b \in V[[x]]$, and, thus, $x^M (a^x b)^t 1 \in V[[t,x]]$. Hence,

$$x^M (a^x b)^t 1 = x^M a^{t+x} (b^t 1)$$

in $V[[t,x]]$, and applying ∂_t to both sides, we obtain

$$x^M \partial_t ((a^x b)^t 1) = x^M \partial_t (a^{t+x}(b^t 1)) = x^M \big(\partial_x (a^{t+x}(b^t 1)) + a^{t+x} \partial_t (b^t 1)\big).$$

Setting $t = 0$, we have $T(a^x b) = \partial_x(a^x b) + a^x(Tb) = (Ta)^x b + a^x(Tb)$. □

We define the following notion for later convenience:

Definition 2. *A* prefield algebra *is a vector space V over $\mathbb{C}$ with an identity $1 \in V$, a translation operator $T : V \to V$, and a bilinear map $\cdot^x\cdot : V \times V \to V((x))$ satisfying the following properties.*
(i) *(identity) For every $a \in V$, $1^x a = a$, and $a^x 1 = e^{xT} a$.*
(ii) *(translation covariance) For every $a, b \in V$,*

$$(Ta)^x b = T(a^x b) - a^x(Tb) = \partial_x(a^x b).$$

By Lemma 1, a field algebra is a prefield algebra. A prefield algebra is a field algebra if it satisfies the associativity axiom in Definition 1.

3. n-Associativity

The associative axiom in Definition 1 can be generalized to the associativity of n vectors. We define some notations that are used throughout the paper.

3.1. Notation $\mathcal{B}_n$

We denote by $\mathcal{B}_n$ the set of all binary bracketings of length n, for $n \geq 1$. For example, for $n = 4$, we have

$$\bullet(\bullet(\bullet\bullet)), \bullet((\bullet\bullet)\bullet), (\bullet\bullet)(\bullet\bullet), (\bullet(\bullet\bullet))\bullet, ((\bullet\bullet)\bullet)\bullet.$$

It is well known that the cardinality of $\mathcal{B}_n$ is given by the Catalan number C_{n-1} [13]. For $B \in \mathcal{B}_n$, let $B(x_1, \cdots, x_n)$ denote the corresponding bracketing of the n letters $x_1, \cdots, x_n$. For example, if $B = \bullet((\bullet\bullet)\bullet)$, then $B(x_1, x_2, x_3, x_4) = x_1((x_2x_3)x_4)$. We denote by $\mathcal{B}_n(x_1, \cdots, x_n)$ the set of $B(x_1, \cdots, x_n)$ for all $B \in \mathcal{B}_n$.

If $B(x_1, \cdots, x_n) \in \mathcal{B}_n(x_1, \cdots, x_n)$, it can be represented by an ordered full binary tree [14], which has n nodes (leaves) $x_1, \cdots, x_n$ having no children, and every other node has both a left child and a right child. There exists a unique node (the root) that has no parent, having the rest of the nodes as its descendants. (When $n = 1$, the single node is both the root and a leaf.) We label the nodes as follows: When a node w has the left child u and the right child v, we write

$$w = uv \tag{5}$$

and say that u and v are siblings. Hence, every node is expressed as a product of leaves, and this notation for a node coincides with the subtree spanned by the node and its descendants. We identify the two concepts. For example, in $x_1((x_2x_3)x_4) \in \mathcal{B}_4(x_1, \cdots, x_4)$, the node $(x_2x_3)x_4$ having descendants x_2x_3, x_2, x_3, x_4 can be considered as the subtree $(x_2x_3)x_4 \in \mathcal{B}_3(x_2, x_3, x_4)$. In particular, the root of a tree is considered to be the tree itself, and we can regard (5) as defining the product w of two trees u and v having disjointed sets of leaves.

Let $B(x_1, \cdots, x_n) \in \mathcal{B}_n(x_1, \cdots, x_n)$. Suppose we have a set of nodes $u_1, \cdots, u_m$ in $B(x_1, \cdots, x_n)$ that are disjoint as subtrees. The *contraction* of $B(x_1, \cdots, x_n)$ by the nodes $u_1, \cdots, u_m$ is the tree obtained by replacing each subtree u_j by the *last* leaf x_{i_j} of u_j. For example, the contraction of $(x_1x_2)(x_3(x_4x_5))$ by the nodes x_1x_2 and $x_3(x_4x_5)$ is the tree $x_2x_5 \in \mathcal{B}_2(x_2, x_5)$. In defining the contraction, we do not lose any generality if we assume that the leaves corresponding to $u_1, \cdots, u_m$ give a partition of the set $\{x_1, \cdots, x_n\}$, as we can assume some of the u_j are leaves themselves without affecting the contraction.

3.2. Maps $\mathcal{X}_B$ and $\mathcal{Y}_B$

Let V be a prefield algebra, and let $B \in \mathcal{B}_n$. We now consider $x_1, \cdots, x_n$ as formal variables, and define $\mathcal{X}_{B(x_1,\cdots,x_n)}$ and $\mathcal{Y}_{B(x_1,\cdots,x_n)}$ as multilinear maps on $V^{\times n}$ depending on the formal variables $x_1, \cdots, x_n$, with the following properties:

(i) For $x_1 \in \mathcal{B}_1(x_1)$,

$$\mathcal{X}_{x_1}(a_1) = a_1$$

for all $a_1 \in V$.

(ii) When the trees u and v have disjoint sets of leaves $x_1, \cdots, x_m$ and $x_{m+1}, \cdots, x_n$, respectively, then

$$\mathcal{X}_{uv}(a_1, \cdots, a_n) = \mathcal{X}_u(a_1, \cdots, a_m)^{x_m - x_n} \mathcal{X}_v(a_{m+1}, \cdots, a_n) \tag{6}$$

for all $a_1, \cdots, a_n \in V$, where x_m and x_n are the last leaves of u and v, respectively. Here, we regard $u \in \mathcal{B}_m(x_1, \cdots, x_m)$, $v \in \mathcal{B}_{n-m}(x_{m+1}, \cdots, x_n)$ and $uv \in \mathcal{B}_n(x_1, \cdots, x_n)$.

(iii) For $a_1, \cdots, a_n \in V$ and $B \in \mathcal{B}_n$, we define

$$\begin{aligned} \mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1, \cdots, a_n) &= \left(\mathcal{X}_{B(x_1,\cdots,x_n)}(a_1, \cdots, a_n)\right)^{x_n} 1 \\ &= \mathrm{e}^{x_n T} \left(\mathcal{X}_{B(x_1,\cdots,x_n)}(a_1, \cdots, a_n)\right). \end{aligned} \tag{7}$$

The result of setting $x_n = 0$ in (7) is denoted by

$$\mathcal{Y}_{B(x_1,\cdots,x_{n-1},0)}(a_1,\cdots,a_n) = \mathcal{X}_{B(x_1,\cdots,x_{n-1},0)}(a_1,\cdots,a_n).$$

Example 1. *Let $B = (\bullet\bullet)(\bullet\bullet)$. Let $a_1,\cdots,a_4$ be vectors in a prefield algebra V. We have*

$$\mathcal{X}_{x_1}(a_1) = a_1, \cdots, \mathcal{X}_{x_4}(a_4) = a_4,$$

and by (6),

$$\mathcal{X}_{x_1x_2}(a_1,a_2) = a_1^{x_1-x_2}a_2$$
$$\mathcal{X}_{x_3x_4}(a_3,a_4) = a_3^{x_3-x_4}a_4,$$

and

$$\mathcal{X}_{(x_1x_2)(x_3x_4)}(a_1,a_2,a_3,a_4) = (a_1^{x_1-x_2}a_2)^{x_2-x_4}(a_3^{x_3-x_4}a_4),$$
$$\mathcal{Y}_{(x_1x_2)(x_3x_4)}(a_1,a_2,a_3,a_4) = \left((a_1^{x_1-x_2}a_2)^{x_2-x_4}(a_3^{x_3-x_4}a_4)\right)^{x_4}1.$$

Setting $x_4 = 0$, we have

$$\mathcal{Y}_{(x_1x_2)(x_30)}(a_1,a_2,a_3,a_4) = (a_1^{x_1-x_2}a_2)^{x_2}(a_3^{x_3}a_4).$$

Proceeding in the same way for the other elements in $\mathcal{B}_4$, we obtain

$$\mathcal{Y}_{x_1(x_2(x_30))}(a_1,a_2,a_3,a_4) = a_1^{x_1}(a_2^{x_2}(a_3^{x_3}a_4))$$
$$\mathcal{Y}_{x_1((x_2x_3)0)}(a_1,a_2,a_3,a_4) = a_1^{x_1}((a_2^{x_2-x_3}a_3)^{x_3}a_4)$$
$$\mathcal{Y}_{(x_1(x_2x_3))0}(a_1,a_2,a_3,a_4) = (a_1^{x_1-x_3}(a_2^{x_2-x_3}a_3))^{x_3}a_4$$
$$\mathcal{Y}_{((x_1x_2)x_3)0}(a_1,a_2,a_3,a_4) = ((a_1^{x_1-x_2}a_2)^{x_2-x_3}a_3)^{x_3}a_4.$$

When $n \leq 2$, we have $\mathcal{Y}_{x_1}(a_1) = a_1^{x_1}1$ and $\mathcal{Y}_{x_1x_2}(a_1,a_2) = (a_1^{x_1-x_2}a_2)^{x_2}1$, and thus $\mathcal{Y}_0(a_1) = a_1$ and $\mathcal{Y}_{x_10}(a_1,a_2) = a_1^{x_1}a_2$.

The dependency of $\mathcal{X}_{B(x_1,\cdots,x_n)}(a_1,\cdots,a_n)$ on the formal variables $x_1,\cdots,x_n$ is only through their differences $x_i - x_j$ for $1 \leq i < j \leq n$, and we can thus recover $\mathcal{X}_{B(x_1,\cdots,x_n)}(a_1,\cdots,a_n)$ from $\mathcal{X}_{B(x_1,\cdots,x_{n-1},0)}(a_1,\cdots,a_n)$ by replacing x_i with $x_i - x_n$ for $1 \leq i \leq n-1$. We thus write (7) in the form

$$\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,a_n) = \mathrm{e}^{x_nT}\,\mathcal{Y}_{B(x_1-x_n,\cdots,x_{n-1}-x_n,0)}(a_1,\cdots,a_n). \tag{8}$$

Lemma 2. *Let V be a prefield algebra. Let $B \in \mathcal{B}_n$, and let $u_1,\cdots,u_m$ be nodes in $B(x_1,\cdots,x_n)$ whose leaves give a partition of $\{x_1,\cdots,x_n\}$. Let $x_{i_1},\cdots,x_{i_m}$ be the last leaves of $u_1,\cdots,u_m$, respectively, with $i_m = n$. Let $\bar{B}(x_{i_1},\cdots,x_{i_m}) \in \mathcal{B}_m(x_{i_1},\cdots,x_{i_m})$ be the contraction of $B(x_1,\cdots,x_n)$ by the nodes $u_1,\cdots,u_m$. Then, we have*

$$\mathcal{X}_{B(x_1,\cdots,x_n)}(a_1,\cdots,a_n) = \mathcal{X}_{\bar{B}(x_{i_1},\cdots,x_{i_m})}\left(\mathcal{X}_{u_1}(a_1,\cdots,a_{i_1}),\cdots,\mathcal{X}_{u_m}(a_{(i_{m-1}+1)},\cdots,a_{i_m})\right) \tag{9}$$

for all $a_1,\cdots,a_n \in V$.

Proof. We can recursively define any function f on the set of all nodes of an ordered full binary tree by defining its values on the leaves and by defining the value $f(w)$ on a node w from the values $f(u)$ and $f(v)$ on its children u and v. Let $a_1,\cdots,a_n \in V$ and $B \in \mathcal{B}_n$. We define a function $F = (F_1,F_2)$ on the set of nodes of $B(x_1,\cdots,x_n)$ in such a way. The value $F(u) = (F_1(u),F_2(u))$ on each node is a tuple, where $F_1(u)$ is some formal expression involving elements $a_1, \cdots, a_n \in V$, and $F_2(u)$ is always one of the formal variables $x_1, \cdots, x_n$. For one of the leaves, let

$$F(x_1) = (a_1, x_1), \cdots, F(x_n) = (a_n, x_n).$$

For every node $w = uv$ with children u and v, if $F(u) = (L, x_l)$ and $F(v) = (R, x_r)$, let

$$F(w) = (L^{x_l - x_r} R,\ x_r).$$

It follows that for every node w of $B(x_1, \cdots, x_n)$ with leaves $\{x_j, \cdots, x_k\}$, we have $F_1(w) = \mathcal{X}_w(a_j, \cdots, a_k)$, and $F_2(w) = x_k$. In particular, $F_1(o) = \mathcal{X}_{B(x_1,\cdots,x_n)}(a_1, \cdots, a_n)$ where o is the root of $B(x_1, \cdots, x_n)$. From the recursive definition, $F_1(o)$ is equivalently computed starting from

$$F(u_1) = (\mathcal{X}_{u_1}(a_1, \cdots, a_{i_1}),\ x_{i_1}),\ \cdots,\ F(u_m) = (\mathcal{X}_{u_m}(a_{(i_{m-1}+1)}, \cdots, a_{i_m}),\ x_{i_m})$$

on the new tree $\bar{B}(x_{i_1}, \cdots, x_{i_m})$ having $x_{i_1}, \cdots, x_{i_m}$ as leaves, and the lemma follows. □

From Lemma 2, it follows that we have

$$\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1, \cdots, a_n) = \mathcal{Y}_{\bar{B}(x_{i_1},\cdots,x_{i_m})}(\mathcal{X}_{u_1}(a_1, \cdots, a_{i_1}), \cdots, \mathcal{X}_{u_m}(a_{(i_{m-1}+1)}, \cdots, a_{i_m})) \tag{10}$$

in the same notation, by applying $(\cdot)^{x_n}1$ on both sides of (9).

Let $B(x_1, \cdots, x_n) \in \mathcal{B}_n(x_1, \cdots, x_n)$ for $n \geq 2$. Suppose leaves x_l and x_{l+1} are siblings for some $1 \leq l \leq n-1$. Let $B^l(x_1, \cdots, \hat{x}_l, \cdots, x_n)$ be the contraction of $B(x_1, \cdots, x_n)$ by the node $x_l x_{l+1}$, where $\hat{x}_l$ means x_l is omitted. By (9) and (10),

$$\begin{aligned} \mathcal{X}_{B(x_1,\cdots,x_n)}(a_1, \cdots, a_n) &= \mathcal{X}_{B^l(x_1,\cdots,\hat{x}_l,\cdots,x_n)}(a_1, \cdots, \mathcal{X}_{x_l x_{l+1}}(a_l, a_{l+1}), \cdots, a_n), \\ \mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1, \cdots, a_n) &= \mathcal{Y}_{B^l(x_1,\cdots,\hat{x}_l,\cdots,x_n)}(a_1, \cdots, \mathcal{X}_{x_l x_{l+1}}(a_l, a_{l+1}), \cdots, a_n). \end{aligned} \tag{11}$$

The relationships (11) give another recursive description of the maps $\mathcal{X}_B$ and $\mathcal{Y}_B$ for $B \in \mathcal{B}_n$ starting with $\mathcal{X}_{x_1}(a_1) = a_1$, $\mathcal{X}_{x_1 x_2}(a_1, a_2) = a_1^{x_1 - x_2} a_2$ and $\mathcal{Y}_{x_1}(a_1) = \mathrm{e}^{x_1 T} a_1$ for all $a_1, a_2 \in V$, because every tree $B(x_1, \cdots, x_n) \in \mathcal{B}_n(x_1, \cdots, x_n)$ for $n \geq 2$ contains at least one pair of leaves x_l and x_{l+1}, which are siblings.

For $B \in \mathcal{B}_n$ and a vector space V over $\mathbb{C}$, let us define the vector spaces $V_{B(x_1,\cdots,x_n)}$ with the following properties:

(i) For $x_1 \in \mathcal{B}_1(x_1)$, $V_{x_1} = V$.

(ii) For $B(x_1, \cdots, x_n) \in \mathcal{B}_n(x_1, \cdots, x_n)$, let $\{x_{l_1}, x_{(l_1+1)}\}, \cdots, \{x_{l_m}, x_{(l_m+1)}\}$ be all pairs of siblings among the leaves of $B(x_1, \cdots, x_n)$. Let $\bar{B}(x_1, \cdots, \hat{x}_{l_1}, \cdots, \hat{x}_{l_m}, \cdots, x_n)$ be the contraction of $B(x_1, \cdots, x_n)$ by the nodes $x_{l_1} x_{(l_1+1)}, \cdots, x_{l_m} x_{(l_m+1)}$. Then,

$$V_{B(x_1,\cdots,x_n)} = V_{\bar{B}(x_1,\cdots,\hat{x}_{l_1},\cdots,\hat{x}_{l_m},\cdots,x_n)}((x_{l_1} - x_{(l_1+1)}, \cdots, x_{l_m} - x_{(l_m+1)})).$$

It follows from Lemma 2 that we have

$$\begin{aligned} \mathcal{X}_{B(x_1,\cdots,x_n)} &: V^{\times n} \to V_{B(x_1,\cdots,x_n)}, \\ \mathcal{Y}_{B(x_1,\cdots,x_n)} &: V^{\times n} \to V_{B(x_1,\cdots,x_n)}[[x_n]]. \end{aligned}$$

We can recursively verify that $V_{B(x_1,\cdots,x_n)}$ contains $V[[x_1 - x_n, x_2 - x_n, \cdots, x_{n-1} - x_n]]$ as a subspace. In particular, $\mathbb{C}_{B(x_1,\cdots,x_n)}$ is an algebra over $\mathbb{C}[[x_1 - x_n, x_2 - x_n, \cdots, x_{n-1} - x_n]]$, and $V_{B(x_1,\cdots,x_n)}$ is a module over $\mathbb{C}_{B(x_1,\cdots,x_n)}$. Hence, $V_{B(x_1,\cdots,x_n)}[[x_n]]$ contains $V[[x_1, \cdots, x_n]]$ as a subspace. It is a module over $\mathbb{C}_{B(x_1,\cdots,x_n)}[[x_n]]$ that contains $\mathbb{C}[[x_1, \cdots, x_n]]$ as a subalgebra.

Example 2. *For a vector space V over $\mathbb{C}$, we have $V_{x_2x_4} = V((x_2 - x_4))$, and*

$$V_{(x_1x_2)(x_3x_4)} = V_{x_2x_4}((x_1 - x_2, x_3 - x_4)) = V((x_2 - x_4))((x_1 - x_2, x_3 - x_4)).$$

Hence, $V_{(x_1x_2)(x_3x_4)}$ contains $V[[x_2 - x_4]][[x_1 - x_2, x_3 - x_4]] = V[[x_1 - x_4, x_2 - x_4, x_3 - x_4]]$, and it is a module over $\mathbb{C}_{(x_1x_2)(x_3x_4)} = \mathbb{C}((x_2 - x_4))((x_1 - x_2, x_3 - x_4))$. If V is a prefield algebra and $a_1, a_2, a_3, a_4 \in V$,

$$\mathcal{X}_{(x_1x_2)(x_3x_4)}(a_1, a_2, a_3, a_4) = (a_1^{x_1-x_2}a_2)^{x_2-x_4}(a_3^{x_3-x_4}a_4) \in V_{(x_1x_2)(x_3x_4)},$$
$$\mathcal{Y}_{(x_1x_2)(x_3x_4)}(a_1, a_2, a_3, a_4) \in V_{(x_1x_2)(x_3x_4)}[[x_4]].$$

$V_{(x_1x_2)(x_3x_4)}[[x_4]]$ is a module over $\mathbb{C}_{(x_1x_2)(x_3x_4)}[[x_4]]$ and contains

$$V[[x_1 - x_4, x_2 - x_4, x_3 - x_4]][[x_4]] = V[[x_1, x_2, x_3, x_4]]$$

as a submodule over $\mathbb{C}[[x_1, x_2, x_3, x_4]]$.

3.3. Definition of n-Associativity

Definition 3. *Let V be a prefield algebra. For $n \geq 1$, we say that the vectors $a_1, \cdots, a_n \in V$ are* n-associative *if there exist $\Psi(x_1, \cdots, x_n) \in V[[x_1, \cdots, x_n]]$ and $N \in \mathbb{N}$ such that*

$$\Big(\prod_{1\leq i<j\leq n}(x_i - x_j)\Big)^N \mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1, \cdots, a_n) = \Psi(x_1, \cdots, x_n) \tag{12}$$

for all $B \in \mathcal{B}_n$. We say that V is n-associative *if every sequence of n vectors in V is n-associative. V is called* ∞-associative *if it is n-associative for every $n \geq 3$.*

Every prefield algebra is automatically 1 and 2-associative. The reason that Definition 3 includes the cases $n \leq 2$ is for later convenience. We use the notation $\mathfrak{s}_n(x_1, \cdots, x_n)$ to denote

$$\mathfrak{s}_n(x_1, \cdots, x_n) = \prod_{1\leq i<j\leq n}(x_i - x_j)$$

with $\mathfrak{s}_1(x_1) = 1$ and any empty product is defined as 1. Then,

$$\mathfrak{s}_n(x_1, \cdots, x_{n-1}, 0) = \prod_{k=1}^{n-1} x_k \prod_{1\leq i<j\leq n-1}(x_i - x_j),$$

and

$$\mathfrak{s}_n(x_1, \cdots, x_n) = \mathfrak{s}_n(x_1 - x_n, \cdots, x_{n-1} - x_n, 0).$$

The following lemma shows that we can set $x_n = 0$ in Definition 3 for simplicity.

Lemma 3. *Let V be a prefield algebra. The vectors $a_1, \cdots, a_n \in V$, and $n \geq 1$ are* n-associative *if and only if there exist $\Phi(x_1, \cdots, x_{n-1}) \in V[[x_1, \cdots, x_{n-1}]]$ ($\Phi \in V$ if $n = 1$), and $N \in \mathbb{N}$ such that*

$$\mathfrak{s}_n(x_1, \cdots, x_{n-1}, 0)^N \mathcal{Y}_{B(x_1,\cdots,x_{n-1},0)}(a_1, \cdots, a_n) = \Phi(x_1, \cdots, x_{n-1}) \tag{13}$$

for all $B \in \mathcal{B}_n$.

Proof. If $a_1, \cdots, a_n \in V$ are n-associative, Equation (13) holds by setting $x_n = 0$ in (12). Conversely, suppose (13) holds. By replacing x_i with $x_i - x_n$ for $1 \leq i \leq n-1$, we have

$$\mathfrak{s}_n(x_1, \cdots, x_n)^N \mathcal{Y}_{B(x_1-x_n,\cdots,x_{n-1}-x_n,0)}(a_1, \cdots, a_n) = \Phi(x_1 - x_n, \cdots, x_{n-1} - x_n)$$

for all $B \in \mathcal{B}_n$. By (8), we see that

$$\begin{aligned}\mathfrak{s}_n(x_1,\cdots,x_n)^N\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,a_n)\\ &=\mathfrak{s}_n(x_1,\cdots,x_n)^N\,e^{x_nT}\,\mathcal{Y}_{B(x_1-x_n,\cdots,x_{n-1}-x_n,0)}(a_1,\cdots,a_n)\\ &=e^{x_nT}\left(\mathfrak{s}_n(x_1,\cdots,x_n)^N\mathcal{Y}_{B(x_1-x_n,\cdots,x_{n-1}-x_n,0)}(a_1,\cdots,a_n)\right)\\ &=e^{x_nT}\,\Phi(x_1-x_n,\cdots,x_{n-1}-x_n)\end{aligned}$$

for all $B\in\mathcal{B}_n$, and (12) holds with $\Psi(x_1,\cdots,x_n)=e^{x_nT}\,\Phi(x_1-x_n,\cdots,x_{n-1}-x_n)$. □

Let V be a prefield algebra. The condition (13) in Lemma 3 with $n=3$ is precisely the associativity axiom in Definition 1, and (13) with $n=4$ is the condition of 4-associativity stated in Section 1.

Lemma 4. *Let V be a prefield algebra. Suppose $a_1,\cdots,a_n\in V$, and $n\geq 2$ are n-associative. Then, for $1\leq l\leq n-1$, the elements $a_1,\cdots,a_{l\,(m)}a_{l+1},\cdots,a_n$ are $(n-1)$-associative for each $m\in\mathbb{Z}$, where $\cdot_{(m)}\cdot$ is given by (1).*

Proof. For $1\leq l\leq n-1$, let $\mathcal{B}_{n,l}(x_1,\cdots,x_n)$ be the set of all $B(x_1,\cdots,x_n)\in\mathcal{B}_n(x_1,\cdots,x_n)$ such that the leaves x_l and x_{l+1} are siblings. Let $B(x_1,\cdots,x_n)\in\mathcal{B}_{n,l}(x_1,\cdots,x_n)$. We have

$$\mathcal{X}_{x_lx_{l+1}}(a_l,a_{l+1})=a_l^{x_l-x_{l+1}}a_{l+1}=\sum_{m\in\mathbb{Z}}(x_l-x_{l+1})^{-1-m}a_{l\,(m)}a_{l+1},$$

and by (11),

$$\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,a_n)=\sum_{m\in\mathbb{Z}}(x_l-x_{l+1})^{-1-m}\mathcal{Y}_{B^l(x_1,\cdots,\hat{x}_l,\cdots,x_n)}(a_1,\cdots,a_{l\,(m)}a_{l+1},\cdots,a_n)$$

where $B^l(x_1,\cdots,\hat{x}_l,\cdots,x_n)\in\mathcal{B}_{n-1}(x_1,\cdots,\hat{x}_l,\cdots,x_n)$ is the contraction of $B(x_1,\cdots,x_n)$ by x_lx_{l+1}. Suppose we have (12) for $a_1,\cdots,a_n$. We have $\Psi(x_1,\cdots,x_n)\in V[[x_1,\cdots,x_n]]$ and $N\in\mathbb{N}$ such that

$$\begin{aligned}\Psi(x_1,\cdots,x_n)&=\mathfrak{s}_n(x_1,\cdots,x_n)^N\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,a_n)\\ &=\sum_{m\in\mathbb{Z}}(x_l-x_{l+1})^{N-1-m}\prod_{1\leq i<l}(x_i-x_l)^N\prod_{l+1<j\leq n}(x_l-x_j)^N\\ &\quad\times\mathfrak{s}_{n-1}(x_1,\cdots,\hat{x}_l,\cdots,x_n)^N\mathcal{Y}_{B^l(x_1,\cdots,\hat{x}_l,\cdots,x_n)}(a_1,\cdots,a_{l\,(m)}a_{l+1},\cdots,a_n)\end{aligned}\tag{14}$$

for all $B(x_1,\cdots,x_n)\in\mathcal{B}_{n,l}(x_1,\cdots,x_n)$, where $N-1-m\geq 0$ in the sum. Let $1\leq i\leq n$ be such that $i\notin\{l,l+1\}$. The last line in (14) is independent of x_l. By writing $x_l-x_j=(x_l-x_i)+(x_i-x_j)$ for $j\notin\{i,l\}$ in (14), we obtain an expansion of $\Psi(x_1,\cdots,x_n)\in V[[x_1,\cdots,x_n]]$ in $V[[x_1,\cdots,\hat{x}_l,\cdots,x_n]][[x_l-x_i]]$, which shows $\Psi(x_1,\cdots,x_n)\in(x_l-x_i)^NV[[x_1,\cdots,x_n]]$. Hence, both sides of (14) as a power series in $V[[x_1,\cdots,x_n]]$ contain the factor $\prod_{1\leq i<l}(x_i-x_l)^N\prod_{l+1<j\leq n}(x_l-x_j)^N$. By canceling this factor in (14), we obtain

$$\begin{aligned}\Phi(x_1,\cdots,x_n)=\sum_{m\in\mathbb{Z}}(x_l-x_{l+1})^{N-1-m}\mathfrak{s}_{n-1}(x_1,\cdots,\hat{x}_l,\cdots,x_n)^N\\ \times\mathcal{Y}_{B^l(x_1,\cdots,\hat{x}_l,\cdots,x_n)}(a_1,\cdots,a_{l\,(m)}a_{l+1},\cdots,a_n)\end{aligned}$$

for some $\Phi(x_1,\cdots,x_n)\in V[[x_1,\cdots,x_n]]$. Expanding

$$\Phi(x_1,\cdots,x_n)=\sum_{k\geq 0}(x_l-x_{l+1})^k\Phi_k(x_1,\cdots,\hat{x}_l,\cdots,x_n)$$

for $\Phi_k(x_1,\cdots,\hat{x}_l,\cdots,x_n)\in V[[x_1,\cdots,\hat{x}_l,\cdots,x_n]]$, we have

$$\begin{aligned}\mathfrak{s}_{n-1}(x_1,\cdots,\hat{x}_l,\cdots,x_n)^N\mathcal{Y}_{B^l(x_1,\cdots,\hat{x}_l,\cdots,x_n)}(a_1,\cdots,a_{l\,(m)}a_{l+1},\cdots,a_n)\\ =\Phi_{N-1-m}(x_1,\cdots,\hat{x}_l,\cdots,x_n)\end{aligned}$$

for all $B(x_1,\cdots,x_n) \in \mathcal{B}_{n,l}(x_1,\cdots,x_n)$, and the contractions $B^l(x_1,\cdots,\hat{x}_l,\cdots,x_n)$ give all elements of $\mathcal{B}_{n-1}(x_1,\cdots,\hat{x}_l,\cdots,x_n)$. □

Lemma 5. *Let V be a prefield algebra. Then,*

$$\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,Ta_l,\cdots,a_n) = \partial_{x_l}\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,a_n)$$

for $a_1,\cdots,a_n \in V$ and $B \in \mathcal{B}_n$ with $1 \le l \le n$ and $n \ge 1$.

Proof. Let us show the claim by induction on n. For $n = 1$, $\mathcal{Y}_{x_1}(Ta_1) = (Ta_1)^{x_1}1 = \partial_{x_1}(a_1^{x_1}1) = \partial_{x_1}\mathcal{Y}_{x_1}(a_1)$. Consider the $n > 1$ case. If there exist siblings x_i and x_{i+1} such that $l \notin \{i, i+1\}$, by (11) and the inductive hypothesis,

$$\begin{aligned}
&\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,Ta_l,\cdots,a_n)\\
&\quad= \mathcal{Y}_{B^i(x_1,\cdots,\hat{x}_i,\cdots,x_n)}(a_1,\cdots,\mathcal{X}_{x_ix_{i+1}}(a_i,a_{i+1}),\cdots,Ta_l,\cdots,a_n)\\
&\quad= \partial_{x_l}\mathcal{Y}_{B^i(x_1,\cdots,\hat{x}_i,\cdots,x_n)}(a_1,\cdots,\mathcal{X}_{x_ix_{i+1}}(a_i,a_{i+1}),\cdots,a_l,\cdots,a_n)\\
&\quad= \partial_{x_l}\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,a_l,\cdots,a_n).
\end{aligned}$$

It only remains to consider the case when x_l has another leaf as a sibling. If x_l and x_{l+1} are siblings, by translation covariance,

$$\begin{aligned}
\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,Ta_l,\cdots,a_n) &= \mathcal{Y}_{B^l(x_1,\cdots,\hat{x}_l,\cdots,x_n)}(a_1,\cdots,(Ta_l)^{x_l-x_{l+1}}a_{l+1},\cdots,a_n)\\
&= \mathcal{Y}_{B^l(x_1,\cdots,\hat{x}_l,\cdots,x_n)}(a_1,\cdots,\partial_{x_l}({a_l}^{x_l-x_{l+1}}a_{l+1}),\cdots,a_n)\\
&= \partial_{x_l}\mathcal{Y}_{B^l(x_1,\cdots,\hat{x}_l,\cdots,x_n)}(a_1,\cdots,{a_l}^{x_l-x_{l+1}}a_{l+1},\cdots,a_n)\\
&= \partial_{x_l}\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,a_l,\cdots,a_n).
\end{aligned}$$

If x_{l-1} and x_l are siblings, by the inductive hypothesis and translation covariance,

$$\begin{aligned}
&\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,Ta_l,\cdots,a_n)\\
&\quad= \mathcal{Y}_{B^{(l-1)}(x_1,\cdots,\hat{x}_{l-1},x_l,\cdots,x_n)}(a_1,\cdots,{a_{l-1}}^{x_{l-1}-x_l}(Ta_l),\cdots,a_n)\\
&\quad= \mathcal{Y}_{B^{(l-1)}(x_1,\cdots,\hat{x}_{l-1},x_l,\cdots,x_n)}(a_1,\cdots,(T-\partial_{x_{l-1}})({a_{l-1}}^{x_{l-1}-x_l}a_l),\cdots,a_n)\\
&\quad= (\partial_{t_l}-\partial_{x_{l-1}})\mathcal{Y}_{B^{(l-1)}(x_1,\cdots,\hat{x}_{l-1},t_l,\cdots,x_n)}(a_1,\cdots,{a_{l-1}}^{x_{l-1}-x_l}a_l,\cdots,a_n)\big|_{t_l=x_l}\\
&\quad= \partial_{x_l}\mathcal{Y}_{B^{(l-1)}(x_1,\cdots,\hat{x}_{l-1},x_l,\cdots,x_n)}(a_1,\cdots,{a_{l-1}}^{x_{l-1}-x_l}a_l,\cdots,a_n)\\
&\quad= \partial_{x_l}\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,a_l,\cdots,a_n),
\end{aligned}$$

and the lemma follows. □

Lemma 6. *Let V be a prefield algebra, and suppose $a_1,\cdots,a_n \in V$ and $n \ge 1$ are n-associative. Then, $a_1,\cdots,Ta_l,\cdots,a_n$ are n-associative, for any $1 \le l \le n$.*

Proof. Suppose we have $\Psi(x_1,\cdots,x_n) \in V[[x_1,\cdots,x_n]]$ and $N \in \mathbb{N}$ such that

$$\Psi(x_1,\cdots,x_n) = \mathfrak{s}_n(x_1,\cdots,x_n)^N\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,a_n) \tag{15}$$

for all $B \in \mathcal{B}_n$. By applying $\mathfrak{s}_n(x_1,\cdots,x_n)\partial_{x_l}$ to both sides, we get

$$\begin{aligned}
&\mathfrak{s}_n(x_1,\cdots,x_n)\partial_{x_l}\Psi(x_1,\cdots,x_n)\\
&\quad= N(\partial_{x_l}\mathfrak{s}_n(x_1,\cdots,x_n))\mathfrak{s}_n(x_1,\cdots,x_n)^N\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,a_n)\\
&\qquad+ \mathfrak{s}_n(x_1,\cdots,x_n)^{N+1}\partial_{x_l}\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,a_n)
\end{aligned}$$

and by (15) and Lemma 5, we have

$$\mathfrak{s}_n(x_1,\cdots,x_n)^{N+1}\mathcal{Y}_{B(x_1,\cdots,x_n)}(a_1,\cdots,Ta_l,\cdots,a_n)$$
$$=\mathfrak{s}_n(x_1,\cdots,x_n)\partial_{x_l}\Psi(x_1,\cdots,x_n)-N(\partial_{x_l}\mathfrak{s}_n(x_1,\cdots,x_n))\Psi(x_1,\cdots,x_n)$$

for all $B\in\mathcal{B}_n$. □

Lemma 7. *Let V be a prefield algebra, and suppose we have $a_1,\cdots,a_n\in V$, $n\geq 2$, with $a_i=1$ for some $1\leq i\leq n$. Then, they are n-associative if and only if $a_1,\cdots,\hat{a}_i,\cdots,a_n$ are $(n-1)$-associative, where $\hat{a}_i$ means that a_i is omitted.*

Proof. The direct implication follows from Lemma 4 because $a_{(-1)}1=1_{(-1)}a=a$ for any $a\in V$. Conversely, suppose $a_1,\cdots,\hat{a}_i,\cdots,a_n$ are $(n-1)$-associative. We have $\Psi(x_1,\cdots,\hat{x}_i,\cdots,x_n)\in V[[x_1,\cdots,\hat{x}_i,\cdots,x_n]]$ and $N\in\mathbb{N}$ such that

$$\mathfrak{s}_{n-1}(x_1,\cdots,\hat{x}_i,\cdots,x_n)^N\mathcal{Y}_{B(x_1,\cdots,\hat{x}_i,\cdots,x_n)}(a_1,\cdots,\hat{a}_i,\cdots,a_n)=\Psi(x_1,\cdots,\hat{x}_i,\cdots,x_n) \tag{16}$$

for all $B\in\mathcal{B}_{n-1}$. Let $C(x_1,\cdots,x_n)\in\mathcal{B}_n(x_1,\cdots,x_n)$ and consider

$$\mathfrak{s}_{n-1}(x_1,\cdots,\hat{x}_i,\cdots,x_n)^N\mathcal{Y}_{C(x_1,\cdots,x_i,\cdots,x_n)}(a_1,\cdots,1,\cdots,a_n). \tag{17}$$

Because $n\geq 2$, x_i is not the root in $C(x_1,\cdots,x_n)$; hence, it is either a left child or a right child. If it is a left child, we have

$$w=x_iv$$

for some nodes w,v in $C(x_1,\cdots,x_n)$, where v has leaves $x_{i+1},\cdots,x_j$. Let $\bar{C}(x_1,\cdots,x_{i-1},x_j,\cdots,x_n)$ be the contraction of $C(x_1,\cdots,x_i,\cdots,x_n)$ by w, and let

$$C^*(x_1,\cdots,\hat{x}_i,\cdots,x_n)\in\mathcal{B}_{n-1}(x_1,\cdots,\hat{x}_i,\cdots,x_n)$$

be the tree obtained by replacing the subtree w by the subtree v in $C(x_1,\cdots,x_n)$. Because

$$\mathcal{X}_w(a_i,a_{i+1},\cdots,a_j)=1^{x_i-x_j}\mathcal{X}_v(a_{i+1},\cdots,a_j)=\mathcal{X}_v(a_{i+1},\cdots,a_j),$$

by Lemma 2, we have

$$\begin{aligned}\mathcal{Y}_{C(x_1,\cdots,x_i,\cdots,x_n)}&(a_1,\cdots,1,\cdots,a_n)\\&=\mathcal{Y}_{\bar{C}(x_1,\cdots,x_{i-1},x_j,\cdots,x_n)}(a_1,\cdots,\mathcal{X}_w(a_i,a_{i+1},\cdots,a_j),\cdots,a_n)\\&=\mathcal{Y}_{\bar{C}(x_1,\cdots,x_{i-1},x_j,\cdots,x_n)}(a_1,\cdots,\mathcal{X}_v(a_{i+1},\cdots,a_j),\cdots,a_n)\\&=\mathcal{Y}_{C^*(x_1,\cdots,\hat{x}_i,\cdots,x_n)}(a_1,\cdots,\hat{a}_i,\cdots,a_n)\end{aligned}$$

and, therefore, Equation (17) equals

$$\mathfrak{s}_{n-1}(x_1,\cdots,\hat{x}_i,\cdots,x_n)^N\mathcal{Y}_{C^*(x_1,\cdots,\hat{x}_i,\cdots,x_n)}(a_1,\cdots,\hat{a}_i,\cdots,a_n)=\Psi(x_1,\cdots,\hat{x}_i,\cdots,x_n)$$

by (16). If x_i is a right child, then

$$w=ux_i$$

for some nodes w,u in $C(x_1,\cdots,x_n)$, where u has leaves $x_k,\cdots,x_{i-1}$. Let $\bar{C}(x_1,\cdots,x_{k-1},x_i,\cdots,x_n)$ be the contraction of $C(x_1,\cdots,x_i,\cdots,x_n)$ by w, and let $C^*(x_1,\cdots,\hat{x}_i,\cdots,x_n)$ be the tree obtained by replacing the subtree w by the subtree u in $C(x_1,\cdots,x_n)$. Because

$$\mathcal{X}_w(a_k,\cdots,a_{i-1},a_i)=\mathcal{X}_u(a_k,\cdots,a_{i-1})^{x_{i-1}-x_i}1=\mathrm{e}^{(x_{i-1}-x_i)T}\,\mathcal{X}_u(a_k,\cdots,a_{i-1}),$$

we have, by Lemmas 2 and 5,

$$\begin{aligned}
&\mathcal{Y}_{C(x_1,\cdots,x_i,\cdots,x_n)}(a_1,\cdots,1,\cdots,a_n)\\
&\quad=\mathcal{Y}_{\bar{C}(x_1,\cdots,x_{k-1},x_i,\cdots,x_n)}(a_1,\cdots,\mathcal{X}_w(a_k,\cdots,a_{i-1},a_i),\cdots,a_n)\\
&\quad=\mathcal{Y}_{\bar{C}(x_1,\cdots,x_{k-1},x_i,\cdots,x_n)}(a_1,\cdots,\mathrm{e}^{(x_{i-1}-x_i)T}\,\mathcal{X}_u(a_k,\cdots,a_{i-1}),\cdots,a_n)\\
&\quad=\mathrm{e}^{(x_{i-1}-x_i)\partial_{x_i}}\,\mathcal{Y}_{\bar{C}(x_1,\cdots,x_{k-1},x_i,\cdots,x_n)}(a_1,\cdots,\mathcal{X}_u(a_k,\cdots,a_{i-1}),\cdots,a_n).
\end{aligned}$$

Because u has leaves $x_k,\cdots,x_{i-1}$, we obtain $\mathcal{X}_{\bar{C}(x_1,\cdots,x_{k-1},x_i,\cdots,x_n)}(a_1,\cdots,\mathcal{X}_u(a_k,\cdots,a_{i-1}),$ $\cdots,a_n)$ from

$$\begin{aligned}
&\mathcal{X}_{\bar{C}(x_1,\cdots,x_{k-1},x_{i-1},\cdots,x_n)}(a_1,\cdots,\mathcal{X}_u(a_k,\cdots,a_{i-1}),\cdots,a_n)\\
&\qquad=\mathcal{X}_{C^*(x_1,\cdots,\hat{x}_i,\cdots,x_n)}(a_1,\cdots,\hat{a}_i,\cdots,a_n)
\end{aligned}\tag{18}$$

by replacing x_j-x_{i-1} to x_j-x_i when x_j is not a leaf of u. The variables x_j for $k\le j<i-1$ only appear inside $\mathcal{X}_u(a_k,\cdots,a_{i-1})$ in (18). Hence, we can replace x_j to $x_j+x_{i-1}-x_i$ for all $j\in J=\{j\mid j<k \text{ or } j>i\}$ in (18). For $j,\ l\in J$, x_j-x_{i-1} becomes x_j-x_i, and x_j-x_l is left-invariant. If $n\in J$, then

$$\begin{aligned}
&\mathcal{Y}_{\bar{C}(x_1,\cdots,x_{k-1},x_i,\cdots,x_n)}(a_1,\cdots,\mathcal{X}_u(a_k,\cdots,a_{i-1}),\cdots,a_n)\\
&\quad=\mathrm{e}^{x_nT}\,\mathcal{X}_{\bar{C}(x_1,\cdots,x_{k-1},x_i,\cdots,x_n)}(a_1,\cdots,\mathcal{X}_u(a_k,\cdots,a_{i-1}),\cdots,a_n)\\
&\quad=\mathrm{e}^{(x_i-x_{i-1})T}\,\big(\mathcal{Y}_{C^*(x_1,\cdots,\hat{x}_i,\cdots,x_n)}(a_1,\cdots,\hat{a}_i,\cdots,a_n)\big|_{x_j\mapsto x_j+x_{i-1}-x_i,\,j\in J}\big),
\end{aligned}\tag{19}$$

and if $n\notin J$ (if $i=n$), $\mathrm{e}^{(x_i-x_{i-1})T}=\mathrm{e}^{(x_n-x_{n-1})T}$ in (19). In both cases, we have

$$\begin{aligned}
&\mathrm{e}^{(x_{i-1}-x_i)\partial_{x_i}}\mathcal{Y}_{\bar{C}(x_1,\cdots,x_{k-1},x_i,\cdots,x_n)}(a_1,\cdots,\mathcal{X}_u(a_k,\cdots,a_{i-1}),\cdots,a_n)\\
&\qquad=\mathrm{e}^{(x_{i-1}-x_i)\partial_{x_i}}\,\big(\mathcal{Y}_{C^*(x_1,\cdots,\hat{x}_i,\cdots,x_n)}(a_1,\cdots,\hat{a}_i,\cdots,a_n)\big|_{x_j\mapsto x_j+x_{i-1}-x_i,\,j\in J}\big).
\end{aligned}$$

Because for any $f(x_1,\cdots,\hat{x}_i,\cdots,x_n)\in V[[x_1,\cdots,\hat{x}_i,\cdots,x_n]]$,

$$\mathrm{e}^{(x_{i-1}-x_i)\partial_{x_i}}\,\big(f(x_1,\cdots,\hat{x}_i,\cdots,x_n)\big|_{x_j\mapsto x_j+x_{i-1}-x_i,\,j\in J}\big)=f(x_1,\cdots,\hat{x}_i,\cdots,x_n),$$

Equation (17) also equals

$$\begin{aligned}
&\mathfrak{s}_{n-1}(x_1,\cdots,\hat{x}_i,\cdots,x_n)^N\mathcal{Y}_{C(x_1,\cdots,x_i,\cdots,x_n)}(a_1,\cdots,1,\cdots,a_n)\\
&\quad=\mathrm{e}^{(x_{i-1}-x_i)\partial_{x_i}}\Big(\big(\mathfrak{s}_{n-1}(x_1,\cdots,\hat{x}_i,\cdots,x_n)^N\\
&\qquad\qquad\times\mathcal{Y}_{C^*(x_1,\cdots,\hat{x}_i,\cdots,x_n)}(a_1,\cdots,\hat{a}_i,\cdots,a_n)\big)\big|_{x_j\mapsto x_j+x_{i-1}-x_i,\,j\in J}\Big)\\
&\quad=\mathrm{e}^{(x_{i-1}-x_i)\partial_{x_i}}\,\big(\Psi(x_1,\cdots,\hat{x}_i,\cdots,x_n)\big|_{x_j\mapsto x_j+x_{i-1}-x_i,\,j\in J}\big)\\
&\quad=\Psi(x_1,\cdots,\hat{x}_i,\cdots,x_n)
\end{aligned}$$

and the lemma follows. □

4. $\mathcal{F}_n\supsetneq\mathcal{F}_{n+1}$

Let $\mathcal{F}_n$ be the class of prefield algebras that are n-associative. We now construct an example of a field algebra V belonging to $\mathcal{F}_n$ but not to $\mathcal{F}_{n+1}$, $n\ge 3$. It is a simple example generated by a single element a.

Theorem 1. *Let $n \geq 3$. Let $\mathbb{C}[T]a$ be a free $\mathbb{C}[T]$-module generated by an element a. Let $\mathcal{A} = \mathcal{T}^*(\mathbb{C}[T]a)$ be the tensor algebra of $\mathbb{C}[T]a$, a free associative $\mathbb{C}$-algebra generated by $\mathbb{C}[T]a$. We define a grading on $\mathcal{A}$ by setting $\deg a = 1$ and $\deg T = 0$, so that*

$$\mathcal{A} = \bigoplus_{k \geq 0} \left(\mathbb{C}[T]a\right)^{\otimes k}$$

and each $\mathcal{A}_k = \left(\mathbb{C}[T]a\right)^{\otimes k}$ is a $\mathbb{C}[T]$-module, where T acts as a derivation of $\mathcal{A}$. Hence, $\mathbb{C}[T]$ acts on $\mathcal{A}_k$ via the k-fold comultiplication map

$$\Delta_k : \mathbb{C}[T] \to \mathbb{C}[T]^{\otimes k}$$

given by $\Delta_k(T) = \sum_{i=1}^k T_i$, with $T_i = 1 \otimes \cdots \otimes T \otimes \cdots \otimes 1$ having T at the ith position, and T is the zero map on $\mathcal{A}_0 = \mathbb{C}$. We consider the ideal

$$\mathcal{I}_{\geq n+2} = \bigoplus_{k \geq n+2} \mathcal{A}_k,$$

consisting of all elements of degree $\geq n+2$, which is closed under T, and define

$$V = \mathcal{A}/\mathcal{I}_{\geq n+2}.$$

V is naturally an associative $\mathbb{C}$ algebra with a derivation T. We define a prefield algebra structure on V as follows: We have the grading on V given by

$$V = \bigoplus_{k=0}^{n+1} V_k$$

and each V_k is identified with $\left(\mathbb{C}[T]a\right)^{\otimes k}$. Choose any element

$$\begin{aligned} &G_1(x_1, \cdots, x_n) \in V_{n+1}((x_1))[[x_2, \cdots, x_n]], \\ &G_k(x_1, \cdots, x_n) \in V_{n+1}((x_k))[[x_1 - x_k, \cdots, x_{k-1} - x_k, x_{k+1}, \cdots, x_n]] \end{aligned} \tag{20}$$

for $2 \leq k \leq n$. Let $1 \in \mathbb{C} = V_0$ be the identity element of the prefield algebra. To give the map

$$\cdot^x \cdot : V \times V \to V((x)),$$

we define it for graded components of V. We have

$$(V \times V)_k = \bigoplus_{l+m=k} V_l \times V_m, \quad 0 \leq l, m \leq n+1.$$

(i) *For $(V \times V)_k$ where $k \leq n$, for all $u \in V_l$ and $v \in V_m$ with $l + m \leq n$, we define*

$$u^x v = \mathrm{e}^{xT}\, u \otimes v \in V_{l+m}((x)).$$

(ii) *For $(V \times V)_k$ with $k \geq n+2$, we let $\cdot^x \cdot : (V \times V)_k \to V((x))$ be the zero map. Hence, when the sum of the degrees exceeds $n+1$, the product vanishes.*

(iii) *It now remains to specify $\cdot^x \cdot$ on*

$$(V \times V)_{n+1} = (V_0 \times V_{n+1}) \oplus (V_1 \times V_n) \oplus \cdots \oplus (V_{n+1} \times V_0).$$

For $(1, v) \in V_0 \times V_{n+1}$, let $1^x v = v$. For $(u, 1) \in V_{n+1} \times V_0$, let $u^x 1 = \mathrm{e}^{xT}\, u$.

(iv) *For the rest of $V_l \times V_m$ with $l + m = n + 1$, $l, m \geq 1$, we define the product on the following $\mathbb{C}[T]$-module generators and let it be generated by the relation $(Ta)^x b = \partial_x(a^x b)$ and $a^x(Tb) = (T - \partial_x)(a^x b)$ to all of $V_l \otimes V_m$. Take*

$$T^{(j_1)}a \otimes \cdots \otimes T^{(j_{k-1})}a \otimes a$$

with $j_1, \cdots, j_{k-1} \geq 0$ as generators of V_k. For $V_1 \otimes V_n$, let

$$a^{x_1}\big(T^{(j_1)}a \otimes \cdots \otimes T^{(j_{n-1})}a \otimes a\big) = c^{(1)}_{j_1 \cdots j_{n-1}}(x_1),$$

where $c^{(1)}_{j_1 \cdots j_{n-1}}(x_1)$ is the coefficient of $x_2^{j_1} \cdots x_n^{j_{n-1}}$ in

$$G_1(x_1, \cdots, x_n) \in V_{n+1}((x_1))[[x_2, \cdots, x_n]].$$

(v) *For $V_k \times V_{n-k+1}$ for $2 \leq k \leq n$, we define*

$$\big(T^{(j_1)}a \otimes \cdots \otimes T^{(j_{k-1})}a \otimes a\big)^{x_k}\big(T^{(j_k)}a \otimes \cdots \otimes T^{(j_{n-1})}a \otimes a\big) = c^{(k)}_{j_1 \cdots j_{n-1}}(x_k)$$

where $c^{(k)}_{j_1 \cdots j_{n-1}}(x_k)$ is the coefficient of $(x_1 - x_k)^{j_1} \cdots (x_{k-1} - x_k)^{j_{k-1}} x_{k+1}^{j_k} \cdots x_n^{j_{n-1}}$ in

$$G_k(x_1, \cdots, x_n) \in V_{n+1}((x_k))[[x_1 - x_k, \cdots, x_{k-1} - x_k, x_{k+1}, \cdots, x_n]].$$

Then, V is a prefield algebra with the following properties:

(a) *The map $\cdot^x\cdot : V \times V \to V((x))$ is degree-preserving.*

(b) *If $u_1, \cdots, u_j \in V$ are such that $\sum_{i=1}^{j} \deg u_i \leq n$, then*

$$\mathcal{X}_{B(x_1,\cdots,x_j)}(u_1, u_2, \cdots, u_j) = e^{(x_1 - x_j)T}\, u_1 \otimes e^{(x_2 - x_j)T}\, u_2 \otimes \cdots \otimes e^{(x_{j-1} - x_j)T}\, u_{j-1} \otimes u_j,$$
$$\mathcal{Y}_{B(x_1,\cdots,x_j)}(u_1, u_2, \cdots, u_j) = e^{x_1 T}\, u_1 \otimes e^{x_2 T}\, u_2 \otimes \cdots \otimes e^{x_j T}\, u_j$$

for all $B \in \mathcal{B}_j$.

(c) *If $u_1, \cdots, u_j \in V$ are such that $\sum_{i=1}^{j} \deg u_i \geq n + 2$, then*

$$\mathcal{X}_{B(x_1,\cdots,x_j)}(u_1, u_2, \cdots, u_j) = \mathcal{Y}_{B(x_1,\cdots,x_j)}(u_1, u_2, \cdots, u_j) = 0$$

for all $B \in \mathcal{B}_j$.

Proof. We check that V is a prefield algebra. By (i) and (ii), the prefield algebra axioms in Definition 2 are seen to hold on

$$(V \times V)_k = \bigoplus_{l+m=k} V_l \times V_m, \quad 0 \leq l, m \leq n + 1,$$

for $k \leq n$ or $k \geq n + 2$, and it follows from (i) and (ii) that (b) and (c) hold. On $(V \times V)_{n+1} = (V_0 \times V_{n+1}) \oplus (V_1 \times V_n) \oplus \cdots \oplus (V_{n+1} \times V_0)$, consider the most general degree-preserving map $\cdot^x\cdot : (V \times V)_{n+1} \to V_{n+1}((x))$ that satisfies the prefield algebra axioms. The product $\cdot^x\cdot$ on $V_0 \times V_{n+1}$ and $V_{n+1} \times V_0$ is dictated by the identity axiom, and thus must be defined as above, as in (iii). Consider $V_l \times V_m$ with $l + m = n + 1$, $l, m \geq 1$. Because each V_l and V_m are free $\mathbb{C}[T]$-modules, the translation covariance is satisfied if we define $\cdot^x\cdot$ on the generators u_i of V_l and v_j of V_m and let the translation covariance determine $\cdot^x\cdot$ uniquely on the rest of $V_l \otimes V_m$ by the formula

$$\Big(\sum_i f_i(T)u_i\Big)^x\Big(\sum_j g_j(T)v_j\Big) = \sum_{i,j} f_i(\partial_x)g_j(T - \partial_x)(u_i^x v_j)$$

for any $f_i(T), g_j(T) \in \mathbb{C}[T]$. We take

$$T^{(j_1)}a \otimes \cdots \otimes T^{(j_{l-1})}a \otimes a \in V_l$$
$$T^{(j_l)}a \otimes \cdots \otimes T^{(j_{n-1})}a \otimes a \in V_m$$

for $j_1, \cdots, j_{n-1} \geq 0$ as the generators of V_l and V_m, respectively, with $l + m = n + 1$ and $l, m \geq 1$. Hence, we are free to choose, for each $T^{(j_1)}a \otimes \cdots \otimes T^{(j_{l-1})}a \otimes a$ and $T^{(j_l)}a \otimes \cdots \otimes T^{(j_{n-1})}a \otimes a$, an element

$$(T^{(j_1)}a \otimes \cdots \otimes T^{(j_{l-1})}a \otimes a)^x (T^{(j_l)}a \otimes \cdots \otimes T^{(j_{n-1})}a \otimes a) \in V_{n+1}((x)).$$

The choices that we make are best organized by generating functions. For $V_1 \times V_n$, replacing x with x_1, we let

$$\begin{aligned} G_1(x_1, \cdots, x_n) = &\sum_{j_1, \cdots, j_{n-1} \geq 0} x_2^{j_1} \cdots x_n^{j_{n-1}} a^{x_1} (T^{(j_1)}a \otimes \cdots \otimes T^{(j_{n-1})}a \otimes a) \\ &\in V_{n+1}((x_1))[[x_2, \cdots, x_n]]. \end{aligned}$$

It also follows that

$$G_1(x_1, \cdots, x_n) = a^{x_1}\left(e^{x_2 T} a \otimes \cdots \otimes e^{x_n T} a \otimes a \right) = a^{x_1}(a^{x_2}(\cdots(a^{x_n}a))). \tag{21}$$

For $V_k \times V_{n-k+1}$ for $2 \leq k \leq n$, we collect the products into generating functions as

$$\begin{aligned} G_k(x_1, \cdots, x_n) = &\sum_{j_1, \cdots, j_{n-1} \geq 0} (x_1 - x_k)^{j_1} \cdots (x_{k-1} - x_k)^{j_{k-1}} x_{k+1}^{j_k} \cdots x_n^{j_{n-1}} \\ &\times (T^{(j_1)}a \otimes \cdots \otimes T^{(j_{k-1})}a \otimes a)^{x_k} (T^{(j_k)}a \otimes \cdots \otimes T^{(j_{n-1})}a \otimes a) \\ &\in V_{n+1}((x_k))[[x_1 - x_k, \cdots, x_{k-1} - x_k, x_{k+1}, \cdots, x_n]]. \end{aligned}$$

Then we have

$$G_k(x_1, \cdots, x_n) = (e^{(x_1 - x_k)T} a \otimes \cdots \otimes e^{(x_{k-1} - x_k)T} a \otimes a)^{x_k} (e^{x_{k+1}T} a \otimes \cdots \otimes e^{x_n T} a \otimes a). \tag{22}$$

In summary, these generating functions determine the product $\cdot^x\cdot$ completely on the generators, and we have freedom to choose any functions

$$G_1(x_1, \cdots, x_n) \in V_{n+1}((x_1))[[x_2, \cdots, x_n]]$$
$$G_k(x_1, \cdots, x_n) \in V_{n+1}((x_k))[[x_1 - x_k, \cdots, x_{k-1} - x_k, x_{k+1}, \cdots, x_n]]$$

for $2 \leq k \leq n$, so that the theorem holds. □

Formal Expansion ι_x^y

For a formal expression $f(x, y, z, \cdots)$, we use the notation ι_x^y to denote

$$\begin{aligned} \iota_x^y f(x, y, z, \cdots) &= e^{(x-y)\partial_t} f(t, y, z, \cdots)\big|_{t=y} \\ &= \sum_{j \geq 0} (x - y)^j \partial_t^{(j)} f(t, y, z, \cdots)\big|_{t=y}, \end{aligned} \tag{23}$$

if the latter expression is well defined. Because the specific domain of the map ι_x^y varies according to the given situation, it is convenient to have a general discussion first. The following properties hold when a suitable domain is specified.

(i) If $f(x,y,z,\cdots)$ is a power series in x, then it is invariant under ι_x^y, because

$$\begin{aligned}\iota_x^y f(x,y,z,\cdots) &= \mathrm{e}^{(x-y)\partial_t} f(t,y,z,\cdots)\big|_{t=y} \\ &= f(x-y+t,y,z,\cdots)\big|_{t=y} = f(x,y,z,\cdots)\end{aligned}$$

if each step of the argument is well defined.

(ii) If the product $f(x,y,z,\cdots)g(x,y,z,\cdots)$ is defined, then ι_x^y respects the product, because

$$\begin{aligned}\iota_x^y(f(x,y,z,\cdots)g(x,y,z,\cdots)) &= \mathrm{e}^{(x-y)\partial_t}(f(t,y,z,\cdots)g(t,y,z,\cdots))\big|_{t=y} \\ &= \mathrm{e}^{(x-y)\partial_t} f(t,y,z,\cdots)\big|_{t=y}\, \mathrm{e}^{(x-y)\partial_t} g(t,y,z,\cdots)\big|_{t=y} \\ &= \iota_x^y f(x,y,z,\cdots)\, \iota_x^y g(x,y,z,\cdots),\end{aligned}$$

if each step of the argument is well defined.

(iii) If the compositions of the expansions are defined, then $\iota_y^z\iota_x^y = \iota_x^z\iota_y^z$, because

$$\begin{aligned}\iota_y^z\iota_x^y f(x,y,z,\cdots) &= \mathrm{e}^{(y-z)\partial_s}\left(\mathrm{e}^{(x-s)\partial_t} f(t,s,z,\cdots)\big|_{t=s}\right)\big|_{s=z} \\ &= \mathrm{e}^{(y-z)\partial_s}\left(\mathrm{e}^{(x-y)\partial_t} f(t,s,z,\cdots)\big|_{t=s}\right)\big|_{s=z} \\ &= \left(\mathrm{e}^{(y-z)(\partial_t+\partial_s)}\, \mathrm{e}^{(x-y)\partial_t} f(t,s,z,\cdots)\right)\big|_{t=z,s=z} \\ &= \left(\mathrm{e}^{(x-z)\partial_t}\, \mathrm{e}^{(y-z)\partial_s} f(t,s,z,\cdots)\right)\big|_{t=z,s=z} \\ &= \iota_x^z\iota_y^z f(x,y,z,\cdots),\end{aligned}$$

if each step of the argument is well defined. In particular, if $f(x,y,z,\cdots)$ is a power series in y, then we have

$$\iota_y^z\iota_x^y f(x,y,z,\cdots) = \iota_x^z f(x,y,z,\cdots). \tag{24}$$

Example 3. *The map*

$$\iota_x^y : \mathbb{C}((x)) \to \mathbb{C}((y))((x-y))$$

gives a field map which is identity on the common subspace $\mathbb{C}[[x]]$*. In particular,*

$$\iota_x^y(x^{-1}) = \sum_{j\geq 0}(-1)^j y^{-1-j}(x-y)^j$$

is the reciprocal of $x = y+(x-y)$ *in* $\mathbb{C}((y))((x-y))$*, and it is the formal expansion of* $x^{-1} = (y+(x-y))^{-1}$ *in the domain* $|x-y| < |y|$ *[15]. The image* $\iota_x^y\mathbb{C}((x))$ *is a subfield of* $\mathbb{C}((y))((x-y))$ *isomorphic to* $\mathbb{C}((x))$*.*

Lemma 8. *Let V be a vector space over $\mathbb{C}$. For $m \geq 0$, we have a well-defined map*

$$\iota_x^y : V[[x,y,z_1,\cdots,z_m]][x^{-1}] \to V((y))[[x-y,z_1-y,\cdots,z_m-y]]$$

given by (23)*.*

Proof. Suppose $f(x,y,z_1,\cdots,z_m) \in V[[x,y,z_1,\cdots,z_m]][x^{-1}]$. Then,

$$f(x,y,z_1,\cdots,z_m) = x^{-N}\phi(x,y,z_1,\cdots,z_m)$$

for $\phi(x,y,z_1,\cdots,z_m) \in V[[x,y,z_1,\cdots,z_m]]$. Hence,

$$\iota_x^y f(x,y,z,\cdots) = (\mathrm{e}^{(x-y)\partial_y}(y^{-N}))\phi(x,y,z_1,\cdots,z_m). \tag{25}$$

By writing $x = (x-y) + y$ and $z_i = (z_i - y) + y$ for $1 \le i \le m$, we see that $\phi(x, y, z_1, \cdots, z_m)$ belongs to $V[[x, y, z_1, \cdots, z_m]] = V[[y, x-y, z_1 - y, \cdots, z_m - y]]$, and (25) belongs to $V((y))[[x-y, z_1 - y, \cdots, z_m - y]]$. Equivalently, because

$$\phi(x, y, z_1, \cdots, z_m) = \mathrm{e}^{(x-y)\partial_s}\, \mathrm{e}^{(z_1-y)\partial_{t_1}} \cdots \mathrm{e}^{(z_m-y)\partial_{t_m}}\, \phi(s, y, t_1, \cdots, t_m)\big|_{s=t_1=\cdots=t_m=y},$$

we can write

$$\iota_x^y f(x, y, z, \cdots) = \mathrm{e}^{(x-y)\partial_s}\, \mathrm{e}^{(z_1-y)\partial_{t_1}} \cdots \mathrm{e}^{(z_m-y)\partial_{t_m}}\, f(s, y, t_1, \cdots, t_m)\big|_{s=t_1=\cdots=t_m=y}$$

which is clear in $V((y))[[x-y, z_1 - y, \cdots, z_m - y]]$. □

Lemma 8 shows that we have a well-defined map

$$\iota_x^y : V[[x, y, z_1, \cdots, z_m]][x^{-1}] \to V((y))[[x-y, z_1 - y, \cdots, z_l - y, z_{l+1}, \cdots, z_m]]$$

for $0 \le l \le m$, because $V[[x, y, z_1, \cdots, z_m]][x^{-1}] \subseteq V[[x, y, z_1, \cdots, z_l]][x^{-1}][[z_{l+1}, \cdots, z_m]]$.

Lemma 9. *Let V be a vector space over $\mathbb{C}$. Two elements*

$$f(x, y, z_1, \cdots, z_m) \in V[[x, y, z_1, \cdots, z_m]][x^{-1}],$$
$$g(x, y, z_1, \cdots, z_m) \in V((y))[[x-y, z_1 - y, \cdots, z_l - y, z_{l+1}, \cdots, z_m]]$$

satisfy

$$x^N f(x, y, z_1, \cdots, z_m) = x^N g(x, y, z_1, \cdots, z_m)$$

for some $N \in \mathbb{N}$ if and only if

$$g(x, y, z_1, \cdots, z_m) = \iota_x^y f(x, y, z_1, \cdots, z_m). \tag{26}$$

Proof. $\mathcal{V} = V[[x, y, z_1, \cdots, z_m]][x^{-1}]$ and $\mathcal{W} = V((y))[[x-y, z_1 - y, \cdots, z_l - y, z_{l+1}, \cdots, z_m]]$ are vector spaces over $\mathbb{C}((x))$ and $\iota_x^y \mathbb{C}((x))$, respectively. If

$$\Psi(x, y, z_1 \cdots, z_m) = x^N f(x, y, z_1 \cdots, z_m) = x^N g(x, y, z_1 \cdots, z_m)$$

for $\Psi(x, y, z_1 \cdots, z_m) \in V[[x, y, z_1 \cdots, z_m]]$, multiplying $\Psi(x, y, z_1, \cdots, z_m) = x^N f(x, y, z_1, \cdots, z_m)$ by x^{-N} in $\mathcal{V}$ gives

$$f(x, y, z_1, \cdots, z_m) = x^{-N} \Psi(x, y, z_1, \cdots, z_m).$$

Multiplying $\Psi(x, y, z_1, \cdots, z_m) = x^N g(x, y, z_1, \cdots, z_m)$ by $\iota_x^y(x^{-N})$ in $\mathcal{W}$ gives

$$g(x, y, z_1, \cdots, z_m) = \iota_x^y(x^{-N}) \Psi(x, y, z_1, \cdots, z_m)$$

and (26) follows. Conversely, if (26) holds, then writing

$$f(x, y, z_1 \cdots, z_m) = x^{-N} \Psi(x, y, z_1 \cdots, z_m)$$

for $\Psi(x, y, z_1 \cdots, z_m) \in V[[x, y, z_1 \cdots, z_m]]$, we have

$$g(x, y, z_1 \cdots, z_m) = \iota_x^y(x^{-N}) \Psi(x, y, z_1 \cdots, z_m),$$

and multiplying by x^N gives

$$x^N g(x, y, z_1 \cdots, z_m) = \Psi(x, y, z_1 \cdots, z_m) = x^N f(x, y, z_1 \cdots, z_m)$$

and the lemma follows. □

Lemma 10. *Let $n \geq 3$. The prefield algebra V defined in Theorem 1 is n-associative if and only if all of the following conditions are satisfied for the generating functions in (20).*

(i) *For $2 \leq i \leq n-1$,*

$$G_1(x_1, \cdots, x_n) \in \big(V_{n+1}[[x_1, \cdots, \hat{x}_i, \cdots x_n]][x_1^{-1}]\big)[[x_i - x_{i+1}]],$$
$$G_1(x_1, \cdots, x_n) \in \big(V_{n+1}[[x_1, \cdots, x_{n-1}]][x_1^{-1}]\big)[[x_n]],$$
$$G_2(x_1, \cdots, x_n) \in \big(V_{n+1}[[x_2, \cdots, x_n]][x_2^{-1}]\big)[[x_1 - x_2]].$$

(ii) *For $2 \leq k \leq n$,*

$$G_k(x_1, \cdots, x_n) = \iota_{x_1}^{x_k} G_1(x_1, \cdots, x_n).$$

(iii) *For $3 \leq k \leq n$,*

$$G_k(x_1, \cdots, x_n) = \iota_{x_2}^{x_k} G_2(x_1, \cdots, x_n).$$

where $\iota_{x_1}^{x_k}$ and $\iota_{x_2}^{x_k}$ are the maps given by the formula (23).

Proof. By Theorem 1, it is only nontrivial to check the n-associativity of the vectors in V when the sum of the degrees is $n+1$. By Lemma 6, it is enough to consider the n-associativity of the generators of V as a $\mathbb{C}[T]$ module. We can take the generators to be

$$1,\ a,\ T^{(j_1)}a \otimes a,\ T^{(j_1)}a \otimes T^{(j_2)}a \otimes a,\ \cdots$$

with $j_i \geq 0$ and consider the cases where the sum of the degrees is $n+1$. It is easy to see that considering the n-associativity of the n vectors

$$a, \cdots, a,\ T^{(j)}a \otimes a,\ a, \cdots, a \tag{27}$$

suffices, where the element $T^{(j)}a \otimes a$ is at the ith position for $1 \leq i \leq n$, $j \geq 0$. Indeed, if (27) are n-associative, then by Lemma 4, the $n-1$ vectors given by

$$a, \cdots, a,\ T^{(j_1)}a \otimes T^{(j_2)}a \otimes a,\ a, \cdots, a, \quad \text{or}$$

$$a, \cdots, a,\ T^{(j_1)}a \otimes a,\ \cdots,\ T^{(j_2)}a \otimes a,\ a, \cdots, a$$

are $(n-1)$-associative, and by Lemma 7, the n vectors

$$a, \cdots, a,\ T^{(j_1)}a \otimes T^{(j_2)}a \otimes a,\ a, \cdots, 1, \cdots, a$$

$$a, \cdots, a,\ T^{(j_1)}a \otimes a, \cdots,\ T^{(j_2)}a \otimes a,\ a, \cdots, 1, \cdots, a$$

are n-associative. Continuing this way, we obtain all possible combinations of generators whose degrees sum to $n+1$ involving more 1s.

Let $B(x_1, \cdots, x_n) \in \mathcal{B}_n(x_1, \cdots, x_n)$. Let o always denote the root, with children u and v. We have a partition of the set of leaves into $\{x_1, \cdots, x_s\} \cup \{x_{s+1}, \cdots, x_n\}$ consisting of the leaves of u and v, respectively, for $1 \leq s \leq n-1$. In this case, we say that B *splits at* s. We define $\mathcal{B}_n(s)$ to be the subset of $\mathcal{B}_n$ consisting of those that split at s. Hence, we have a partition of $\mathcal{B}_n$ into $\bigcup_{1 \leq s \leq n-1} \mathcal{B}_n(s)$.

Let i be the position of the vector $T^{(j)}a \otimes a$ in (27) for a fixed $j \geq 0$. First, we consider the case $i = 1$, namely, the n-associativity of

$$T^{(j)}a \otimes a,\ a, \cdots, a. \tag{28}$$

For each $j \geq 0$, the expression $\mathcal{Y}_{B(x_1, \cdots, x_n)}(T^{(j)}a \otimes a,\ a, \cdots, a)$ for $B \in \mathcal{B}_n$ only depends on $1 \leq s \leq n-1$ where B splits.

(i) Case $i = 1$ and $s = 1$.

Suppose B splits at $s = 1$. Then, for $o = uv$ in $B(x_1, \cdots, x_n)$, we have

$$\begin{aligned} \mathcal{X}_u(T^{(j)}a \otimes a) &= T^{(j)}a \otimes a, \\ \mathcal{X}_v(a, \cdots, a) &= \mathrm{e}^{(x_2-x_n)T}\, a \otimes \mathrm{e}^{(x_3-x_n)T}\, a \otimes \cdots \otimes \mathrm{e}^{(x_{n-1}-x_n)T}\, a \otimes a, \end{aligned}$$

and, therefore,

$$\mathcal{Y}_{B(x_1,\cdots,x_{n-1},0)}(T^{(j)}a \otimes a, a, \cdots, a) = (T^{(j)}a \otimes a)^{x_1}(\mathrm{e}^{x_2 T}\, a \otimes \mathrm{e}^{x_3 T}\, a \otimes \cdots \otimes \mathrm{e}^{x_{n-1} T}\, a \otimes a).$$

By a change in variables,

$$\mathcal{Y}_{B(x_2,\cdots,x_n,0)}(T^{(j)}a \otimes a, a, \cdots, a) = (T^{(j)}a \otimes a)^{x_2}(\mathrm{e}^{x_3 T}\, a \otimes \mathrm{e}^{x_4 T}\, a \otimes \cdots \otimes \mathrm{e}^{x_n T}\, a \otimes a).$$

By (22), this is the coefficient of $(x_1 - x_2)^j$ of

$$G_2(x_1, x_2, \cdots, x_n) = (\mathrm{e}^{(x_1-x_2)T}\, a \otimes a)^{x_2}(\mathrm{e}^{x_3 T}\, a \otimes \mathrm{e}^{x_4 T}\, a \otimes \cdots \otimes \mathrm{e}^{x_n T}\, a \otimes a)$$

which is an element in

$$V_{n+1}((x_2))[[x_1 - x_2, x_3, \cdots, x_n]].$$

Writing $G_2(x_1, x_2, \cdots, x_n) = \sum_{j \geq 0}(x_1 - x_2)^j G_{2,j}(x_2, \cdots, x_n)$, we have

$$G_{2,j}(x_2, \cdots, x_n) \in V_{n+1}((x_2))[[x_3, \cdots, x_n]].$$

(ii) Case $i = 1$ and $2 \leq s \leq n-1$.

Suppose now $B \in \mathcal{B}_n(s)$ for $2 \leq s \leq n-1$. Let $k = s+1$, $3 \leq k \leq n$. For $o = uv$, we have

$$\begin{aligned} &\mathcal{X}_u(T^{(j)}a \otimes a,\ a, \cdots, a) \\ &\qquad = \mathrm{e}^{(x_1-x_{k-1})T}(T^{(j)}a \otimes a) \otimes \mathrm{e}^{(x_2-x_{k-1})T}\, a \otimes \cdots \otimes \mathrm{e}^{(x_{k-2}-x_{k-1})T}\, a \otimes a, \\ &\mathcal{X}_v(a, \cdots, a) = \mathrm{e}^{(x_k-x_n)T}\, a \otimes \mathrm{e}^{(x_{k+1}-x_n)T}\, a \otimes \cdots \otimes \mathrm{e}^{(x_{n-1}-x_n)T}\, a \otimes a. \end{aligned}$$

Thus,

$$\begin{aligned} &\mathcal{Y}_{B(x_1,\cdots,x_{n-1},0)}(T^{(j)}a \otimes a, a, \cdots, a) \\ &\qquad = (\mathrm{e}^{(x_1-x_{k-1})T}(T^{(j)}a \otimes a) \otimes \mathrm{e}^{(x_2-x_{k-1})T}\, a \otimes \cdots \otimes \mathrm{e}^{(x_{k-2}-x_{k-1})T}\, a \otimes a)^{x_{k-1}} \\ &\qquad\quad \times (\mathrm{e}^{x_k T}\, a \otimes \mathrm{e}^{x_{k+1} T}\, a \otimes \cdots \otimes \mathrm{e}^{x_{n-1} T}\, a \otimes a), \end{aligned}$$

and with a change in variables,

$$\begin{aligned} &\mathcal{Y}_{B(x_2,\cdots,x_n,0)}(T^{(j)}a \otimes a,a, \cdots, a) \\ &\qquad = (\mathrm{e}^{(x_2-x_k)T}(T^{(j)}a \otimes a) \otimes \mathrm{e}^{(x_3-x_k)T}\, a \otimes \cdots \otimes \mathrm{e}^{(x_{k-1}-x_k)T}\, a \otimes a)^{x_k} \\ &\qquad\quad \times (\mathrm{e}^{x_{k+1} T}\, a \otimes \mathrm{e}^{x_{k+2} T}\, a \otimes \cdots \otimes \mathrm{e}^{x_n T}\, a \otimes a). \end{aligned}$$

Hence, it is obtained as the $(x_1 - x_2)^j$coefficient of

$$\begin{aligned} G_k(x_1, x_2 \cdots, x_n) &= (\mathrm{e}^{(x_1-x_k)T}\, a \otimes \mathrm{e}^{(x_2-x_k)T}\, a \otimes \cdots \otimes \mathrm{e}^{(x_{k-1}-x_k)T}\, a \otimes a)^{x_k} \\ &\quad \times (\mathrm{e}^{x_{k+1} T}\, a \otimes \mathrm{e}^{x_{k+2} T}\, a \otimes \cdots \otimes \mathrm{e}^{x_n T}\, a \otimes a) \\ &= (\mathrm{e}^{(x_2-x_k)T}(\mathrm{e}^{(x_1-x_2)T}\, a \otimes a) \otimes \mathrm{e}^{(x_3-x_k)T}\, a \otimes \cdots \otimes \mathrm{e}^{(x_{k-1}-x_k)T}\, a \otimes a)^{x_k} \\ &\quad \times (\mathrm{e}^{x_{k+1} T}\, a \otimes \mathrm{e}^{x_{k+2} T}\, a \otimes \cdots \otimes \mathrm{e}^{x_n T}\, a \otimes a) \end{aligned}$$

which belongs to

$$\begin{aligned}&V_{n+1}((x_k))[[x_1-x_k,\cdots,x_{k-1}-x_k,x_{k+1},\cdots,x_n]]\\&=V_{n+1}((x_k))[[x_2-x_k,\cdots,x_{k-1}-x_k,x_{k+1},\cdots,x_n]][[x_1-x_2]]\end{aligned}$$

by writing x_1-x_k as $(x_1-x_2)+(x_2-x_k)$. Writing $G_k(x_1,\cdots,x_n)=\sum_{j\geq 0}(x_1-x_2)^j G_{k,j}(x_2,\cdots,x_n)$, we have

$$G_{k,j}(x_2,\cdots,x_n)\in V_{n+1}((x_k))[[x_2-x_k,\cdots,x_{k-1}-x_k,x_{k+1},\cdots,x_n]]. \tag{29}$$

We now combine the cases (i) and (ii). Suppose the vectors in (28) are n-associative. We have $\Psi_j(x_2,\cdots,x_n)\in V_{n+1}[[x_2,\cdots,x_n]]$ and $N_j\in\mathbb{N}$ such that

$$\Psi_j(x_2,\cdots,x_n)=\mathfrak{s}_n(x_2,\cdots,x_n,0)^{N_j}G_{k,j}(x_2,\cdots,x_n) \tag{30}$$

for all $2\leq k\leq n$. Because $G_{2,j}(x_2,\cdots,x_n)\in V_{n+1}((x_2))[[x_3,\cdots,x_n]]$, $\Psi_j(x_2,\cdots,x_n)$ must have a factor

$$\left(x_3\cdots x_n\prod_{3\leq p<q\leq n}(x_p-x_q)\right)^{N_j}.$$

Because $G_{k,j}(x_2,\cdots,x_n)\in V((x_k))[[x_2-x_k,\cdots,x_{k-1}-x_k,x_{k+1},\cdots,x_n]]$, $\Psi_j(x_2,\cdots,x_n)$ must also have a factor $(x_2-x_k)^{N_j}$ for all $3\leq k\leq n$. Canceling these factors in (30) for $k=2$, we conclude that

$$x_2^{N_j}G_{2,j}(x_2,\cdots,x_n)\in V_{n+1}[[x_2,\cdots,x_n]],$$

and thus

$$G_{2,j}(x_2,\cdots,x_n)\in V_{n+1}[[x_2,\cdots,x_n]][x_2^{-1}], \tag{31}$$

and

$$x_2^{N_j}G_{2,j}(x_2,\cdots,x_n)=x_2^{N_j}G_{k,j}(x_2,\cdots,x_n) \tag{32}$$

for all $3\leq k\leq n$. Therefore, by Lemma 9, we must have

$$G_{k,j}(x_2,\cdots,x_n)=\iota_{x_2}^{x_k}G_{2,j}(x_2,\cdots,x_n) \tag{33}$$

for all $3\leq k\leq n$. Conversely, if (31) and (33) hold, it follows from Lemma 9 that (29) and (32) hold, and the vectors in (28) are n-associative. The conditions for all $j\geq 0$ can equivalently be written as

$$\begin{aligned}&G_2(x_1,\cdots,x_n)\in\left(V_{n+1}[[x_2,\cdots,x_n]][x_2^{-1}]\right)[[x_1-x_2]]\\&G_k(x_1,\cdots,x_n)=\iota_{x_2}^{x_k}G_2(x_1,\cdots,x_n)\end{aligned} \tag{34}$$

for all $3\leq k\leq n$.

Now consider the n-associativity of

$$a,\cdots,a,\ T^{(j)}a\otimes a,\ a,\cdots,a$$

where $T^{(j)}a\otimes a$ is at the ith position, for a fixed $2\leq i\leq n$ and $j\geq 0$.

(iii) Case $2\leq i<n$ and $s=1$.

Suppose B splits at $s=1$. For $o=uv$,

$$\begin{aligned}&\mathcal{X}_u(a)=a,\\&\mathcal{X}_v(a,\cdots,T^{(j)}a\otimes a,\cdots,a)\\&\quad=\mathrm{e}^{(x_2-x_n)T}a\otimes\cdots\otimes\mathrm{e}^{(x_i-x_n)T}(T^{(j)}a\otimes a)\otimes\cdots\otimes\mathrm{e}^{(x_{n-1}-x_n)T}a\otimes a.\end{aligned}$$

By changing variables to $x_1, \cdots, \hat{x}_i, \cdots, x_n, 0$, we obtain

$$\begin{aligned}&\mathcal{Y}_{B(x_1,\cdots,\hat{x}_i,\cdots,x_n,0)}(a,\cdots,a,T^{(j)}a\otimes a,a,\cdots,a)\\&\qquad=a^{x_1}(\mathrm{e}^{x_2T}a\otimes\cdots\otimes\mathrm{e}^{x_{i-1}T}a\otimes\mathrm{e}^{x_{i+1}T}(T^{(j)}a\otimes a)\otimes\cdots\otimes\mathrm{e}^{x_nT}a\otimes a).\end{aligned}$$

We obtain this by taking $(x_i - x_{i+1})^j$ coefficient of

$$\begin{aligned}G_1(x_1,\cdots,x_n)&=a^{x_1}(\mathrm{e}^{x_2T}a\otimes\cdots\otimes\mathrm{e}^{x_iT}a\otimes\mathrm{e}^{x_{i+1}T}a\otimes\cdots\otimes\mathrm{e}^{x_nT}a\otimes a)\\&=a^{x_1}(\mathrm{e}^{x_2T}a\otimes\cdots\otimes\mathrm{e}^{x_{i+1}T}(\mathrm{e}^{(x_i-x_{i+1})T}a\otimes a)\otimes\cdots\otimes\mathrm{e}^{x_nT}a\otimes a)\\&\in V_{n+1}((x_1))[[x_2,\cdots,x_n]]\end{aligned}$$

where we have

$$V_{n+1}((x_1))[[x_2,\cdots,x_n]]=V_{n+1}((x_1))[[x_2,\cdots,\hat{x}_i,\cdots,x_n]][[x_i-x_{i+1}]].$$

We write

$$G_1(x_1,\cdots,x_n)=\sum_{j\ge 0}(x_i-x_{i+1})^jG_{1,j}^{[i]}(x_1,\cdots,\hat{x}_i,\cdots,x_n)$$

where

$$G_{1,j}^{[i]}(x_1,\cdots,\hat{x}_i,\cdots,x_n)\in V_{n+1}((x_1))[[x_2,\cdots,\hat{x}_i,\cdots,x_n]].$$

(iv) Case $3 \le i < n$ and $2 \le s \le i-1$.

Here, we have, for $o = uv$,

$$\begin{aligned}\mathcal{X}_u(a,\cdots,a)&=\mathrm{e}^{(x_1-x_s)T}a\otimes\cdots\otimes\mathrm{e}^{(x_{s-1}-x_s)T}a\otimes a,\\\mathcal{X}_v(a,\cdots,T^{(j)}a\otimes a,\cdots,a)&\\&=\mathrm{e}^{(x_{s+1}-x_n)T}a\otimes\cdots\otimes\mathrm{e}^{(x_{i-1}-x_n)T}a\otimes\mathrm{e}^{(x_i-x_n)T}(T^{(j)}a\otimes a)\otimes\\&\qquad\times\cdots\otimes\mathrm{e}^{(x_{n-1}-x_n)T}a\otimes a.\end{aligned}$$

By a change in variables,

$$\begin{aligned}&\mathcal{Y}_{B(x_1,\cdots,\hat{x}_i,\cdots,x_n,0)}(a,\cdots,a,T^{(j)}a\otimes a,a,\cdots,a)\\&\qquad=(\mathrm{e}^{(x_1-x_s)T}a\otimes\cdots\otimes\mathrm{e}^{(x_{s-1}-x_s)T}a\otimes a)^{x_s}\\&\qquad\quad\times(\mathrm{e}^{x_{s+1}T}a\otimes\cdots\otimes\mathrm{e}^{x_{i-1}T}a\otimes\mathrm{e}^{x_{i+1}T}(T^{(j)}a\otimes a)\otimes\cdots\otimes\mathrm{e}^{x_nT}a\otimes a).\end{aligned}$$

With $k = s$, this is obtained as the $(x_i - x_{i+1})^j$ coefficient of

$$\begin{aligned}G_k(x_1,\cdots,x_n)&=(\mathrm{e}^{(x_1-x_k)T}a\otimes\cdots\otimes\mathrm{e}^{(x_{k-1}-x_k)T}a\otimes a)^{x_k}(\mathrm{e}^{x_{k+1}T}a\otimes\cdots\otimes\mathrm{e}^{x_nT}a\otimes a)\\&=(\mathrm{e}^{(x_1-x_k)T}a\otimes\cdots\otimes\mathrm{e}^{(x_{k-1}-x_k)T}a\otimes a)^{x_k}\\&\quad\times(\mathrm{e}^{x_{k+1}T}a\otimes\cdots\otimes\mathrm{e}^{x_{i+1}T}(\mathrm{e}^{(x_i-x_{i+1})T}a\otimes a)\otimes\cdots\otimes\mathrm{e}^{x_nT}a\otimes a)\end{aligned}$$

which is in

$$\begin{aligned}&V_{n+1}((x_k))[[x_1-x_k,\cdots,x_{k-1}-x_k,x_{k+1},\cdots,x_n]]\\&\qquad=V_{n+1}((x_k))[[x_1-x_k,\cdots,x_{k-1}-x_k,x_{k+1},\cdots,\hat{x}_i,\cdots,x_n]][[x_i-x_{i+1}]].\end{aligned}$$

Expanding $G_k(x_1,\cdots,x_n)=\sum_{j\ge 0}(x_i-x_{i+1})^jG_{k,j}^{[i]}(x_1,\cdots,\hat{x}_i,\cdots,x_n)$, we have

$$G_{k,j}^{[i]}(x_1,\cdots,\hat{x}_i,\cdots,x_n)\in V_{n+1}((x_k))[[x_1-x_k,\cdots,x_{k-1}-x_k,x_{k+1},\cdots,\hat{x}_i,\cdots,x_n]].$$

(v) Case $i = n$ and $1 \leq s \leq i - 1$.

Here, we have, for $o = uv$,

$$\mathcal{X}_u(a, \cdots, a) = e^{(x_1 - x_s)T} a \otimes \cdots \otimes e^{(x_{s-1} - x_s)T} a \otimes a,$$
$$\mathcal{X}_v(a, \cdots, a, T^{(j)} a \otimes a) = e^{(x_{s+1} - x_n)T} a \otimes \cdots \otimes e^{(x_{n-1} - x_n)T} a \otimes (T^{(j)} a \otimes a).$$

By a change in variables to $x_1, \cdots, \hat{x}_i, \cdots, x_n, 0$, which is $x_1, \cdots, x_{n-1}, 0$ because $i = n$,

$$\mathcal{Y}_{B(x_1, \cdots, x_{n-1}, 0)}(a, \cdots, a, T^{(j)} a \otimes a) = (e^{(x_1 - x_s)T} a \otimes \cdots \otimes e^{(x_{s-1} - x_s)T} a \otimes a)^{x_s} \times (e^{x_{s+1}T} a \otimes \cdots \otimes e^{x_{n-1}T} a \otimes (T^{(j)} a \otimes a)).$$

With $k = s$, this is obtained as the x_n^j coefficient of

$$\begin{aligned} G_k(x_1, \cdots, x_n) &= (e^{(x_1 - x_k)} a \otimes \cdots \otimes e^{(x_{k-1} - x_k)T} a \otimes a)^{x_k} (e^{x_{k+1}T} a \otimes \cdots \otimes e^{x_n T} a \otimes a) \\ &\in V_{n+1}((x_k))[[x_1 - x_k, \cdots, x_{k-1} - x_k, x_{k+1}, \cdots, x_n]] \\ &= V_{n+1}((x_k))[[x_1 - x_k, \cdots, x_{k-1} - x_k, x_{k+1}, \cdots, x_{n-1}]][[x_n]]. \end{aligned}$$

Expanding $G_k(x_1, \cdots, x_n) = \sum_{j \geq 0} x_n^j G_{k,j}^{[n]}(x_1, \cdots, x_{n-1})$, we have

$$G_{k,j}^{[n]}(x_1, \cdots, x_{n-1}) \in V_{n+1}((x_k))[[x_1 - x_k, \cdots, x_{k-1} - x_k, x_{k+1}, \cdots, x_{n-1}]].$$

(vi) Case $2 \leq i \leq n - 1$ and $s = i$.

In this case, for $o = uv$, we have

$$\mathcal{X}_u(a, \cdots, a, T^{(j)} a \otimes a) = e^{(x_1 - x_s)T} a \otimes \cdots \otimes e^{(x_{s-1} - x_s)T} a \otimes (T^{(j)} a \otimes a),$$
$$\mathcal{X}_v(a, \cdots, a) = e^{(x_{s+1} - x_n)T} a \otimes \cdots \otimes e^{(x_{n-1} - x_n)T} a \otimes a.$$

By a change in variables,

$$\begin{aligned} &\mathcal{Y}_{B(x_1, \cdots, \hat{x}_i, \cdots, x_n, 0)}(a, \cdots, a, T^{(j)} a \otimes a, a, \cdots, a) \\ &= (e^{(x_1 - x_{s+1})T} a \otimes \cdots \otimes e^{(x_{s-1} - x_{s+1})T} a \otimes (T^{(j)} a \otimes a))^{x_{s+1}} \\ &\quad \times (e^{x_{s+2}T} a \otimes \cdots \otimes e^{x_n T} a \otimes a). \end{aligned}$$

With $k = s + 1 = i + 1$, this is obtained as the $(x_i - x_{i+1})^j$ coefficient of

$$\begin{aligned} G_k(x_1, \cdots, x_n) &= (e^{(x_1 - x_k)T} a \otimes \cdots \otimes e^{(x_{k-1} - x_k)T} a \otimes a)^{x_k} (e^{x_{k+1}T} a \otimes \cdots \otimes e^{x_n T} a \otimes a) \\ &= (e^{(x_1 - x_k)T} a \otimes \cdots \otimes e^{(x_i - x_{i+1})T} a \otimes a)^{x_k} (e^{x_{k+1}T} a \otimes \cdots \otimes e^{x_n T} a \otimes a) \\ &\in V_{n+1}((x_k))[[x_1 - x_k, \cdots, x_{k-1} - x_k, x_{k+1}, \cdots, x_n]] \\ &= V_{n+1}((x_k))[[x_1 - x_k, \cdots, x_{k-2} - x_k, x_{k+1}, \cdots, x_n]][[x_i - x_{i+1}]]. \end{aligned}$$

We write $G_k(x_1, \cdots, x_n) = \sum_{j \geq 0} (x_i - x_{i+1})^j G_{k,j}^{[i]}(x_1, \cdots, \hat{x}_i, \cdots, x_n)$, with

$$G_{k,j}^{[i]}(x_1, \cdots, \hat{x}_i, \cdots, x_n) \in V_{n+1}((x_k))[[x_1 - x_k, \cdots, x_{k-2} - x_k, x_{k+1}, \cdots, x_n]].$$

(vii) Case $2 \leq i \leq n - 2$ and $i + 1 \leq s \leq n - 1$.

In this case, for $o = uv$, we have

$$\begin{aligned}\mathcal{X}_u(a, \cdots, T^{(j)}a \otimes a, \cdots, a)& \\ &= e^{(x_1-x_s)T} a \otimes \cdots \otimes e^{(x_i-x_s)T}(T^{(j)}a \otimes a) \otimes \cdots \otimes e^{(x_{s-1}-x_s)T} a \otimes a,\\ \mathcal{X}_v(a, \cdots, a) &= e^{(x_{s+1}-x_n)T} a \otimes \cdots \otimes e^{(x_{n-1}-x_n)T} a \otimes a.\end{aligned}$$

By a change in variables,

$$\begin{aligned}\mathcal{Y}_{B(x_1,\cdots,\hat{x}_i,\cdots,x_n,0)}&(a, \cdots, a, T^{(j)}a \otimes a, a, \cdots, a)\\ &= \big(e^{(x_1-x_{s+1})T} a \otimes \cdots \otimes e^{(x_{i-1}-x_{s+1})T} a \otimes e^{(x_{i+1}-x_{s+1})T}(T^{(j)}a \otimes a) \otimes \\ &\qquad \cdots \otimes e^{(x_s-x_{s+1})T} a \otimes a\big)^{x_{s+1}} (e^{x_{s+2}T} a \otimes \cdots \otimes e^{x_nT} a \otimes a).\end{aligned}$$

With $k = s + 1$, this is obtained as the $(x_i - x_{i+1})^j$ coefficient of

$$\begin{aligned}G_k&(x_1, \cdots, x_n)\\ &= (e^{(x_1-x_k)T} a \otimes \cdots \otimes e^{(x_{k-1}-x_k)T} a \otimes a)^{x_k} (e^{x_{k+1}T} a \otimes \cdots \otimes e^{x_nT} a \otimes a)\\ &= (e^{(x_1-x_k)T} a \otimes \cdots \otimes e^{(x_{i+1}-x_k)T}(e^{(x_i-x_{i+1})T} a \otimes a) \otimes \cdots \otimes e^{(x_{k-1}-x_k)T} a \otimes a)^{x_k}\\ &\qquad \times (e^{x_{k+1}T} a \otimes \cdots \otimes e^{x_nT} a \otimes a)\\ &\in V_{n+1}((x_k))[[x_1 - x_k, \cdots, x_{k-1} - x_k, x_{k+1}, \cdots, x_n]]\\ &\quad = V_{n+1}((x_k))[[x_1 - x_k, \cdots, \widehat{x_i - x_k}, \cdots, x_{k-1} - x_k, x_{k+1}, \cdots, x_n]][[x_i - x_{i+1}]].\end{aligned}$$

We write $G_k(x_1, \cdots, x_n) = \sum_{j\geq 0}(x_i - x_{i+1})^j G^{[i]}_{k,j}(x_1, \cdots, \hat{x}_i, \cdots, x_n)$, with

$$G^{[i]}_{k,j}(x_1, \cdots, \hat{x}_i, \cdots, x_n) \in V_{n+1}((x_k))[[x_1 - x_k, \cdots, \widehat{x_i - x_k}, \cdots, x_{k-1} - x_k, x_{k+1}, \cdots, x_n]].$$

Cases (iii)–(vii) cover all points in the rectangle $2 \leq i \leq n$ and $1 \leq s \leq n - 1$. Cases (iii)–(v) cover all the cases $s < i$; in this case, with $k = s$, $\mathcal{Y}_{B(x_1,\cdots,\hat{x}_i,\cdots,x_n,0)}(a, \cdots, a, T^{(j)}a \otimes a, a, \cdots, a)$ is obtained from the $(x_i - x_{i+1})^j$ coefficient of $G_k(x_1, \cdots, x_n)$, with the understanding that $x_i - x_{i+1} = x_n$ if $i = n$. Cases (vi) and (vii) cover all cases $s \geq i$, and in this case, the same is true except we should take $k = s + 1$. Hence, for a fixed $2 \leq i \leq n$, we have $1 \leq k \leq n$ with $k \neq i$.

Suppose (27) is n-associative for a fixed $2 \leq i \leq n$ and $j \geq 0$. We have $\Psi_{ij}(x_1, \cdots, \hat{x}_i, \cdots, x_n) \in V_{n+1}[[x_1, \cdots, \hat{x}_i, \cdots, x_n]]$ and $N_{ij} \in \mathbb{N}$ such that

$$\Psi_{ij}(x_1, \cdots, \hat{x}_i, \cdots, x_n) = \mathfrak{s}_n(x_1, \cdots, \hat{x}_i, \cdots, x_n, 0)^{N_{ij}} G^{[i]}_{k,j}(x_1, \cdots, \hat{x}_i, \cdots, x_n)$$

for all $1 \leq k \leq n$ with $k \neq i$. Because $G^{[i]}_{1,j}(x_1, \cdots, \hat{x}_i, \cdots, x_n) \in V_{n+1}((x_1))[[x_2, \cdots, \hat{x}_i, \cdots, x_n]]$, $\Psi_{ij}(x_1, \cdots, \hat{x}_i, \cdots, x_n)$ must have a factor

$$\Big(x_2 \cdots \hat{x}_i \cdots x_n \prod_{\substack{2\leq p<q\leq n\\ p,q\neq i}} (x_p - x_q)\Big)^{N_{ij}}.$$

Because $G^{[i]}_{k,j}(x_1, \cdots, \hat{x}_i, \cdots, x_n)$ belongs to $V_{n+1}((x_k))[[x_1 - x_k, \cdots, \widehat{x_i - x_k}, \cdots, x_{k-1} - x_k, x_{k+1}, \cdots, x_n]]$ for $k > i$ and to $V_{n+1}((x_k))[[x_1 - x_k, \cdots, x_{k-1} - x_k, x_{k+1}, \cdots, \hat{x}_i, \cdots, x_n]]$ for $k < i$, we see that $\Psi_{ij}(x_1, \cdots, \hat{x}_i, \cdots, x_n)$ must also have a factor $(x_1 - x_k)^{N_{ij}}$ for all $2 \leq k \leq n$, $k \neq i$. Canceling these factors for $k = 1$, we conclude that

$$x_1^{N_{ij}} G^{[i]}_{1,j}(x_1, \cdots, \hat{x}_i, \cdots, x_n) \in V_{n+1}[[x_1, \cdots, \hat{x}_i, \cdots, x_n]] \tag{35}$$

and thus

$$G^{[i]}_{1,j}(x_1,\cdots,\hat{x}_i,\cdots,x_n)\in V_{n+1}[[x_1,\cdots,\hat{x}_i,\cdots,x_n]][x_1^{-1}] \tag{36}$$

and

$$x_1^{N_{ij}}G^{[i]}_{1,j}(x_1,\cdots,\hat{x}_i,\cdots,x_n)=x_1^{N_{ij}}G^{[i]}_{k,j}(x_1,\cdots,\hat{x}_i,\cdots,x_n)$$

for all $2\le k\le n$, $k\neq i$. Therefore, by Lemma 9, we must have

$$G^{[i]}_{k,j}(x_1,\cdots,\hat{x}_i,\cdots,x_n)=\iota^{x_k}_{x_1}G^{[i]}_{1,j}(x_1,\cdots,\hat{x}_i,\cdots,x_n)$$

for all $2\le k\le n$, $k\neq i$. Conversely, if (35) and (36) hold, it also follows from Lemma 9 that (27) are n-associative. Because $G^{[i]}_{1,j}(x_1,\cdots,\hat{x}_i,\cdots,x_n)$ for $j\ge 0$ are obtained as the $(x_i-x_{i+1})^j$ coefficient of $G_1(x_1,\cdots,x_n)$, the conditions for n-associativity for all $j\ge 0$ can be written as

$$\begin{aligned}&G_1(x_1,\cdots,x_n)\in\big(V_{n+1}[[x_1,\cdots,\hat{x}_i,\cdots x_n]][x_1^{-1}]\big)[[x_i-x_{i+1}]]\\&G_k(x_1,\cdots,x_n)=\iota^{x_k}_{x_1}G_1(x_1,\cdots,x_n)\end{aligned} \tag{37}$$

for all $2\le k\le n$ with $k\neq i$, with $x_i-x_{i+1}=x_n$ if $i=n$. Requiring these conditions for all $2\le i\le n$ implies that (37) should hold for all $2\le k\le n$. Combining with (34), the lemma follows. □

Theorem 2. *Let $n\ge 3$. Let V be a prefield algebra in Theorem 1 defined with the functions*

$$\begin{aligned}G_1(x_1,\cdots,x_n)&=\exp\left(\frac{(x_1-x_2)(x_2-x_3)\cdots(x_{n-1}-x_n)x_n}{x_1}\right)a^{\otimes(n+1)},\\G_k(x_1,\cdots,x_n)&=\iota^{x_k}_{x_1}\exp\left(\frac{(x_1-x_2)(x_2-x_3)\cdots(x_{n-1}-x_n)x_n}{x_1}\right)a^{\otimes(n+1)}\end{aligned}$$

for $2\le k\le n$. Then, V is n-associative, but it is not $(n+1)$-associative. Hence, the condition of $(n+1)$-associativity is strictly stronger than n-associativity for all $n\ge 3$.

Proof. We use the fact that for any commutative associative $\mathbb{C}$-algebra A, if $u(x_1,\cdots,x_k)\in A[[x_1,\cdots,x_k]]$ has the vanishing constant term, then for any $f(x)\in\mathbb{C}[[x]]$, the composition $f(u(x_1,\cdots,x_k))$ is well defined as an element of $A[[x_1,\cdots,x_k]]$. We verify that $G_k(x_1,\cdots,x_n)$, $1\le k\le n$, satisfy the conditions for n-associativity of V given in Lemma 10. Let

$$P(x_1,\cdots,x_n)=\frac{(x_1-x_2)(x_2-x_3)\cdots(x_{n-1}-x_n)x_n}{x_1}.$$

With the understanding that $x_{n+1}=0$, we have

$$\begin{aligned}P(x_1,\cdots,x_n)&=\frac{(x_1-x_2)\cdots\big((x_{i-1}-x_{i+1})-(x_i-x_{i+1})\big)(x_i-x_{i+1})\cdots(x_{n-1}-x_n)x_n}{x_1}\\&\in\big(\mathbb{C}[[x_1,\cdots,\hat{x}_i,\cdots,x_n]][x_1^{-1}]\big)[[x_i-x_{i+1}]]\end{aligned}$$

in the ideal generated by x_i-x_{i+1} for all $2\le i\le n$. Hence,

$$\begin{aligned}G_1(x_1,\cdots,x_n)&=(\exp P(x_1,\cdots,x_n))a^{\otimes(n+1)}\\&\in\big(V_{n+1}[[x_i,\cdots,\hat{x}_i,\cdots x_n]][x_1^{-1}]\big)[[x_i-x_{i+1}]]\end{aligned}$$

for all $2\le i\le n$, and the identities

$$G_k(x_1,\cdots,x_n)=\iota^{x_k}_{x_1}G_1(x_1,\cdots,x_n)$$

for all $2 \leq k \leq n$ in Lemma 10 are satisfied by definition. We also need to verify

$$G_2(x_1, \cdots, x_n) \in \big(V_{n+1}[[x_2, \cdots, x_n]][x_2^{-1}]\big)[[x_1 - x_2]].$$

This follows because we have $G_2(x_1, \cdots, x_n) = \exp\big(\iota_{x_1}^{x_2} P(x_1, \cdots, x_n)\big) a^{\otimes(n+1)}$, where

$$\iota_{x_1}^{x_2} P(x_1, \cdots, x_n) = \big(\sum_{j \geq 0} (-1)^j x_2^{-1-j} (x_1 - x_2)^j\big)(x_1 - x_2)(x_2 - x_3) \cdots (x_{n-1} - x_n) x_n$$

which belongs to the ideal generated by $x_1 - x_2$ in $\big(\mathbb{C}[[x_2, \cdots x_n]][x_2^{-1}]\big)[[x_1 - x_2]]$. Finally, the identities

$$G_k(x_1, \cdots, x_n) = \iota_{x_2}^{x_k} G_2(x_1, \cdots, x_n)$$

for $3 \leq k \leq n$ are the consequences of the fact that $\iota_{x_2}^{x_k} \iota_{x_1}^{x_2} P(x_1, \cdots, x_n) = \iota_{x_1}^{x_k} P(x_1, \cdots, x_n)$, which holds by (24). Hence, V is n-associative. By (21), we have

$$G_1(x_1, \cdots, x_n) = a^{x_1}(a^{x_2}(\cdots (a^{x_n} a))) = \mathcal{Y}_{x_1(x_2(\cdots(x_n 0)))}(a, a \cdots, a)$$

for the $n+1$ vectors $a, a, \cdots, a \in V$. They do not satisfy $(n+1)$-associativity because the presence of an "essential singularity" along x_1 in $G_1(x_1, \cdots, x_n)$ shows

$$\mathfrak{s}_{n+1}(x_1, \cdots, x_n, 0)^N \mathcal{Y}_{x_1(x_2(\cdots(x_n 0)))}(a, a \cdots, a) \notin V[[x_1, \cdots, x_n]]$$

for any $N \in \mathbb{N}$. □

5. Conclusions

Vertex algebras are analogous to the commutative and associative algebras, and field algebras generalize vertex algebras by only requiring the associative properties. We defined the notion of higher-order associativity of field algebras. If $\mathcal{F}_n$ is the class of field algebras that are n-associative, then by Theorem 2, the inclusions (2) can be written as proper inclusions

$$\mathcal{F}_3 \supsetneq \mathcal{F}_4 \supsetneq \mathcal{F}_5 \supsetneq \cdots \supsetneq \mathcal{F}_\infty. \quad (38)$$

We may phrase this phenomenon by saying that even if the product of every n fields is associative and has only meromorphic operator product expansion, the product of more fields may develop an essential singularity. One may wonder if this can be used to an advantage to find some strange but interesting examples.

On the other hand, we may want to specify the class $\mathcal{F}_\infty$ in an efficient way. We can certainly require the ∞-associativity in Definition 3. The notion of meromorphic field algebras was given in [12] with an equivalent definition as a formally rational deformation operad. Whether there exists a simpler description remains a question.

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References

1. Borcherds, R.E. Vertex algebras, Kac-Moody algebras, and the Monster. *Proc. Natl. Acad. Sci. USA* **1986**, *83*, 3068–3071. [CrossRef] [PubMed]
2. Borcherds, R.E. Vertex algebras. In *Topological Field Theory, Primitive Forms and Related Topics (Kyoto, 1996)*; Birkhäuser Boston: Boston, MA, USA, 1998; Volume 160, pp. 35–77. [CrossRef]
3. Borcherds, R.E. Quantum vertex algebras. In *Taniguchi Conference on Mathematics Nara '98*; Math. Soc. Japan : Tokyo, Japan, 2001; Volume 31, pp. 51–74. [CrossRef]
4. Wilson, K.G. Non-Lagrangian Models of Current Algebra. *Phys. Rev.* **1969**, *179*, 1499–1512. [CrossRef]

5. Wilson, K.G. Operator-Product Expansions and Anomalous Dimensions in the Thirring Model. *Phys. Rev. D* **1970**, *2*, 1473–1477. [CrossRef]
6. Belavin, A.A.; Polyakov, A.M.; Zamolodchikov, A.B. Infinite conformal symmetry in two-dimensional quantum field theory. *Nucl. Phys. B* **1984**, *241*, 333–380. [CrossRef]
7. Zamolodchikov, A.B. Infinite additional symmetries in two-dimensional conformal quantum field theory. *Theor. Math. Phys.* **1985**, *65*, 1205–1213. [CrossRef]
8. Bakalov, B.; Kac, V.G. Field algebras. *Int. Math. Res. Not.* **2003**, *3*, 123–159. [CrossRef]
9. Li, H. Axiomatic G_1-vertex algebras. *Commun. Contemp. Math.* **2003**, *5*, 281–327. [CrossRef]
10. Li, H. Nonlocal vertex algebras generated by formal vertex operators. *Sel. Math.* **2005**, *11*, 349–397. [CrossRef]
11. Etingof, P.; Kazhdan, D. Quantization of Lie bialgebras. V. Quantum vertex operator algebras. *Sel. Math.* **2000**, *6*, 105–130. [CrossRef]
12. Kim, N. Associativity of field algebras. *Ann. Henri Poincaré* **2011**, *12*, 1145–1168. [CrossRef]
13. Stanley, R.P. Hipparchus, Plutarch, Schröder, and Hough. *Am. Math. Mon.* **1997**, *104*, 344–350. [CrossRef]
14. Rosen, K.H. *Discrete Mathematics and Its Applications*, 5th ed.; McGraw-Hill Higher Education: New York, NY, USA, 2002.
15. Kac, V. *Vertex Algebras for Beginners*, 2nd ed.; University Lecture Series; American Mathematical Society: Providence, RI, USA, 1998; Volume 10, pp. vi+201. [CrossRef]