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Fixed Points for (ξ, ω)-Weakly Cyclic Type Generalized Contraction Condition in Metric Spaces with an Application

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Abstract: In the present work, we have introduced a new type of (ξ, ω)-weakly cyclic generalized contraction in the setting of metric spaces and established some fixed-point results. Fixed-point results are useful in establishing the existence of unique solution to differential equations. We have supplemented the derived results with suitable non-trivial examples with an application to the Boundary Value Problem, generalizing some known results. The analytical result has been verified with numerical simulation.

Keywords: fixed point; cyclic representation; altering distance function; (ξ, ω)-weakly cyclic generalized contraction; boundary value problem

MSC: 47H10; 54H25

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1. Introduction and Prelimnaries

The foundation of Metric Fixed-Point Theory was laid in the year 1922, when S. Banach gave the famous Contraction Mapping Theorem (CMT) [1], in the setting of complete Metric space (M, ρ) by making rich hypothesis than of Brouwer's [2] particularly to obtain a unique fixed point. In real-life situations as well, uniqueness plays an essential role in our life. The past few decades have seen many generalizations of the CMT being reported in the literature by various mathematicians using different contractive conditions in the setting of metric and metric-like spaces such as dislocated metric space, quasi metric space, rectangular metric space, b metric space, fuzzy metric space, to name a few.

Throughout this paper, $\mathbb{R}^+$ denotes non-negative reals, while $\mathbb{N}$ denotes positive integers.

In the sequel, Khan, Swaleh, and Sessa [3] introduced the altering distance function ξ in the year 1984, extending the condition used by Massa [4] in the setting of complete metric space (M, ρ). Khan et al. [3] furnished the following definition:

Definition 1 ([3])**.** *A function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function if the following properties are satisfied:*

1. *ξ is continuous and strictly increasing in $\mathbb{R}^+$;*
2. *$\xi(t) = 0$ if and only if $t = 0$.*

Later, Kirk et al. [5] presented a fixed-point result using cyclic maps, which is as follows:

Definition 2 ([5]). *Consider the metric space (M,ρ). Let $q \in \mathbb{N}$, $\wp_1, \wp_2, \dots, \wp_q$ be subsets of M, $Z = \bigcup_{i=1}^{q} \wp_i$ and $\Omega : Z \to Z$. Then, Z is a cyclic representation of Z with respect to Ω if*

1. *$\wp_i$, $i = 1, 2, \dots, q$ are non-void and closed, and*
2. *$\Omega(\wp_1) \subseteq \wp_2, \dots,\ \Omega(\wp_{q-1}) \subseteq \wp_q,\ \Omega(\wp_q) \subseteq \wp_1$.*

Subsequently, many mathematicians proved fixed-point results using different types cyclic contractions in various topological spaces, see [6–10].

The concept of weak contractive condition was first proposed at WCNA-2000 by Billy Rhoades [11]. He proposed the following:

Definition 3 ([11]). *Let (M,ρ) be a metric space. The map $\Omega : M \to M$ is weakly contractive if*

$$\rho(\Omega\eta, \Omega\delta) \le \rho(\eta,\delta) - \xi(\rho(\eta,\delta)),\ \text{for all}\ \eta,\delta \in M,$$

where ξ is an altering distance function.

Subsequently, various fixed-point results using weak contraction conditions were reported in various types of topological spaces. Readers may consult the following for more knowledge on weak contractions and their generalizations, [12–20], etc.

Later, Murthy et al. [21] generalized the weakly contractive map for two pairs of maps and established fixed-point result as follows:

Theorem 1 ([21]). *Let (M,ρ) be a complete metric space, and let Υ, Γ, ζ, and $\Omega : M \to M$ be mappings satisfying*

$$\omega(\rho(\Upsilon\eta, \Gamma\delta)) \le \omega(\sigma_1(\eta,\delta)) - \xi(\sigma_2(\eta,\delta)) \tag{1}$$

for all $\eta, \delta \in M$, with $\eta \neq \delta$ and

$$\sigma_1(\eta,\delta) = \max\left\{\rho(\zeta\eta,\Omega\delta), \frac{1}{2}(\rho(\zeta\eta,\Upsilon\eta) + \rho(\Omega\delta,\Gamma\delta)), \frac{1}{2}(\rho(\zeta\eta,\Gamma\delta) + \rho(\Omega\delta,\Upsilon\eta))\right\}$$

and

$$\sigma_2(\eta,\delta) = \min\left\{\rho(\zeta\eta,\Omega\delta), \frac{1}{2}(\rho(\zeta\eta,\Upsilon\eta) + \rho(\Omega\delta,\Gamma)), \frac{1}{2}(\rho(\zeta\eta,\Gamma\delta) + \rho(\Omega\delta,\Upsilon\eta))\right\}$$

$\Upsilon(M) \subseteq \Omega(M)$ and $\Gamma(M) \subseteq \zeta(M)$, (Υ,ζ) and (Γ,Ω) are weak compatible pairs, $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is lower semi-continuous, such that $\xi(t) > 0$, for all $t > 0$ and discontinuous at $t = 0$ with $\xi(0) = 0$, $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is an altering distance function.

Then Υ, Γ, ζ, and Ω have a unique common fixed point in M.

The authors of this note recommend the readers to refer to [22–24] for various new concepts of fixed-point theory and applications such as Volterra–Fredholm integral equations, solving nonlinear integral equations, etc.

In the present work, we extend the contraction condition (1) for cyclic maps in complete metric space and present our work as follows. In Section 2, we present our main results by defining (ξ,ω)-weakly cyclic generalized contraction in metric spaces and establish fixed-point results therein. In Section 3, we apply the derived result to find a unique analytical solution to the boundary value problem. We also present a numerical simulation of the derived result.

2. Main Results

We present the following definition:

Definition 4. *Let (M, ρ) be a metric space. Let $q \in \mathbb{N}$, $\wp_1, \wp_2, \ldots, \wp_q$ be non-void subsets of M and $Z = \bigcup_{i=1}^{q} \wp_i$. An operator $\Omega : Z \to Z$ is a (ξ, ω)-weakly cyclic generalized contraction if*

1. *$Z = \bigcup_{i=1}^{q} \wp_i$ is a cyclic representation of Z with respect to Ω;*
2. *for any $(\eta, \delta) \in \wp_i \times \wp_{i+1}, i = 1, 2, \ldots, q$ (with $\wp_{q+1} = \wp_1$),*

$$\xi(\rho(\Omega\eta, \Omega\delta)) \quad \leq \quad \xi(\Delta_1(\eta, \delta)) - \omega(\Delta_2(\eta, \delta)), \tag{2}$$

where

$$\Delta_1(\eta, \delta) = \max\left\{\rho(\eta, \delta), \frac{1}{2}[\rho(\eta, \Omega\eta) + \rho(\delta, \Omega\delta)], \frac{1}{2}[\rho(\eta, \Omega\delta) + \rho(\delta, \Omega\eta)]\right\} \tag{3}$$

and

$$\Delta_2(\eta, \delta) = \min\left\{\rho(\eta, \delta), \frac{1}{2}[\rho(\eta, \Omega\delta) + \rho(\delta, \Omega\eta)]\right\}; \tag{4}$$

$\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is an altering distance function and $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with $\omega(t) = 0$ if and only if $t = 0$.

We now present our main result:

Theorem 2. *Let (M, ρ) be a complete metric space, $q \in \mathbb{N}$, $\wp_1$, $\wp_2, \ldots,$ $\wp_q$ be non-void closed subsets of M and $Z = \bigcup_{i=1}^{q} \wp_i$. Suppose $\Omega : Z \to Z$ is a (ξ, ω)-weakly cyclic generalized contraction. Then there exists a unique fixed point for Ω in $\bigcap_{i=1}^{q} \wp_i$.*

Proof. Let $\eta_0 \in \wp_1$($\wp_i$ is non-void for all i). Consider the sequence $\{\eta_n\}$ in M given by

$$\eta_{n+1} = \Omega\eta_n, \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

If $\eta_n = \eta_{n+1}$, then η_n is a fixed point of Ω.
Now, we can assume that

$$\eta_n \neq \eta_{n+1}, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{5}$$

We shall prove that

$$\lim_{n \to +\infty} \rho(\eta_n, \eta_{n+1}) = 0. \tag{6}$$

By the assumption, $\rho(\eta_n, \eta_{n+1}) > 0$ for all n. From (1), we have $i = i(n) \in \{1, 2, \ldots, q\}$ for all n, such that $(\eta_n, \eta_{n+1}) \in \wp_i \times \wp_{i+1}$. Putting $\eta = \eta_n$ and $\delta = \eta_{n+1}$ in the condition (2), we have

$$\begin{aligned} \xi(\rho(\Omega\eta_n, \Omega\eta_{n+1})) \quad &= \quad \xi(\rho(\eta_{n+1}, \eta_{n+2})) \\ &\leq \quad \xi(\Delta_1(\eta_n, \eta_{n+1})) - \omega(\Delta_2(\eta_n, \eta_{n+1})), \end{aligned} \tag{7}$$

where

$$\Delta_1(\eta_n,\eta_{n+1}) = \max\left\{\rho(\eta_n,\eta_{n+1}), \frac{1}{2}[\rho(\eta_n,\Omega\eta_n)+\rho(\eta_{n+1},\Omega\eta_{n+1})],\right.$$
$$\left.\frac{1}{2}[\rho(\eta_n,\Omega\eta_{n+1})+\rho(\eta_{n+1},\Omega\eta_n)]\right\}$$
$$= \max\left\{\rho(\eta_n,\eta_{n+1}), \frac{1}{2}[\rho(\eta_n,\eta_{n+1})+\rho(\eta_{n+1},\eta_{n+2})],\right.$$
$$\left.\frac{1}{2}[\rho(\eta_n,\eta_{n+2})+\rho(\eta_{n+1},\eta_{n+1})]\right\}$$

and

$$\begin{aligned}\Delta_2(\eta_n,\eta_{n+1}) &= \min\left\{\rho(\eta_n,\eta_{n+1}), \frac{1}{2}[\rho(\eta_n,\Omega\eta_{n+1})+\rho(\eta_{n+1},\Omega\eta_n)]\right\}\\ &= \min\left\{\rho(\eta_n,\eta_{n+1}), \frac{1}{2}[\rho(\eta_n,\eta_{n+2})+\rho(\eta_{n+1},\eta_{n+1})]\right\}\end{aligned}$$

Then, by the triangle inequality, we have

$$\Delta_1(\eta_n,\eta_{n+1}) \le \max\left\{\rho(\eta_n,\eta_{n+1}), \frac{1}{2}[\rho(\eta_n,\eta_{n+1})+\rho(\eta_{n+1},\eta_{n+2})],\right.$$
$$\left.\frac{1}{2}[\rho(\eta_n,\eta_{n+1})+\rho(\eta_{n+1},\eta_{n+2})]\right\}$$

$$\Delta_1(\eta_n,\eta_{n+1}) \le \max\left\{\rho(\eta_n,\eta_{n+1}), \tfrac{1}{2}[\rho(\eta_n,\eta_{n+1})+\rho(\eta_{n+1},\eta_{n+2})]\right\} \tag{8}$$

and

$$\Delta_2(\eta_n,\eta_{n+1}) = \min\left\{\rho(\eta_n,\eta_{n+1}), \frac{1}{2}[\rho(\eta_n,\eta_{n+2})]\right\} \tag{9}$$

If possible, let for some n, $\rho(\eta_n,\eta_{n+1}) < \rho(\eta_{n+1},\eta_{n+2})$.

Then, by the triangle inequality, $0 < \rho(\eta_{n+1},\eta_{n+2}) - \rho(\eta_n,\eta_{n+1}) \le \rho(\eta_n,\eta_{n+2})$. Hence, by (5), we have $\Delta_2(\eta_n,\eta_{n+1}) > 0$.

Now, if $\rho(\eta_n,\eta_{n+1}) < \rho(\eta_{n+1},\eta_{n+2})$ then we obtain

$$\Delta_1(\eta_n,\eta_{n+1}) \le \rho(\eta_{n+1},\eta_{n+2}), \tag{10}$$

and (7) implies that this inequality

$$\begin{aligned}\xi(\rho(\eta_{n+1},\eta_{n+2})) &\le \xi(\Delta_1(\eta_n,\eta_{n+1})) - \omega(\Delta_2(\eta_n,\eta_{n+1}))\\ &\le \xi(\Delta_1(\eta_n,\eta_{n+1})).\end{aligned}$$

As ξ is monotonically increasing, we have

$$\rho(\eta_{n+1},\eta_{n+2}) \le \Delta_1(\eta_n,\eta_{n+1}). \tag{11}$$

From (10) and (11), we have

$$\Delta_1(\eta_n, \eta_{n+1}) = \rho(\eta_{n+1}, \eta_{n+2}).$$

Then, from (7)–(9) and our assumption, we have

$$\begin{aligned} \xi(\rho(\eta_{n+1}, \eta_{n+2})) &\leq \xi(\rho(\eta_{n+1}, \eta_{n+2})) - \omega(\Delta_2(\eta_n, \eta_{n+1})) \\ &< \xi(\rho(\eta_{n+1}, \eta_{n+2})) \end{aligned}$$

which is a contradiction. Hence, for all $n \geq 0$,

$$\rho(\eta_{n+1}, \eta_{n+2}) \leq \rho(\eta_n, \eta_{n+1}) \tag{12}$$

By (12), we obtain from (8) and (9), for all $n \geq 0$,

$$\Delta_1(\eta_n, \eta_{n+1}) = \rho(\eta_n, \eta_{n+1}) \tag{13}$$

$$\Delta_2(\eta_n, \eta_{n+1}) = \frac{1}{2}\rho(\eta_n, \eta_{n+2}) \tag{14}$$

Putting (12) and (13) in (7), we have for all $n \geq 0$,

$$\xi(\rho(\eta_{n+1}, \eta_{n+2})) \leq \xi(\rho(\eta_n, \eta_{n+1})) - \omega\left(\frac{1}{2}\rho(\eta_n, \eta_{n+2})\right). \tag{15}$$

Again (12) implies that the sequence $\{\rho(\eta_n, \eta_{n+1})\}$ decreases monotonically. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to+\infty} \rho(\eta_n, \eta_{n+1}) = r.$$

Again, by the triangle inequality, we have

$$\begin{aligned} \rho(\eta_n, \eta_{n+2}) &\leq \rho(\eta_n, \eta_{n+1}) + \rho(x_{n+1,\eta_{n+2}}) \\ \rho(\eta_n, \eta_{n+2}) - 2r &\leq \rho(\eta_n, \eta_{n+1}) + \rho(\eta_{n+1}, \eta_{n+2}) - 2r \\ \rho(\eta_n, \eta_{n+2}) - 2r &\leq [\rho(\eta_n, \eta_{n+1}) - r] + [\rho(\eta_{n+1}, \eta_{n+2}) - r]. \end{aligned}$$

As $n \to +\infty$, we have

$$\lim_{n\to+\infty} \rho(\eta_n, \eta_{n+2}) = 2r.$$

and as $n \to +\infty$ in (15), we have

$$\lim_{n\to+\infty} \xi(\rho(\eta_{n+1}, \eta_{n+2})) \leq \lim_{n\to+\infty} \xi(\rho(\eta_n, \eta_{n+1})) - \lim_{n\to+\infty} \omega\left(\frac{1}{2}\rho(\eta_n, \eta_{n+2})\right).$$

As ξ and ω are continuous,

$$\xi(r) \leq \xi(r) - \omega(r).$$

Therefore, $\omega(r) = 0$ and hence $r = 0$. Thus,

$$\lim_{n\to+\infty} \rho(\eta_n, \eta_{n+1}) = 0. \tag{16}$$

We now claim $\{\eta_n\}$ is a Cauchy sequence in M. Suppose not.
Then, for some $\epsilon > 0$, we can find two subsequences $\{\eta_{m(k)}\}$ and $\{\eta_{n(k)}\}$ of $\{\eta_n\}$ such that $n(k) > m(k) > k$ for $k \in \mathbb{N}$ and

$$\rho(\eta_{m(k)}, \eta_{n(k)}) \geq \epsilon. \tag{17}$$

which implies,

$$\rho(\eta_{m(k)}, \eta_{n(k)-1}) < \epsilon. \tag{18}$$

From (17) and (18) and by the triangle inequality, we have

$$\begin{aligned}\epsilon &\leq \rho(\eta_{m(k)}, \eta_{n(k)}) \\ &\leq \rho(\eta_{m(k)}, \eta_{n(k)-1}) + \rho(\eta_{n(k)-1}, \eta_{n(k)}) \\ &< \epsilon + \rho(\eta_{n(k)-1}, \eta_{n(k)})\end{aligned}$$

Letting $k \to +\infty$ and using (16), we obtain

$$\lim_{k\to+\infty} \rho(\eta_{m(k)}, \eta_{n(k)}) = \epsilon. \tag{19}$$

Putting $\eta = \eta_{m(k)-1}$ and $\delta = \eta_{n(k)-1}$ in (2)–(4) respectively, for all $k \in \mathbb{N}$,

$$\begin{aligned}\xi(\rho(\Omega\eta_{m(k)-1}, \Omega\eta_{n(k)-1})) &= \xi(\rho(\eta_{m(k)}, \eta_{n(k)})) \\ &\leq \xi(\Delta_1(\eta_{m(k)-1}, \eta_{n(k)-1})) - \omega(\Delta_2(\eta_{m(k)-1}, \eta_{n(k)-1})),\end{aligned} \tag{20}$$

where

$$\begin{aligned}\Delta_1(\eta_{m(k)-1}, \eta_{n(k)-1}) =&\max\left\{\rho(\eta_{m(k)-1}, \eta_{n(k)-1}), \frac{1}{2}\left[\rho(\eta_{m(k)-1}, \Omega\eta_{m(k)-1}) + \rho(\eta_{n(k)-1}, \Omega\eta_{n(k)-1})\right],\right. \\ &\left.\frac{1}{2}\left[\rho(\eta_{m(k)-1}, \Omega\eta_{n(k)-1}) + \rho(\eta_{n(k)-1}, \Omega\eta_{m(k)-1})\right]\right\} \\ =&\max\left\{\rho(\eta_{m(k)-1}, \eta_{n(k)-1}), \frac{1}{2}\left[\rho(\eta_{m(k)-1}, \eta_{m(k)}) + \rho(\eta_{n(k)-1}, \eta_{n(k)})\right],\right. \\ &\left.\frac{1}{2}\left[\rho(\eta_{m(k)-1}, \eta_{n(k)}) + \rho(\eta_{n(k)-1}, \eta_{m(k)})\right]\right\}\end{aligned}$$

and

$$\begin{aligned}\Delta_2(\eta_{m(k)-1}, \eta_{n(k)-1}) =&\min\left\{\rho(\eta_{m(k)-1}, \eta_{n(k)-1}), \frac{1}{2}\left[\rho(\eta_{m(k)-1}, \Omega\eta_{n(k)-!1}) + \rho(\eta_{n(k)-1}, \Omega\eta_{m(k)-1})\right]\right\} \\ =&\min\left\{\rho(\eta_{m(k)-1}, \eta_{n(k)-1}), \frac{1}{2}\left[\rho(\eta_{m(k)-1}, \eta_{n(k)}) + \rho(\eta_{n(k)-1}, \eta_{m(k)})\right]\right\}.\end{aligned}$$

Now, for all $k \in \mathbb{N}$, we have

$$\rho(\eta_{m(k)-1}, \eta_{n(k)-1}) \leq \rho(\eta_{m(k)-1}, \eta_{m(k)}) + \rho(\eta_{m(k)}, \eta_{n(k)}) + \rho(\eta_{n(k)}, \eta_{n(k)-1})$$

and

$$\rho(\eta_{m(k)}, \eta_{n(k)}) \leq \rho(\eta_{m(k)}, \eta_{m(k)-1}) + \rho(\eta_{m(k)-1}, \eta_{n(k)-1}) + \rho(\eta_{n(k)-1}, \eta_{n(k)}).$$

Letting $k \to +\infty$ and using (16) and (19), we have

$$\lim_{k\to+\infty} \rho(\eta_{m(k)-1}, \eta_{n(k)-1}) = \epsilon. \tag{21}$$

Now, for any $k \in \mathbb{N}$, we have

$$\rho(\eta_{m(k)-1}, \eta_{n(k)}) \leq \rho(\eta_{m(k)-1}, \eta_{m(k)}) + \rho(\eta_{m(k)}, \eta_{n(k)})$$

and

$$\rho(\eta_{m(k)}, \eta_{n(k)}) \leq \rho(\eta_{m(k)}, \eta_{m(k)-1}) + \rho(\eta_{m(k)-1}, \eta_{n(k)}).$$

As $k \to +\infty$, from (16) and (19), we have

$$\lim_{k\to+\infty} \rho(\eta_{m(k)-1}, \eta_{n(k)}) = \epsilon. \tag{22}$$

For $k \in \mathbb{N}$, we have

$$\rho(\eta_{n(k)-1}, \eta_{m(k)}) \leq \rho(\eta_{n(k)-1}, \eta_{n(k)}) + \rho(\eta_{n(k)}, \eta_{m(k)})$$

and

$$\rho(\eta_{n(k)}, \eta_{m(k)}) \leq \rho(\eta_{n(k)}, \eta_{n(k)-1}) + \rho(\eta_{n(k)-1}, \eta_{m(k)})$$

Letting $k \to +\infty$ and using (16) and (19), we have

$$\lim_{k\to+\infty} \rho(\eta_{n(k)-1}, \eta_{m(k)}) = \epsilon. \tag{23}$$

As $k \to +\infty$ in (20) and from (16), (19)–(23) and with ξ and ω being continuous, we have

$$\xi(\epsilon) \leq \xi(\epsilon) - \omega(\epsilon),$$

a contradiction as $\epsilon > 0$. Hence, $\{\eta_n\}$ is Cauchy.

By completeness of (M, ρ), there exists some $\eta^* \in M$ such that

$$\lim_{n\to+\infty} \eta_n = \eta^*. \tag{24}$$

We now claim

$$\eta^* \in \bigcap_{i=1}^{q} \wp_i.$$

From condition (1), and since $\eta_0 \in \wp_1$, we have $(\eta_{nq})_{n\geq 0} \subseteq \wp_1$. Since $\wp_1$ is closed, from (24), we obtain $\eta^* \in \wp_1$. Again, from condition (1), we have $(\eta_{nq+1})_{n\geq 0} \subseteq \wp_2$. Since $\wp_2$ is closed, from (24), we obtain $\eta^* \in \wp_2$. Continuing this process, we obtain

$$\eta^* \in \bigcap_{i=1}^{q} \wp_i. \tag{25}$$

We now prove η^* is a fixed point of Ω. Clearly, from (25), since for all n there exists $i(n) \in \{1, 2, \ldots, q\}$ such that $\eta_n \in \wp_{i(n)}$, applying (2) with $\eta = \eta_n$ and $\delta = \eta^*$, we have

$$\begin{aligned} \xi(\rho(\Omega\eta_n, \Omega\eta^*)) &= \xi(\rho(\eta_{n+1}, \Omega\eta^*)) \\ &\leq \xi(\Delta_1(\eta_n, \eta^*)) - \omega(\Delta_2(\eta_n, \eta^*)) \end{aligned}$$

where

$$\begin{aligned} \Delta_1(\eta_n, \eta^*) &= \max\left\{\rho(\eta_n, \eta^*), \frac{1}{2}[\rho(\eta_n, \Omega\eta_n) + \rho(\eta^*, \Omega\eta^*)], \frac{1}{2}[\rho(\eta_n, \Omega\eta^*) + \rho(\eta^*, \Omega\eta_n)]\right\} \\ &= \max\left\{\rho(\eta_n, \eta^*), \frac{1}{2}[\rho(\eta_n, \eta_{n+1}) + \rho(\eta^*, \Omega\eta^*)], \frac{1}{2}[\rho(\eta_n, \Omega\eta^*) + \rho(\eta^*, \eta_{n+1})]\right\} \end{aligned}$$

and

$$\begin{aligned} \Delta_2(\eta_n, \eta^*) &= \min\left\{\rho(\eta_n, \eta^*), \frac{1}{2}[\rho(\eta_n, \Omega\eta^*) + \rho(\eta^*, \Omega\eta_n)]\right\} \\ &= \min\left\{\rho(\eta_n, \eta^*), \frac{1}{2}[\rho(\eta_n, \Omega\eta^*) + \rho(\eta^*, \eta_{n+1})]\right\} \end{aligned}$$

Now,
$$\xi(\rho(\eta^*,\Omega\eta^*)) \leq \xi\left(\frac{1}{2}\rho(\eta^*,\Omega\eta^*)\right) - \omega(0).$$
as $n \to +\infty$ Therefore,
$$\xi(\rho(\eta^*,\Omega\eta^*)) \leq \xi\left(\frac{1}{2}\rho(\eta^*,\Omega\eta^*)\right).$$
ξ is monotonically increasing. Hence,
$$\rho(\eta^*,\Omega\eta^*) \leq \frac{1}{2}\rho(\eta^*,\Omega\eta^*)$$
a contradiction. Therefore, $\rho(\eta^*,\Omega\eta^*) = 0$ and hence $\Omega\eta^* = \eta^*$. Thus, η^* is a fixed point of Ω.

For uniqueness, suppose that δ^* is another fixed point with $\Omega\delta^* = \delta^*$. By condition (1), this implies that $\delta^* \in \bigcap_{i=1}^{q} \wp_i$. By using (2) for $\eta = \eta^*$ and $\delta = \delta^*$,
$$\begin{aligned} \xi(\rho(\Omega\eta^*, T\delta^*)) &= \xi(\rho(\eta^*,\delta^*)) \\ &\leq \xi(\Delta_1(\eta^*,\delta^*)) - \omega(\Delta_2(\eta^*,\delta^*)) \end{aligned} \tag{26}$$
where
$$\Delta_1(\eta^*,\delta^*) = \max\left\{\rho(\eta^*,\delta^*), \frac{1}{2}[\rho(\eta^*,\Omega\eta^*) + \rho(\delta^*,\Omega\delta^*)], \frac{1}{2}[\rho(\eta^*,\Omega\delta^*) + \rho(\delta^*,\Omega\eta^*)]\right\} \tag{27}$$
and
$$\Delta_2(\eta^*,\delta^*) = \min\left\{\rho(\eta^*,\delta^*), \frac{1}{2}[\rho(\eta^*,\Omega\delta^*) + \rho(\delta^*,\Omega\eta^*)]\right\} \tag{28}$$
Since η^* and δ^* are both fixed points of Ω, from (26)–(28), we have
$$\xi(\rho(\eta^*,\delta^*)) \leq \xi(\rho(\eta^*,\delta^*)) - \omega(\rho(\eta^*,\delta^*))$$
which implies $\rho(\eta^*,\delta^*) = 0$, Hence, $\eta^* = \delta^*$. □

Remark 1. *Letting $\xi(t) = t$ in Theorem 2 results in the following:*

Corollary 1. *Let (M,ρ) be a complete metric space, $q \in \mathbb{N}$, $\wp_1, \wp_2, \ldots, \wp_q$ be non-empty closed subsets of M and $Z = \bigcup_{i=1}^{q} \wp_i$. Suppose $\Omega : Z \to Z$ such that*
$$\rho(\Omega\eta,\Omega\delta) \leq \Delta_1(\eta,\delta) - \omega(\Delta_2(\eta,\delta)),$$
where $\eta,\delta \in M, \eta \neq \delta$ and
$$\begin{aligned} \Delta_1(\eta,\delta) &= \max\left\{\rho(\eta,\delta), \frac{1}{2}[\rho(\eta,\Omega\eta) + \rho(\delta,\Omega\delta)], \frac{1}{2}[\rho(\eta,\Omega\delta) + \rho(\delta,\Omega\eta)]\right\} \\ \Delta_2(\eta,\delta) &= \min\left\{\rho(\eta,\delta), \frac{1}{2}[\rho(\eta,\Omega\delta) + \rho(\delta,\Omega\eta)]\right\}; \end{aligned}$$
$\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with $\omega(t) = 0$ if and only if $t = 0$.

Then, Ω has a unique fixed point in $\bigcap_{i=1}^{q} \wp_i$.

We supplement our derived result with the following example:

Example 1. *Let* $M = \mathbb{R}^2$ *and let* $\rho : M \times M \to \mathbb{R}^+$ *be given as*

$$\rho((\eta_1, \delta_1), (\eta_2, \delta_2)) = |\eta_1 - \eta_2| + |\delta_1 - \delta_2|.$$

(M, ρ) is complete and let

$$\wp_1 = \{(a, 0) : 0 \leq a \leq 1\} \text{ and } \wp_2 = \{(0, b) : 0 \leq b \leq 1\},$$

be the closed subsets of M. Let $\Omega : \wp_1 \cup \wp_2 \to \wp_1 \cup \wp_2$ be a map, such that

$$\Omega((a, 0)) = \left(0, \frac{a}{2}\right), \Omega((0, b)) = \left(\frac{b}{2}, 0\right).$$

Clearly, $Z = \wp_1 \cup \wp_2$ is a cyclic representation of Z with respect to Ω.

Claim: Ω satisfies (2).

Let $\xi(t) = t$ and $\omega(t) = \frac{1}{2}t$ such that $\omega(0) = 0$. Let $\eta = (a, 0)$ and $\delta = (0, b)$ then $\Omega\eta = \left(0, \frac{a}{2}\right)$, $\Omega\delta = \left(\frac{b}{2}, 0\right)$ and

$$\begin{aligned}
\rho(\Omega\eta, \Omega\delta) &= \frac{1}{2}(a + b) \\
\rho(\eta, \delta) &= a + b \\
\frac{1}{2}[\rho(\eta, \Omega\eta) + \rho(\delta, \Omega\delta)] &= \frac{3}{4}(a + b) \\
\frac{1}{2}[\rho(\eta, \Omega\delta) + \rho(\delta, \Omega\eta)] &= \frac{1}{2}\left[\left|a - \frac{b}{2}\right| + \left|b - \frac{a}{2}\right|\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
\xi(\rho(\Omega\eta, \Omega\delta)) &= \frac{1}{2}(a + b) \\
&= \frac{1}{2}\xi(\rho(\eta, \delta)) \\
&\leq \xi(\Delta_1(\eta, \delta)) - \omega(\Delta_2(\eta, \delta)).
\end{aligned}$$

All conditions of Theorem 2 are satisfied and Ω has a unique fixed point $(0, 0) \in \wp_1 \cap \wp_2$.

3. An Application to Boundary Value Problem

The derived result is applied to the following Boundary Value Problem to find an analytical solution:

$$\left.\begin{aligned}
-\frac{d^2\delta}{d\eta^2} = F(\eta, \delta(\eta)), \quad \eta \in [0, 1], \\
\delta(0) = \delta(1) = 0,
\end{aligned}\right\} \tag{29}$$

where $F : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous function. This problem is equivalent to the integral equation

$$\delta(\eta) = \int_0^1 G(\eta, \gamma) F(\gamma, \delta(\gamma)) d\gamma, \text{ for } \eta \in [0, 1], \tag{30}$$

where $G(\eta, \gamma)$ is the Green's function defined as

$$G(\eta, \gamma) = \begin{cases} \gamma(1 - \eta); & 0 \leq \gamma < \eta, \\ \eta(1 - \gamma); & \eta < \gamma \leq 1. \end{cases}$$

Let $M = C([0,1], \mathbb{R}^+)$ be a space of non-negative continuous real-valued functions defined on $[0,1]$. Now, we define the metric ρ on M, that is,

$$\rho(\delta, \mu) = \max_{\eta \in [0,1]} |\delta(\eta) - \mu(\eta)|$$

for $\delta, \mu \in M$. Then, (M, ρ) is a complete metric space.

Let $\wp_1 = \wp_2 = M = C([0,1], \mathbb{R}^+)$. It is clear that $\wp_1$ and $\wp_2$ are closed subsets of M. Consider the self mapping $\Omega : \wp_1 \cup \wp_2 \to \wp_1 \cup \wp_2$ is defined by

$$\Omega\delta(\eta) = \int_0^1 G(\eta, \gamma) F(\gamma, \delta(\gamma)) d\gamma, \quad \eta \in [0,1].$$

Clearly, $\Omega(\wp_1) \subseteq \wp_2$ and $\Omega(\wp_2) \subseteq \wp_1$. Thus, Ω is cyclic map on $\wp_1 \cup \wp_2$.
Suppose the following condition hold:

$$|F(\gamma, \alpha) - F(\gamma, \beta)| \leq |\alpha - \beta|, \text{ for all } \gamma \in [0,1] \text{ and } \alpha, \beta \in \mathbb{R}^+.$$

Then, (30) has a unique solution $\delta^* \in M$.
Finally, we will show that, for each $\delta \in \wp_1$ and $\mu \in \wp_2$, we have

$$\xi(\rho(\Omega\delta, \Omega\mu)) \leq \xi(\Delta_1(\delta, \mu)) - \omega(\Delta_2(\delta, \mu))$$

for $\xi(\eta) = \eta$ and $\omega(\eta) = \frac{1}{2}\eta$. Now, let $(\delta, \mu) \in \wp_1 \times \wp_2$. Therefore, by (2) we deduce that for each $\eta \in [0,1]$.

$$\begin{aligned}
\rho(\Omega\delta, \Omega\mu) &= \max_{\eta \in [0,1]} |\Omega\delta(\eta) - \Omega\mu(\eta)| \\
&= \max_{\eta \in [0,1]} \left| \int_0^1 G(\eta, \gamma) F(\gamma, \delta(\gamma)) d\gamma - \int_0^1 G(\eta, \gamma) F(\gamma, \mu(\gamma)) d\gamma \right| \\
&= \max_{\eta \in [0,1]} \left| \int_0^1 G(\eta, \gamma) [F(\gamma, \delta(\gamma)) - F(\gamma, \mu(\gamma))] d\gamma \right| \\
&\leq \max_{\eta \in [0,1]} \int_0^1 G(\eta, \gamma) |F(\gamma, \delta(\gamma)) - F(\gamma, \mu(\gamma))| d\gamma \\
&\leq \max_{\eta \in [0,1]} \int_0^1 G(\eta, \gamma) |\delta(\gamma) - \mu(\gamma)| d\gamma \\
&\leq \max_{\eta \in [0,1]} \int_0^1 G(\eta, \gamma) \max_{\gamma \in [0,1]} |\delta(\gamma) - \mu(\gamma)| d\gamma \\
&= \rho(\delta, \mu) \max_{\eta \in [0,1]} \int_0^1 G(\eta, \gamma) d\gamma. \qquad (31)
\end{aligned}$$

It is easy to verify that $\int_0^1 G(\eta, \gamma) d\gamma = \frac{\eta}{2} - \frac{\eta^2}{2}$ and thus, $\max_{\eta \in [0,1]} \int_0^1 G(\eta, \gamma) d\gamma = \frac{1}{8}$.
Considering the above facts, the inequality (31) gives us

$$\rho(\Omega\delta, \Omega\mu) \leq \frac{1}{8} \rho(\delta, \mu). \qquad (32)$$

Now,

$$
\begin{aligned}
\Delta_1(\delta,\mu) = \max\Bigg\{ & \max_{\eta\in[0,1]}|\delta(\eta)-\mu(\eta)|, \frac{1}{2}\Bigg[\max_{\eta\in[0,1]}\left|\delta(\eta)-\int_0^1 G(\eta,\gamma)F(\gamma,\delta(\gamma))d\gamma\right| \\
& + \max_{\eta\in[0,1]}\left|\mu(\eta)-\int_0^1 G(\eta,\gamma)F(\gamma,\mu(\gamma))d\gamma\right|\Bigg], \\
& \frac{1}{2}\Bigg[\max_{\eta\in[0,1]}\left|\delta(\eta)-\int_0^1 G(\eta,\gamma)F(\gamma,\mu(\gamma))d\gamma\right| \\
& + \max_{\eta\in[0,1]}\left|\mu(\eta)-\int_0^1 G(\eta,\gamma)F(\gamma,\delta(\gamma))d\gamma\right|\Bigg]\Bigg\} \\
= \max\Bigg\{ & \max_{\eta\in[0,1]}|\delta(\eta)-\mu(\eta)|, \frac{1}{2}\Bigg[\max_{\eta\in[0,1]}\left|\delta(\eta)-\int_0^1 G(\eta,\gamma)F(\gamma,\delta(\gamma))d\gamma\right| \\
& + \max_{\eta\in[0,1]}\left|\int_0^1 G(\eta,\gamma)F(\gamma,\mu(\gamma))d\gamma-\mu(\eta)\right|\Bigg], \frac{1}{2}\Bigg[\max_{\eta\in[0,1]}\left|\delta(\eta)-\int_0^1 G(\eta,\gamma)F(\gamma,\mu(\gamma))d\gamma\right| \\
& + \max_{\eta\in[0,1]}\left|\int_0^1 G(\eta,\gamma)F(\gamma,\delta(\gamma))d\gamma-\mu(\eta)\right|\Bigg]\Bigg\} \\
\geq \max\Bigg\{ & \max_{\eta\in[0,1]}|\delta(\eta)-\mu(\eta)|, \frac{1}{2}\Bigg[\max_{\eta\in[0,1]}\Bigg|\delta(\eta)-\int_0^1 G(\eta,\gamma)F(\gamma,\delta(\gamma))d\gamma \\
& + \int_0^1 G(\eta,\gamma)F(\gamma,\mu(\gamma))d\gamma-\mu(\eta)\Bigg|\Bigg], \frac{1}{2}\Bigg[\max_{\eta\in[0,1]}\Bigg|\delta(\eta)-\int_0^1 G(\eta,\gamma)F(\gamma,\mu(\gamma))d\gamma \\
& + \int_0^1 G(\eta,\gamma)F(\gamma,\delta(\gamma))d\gamma-\mu(\eta)\Bigg|\Bigg]\Bigg\} \\
= \max\Bigg\{ & \max_{\eta\in[0,1]}|\delta(\eta)-\mu(\eta)|, \frac{1}{2}\Bigg[\max_{\eta\in[0,1]}\Bigg|\delta(\eta)-\mu(\eta)-\Bigg(\int_0^1 G(\eta,\gamma)F(\gamma,\delta(\gamma))d\gamma \\
& - \int_0^1 G(\eta,\gamma)F(\gamma,\mu(\gamma))d\gamma\Bigg)\Bigg|\Bigg], \frac{1}{2}\Bigg[\max_{\eta\in[0,1]}\Bigg|\delta(\eta)-\mu(\eta)-\Bigg(\int_0^1 G(\eta,\gamma)F(\gamma,\mu(\gamma))d\gamma \\
& - \int_0^1 G(\eta,\gamma)F(\gamma,\delta(\gamma))d\gamma\Bigg)\Bigg|\Bigg]\Bigg\} \\
\geq \max\Bigg\{ & \max_{\eta\in[0,1]}|\delta(\eta)-\mu(\eta)|, \frac{1}{2}\Bigg[\max_{\eta\in[0,1]}\Bigg||\delta(\eta)-\mu(\eta)|-\Bigg|\int_0^1 G(\eta,\gamma)F(\gamma,\delta(\gamma))d\gamma \\
& - \int_0^1 G(\eta,\gamma)F(\gamma,\mu(\gamma))d\gamma\Bigg|\Bigg|\Bigg], \frac{1}{2}\Bigg[\max_{\eta\in[0,1]}\Bigg||\delta(\eta)-\mu(\eta)|-\Bigg|\int_0^1 G(\eta,\gamma)F(\gamma,\mu(\gamma))d\gamma \\
& - \int_0^1 G(\eta,\gamma)F(\gamma,\delta(\gamma))d\gamma\Bigg|\Bigg|\Bigg]\Bigg\} \\
= \max\Bigg\{ & \max_{\eta\in[0,1]}|\delta(\eta)-\mu(\eta)|, \frac{1}{2}\Bigg[\max_{\eta\in[0,1]}\Bigg||\delta(\eta)-\mu(\eta)|-\Bigg|\int_0^1 G(\eta,\gamma)F(\gamma,\delta(\gamma))d\gamma \\
& - \int_0^1 G(\eta,\gamma)F(\gamma,\mu(\gamma))d\gamma\Bigg|\Bigg|\Bigg], \frac{1}{2}\Bigg[\max_{\eta\in[0,1]}\Bigg||\delta(\eta)-\mu(\eta)|-\Bigg|\int_0^1 G(\eta,\gamma)F(\gamma,\delta(\gamma))d\gamma \\
& - \int_0^1 G(\eta,\gamma)F(\gamma,\mu(\gamma))d\gamma\Bigg|\Bigg|\Bigg]\Bigg\}
\end{aligned}
$$

$$= \max\left\{\max_{\eta\in[0,1]}|\delta(\eta)-\mu(\eta)|, \frac{1}{2}\left[\max_{\eta\in[0,1]}\left||\delta(\eta)-\mu(\eta)| - \left|\int_0^1 G(\eta,\gamma)F(\gamma,\delta(\gamma))d\gamma\right.\right.\right.\right.$$

$$\left.\left.\left.\left. - \int_0^1 G(\eta,\gamma)F(\gamma,\mu(\gamma))d\gamma\right|\right|\right]\right\}$$

$$\geq \max\left\{\max_{\eta\in[0,1]}|\delta(\eta)-\mu(\eta)|, \frac{1}{2}\left[\max_{\eta\in[0,1]}\left||\delta(\eta)-\mu(\eta)| - \int_0^1 G(\eta,\gamma)|F(\gamma,\delta(\gamma))-F(\gamma,\mu(\gamma))|d\gamma\right|\right]\right\}$$

$$\geq \max\left\{\max_{\eta\in[0,1]}|\delta(\eta)-\mu(\eta)|, \frac{1}{2}\left[\max_{\eta\in[0,1]}\left||\delta(\eta)-\mu(\eta)| - \int_0^1 G(\eta,\gamma)|\delta(\gamma)-\mu(\gamma)|d\gamma\right|\right]\right\}$$

$$\geq \max\left\{\max_{\eta\in[0,1]}|\delta(\eta)-\mu(\eta)|, \frac{1}{2}\left[\max_{\eta\in[0,1]}\left||\delta(\eta)-\mu(\eta)| - \int_0^1 G(\eta,\gamma)\max_{\gamma\in[0,1]}|\delta(\gamma)-\mu(\gamma)|d\gamma\right|\right]\right\}$$

$$= \max\left\{\rho(\delta,\mu), \frac{1}{2}\left[\max_{\eta\in[0,1]}\left||\delta(\eta)-\mu(\eta)| - \rho(\delta,\mu)\int_0^1 G(\eta,\gamma)d\gamma\right|\right]\right\}$$

$$\geq \max\left\{\rho(\delta,\mu), \frac{1}{2}\left[\rho(\delta,\mu) - \frac{1}{8}\rho(\delta,\mu)\right]\right\}$$

$$= \max\left\{\rho(\delta,\mu), \frac{7}{16}\rho(\delta,\mu)\right\}.$$

Therefore, we have

$$\Delta_1(\delta,\mu) \geq \rho(\delta,\mu). \tag{33}$$

Additionally, we have

$$\Delta_2(\delta,\mu) \leq \frac{7}{16}\rho(\delta,\mu). \tag{34}$$

Moreover, from (32)–(34), we have

$$\xi(\rho(\delta,\mu)) \leq \xi(\Delta_1(\delta,\mu)) - \omega(\Delta_2(\delta,\mu)).$$

Using the same procedure, we can show that the above inequality also holds if we take $(\delta,\mu) \in \wp_2 \times \wp_1$. Thus, Ω satisfies the contraction condition of Theorem 2.

Hence, by Theorem 2, Ω has a unique fixed point $\delta^* \in \wp_1 \cap \wp_2$. Hence, the integral equation (30) has a unique solution.

Now, we present a numerical solution of an ODE of Green's function to supplement the derived results.

Example 2. *As an example, suppose that $F(\eta,\delta(\eta))$ takes the form of $F(\eta,\delta(\eta)) = 0.2sin(\pi\eta) - 0.5\delta(\eta)$. The exact form of the unique solution for this case can be obtained as follows:*

$$\delta(\eta) = 0.0192871\sin(\pi\eta).$$

The graph of the solution is shown in Figure 1.

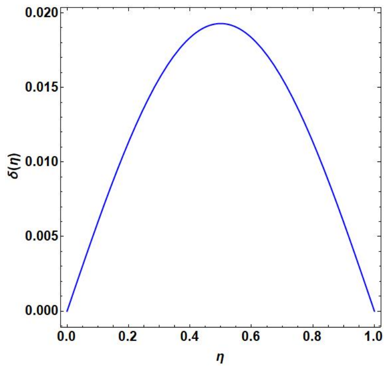

Figure 1. Solution of the boundary value problem.

4. Conclusions

In this paper, we have established fixed-point results by introducing a new type of (ξ, ω)-weakly cyclic generalized contraction in the setting of metric spaces. Our results are extensions or generalizations of the results proven in the past. We have also provided a non-trivial example to substantiate the derived result. To supplement the example, we have provided an application to find the analytical solution to the boundary value problem of second-order differential equations. There is scope for applying the proposed contraction in this paper in the setting of various topological spaces and their extensions.

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References

1. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundam. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Brouwer, L.E.J. Über Abbildungen von Mannigfaltigkeiten. *Math. Ann.* **1912**, *71*, 97–115. [CrossRef]
3. Khan, M.S.; Swaleh, M.; Sessa, S. Fixed point theorems by altering distances between the points. *Bull. Aust. Math. Soc.* **1984**, *30*, 1–9. [CrossRef]
4. Massa, S. Uńosservazione su un teorema di Browder-Roux-Soardi. *Boll. U. M. I.* **1973**, *4*, 151–155.
5. Kirk, W.A.; Srinivasan, P.S.; Veeramani, P. Fixed points for mapping satisfying cyclical contractive conditions. *Fixed Point Theory* **2003**, *4*, 79–89.
6. Petrushel, G. Cyclic representations and Periodic points. *Stud. Univ. Babes-Bloyai Math* **2005**, *50*, 107–112.
7. Karpagam, S.; Agrawal, S. Best Proximity Points theorems for Cyclic Meir Keeler Contraction Maps. *Nonlinear Anal.* **2011**, *74*, 1040–1046. [CrossRef]
8. Pacurar M.; Rus, I.A. Fixed Point Theory for ϕ-contractions. *Nonlinear Anal.* **2010**, *72*, 1181–1187. [CrossRef]
9. Derafshpour, M.; Rezapur, S.; Shahzad, N. Best Proximity point of cyclic ϕ contractions in ordered metric spaces. *Topol. Methods Nonlinear Anal.* **2011**, *37*, 193–202.
10. George, R.; Rajagopalan, R. Cyclic Contractions and fixed points in a dislocated metric space. *Int. J. Math. Anal.* **2013**, *7*, 403– 411. [CrossRef]
11. Rhoades, B.E. Some theorems on weakly contractive maps. *Nonlinear Anal.* **2001**, *47*, 2683–2693. [CrossRef]
12. Dutta, P.N.; Choudhury, B.S. A generalisation of contraction principle in metric spaces. *Fixed Point Theory Appl.* **2008**, *2008*, 406368. [CrossRef]
13. Đorić, D. Common fixed point for generalized (ψ, ϕ)-weak contractions. *Appl. Math. Lett.* **2009**, *22*, 1896–1900. [CrossRef]

14. Zhang, Q.; Song, Y. Fixed point theory for generalized φ-weak contractions. *Appl. Math. Lett.* **2009**, *22*, 75–78. [CrossRef]
15. Hosseini, V.R. Common fixed for generalized (ϕ-ψ)-weak contractions mappings condition of integral type. *Int. J. Math. Anal.* **2010**, *4*, 1535–1543.
16. Popescu, O. Fixed points for (ψ, ϕ)-weak contractions. *Appl. Math. Lett.* **2011**, *24*, 1–4. [CrossRef]
17. Nashine, H.K.; Golubović, Z.; Kadelburg, Z. Nonlinear cyclic weak contractions in G-metric spaces and applications to boundary value problems. *Fixed Point Theory Appl.* **2012**, *2012*, 227. [CrossRef]
18. Abkar, A.; Choudhury, B.S. Fixed point result in partially ordered metric spaces using weak contractive inequalities. *Facta Univ. Ser. Math. Inform.* **2012**, *27*, 1–11.
19. Ahmad, J.; Arshad, M.; Vetro, P. Coupled coincidence point results for (φ, ψ)-contractive mappings in partially ordered metric spaces. *Georgian Math. J.* **2014**, *21*, 113–124. [CrossRef]
20. Hussain, N.; Arshad, M.; Shoaib, A. Fahimuddin Common fixed point results for α-ψ-contractions on a metric space endowed with graph. *J. Inequal. Appl.* **2014**, *1*, 1–14.
21. Murthy, P.P.; Tas, K.; Patel, U.D. Common fixed point theorems for generalized (ϕ, ψ)-weak contraction condition in complete metric spaces. *J. Inequal. Appl.* **2015**, *2015*, 139. [CrossRef]
22. Okeke, G.A.; Francis, D.; Gibali, A. On fixed point theorems for a class of α-ν-Meir-Keeler-type contraction mapping in modular extended b-metric spaces. *J. Anal.* **2022**, *30*, 1257–1282. [CrossRef]
23. Okeke, G.A.; Francis, D. Fixed point theorems for Geraghty-type mappings applied to solving nonlinear Volterra-Fredholm integral equations in modular G-metric spaces. *Arab. J. Math. Sci.* **2021**, *27*, 214–234. [CrossRef]
24. Okeke, G.A.; Francis, D. Fixed point theorems for asymptotically T-regular mappings in preordered modular G-metric spaces applied to solving nonlinear integral equations. *J. Anal.* **2022**, *30*, 501–545. [CrossRef]