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A New Result Concerning Nonlocal Controllability of Hilfer Fractional Stochastic Differential Equations via almost Sectorial Operators

Sivajiganesan Sivasankar [1], Ramalingam Udhayakumar [1,*], Muchenedi Hari Kishor [2], Sharifah E. Alhazmi [3] and Shrideh Al-Omari [4]

[1] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India
[2] Department of Analytics, School of Computer Science and Engineering, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India
[3] Mathematics Department, Al-Qunfudah University College, Umm Al-Qura University, Mecca 24382, Saudi Arabia
[4] Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Amman 11134, Jordan
* Correspondence: udhayaram.v@gmail.com

Abstract: This manuscript mainly focused on the nonlocal controllability of Hilfer fractional stochastic differential equations via almost sectorial operators. The key ideas of the study are illustrated by using ideas from fractional calculus, the fixed point technique, and measures of noncompactness. Then, the authors establish new criteria for the mild existence of solutions and derive fundamental characteristics of the nonlocal controllability of a system. In addition, researchers offer theoretical and real-world examples to demonstrate the effectiveness and suitability of our suggested solutions.

Keywords: Hilfer fractional derivative; stochastic differential system; nonlocal controllability; measures of noncompactness; Darbo–Sadovskii fixed-point theorem; almost sectorial operators

MSC: 34K30; 34K50; 47H08; 47H10

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1. Introduction

The monographs of Ahmad et al. [1], Diethelm [2], Lakshmikantham and Vatsala [3], Miller and Ross [4], Podlubny [5], and Zhou [6,7] present several approaches to the subject of fractional calculus. On the other hand, differential equations with arbitrary order derivatives are presented as an extension of fractional differential equations. They are often used to explain the behavior of phase evolution and temporal interactions in a variety of applied sciences domains; for a list of references, see [8–19].

As far as we know, the concept of controllability is a vital qualitative and quantitative characterization of the control system, whereas the controllability feature is significant in a variety of control problems in both limited and unlimited systems. The controllability of a fractional wave equation has recently gained a lot of attention in this research. Using the Mönch fixed-point approach and measures of noncompactness (MNC), the investigator's Wang and Zhou [20] recently discovered a few requirements providing the complete controllability of fractional evolution systems (FES) without assuming the adaptability of specific response technicians. Wang et al. [21] identified two essential requirements for nonlocal controllability in a fractional evolution system. For certain weak noncompactness criteria, these theorems ensure that the controllability findings will work as intended. Ji et al. [22] deduced the nonlocal controllability of an impulsive differential evolution system using the Mönch fixed-point theorem via the measures of noncompactness. Numerous writers have made important contributions to the exact and approximate controllability of various nonlinear dynamical systems with or without delays. In [22–24], researchers developed a new set of necessary requirements for the exact controllability of differential systems. For a summary of recent studies on the existence and controllability of the differential system with the fractional number $1 < \alpha < 2$, see [12,25].

Stochastic models should be examined instead of deterministic ones since noise and uncontrolled fluctuations are common and inherent in both natural and artificial systems. Unpredictability is reflected in the mathematical depiction of some events in stochastic differential equations. The use of stochastic differential equations (SDEs) to describe various phenomena in population dynamics, physics, electrical engineering, geography, psychology, biochemistry, and some other fields of physics and technology has recently attracted a lot of interest. SDEs can be applied in both finite and infinite dimensions. See, for example, [9,26–28] for a thorough introduction to stochastic differential equations and their applications.

Researchers employed almost sectorial operators to advance the fractional existence of fractional calculus. A new method was developed to find a mild solution for the system under investigation. Additionally, researchers developed a theory to analyze different attributes of connected semigroups produced by almost sectorial operators applying semigroups, fractional calculus, MNC, Wright-type functions, multivalued analysis, Laplace transforms, and a fixed-point approach. For further details, see [24,29–33].

Another sort of fractional derivative, including the R-L and Caputo fractional derivatives, was given by Hilfer [34]. Currently, investigators place a high value on Hilfer fractional differential calculus. Recently, many academics have shown considerable interest in this area, which has inspired work in [35–40]. The researchers in [41–43] employed Schauder's fixed-point theorem to obtain their results via almost sectorial operators. The author used the Mönch fixed-point principle via the MNC to establish their conclusions in [21,22,44,45]. The Darbo–Sadovskii fixed-point approach via MNC was used by the authors to construct the concepts of differential systems [46,47]. Further, [48] investigated whether there is a mild solution for HF differential systems using almost sectorial operators. Inspired by the above article, we have developed the nonlocal controllability of the Hilfer fractional (HF) stochastic differential equations via almost sectorial operators by using the Darbo–Sadovskii fixed-point theorem via measures of noncompactness. However, to the best of our knowledge and investigation, no research has been conducted on this research problem.

In this article, researchers describe their latest finding on the nonlocal controllability of the HF stochastic differential equations via almost sectorial operators of the form

$$D_{0^+}^{\gamma,\delta}x(t) = Ax(t) + Bu(t) + G\big(t, x(t)\big)\frac{dW(t)}{dt}, \qquad t \in V' = (0, c], \tag{1}$$

$$I_{0^+}^{(1-\gamma)(1-\delta)}\big[x(0) + N(x)\big] = x_0, \tag{2}$$

where A is an almost sectorial operator that represents an analytic semigroup $\{T(t), t \geq 0\}$ on Z. $D_{0^+}^{\gamma,\delta}$ denotes the Hilfer fractional derivative (HFD) of order γ, $0 < \gamma < 1$ and type δ, $0 \leq \delta \leq 1$. Let $x(\cdot)$ be the state in a Hilbert space Z with $\|\cdot\|$ and $u(\cdot)$ be the control function in $L^2(V, U)$, where U is the Hilbert space. Here, $B : U \to Z$ is the bounded linear operator. Set $V = [0, c]$, and let $G : V \times Z \to Z$ be the Z-valued function, and the nonlocal term is given by $N : C(V, Z) \to Z$.

The article's framework may be divided into the following categories: In Section 2, researchers propose the fundamental characteristics of fractional calculus, semigroups, almost sectorial operators, and measures of non-compactness. The authors describe the system's nonlocal controllability in Section 3. Finally, in Section 4, the authors give theoretical and practical implementations to make the conversation as successful as possible.

2. Preliminaries

In this chapter, we discuss fundamental definitions, theorems, and lemmas that are utilized throughout the whole work.

The two real Hilbert spaces are represented by the symbols $(Z, \|\cdot\|)$ and $(U, \|\cdot\|)$. Let $(\Omega, \mathscr{E}, P)$ be a complete probability space associated with a complete family of right continuous growing sub σ-algebra $\{\mathscr{E}_t : t \in V\}$ such that $\mathscr{E}_t \subset \mathscr{E}$. Let $W = (W_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathscr{E}, P)$ with the covariance operator Q such that $\mathrm{Tr}(Q) < \infty$. Then, we claim that a complete orthonormal system e_k, $k \geq 1$ exists in U, a finite sequence of positive real integers χ_k such that $Qe_k = \chi_k e_k$, $k = 1, 2, \cdots$ and $\{\beta_k\}$ of independent Brownian motion such that

$$\big(W(t), e\big)_U = \sum_{k=1}^{\infty} \sqrt{\chi_k}(e_k, e)\beta_k(t), \ \ e \in U \ t \geq 0.$$

Assume that the space of all Q-Hilbert-Schmidt operators $\varphi : Q^{\frac{1}{2}}U \to Z$ with the scalar product $\|\varphi\|_Q^2 = \langle \varphi, \varphi \rangle = \mathrm{Tr}(\varphi Q \varphi)$ is represented by the notation $L_2^0 = L_2\big(Q^{\frac{1}{2}}U, Z\big)$. Consider the resolvent operator of A, $0 \in \rho(A)$, where $\mathcal{S}(\cdot)$ is uniformly bounded, that is, $\|\mathcal{S}(t)\| \leq M$, $M \geq 1$, and $t \geq 0$. The fractional power operator A^η on its domain $D(A^\eta)$ may therefore be derived for $\eta \in (0, 1]$. Furthermore, $D(A^\eta)$ is dense in Z.

The following important properties of A^η will be discussed as follows.

Theorem 1.

1. *Suppose* $0 < \eta \leq 1$, *then* $Z_\eta = D(A^\eta)$ *is a Banach space with* $\|x\|_\eta = \|A^\eta x\|$, $x \in Z_\chi$.
2. *Assume that* $0 < \gamma < \eta \leq 1$, *then* $D(A^\eta) \to D(A^\gamma)$, *$A$ is compact and the embedding is also compact.*
3. *For all* $0 < \eta \leq 1$, $C_\eta > 0$ *exists such that*

$$\|A^\eta \mathcal{S}(t)\| \leq \frac{C_\eta}{t^\eta}, \ 0 < t \leq c.$$

The set of all strongly continuous, square-integrable, Z-valued random variables, denoted by $L_2(\Omega, Z)$, is a Banach space associated with $\|x(\cdot)\|_{L_2(\Omega,Z)} = \big(E\|x(., W)\|^2\big)^{\frac{1}{2}}$, where E is defined as $E(x) = \int_\Omega x(W)dP$. A necessary subspace of $L_2(\Omega, Z)$ is given by

$$L_2^0(s, Z) = \{x \in L_2(\Omega, Z), \ x \text{ is } \mathscr{E}_0 - \text{measurable}\}.$$

Let $C(V, Z) = \complement$ be the set of all continuous functions from V to Z, where $V = [0, c]$ and $V' = (0, c]$ with $c > 0$. Let the Banach space $\mathcal{X} = \{x \in C(V', Z) : \lim_{t\to 0} t^{1-\delta+\gamma\delta-\gamma\vartheta}x(t)\}$ exist and be finite and its $\|\cdot\|_\mathcal{X}$ be denoted by $\|x\|_\mathcal{X} = \sup_{t\in V'}\big\{t^{1-\delta+\gamma\delta-\gamma\vartheta}\|x(t)\|\big\}$. Set $B_P(V) = \{u \in \complement$ such that $\|u\| \leq P\}$. Let $x(t) = t^{-1+\delta-\gamma\delta+\gamma\vartheta}Z(t)$, $t \in (0, c]$ be noted. Consequently, $x \in \mathcal{X}$ if and only if $Z \in \complement$ and $\|x\|_\mathcal{X} = \|y\|$.

Definition 1 ([31])**.** *For* $0 < \vartheta < 1$, $0 < \omega < \frac{\pi}{2}$, *we denote the family of closed linear operators* $\Theta_\omega^{-\vartheta}$, *the sector* $S_\omega = \{v \in \mathbb{C}\backslash\{0\}$ *with* $|\arg v| \leq \omega\}$ *and* $A : D(A) \subset Z \to Z$ *that satisfies*

(i) $\sigma(A) \subseteq S_\omega$;
(ii) $\big\|(vI - A)^{-1}\big\| \leq \mathcal{K}_\delta|v|^{-\vartheta}$, *for all* $\omega < \delta < \pi$ *and let* $\mathcal{K}_\delta$ *be a constant,*

then $A \in \Theta_\omega^{-\vartheta}$ *is called an almost sectorial operator on* Z.

Proposition 1 ([31])**.** *Suppose* $x \in \Theta_\omega^{-\vartheta}$ *for* $0 < \vartheta < 1$ *and* $0 < \omega < \frac{\pi}{2}$. *Then, the following are satisfied:*

(a) $T(t + s) = T(t)T(s)$, *for all* $s, t \in S_{\frac{\pi}{2}-\omega}$;
(b) $\|T(t)\|_{L(Z)} \leq \kappa_0 t^{\vartheta-1}$, $t > 0$; *where the constant* $\kappa_0 > 0$;

(c) *The range $R(T(t))$ of $T(t)$, $t \in S^0_{\frac{\pi}{2}-\omega}$ is contained in $D(A^\infty)$. Particularly, $R(T(t)) \subset D(A^\theta)$ for all $\theta \in \mathbb{C}$ with $Re(\theta) > 0$*

$$A^\theta T(t)x = \frac{1}{2\pi i}\int_{\Gamma_\delta} v^\theta e^{-tv} R(v; A)xdz, \text{ for all } x \in Z,$$

and hence there exists a constant $C' = C'(\gamma, \theta) > 0$ such that

$$\|A^\theta T(t)\|_{L(Z)} \leq C' t^{-\gamma - Re(\theta) - 1}, \text{ for all } t > 0;$$

(d) *Suppose that $\Sigma_T = \{x \in Z : \lim_{t\to 0^+} T(t)x = x\}$, then $D(A^\theta) \subset \Sigma_T$ provided $\theta > 1 - \vartheta$;*
(e) *$(vI - A)^{-1} = \int_0^\infty e^{-vs} T(s)ds$, $v \in \mathbb{C}$ and $Re(v) > 0$.*

Definition 2 ([6]). *For the function $G : [c, \infty) \to \mathbb{R}$ with lower limit c, the R-L fractional integral of order γ is provided by*

$$I^\gamma_{c+} G(t) = \frac{1}{\Gamma(\gamma)} \int_c^t \frac{G(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0;\ \gamma \in \mathbb{R}^+.$$

Definition 3 ([6]). *The R-L derivative of order $\gamma > 0$, $k - 1 \leq \gamma < k$, $k \in \mathbb{N}$, of the function $G : [c, +\infty) \to \mathbb{R}$ is presented by*

$${}^{RL}D^\gamma_{c+} G(t) = \frac{1}{\Gamma(k-\gamma)} \frac{d^k}{dt^k} \int_c^t \frac{G(s)}{(t-s)^{\gamma+1-k}} ds,\ t > c,\ s \in \mathbb{R}^+.$$

Definition 4 ([6]). *The Caputo derivative of order $\gamma > 0$, $k - 1 \leq \gamma < k$, $k \in \mathbb{N}$ for a function $G : [c, +\infty) \to \mathbb{R}$ is denoted by*

$${}^{C}D^\gamma_{c+} G(t) = \frac{1}{\Gamma(k-\gamma)} \int_c^t \frac{G^k(s)}{(t-s)^{\gamma+1-k}} ds = I^{k-\gamma}_{c+} G^k(t),\ t > c,\ s \in \mathbb{R}^+.$$

Definition 5 ([34]). *The HFD of order $0 < \gamma < 1$ and type $\delta \in [0, 1]$ for the function $G : [c, +\infty) \to \mathbb{R}$ is presented by*

$$D^{\gamma,\delta}_{c+} G(t) = [I^{(1-\gamma)\delta}_{c+} D(I^{(1-\gamma)(1-\delta)}_{c+} G)](t).$$

Definition 6 ([48]). *Define the Wright function $M_\gamma(\beta)$ by*

$$M_\gamma(\beta) = \sum_{k\in\mathbb{N}} \frac{(-\beta)^{k-1}}{\Gamma(1-\gamma k)(k-1)!},\ \beta \in \mathbb{C}, \tag{3}$$

such that

$$\int_0^\infty \theta^\iota M_\gamma(\theta) d\theta = \frac{\Gamma(1+\iota)}{\Gamma(1+\gamma\iota)}, \quad \text{for } \iota \geq 0.$$

Theorem 2 ([6]). *In the uniform operator topology, $\mathcal{S}_{\gamma,\delta}(t)$ and $\mathcal{Q}_\gamma(t)$ are continuous, for $t > 0$, for every $c > 0$, and the continuity is uniform on $[c, \infty)$.*

Lemma 1 ([48]). *If $\{T_\gamma(t)\}_{t>0}$ is a compact operator, then $\{\mathcal{S}_{\gamma,\delta}(t)\}$ and $\{\mathcal{Q}_\gamma(t)\}_{t>0}$ are also compact linear operators.*

Lemma 2 ([48]). *Assume that $\{T_\gamma(t)\}_{t>0}$ is a compact operator. Then, $\{T_\gamma(t)\}_{t>0}$ is equicontinuous.*

Lemma 3 ([36]). *System (1) and (2) is equivalent to an integral equation given by*

$$x(t) = \frac{x(0) - N(x))}{\Gamma(\delta(1-\gamma)}t^{(1-\gamma)(1-\delta)} + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\big[Ax(s) + Bu(s)\big]ds + G\big(s, x(s)\big)dW(s).$$

Lemma 4 ([36]). *Let $x(t)$ be a solution of the integral equation given by Lemma 3 then $x(t)$ satisfies*

$$x(t) = \mathcal{S}_{\gamma,\delta}(t)\big[x(0) - N(x)\big] + \int_0^t \mathcal{K}_\gamma(t-s)G\big(s, x(s)\big)dW(s) + \int_0^t \mathcal{K}_\gamma(t-s)Bu(s)ds,\ t \in V,$$

where $\mathcal{S}_{\gamma,\delta}(t) = I_0^{\delta(1-\gamma)}\mathcal{K}_\gamma(t)$, $\mathcal{K}_\gamma(t) = t^{\gamma-1}\mathcal{Q}_\gamma(t)$ and $\mathcal{Q}_\gamma(t) = \int_0^\infty \gamma\xi M_\gamma(\xi)T(t^\gamma\xi)d\xi$.

Definition 7. *An $\mathscr{E}_t$-adapted stochastic process $x(t) \in C(V', Z)$ is called a mild solution of the Cauchy system (1) and (2), given $I_{0^+}^{(1-\gamma)(1-\delta)}\big[x(0) + N(x)\big] = x_0$; $x_0 \in L_2^0(\Omega, Y)$, and $g \in L^2(\Omega, Z)$ exists such that $g(t) \in G(t, x(t))$ on $t \in V'$ and satisfies*

$$x(t) = \mathcal{S}_{\gamma,\delta}(t)\big[x(0) - N(x)\big] + \int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)G\big(s, x(s)\big)dW(s) + \int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)Bu(s)ds,\ t \in V.$$

Lemma 5 ([48]).

1. *$\mathcal{K}_\gamma(t)$, $\mathcal{Q}_\gamma(t)$ and $\mathcal{S}_{\gamma,\delta}(t)$ are strongly continuous, for $t > 0$.*
2. *If $\mathcal{K}_\gamma(t)$, $\mathcal{Q}_\gamma(t)$ and $\mathcal{S}_{\gamma,\delta}(t)$ are bounded linear operators on Z, for any fixed $t \in S_{\frac{\pi}{2}-\omega}$, then we obtain*

$$\big\|\mathcal{K}_\gamma(t)x\big\| \le \kappa_p t^{-1+\gamma\vartheta}\|x\|,\ \big\|\mathcal{Q}_\gamma(t)x\big\| \le \kappa_p t^{-\gamma+\gamma\vartheta}\|x\|,$$
$$\big\|\mathcal{S}_{\gamma,\delta}(t)x\big\| \le \frac{\Gamma(\gamma)}{\Gamma(\delta(1-\gamma)+\gamma\vartheta)}\kappa_0 t^{-1+\delta-\gamma\delta+\gamma\vartheta}\|x\|,$$

where, $\kappa_p = \frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\gamma\vartheta)}$.

Definition 8 ([21]). *Suppose E^+ is the positive cone of an order Banach space $(E, \le)$. Let Φ be the function defined on the set of all bounded subsets of the Banach space Z with values in E^+ known as* MNC *on Z if and only if $\Phi(conv(\Omega)) = \Phi(\Omega)$ for very bounded subset $\Omega \subset Z$, where $conv(\Omega)$ denoted the closed convex hull of Ω.*

Definition 9. *The system (1) and (2) is called nonlocally controllable on the interval V if and only if, for all $x_0, x_1 \in Z$, a control $u \in L(V, U)$ exists such that the mild solution $x(\cdot)$ of the system (1) and (2) satisfies $x(b) + N(x) = x_1$.*

We will now review a few concepts related to the Hausdorff MNC.

Definition 10 ([22]). *For a bounded set X in a Banach space Z, the Hausdorff* MNC *χ is denoted as*

$$\chi(\mathsf{X}) = \inf\{\epsilon > 0 : \mathsf{X} \text{ can be related by a finite number of balls with radii } \epsilon\}. \tag{4}$$

Lemma 6 ([22]). *Suppose Z is a Banach space and $\mathsf{X}, \mathsf{Y} \subseteq Z$ are bounded. Then, the following properties satisfy:*

(i) X *is precompact if and only if* $\chi(\mathsf{X}) = 0$*;*
(ii) $\chi(\mathsf{X}) = \chi(\overline{\mathsf{X}}) = \chi(conv(\mathsf{X}))$*, where* $\overline{\mathsf{X}}$ *and* $conv(\mathsf{X})$ *denotes the closure and convex hull of* X*, respectively;*
(iii) *If* $\mathsf{X} \subseteq \mathsf{Y}$ *then* $\chi(\mathsf{X}) \leq \chi(\mathsf{Y})$*;*
(iv) $\chi(\mathsf{X} + \mathsf{Y}) \leq \chi(\mathsf{X}) + \chi(\mathsf{Y})$*, such that* $\mathsf{X} + \mathsf{Y} = \{a_1 + a_2 : a_1 \in \mathsf{X}, a_2 \in \mathsf{Y}\}$*;*
(v) $\chi(\mathsf{X} \cup \mathsf{Y}) \leq \max\{\chi(\mathsf{X}), \chi(\mathsf{Y})\}$*;*
(vi) $\chi(\gamma \mathsf{X}) = |\gamma| \chi(\mathsf{X})$ *for all* $\gamma \in \mathbb{R}$*, when* Z *be a real Banach space;*
(vii) *If the operator* $\Psi : D(\Psi) \subseteq Z \to Z_1$ *is Lipschitz-continuous with constant* κ_1*, then we know* $t(\Psi(\mathsf{X})) \leq \kappa_1 \chi(\mathsf{X})$ *for all bounded subset* $\mathsf{X} \subset D(\Psi)$*, where* Z_1 *is a Banach space and* t *represents the Hausdorff MNC in* Z_1*.*

Definition 11 ([22])**.** *The operator* $\Psi : D(\Psi) \subseteq Z \to Z$ *is said to be an* $\chi -$ *contraction if a positive constant* $\kappa_1 < 1$ *exists such that* $\chi(\Psi(\mathsf{X})) \leq \kappa_1 \chi(\mathsf{X})$ *for all bounded closed subsets* $\mathsf{X} \subseteq D(\Psi)$*.*

Theorem 3 ([21])**.** *If* $\{x_k\}_{k=1}^{\infty}$ *is a sequence of Bochner integrable functions from* V *to* Z *with the measurement* $\|x_k(t)\| \leq \beta(t)$*, for every* $t \in V$ *and for any* $k \geq 1$*, where* $\beta \in L^1(V, \mathbb{R})$*, then the function* $\omega(t) = \chi(\{x_k(t) : k \geq 1\})$ *is in* $L^1(V, \mathbb{R})$ *and satisfies*

$$\chi\left(\left\{ \int_0^t x_k(s)ds : \ k \geq 1\right\}\right) \leq 2 \int_0^t \omega(s)ds.$$

Lemma 7 ([21])**.** *Let* $\mathsf{X} \subset Z$ *be a bounded set; then, a countable set* $\mathsf{X}_0 \subset \mathsf{X}$ *exists such that* $\chi(\mathsf{X}) \leq 2\chi(\mathsf{X}_0)$*.*

We mean by χ_c the Hausdorff MNC in the space $\mathtt{C}$.

Lemma 8 ([21])**.** *Let* $B \subset \mathtt{C}$ *be bounded and equicontinuous; then,*

1. $\chi(\mathsf{X}(t))$ *is continuous on* $[0, c]$*,*
2. $\chi_c(\mathsf{X}) = \max_{t \in [0,c]} (\chi(\mathsf{X}(t)))$*.*

Lemma 9 ([6])**.** *(The Darbo–Sadovskii theorem) Let* $\mathsf{X} \subset Z$ *be a bounded, convex, and closed set.* $\Psi : D(\Psi) \subseteq Z \to Z$ *is a continuous and* $\chi -$ *contraction operator. Then,* Ψ *has at least one fixed point in* X*.*

3. Controllability

We require the following hypotheses:

(H_1) Let A be the almost sectorial operator of the analytic semigroup $T(t)$, $t > 0$ in Z such that $\|T(t)\| \leq \mathcal{K}_1$ where $\mathcal{K}_1 \geq 0$ is a constant.

(H_2) The function $G : V \times Z \to Z$ satisfies:

- (a) The Caratheodory condition: $G(\cdot, x)$ is strongly measurable for all $x \in Z$, and $G(t, \cdot)$ is continuous for a.e. $t \in V$;
- (b) There is a constant $0 < \gamma_1 < \gamma$ and $m \in L^{\frac{1}{\gamma_1}}(V, \mathbb{R}^+)$ and non-decreasing continuous function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|G(t, x)\| \leq m(t) g(\|x\|)$, $x \in Z$, $t \in V$, where g satisfies $\liminf_{k \to \infty} \frac{g(k)}{k} = 0$;
- (c) There is a constant $0 < \gamma_2 < \gamma$ and $h \in L^{\frac{1}{\gamma_2}}(V, \mathbb{R}^+)$ such that, for all bounded subsets $M \subset Z$, $\chi(G(t, D)) \leq h(t) \chi(D)$ for a.e. $t \in V$.

(H_3) (a) The linear operator $B : L^2(V, U) \to L^1(V, Z)$ is bounded, $W : L^2(V, U) \to Z$ denoted by $Wu = \int_0^c (c - s)^{\gamma - 1} \mathcal{Q}_\gamma(c - s) Bu(s) ds$, and it has an inverse operator W^{-1}, which take the values in $L^2(V, U)/\ker W$, and there are two positive values $\mathcal{K}_2$ and $\mathcal{K}_3$ such that $\|B\|_{L_c(U,Z)} \leq \mathcal{K}_2$, $\|W^{-1}\|_{L_c(Z, U/\ker W)} \leq \mathcal{K}_3$;

(b) There is a constant $\gamma_0 \in (0,\gamma)$ and $K_W \in L^{\frac{1}{\gamma_0}}(V,\mathbb{R}^+)$ such that, for all bounded sets $Q \subset Z$, $\chi((W^{-1}Q)(t)) \leq K_W(t)\chi(Q)$.

(H_4) The function $N : C(V,Z) \to Z$ is a continuous, compact operator, and there $L_1 > 0$ exists such that $\|N(x_1) - N(x_2)\| \leq L_1\|x_1 - x_2\|$.

For our convenience, we introduce
$K_{\gamma_i} = \left[\left(\frac{1-\gamma_i}{\gamma\vartheta-1}\right)c^{\left(\frac{\gamma\vartheta-1}{1-\gamma_i}\right)}\right]$, $i = 1,2$, $\mathcal{K}_4 = K_{\gamma_1}\|K_W\|_{L^{\frac{1}{\gamma_1}}(\mathcal{I},\mathbb{R}^+)}$ *and* $\mathcal{K}_5 = K_{\gamma_2}\|h\|_{L^{\frac{1}{\gamma_2}}(\mathcal{I},\mathbb{R}^+)}$.

Theorem 4. *Suppose that the hypotheses (H_1)–(H_4) hold; then, the HF stochastic differential system (1) and (2) has a solution on V provided $x(0) \in D(A^\theta)$ with $\theta > 1 - \vartheta$.*

Proof. The operator $\Psi : \mathcal{X} \to \mathcal{X}$ is defined as

$$\begin{aligned}\Psi(x(t)) = \Big\{z \in \mathcal{X} : z(t) =& t^{1-\delta+\gamma\delta-\gamma\vartheta}\Big[\mathcal{S}_{\gamma,\delta}(t)\left[x_0 - N(x)\right]\\ &+ \int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)G\big(s,x(s)\big)dW(s)\\ &+ \int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)Bu(s)ds\Big],\ t \in (0,c]\Big\}.\end{aligned}$$

To prove that Ψ has a fixed point.

From hypotheses (H_3), for an arbitrary function $x \in \mathcal{X}$, we describe the control $u_x(t)$ by

$$u_x(t) = W^{-1}\left(x_1 - N(x) - \mathcal{S}_{\gamma,\delta}(c)(x_0 - N(x)) - \int_0^c (c-r)^{\gamma-1}\mathcal{Q}_\gamma(c-r)G(r,x(r))dW(r)\right)(t).$$

As we can see, $\Psi x(c) = x_1 - N(x)$, which means that u_x steer the Hilfer fractional stochastic differential system (1) and (2) x_0 to x_1 in the finite time c. This suggests that the system (1) and (2) can be nonlocally controllable on V.

Step 1: We have to show that a positive value P exists such that $\Psi(B_P(V)) \subseteq B_P(V)$. Assume that the statement is false, i.e., for all $P > 0$, $x^p \in B_P(V)$ exists, but $\Psi(x^p)$ is not in $B_P(V)$,

$$\begin{aligned}E\|x^p\|^2 \leq P < E\|(\Psi x^p(t))\|^2\\ \leq E\Big\|&t^{1-\delta+\gamma\delta-\gamma\vartheta}\Big[\mathcal{S}_{\gamma,\delta}(t)\left[x_0 - N(x^p)\right]\\ &+ \int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)G\big(s,x^p(s)\big)dW(s)\\ &+ \int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)BW^{-1}\Big(x_1 - N(x^p) - \mathcal{S}_{\gamma,\delta}(c)(x_0 - N(x^p))\\ &- \int_0^c (c-r)^{\gamma-1}\mathcal{Q}_\gamma(c-r)G(r,x^p(r))dW(r)\Big)(t)ds\Big\|^2\\ \leq t^{2(1-\delta+\gamma\delta-\gamma\vartheta)}&\Big\{3E\Big\|\Big[\mathcal{S}_{\gamma,\delta}(t)\left[x_0 - N(x^p)\right]\Big\|^2\\ &+ 3E\Big\|\int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)G\big(s,x^p(s)\big)dW(s)\Big\|^2\\ &+ 3E\Big\|\int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)BW^{-1}\Big(x_1 - N(x^p) - \mathcal{S}_{\gamma,\delta}(c)(x_0 - N(x^p))\\ &- \int_0^c (c-r)^{\gamma-1}\mathcal{Q}_\gamma(c-r)G(r,x^p(r))dW(r)\Big)(t)ds\Big\|^2\Big\}.\end{aligned}$$

Hence,

$$
\begin{aligned}
E\|x^p\|^2 \leq{}& t^{2(1-\delta+\gamma\delta-\gamma\vartheta)}\bigg\{3E\|\mathcal{S}_{\gamma,\delta}(t)[x_0 - N(x^p)]\|^2 \\
&+ 3\mathrm{Tr}(Q)\int_0^t (t-s)^{2(\gamma-1)}\|\mathcal{Q}_\gamma(t-s)\|^2 E\|G(s,x^p(s))\|^2 ds \\
&+ 9\int_0^t (t-s)^{2(\gamma-1)}\|\mathcal{Q}_\gamma(t-s)\|^2\|B\|^2\|W^{-1}\|\bigg(\|x_1 - N(x^p)\|^2 \\
&+ \|\mathcal{S}_{\gamma,\delta}(c)(x_0 - N(x^p))\|^2 \\
&+ \mathrm{Tr}(Q)\int_0^c (c-r)^{2(\gamma-1)}\|\mathcal{Q}_\gamma(c-r)\|^2 E\|G(r,x^p(r))\|^2 dr\bigg)(t)ds\bigg\} \\
\leq{}& c^{2(1-\delta+\gamma\delta-\gamma\vartheta)}\left[3M^* + 9\frac{c^{2\gamma\vartheta}}{(\gamma\vartheta)^2}\kappa_p^2\mathcal{K}_2^2\mathcal{K}_3^2[\|x_1\|^2 + L_1^2\|x\|^2 + \|N(0)\|^2 - M^*]\right],
\end{aligned}
$$

where

$$
\begin{aligned}
M^* = \bigg[&\left(\frac{\Gamma(\vartheta)}{\Gamma(\delta(1-\gamma)+\gamma\vartheta)}\right)^2\kappa_0^2 c^{2(-1+\delta-\gamma\delta+\gamma\vartheta)}\|x_0\|^2 + L_1^2\|x\|^2 + \|N(0)\|^2 \\
&+ \mathrm{Tr}(Q)\frac{c^{2\gamma\vartheta}}{(\gamma\vartheta)^2}\kappa_p^2 m^2(c)g(\|x\|^2)\bigg].
\end{aligned}
$$

The above inequality is divided by $\|x^p\|$, and by applying the limit as $\|x^p\| \to \infty$, we obtain $0 \geq 1$, which is the contradiction. Therefore, $\Psi(B_P(V)) \subset B_P(V)$.

Step 2: Prove that $\Psi : B_P \to B_P$ is continuous. Let $x_k \subset B_P$ such that $x_k \to x$ in B_P. Then, we have

$$
\begin{aligned}
\Psi(x_k) - \Psi(x) ={}& \bigg\{t^{1-\delta+\gamma\delta-\gamma\vartheta}\bigg[\mathcal{S}_{\gamma,\delta}(t)[x_0 - N(x_k)] \\
&+ \int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)G(s,x_k(s))dW(s) \\
&+ \int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)Bu_{x_k}(s)ds\bigg] \\
&- t^{1-\delta+\gamma\delta-\gamma\vartheta}\bigg[\mathcal{S}_{\gamma,\delta}(t)[x_0 - N(x)] \\
&+ \int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)G(s,x(s))dW(s) \\
&+ \int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)Bu(s)ds\bigg]\bigg\} \\
={}& t^{1-\delta+\gamma\delta-\gamma\vartheta}\bigg\{\mathcal{S}_{\gamma,\delta}(t)[N(x_k) - N(x)] \\
&+ \int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)[G(s,x_k(s)) - G(s,x(s))]dW(s) \\
&+ \int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)B[u_{x_k}(s) - u(s)]ds.
\end{aligned}
$$

From the hypotheses (H_1) and (H_4), and the Lebesgue dominated convergence theorem [49], we write

$$\begin{aligned}
E\|\Psi(x_k)-\Psi(x)\|^2 \leq{}& 3c^{2(1-\delta+\gamma\delta-\gamma\vartheta)}\bigg\{\|\mathcal{S}_{\gamma,\delta}(t)\|^2 E\|N(x_k)-N(x)\|^2 \\
&+\mathrm{Tr}(Q)\int_0^c \big[(c-s)^{\gamma\vartheta-1}\big]^2 E\|G\big(s,x_k(s)\big)-G\big(s,x(s)\big)\|^2 ds \\
&+\int_0^c \big[(c-s)^{\gamma\vartheta-1}\big]^2\|B\|^2 E\|u_{x_k}(s)-u(s)\|^2\bigg\} \\
\leq{}& 3c^{2(1-\delta+\gamma\delta-\gamma\vartheta)}\bigg\{\kappa_0^2 c^{2(-1+\delta-\gamma\delta+\gamma\vartheta)}\bigg(\frac{\Gamma(\vartheta)}{\Gamma(\delta(1-\gamma)-\gamma\vartheta)}\bigg)^2 \\
&\times E\|N(x_k)-N(x)\|^2 \\
&+\mathrm{Tr}(Q)\bigg(\frac{c^{\gamma\vartheta}}{\gamma\vartheta}\bigg)^2\kappa_p^2 E\|G(s,x_k(s))-G(s,x(s))\|^2 \\
&+\bigg(\frac{c^{\gamma\vartheta}}{\gamma\vartheta}\bigg)^2\kappa_p^2\mathcal{K}_2^2 E\|u_{x_k}(s)-u_x(s)\|^2\bigg\},
\end{aligned}$$

where

$$\begin{aligned}
E\big\|u_{x_k}(s)-u_x(s)\big\|^2 \leq 3\mathcal{K}_3^2\bigg(&1+\kappa_0^2 c^{2(-1+\delta-\gamma\delta+\gamma\vartheta)}\bigg(\frac{\Gamma(\vartheta)}{\Gamma(\delta(1-\gamma)-\gamma\vartheta)}\bigg)^2 E\big\|N(x_k)-N(x)\big\|^2 \\
&+\mathrm{Tr}(Q)\bigg(\frac{c^{\gamma\vartheta}}{\gamma\vartheta}\bigg)^2\kappa_p^2 E\big\|G(s,x_k(s))-G(s,x(s))\big\|^2\bigg).
\end{aligned}$$

From the above equations, we obtain $E\|\Psi(x_k)-\Psi(x)\|^2 \to 0\ as\ k\to\infty$. So, Ψ is continuous on $B_P(V)$.

Step 3: To demonstrate that Ψ is equicontinuous, let $z(t)\in\Psi(M)$, and $0\leq t_1<t_2\leq c$; then, $x\in M$ exists such that

$$\begin{aligned}
E\|z(t_2)-z(t_1)\|^2 \leq E\bigg\|&t_2^{1-\delta+\gamma\delta-\gamma\vartheta}\bigg[\mathcal{S}_{\gamma,\delta}(t_2)[x_0-N(x)] \\
&+\int_0^{t_2}(t_2-s)^{\gamma-1}\mathcal{Q}_\gamma(t_2-s)G\big(s,x(s)\big)dW(s) \\
&+\int_0^{t_2}(t_2-s)^{\gamma-1}\mathcal{Q}_\gamma(t_2-s)Bu(s)ds\bigg] \\
&-t_1^{1-\delta+\gamma\delta-\gamma\vartheta}\bigg[\mathcal{S}_{\gamma,\delta}(t_1)[x_0-N(x)] \\
&+\int_0^{t_1}(t_1-s)^{\gamma-1}\mathcal{Q}_\gamma(t_1-s)G\big(s,x(s)\big)dW(s) \\
&+\int_0^{t_1}(t_1-s)^{\gamma-1}\mathcal{Q}_\gamma(t_1-s)Bu(s)ds\bigg]\bigg\|^2.
\end{aligned}$$

Thus, we write

$$
\begin{aligned}
E\|z(t_2)-z(t_1)\|^2 \leq\ & 3E\left\|t_2^{1-\delta+\gamma\delta-\gamma\vartheta}\mathcal{S}_{\gamma,\delta}(t_2)-t_1^{1-\delta+\gamma\delta-\gamma\vartheta}\mathcal{S}_{\gamma,\delta}(t_1)\right\|^2\|x_0-N(x)\|^2 \\
&+3E\left\|t_2^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^{t_1}(t_2-s)^{\gamma-1}\mathcal{Q}_\gamma(t_2-s)G(s,x(s))dW(s)\right. \\
&+t_2^{1-\delta+\gamma\delta-\gamma\vartheta}\int_{t_1}^{t_2}(t_2-s)^{\gamma-1}\mathcal{Q}_\gamma(t_2-s)G(s,x(s))dW(s) \\
&\left.-t_1^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^{t_1}(t_1-s)^{\gamma-1}\mathcal{Q}_\gamma(t_1-s)G(s,x(s))dW(s)\right\|^2 \\
&+3E\left\|t_2^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^{t_1}(t_2-s)^{\gamma-1}\mathcal{Q}_\gamma(t_2-s)Bu(s)ds\right. \\
&+t_2^{1-\delta+\gamma\delta-\gamma\vartheta}\int_{t_1}^{t_2}(t_2-s)^{\gamma-1}\mathcal{Q}_\gamma(t_2-s)Bu(s)ds \\
&\left.-t_1^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^{t_1}(t_1-s)^{\gamma-1}\mathcal{Q}_\gamma(t_1-s)Bu(s)ds\right\|^2 \\
\leq\ & 3E\|t_2^{1-\delta+\gamma\delta-\gamma\vartheta}\mathcal{S}_{\gamma,\delta}(t_2)-t_1^{1-\delta+\gamma\delta-\gamma\vartheta}\mathcal{S}_{\gamma,\delta}(t_1)\|^2\|x_0-N(x)\|^2 \\
&+9E\left\|t_2^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^{t_1}(t_2-s)^{\gamma-1}\mathcal{Q}_\gamma(t_2-s)G(s,x(s))dW(s)\right. \\
&\left.-t_1^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^{t_1}(t_1-s)^{\gamma-1}\mathcal{Q}_\gamma(t_2-s)G(s,x(s))dW(s)\right\|^2 \\
&+9E\left\|t_1^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^{t_1}(t_1-s)^{\gamma-1}\mathcal{Q}_\gamma(t_2-s)G(s,x(s))dW(s)\right. \\
&\left.-t_1^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^{t_1}(t_1-s)^{\gamma-1}\mathcal{Q}_\gamma(t_1-s)G(s,x(s))dW(s)\right\|^2 \\
&+9E\left\|t_2^{1-\delta+\gamma\delta-\gamma\vartheta}\int_{t_1}^{t_2}(t_2-s)^{\gamma-1}\mathcal{Q}_\gamma(t_2-s)G(s,x(s))dW(s)\right\|^2 \\
&+9E\left\|t_2^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^{t_1}(t_2-s)^{\gamma-1}\mathcal{Q}_\gamma(t_2-s)Bu(s)ds\right. \\
&\left.-t_1^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^{t_1}(t_1-s)^{\gamma-1}\mathcal{Q}_\gamma(t_2-s)Bu(s)ds\right\|^2 \\
&+9E\left\|t_1^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^{t_1}(t_1-s)^{\gamma-1}\mathcal{Q}_\gamma(t_2-s)Bu(s)ds\right. \\
&\left.-t_1^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^{t_1}(t_1-s)^{\gamma-1}\mathcal{Q}_\gamma(t_1-s)Bu(s)ds\right\|^2 \\
&+9E\left\|t_2^{1-\delta+\gamma\delta-\gamma\vartheta}\int_{t_1}^{t_2}(t_2-s)^{\gamma-1}\mathcal{Q}_\gamma(t_2-s)Bu(s)ds\right\|^2 \\
=\ & \sum_{i=1}^{7} I_i.
\end{aligned}
$$

By the strong continuity of $\mathcal{S}_{\gamma,\delta}(t)$, we obtain

I_1 tends to zero as $t_2 \to t_1$.

$$\begin{aligned} I_2 &= 9E\left\| t_2^{1-\delta+\gamma\delta-\gamma\vartheta} \int_0^{t_1} (t_2-s)^{\gamma-1} \mathcal{Q}_\gamma(t_2-s) G(s,x(s)) dW(s) \right. \\ &\quad \left. - t_1^{1-\delta+\gamma\delta-\gamma\vartheta} \int_0^{t_1} (t_1-s)^{\gamma-1} \mathcal{Q}_\gamma(t_2-s) G(s,x(s)) dW(s) \right\|^2 \\ &\leq 9\mathrm{Tr}(Q)\kappa_p^2 \int_0^{t_1} (t_2-s)^{-2\gamma(1-\vartheta)} \Big[t_2^{2(1-\delta+\gamma\delta-\gamma\vartheta)} (t_2-s)^{2(\gamma-1)} \\ &\quad - t_1^{2(1-\delta+\gamma\delta-\gamma\vartheta)} (t_1-s)^{2(\gamma-1)} \Big] m^2(s) g(\|x\|^2) ds. \end{aligned}$$

From the Lebesgue's dominated convergence theorem [49], we obtain $\lim_{t_2 \to t_1} I_2 = 0$.

$$\begin{aligned} I_3 &= 9E\left\| t_1^{1-\delta+\gamma\delta-\gamma\vartheta} \int_0^{t_1} (t_1-s)^{\gamma-1} \mathcal{Q}_\gamma(t_2-s) G(s,x(s)) dW(s) \right. \\ &\quad \left. - t_1^{1-\delta+\gamma\delta-\gamma\vartheta} \int_0^{t_1} (t_1-s)^{\gamma-1} \mathcal{Q}_\gamma(t_1-s) G(s,x(s)) dW(s) \right\|^2 \\ &\leq 9\mathrm{Tr}(Q) t_1^{2(1-\delta+\gamma\delta-\gamma\vartheta)} \int_0^{t_1} (t_1-s)^{2(\gamma-1)} \left\| \mathcal{Q}_\gamma(t_2-s) - \mathcal{Q}_\gamma(t_1-s) \right\|^2 m^2(s) g(\|x\|^2) ds. \end{aligned}$$

By Theorem 2, $\mathcal{Q}_\gamma(t)$ is uniformly continuous in the operator norm topology. So, we obtain $I_3 \to 0$ as $t_2 \to t_1$.

$$\begin{aligned} I_4 &= 9E\left\| t_2^{1-\delta+\gamma\delta-\gamma\vartheta} \int_{t_1}^{t_2} (t_2-s)^{\gamma-1} \mathcal{Q}_\gamma(t_2-s) G(s,x(s)) dW(s) \right\|^2 \\ &\leq 9\mathrm{Tr}(Q)\kappa_p^2 \Big[t_2^{2(1-\delta+\gamma\delta-\gamma\vartheta)} \int_0^{t_2} (t_2-s)^{2(\gamma\vartheta-1)} \\ &\quad - t_1^{2(1-\delta+\gamma\delta-\gamma\vartheta)} \int_0^{t_1} (t_1-s)^{2(\gamma\vartheta-1)} \Big] m^2(s) g(\|x\|^2) ds. \end{aligned}$$

Then, $I_4 \to 0$ as $t_2 \to t_1$, by using the dominated convergence theorem. Next,

$$\begin{aligned} I_5 &= 9E\left\| t_2^{1-\delta+\gamma\delta-\gamma\vartheta} \int_0^{t_1} (t_2-s)^{\gamma-1} \mathcal{Q}_\gamma(t_2-s) Bu(s) ds \right. \\ &\quad \left. - t_1^{1-\delta+\gamma\delta-\gamma\vartheta} \int_0^{t_1} (t_1-s)^{\gamma-1} \mathcal{Q}_\gamma(t_2-s) Bu(s) ds \right\|^2 \\ &\leq 9\kappa_p^2 \mathcal{K}_2^2 \int_0^{t_1} \Big(t_2^{2(1-\delta+\gamma\delta-\gamma\vartheta)} (t_2-s)^{2(\gamma-1)} \\ &\quad - t_1^{2(1-\delta+\gamma\delta-\gamma\vartheta)} (t_1-s)^{2(\gamma-1)} \Big) (t_2-s)^{-2\gamma(1+\vartheta)} u(s) ds. \\ I_6 &= 9E\left\| t_1^{1-\delta+\gamma\delta-\gamma\vartheta} \int_0^{t_1} (t_1-s)^{\gamma-1} \mathcal{Q}_\gamma(t_2-s) Bu(s) ds \right. \\ &\quad \left. - t_1^{1-\delta+\gamma\delta-\gamma\vartheta} \int_0^{t_1} (t_1-s)^{\gamma-1} \mathcal{Q}_\gamma(t_1-s) Bu(s) ds \right\|^2 \\ &\leq 9\mathcal{K}_2^2\, t_1^{2(1-\delta+\gamma\delta-\gamma\vartheta)} \int_0^{t_1} (t_1-s)^{2(\gamma-1)} \left\| \mathcal{Q}_\gamma(t_2-s) - \mathcal{Q}_\gamma(t_1-s) \right\|^2 u(s) ds. \\ I_7 &= 9E\left\| t_2^{1-\delta+\gamma\delta-\gamma\vartheta} \int_{t_1}^{t_2} (t_2-s)^{\gamma-1} \mathcal{Q}_\gamma(t_2-s) Bu(s) ds \right\|^2 \\ &\leq 9\kappa_p^2 \mathcal{K}_2^2 t_2^{2(1-\delta+\gamma\delta-\gamma\vartheta)} \left\| \int_{t_1}^{t_2} (t_2-s)^{\gamma\vartheta-1} u(s) ds \right\|^2. \end{aligned}$$

Similar to the proof of I_2 and I_3, we obtain that I_5 and I_6 tend to zero as $t_2 \to t_1$, and I_7 tends to zero as $t_2 \to t_1$. Therefore, $\Psi(M)$ is equicontinuous on V.

Step 4: To show $\Psi : B_P(V) \to B_P(V)$ is a χ-contraction operator.

Let $D \subseteq B_P$; then, from Lemma 7, there exists a countable set $D_0 = \{x_k\}_{k=1}^{\infty} \subset D$ such that $\chi(G(D)(t)) \leq 2\chi(\{x_k\}_{k=1}^{\infty})$. By the equicontinuous of B_P, we know that D is also equicontinuous. Therefore, from Lemma 7, we obtain

$$\chi_c(\Psi(D_0)) = \max_{t\in[0,c]} \chi(\Psi(D_0)). \tag{5}$$

Hence, we have

$$\begin{aligned}
E\|\chi_c(\Psi(D))\|^2 &\leq 2E\|\chi_c(\Psi(D_0))\|^2 \\
&= 2\max_{t\in[0,c]} E\|\big(\chi(\Psi(D_0)(t))\big)\|^2 \\
&\leq 2\max_{t\in[0,c]} E\bigg\|\chi t^{1-\delta+\gamma\delta-\gamma\vartheta}\bigg(-\mathcal{S}_{\gamma,\delta}(t)N(D_0) \\
&\quad + \int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)G(s,D_0(s))dW(s) \\
&\quad + \int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)Bu_{D_0}(s)ds\bigg)\bigg\|^2 \\
&\leq 2\max_{t\in[0,c]} E\bigg\|\chi t^{1-\delta+\gamma\delta-\gamma\vartheta}\bigg(-\mathcal{S}_{\gamma,\delta}(t)N(D_0)\bigg) \\
&\quad + \chi t^{1-\delta+\gamma\delta-\gamma\vartheta}\bigg(\int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)G(s,D_0(s))dW(s)\bigg) \\
&\quad + \chi t^{1-\delta+\gamma\delta-\gamma\vartheta}\bigg(\int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)Bu_{D_0}(s)ds\bigg)\bigg\|^2.
\end{aligned}$$

Since N is compact and $\mathcal{S}_{\gamma,\delta}(t)N(D_0)$ is relatively compact, we obtain

$$\begin{aligned}
E\|\chi_c(\Psi(D))\|^2 &\leq 2\max_{t\in[0,c]} E\bigg\|t^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^t \chi\Big((t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)G(s,D_0(s))dW(s)\Big) \\
&\quad + t^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^t \chi\Big((t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)Bu_{D_0}(s)ds\Big)\bigg\|^2 \\
&\leq 4\max_{t\in[0,c]} E\bigg\|t^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)\chi\Big(G(s,D_0(s))dW(s)\Big) \\
&\quad + t^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)\chi\Big(Bu_{D_0}(s)ds\Big)\bigg\|^2 \\
&\leq 8\max_{t\in[0,c]} \bigg[\mathrm{Tr}(Q)E\bigg\|t^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)\chi\Big(G(s,D_0(s))ds\Big)\bigg\|^2 \\
&\quad + E\bigg\|t^{1-\delta+\gamma\delta-\gamma\vartheta}\int_0^t (t-s)^{\gamma-1}\mathcal{Q}_\gamma(t-s)\chi\Big(Bu_{D_0}(s)ds\Big)\bigg\|^2\bigg] \\
&\leq 8\max_{t\in[0,c]} \bigg[\mathrm{Tr}(Q)t^{2(1-\delta+\gamma\delta-\gamma\vartheta)}\int_0^t (t-s)^{2(\gamma-1)}\|\mathcal{Q}_\gamma(t-s)\|^2 \\
&\quad \times E\bigg\|\chi\Big(G(s,D_0(s))\Big)\bigg\|^2 ds \\
&\quad + t^{2(1-\delta+\gamma\delta-\gamma\vartheta)}\int_0^t (t-s)^{2(\gamma-1)}\|\mathcal{Q}_\gamma(t-s)\|^2 E\bigg\|\chi\Big(Bu_{D_0}(s)\Big)\bigg\|^2 ds\bigg].
\end{aligned}$$

That is,

$$
\begin{aligned}
E\|\chi_c(\Psi(D))\|^2 \leq{}& 8\kappa_p^2 \max_{t\in[0,c]} \Bigg(\mathrm{Tr}(Q) t^{2(1-\delta+\gamma\delta-\gamma\vartheta)} \int_0^t (t-s)^{2(\gamma\vartheta-1)} h^2(s) ds \chi^2(D(t)) \\
&+ t^{2(1-\delta+\gamma\delta-\gamma\vartheta)} \int_0^t (t-s)^{2(\gamma\vartheta-1)} ds \|B\|^2 E\|u_{D_0}(s)\|^2 \Bigg) \\
\leq{}& 8\kappa_p^2 \max_{t\in[0,c]} \Bigg(\mathrm{Tr}(Q) t^{2(1-\delta+\gamma\delta-\gamma\vartheta)} \int_0^t (t-s)^{2(\gamma\vartheta-1)} h^2(s) ds \chi^2(D(t)) \\
&+ 3t^{2(1-\delta+\gamma\delta-\gamma\vartheta)} \int_0^t (t-s)^{2(\gamma\vartheta-1)} ds \mathcal{K}_2^2 \chi^2 \Bigg[\|W^{-1}\|^2 \Bigg\{ E\|x_1 - N(D_0)\|^2 \\
&+ E\|\mathcal{S}_{\gamma,\delta}(c)(x_0 - N(x))\|^2 \\
&+ E\| \int_0^c (c-r)^{\gamma-1} \mathcal{Q}_\gamma(c-r) G(r, D_0(r)) dr\|^2 \Bigg\}(s) \Bigg] \Bigg) \\
\leq{}& 8\kappa_p^2 \max_{t\in[0,c]} \Bigg(\mathrm{Tr}(Q) t^{2(1-\delta+\gamma\delta-\gamma\vartheta)} \int_0^t (t-s)^{2(\gamma\vartheta-1)} h^2(s) ds \chi^2(D(t)) \\
&+ 3t^{2(1-\delta+\gamma\delta-\gamma\vartheta)} \int_0^t (t-s)^{2(\gamma\vartheta-1)} \mathcal{K}_2^2 \|K_W(s)\|^2 ds \kappa_p^2 \\
&\times \int_0^c (c-r)^{2(\gamma\vartheta-1)} h^2(r) dr \chi^2(D(t)) \Bigg) \\
\leq{}& 8\kappa_p^2 c^{2(1-\delta+\gamma\delta-\gamma\vartheta)} \Bigg[\mathrm{Tr}(Q) K_{\gamma_2}^2 \|h\|^2_{L^{\frac{1}{\gamma_2}}(V,\mathbb{R}^+)} \chi^2(D) \\
&+ 3\kappa_p^2 \mathcal{K}_2^2 K_{\gamma_1}^2 \|K_W\|^2_{L^{\frac{1}{\gamma_1}}(V,\mathbb{R}^+)} K_{\gamma_2}^2 \|h\|^2_{L^{\frac{1}{\gamma_2}}(V,\mathbb{R}^+)} \chi^2(D) \Bigg] \\
\leq{}& 8\kappa_p^2 c^{2(1-\delta+\gamma\delta-\gamma\vartheta)} \left[\mathrm{Tr}(Q) \mathcal{K}_5^2 \chi^2(D) + 3\kappa_p^2 \mathcal{K}_2^2 \mathcal{K}_4^2 \mathcal{K}_5^2 \chi^2(D) \right] \\
\leq{}& 8c^{2(1-\delta+\gamma\delta-\gamma\vartheta)} \left[\mathrm{Tr}(Q) + 3\kappa_p^2 \mathcal{K}_2^2 \mathcal{K}_4^2 \right] \kappa_p^2 \mathcal{K}_5^2 \chi^2(D) \\
E\|\chi_c(\Psi(D))\|^2 \leq{}& \kappa_1 \chi_c^2(D),
\end{aligned}
$$

where $\kappa_1 = 8c^{2(1-\delta+\gamma\delta-\gamma\vartheta)} \left[\mathrm{Tr}(Q) + 3\kappa_p^2 \mathcal{K}_2^2 \mathcal{K}_4^2 \right] \kappa_p^2 \mathcal{K}_5^2$.

Therefore, from Definition (11), Ψ is a χ_c-contraction operator. As a result, Ψ has at least one fixed point according to Lemma 7, and the mild solution also exists. □

4. Example

4.1. Example-I

Assume that the following HF stochastic differential equation is in the form:

$$
\begin{cases}
D_{0^+}^{\frac{2}{3},\delta} x(t,y) = x_{yy}(t,y) + \beta\omega(t,y) + \frac{e^{-t}}{q+e^t} \sin\left(x(t,y)\right) \frac{dW(t)}{dt}, \ t \in V = [0,1], \\
x(t,0) = x(t,1) = 0, \\
I_{0^+}^{(1-\frac{2}{3})(1-\delta)}[x(0,y)] + \int_0^c h(s) \ln\left(1 + |x(s,y)|^{\frac{1}{2}}\right) ds = x_0, \ 0 < y < 1,
\end{cases}
\tag{6}
$$

where $\beta > 0$, $q \geq 1$, and $D_{0^+}^{\frac{2}{3},\delta}$ are the HFD of order $\frac{2}{3}$, type δ, $I_{0^+}^{(1-\frac{2}{3})(1-\delta)}$ is the Riemann–Liouville integral, and the function $\omega : V \times (0,1) \to (0,1)$ is continuous in t and $h \in L^1(V,\mathbb{R})$. Let $W(t)$ be a one-dimensional standard Brownian motions in Z denoted by $\|\cdot\|_Z$ on the filtered probability space $(\Omega, \mathscr{E}, P)$. Let $Z = L^2([0,1])$, $U = L^2([0,1])$; then, the operator $A : D(A) \subset Z \to Z$ is determined by $Ax = x''$, $x \in D(A)$, where $D(A) = \{x \in Z : x,\ x'$ are absolutely continuous, $x'' \in Z, x(t,0) = x(t,1) = 0\}$ and

$$
Ax = \sum_{k=1}^{\infty} k^2 \langle x, e_k \rangle e_k,\ e \in D(A),
$$

where $e_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx)$, $k \in \mathbb{N}$ is the orthogonal set of eigenvectors of A. Here, A is the almost sectorial operator of the semigroup $\{T(t),\ t \geq 0\}$ in Z and given by $T(t)(x(s)) = x(t+s)$, $s \in Z$, $T(t)$ is not a compact semigroup on Z with $\chi\big(T(t)M\big) \leq \chi(M)$ where χ is the Hausdorff MNC and $\mathcal{K}_1 \geq 1$ exists such that $\sup_{t\in V} \|T(t)\| \leq \mathcal{K}_1$. Furthermore, $t \to x(t^{\frac{2}{3}}\theta + s)x$ is equicontinuous for $t > 0$ and $\theta \in (0, \infty)$.

Let $x(t)(y) = x(t, y)$,

$$G(t, x(t))(y) = \frac{e^{-t}}{q + e^{t}} \sin\big(x(t, y)\big).$$

It is clear that G is Lipschitz-continuous for the second variable with constant $\frac{1}{1+q}$ and satisfies (H_2). Let the function $B : U \to Z$ be defined by $(Bu)(t)(y) = \beta\omega(t, y)$, $y \in (0, 1)$ and the nonlocal term $N : C(V, Z) \to Z$ be defined as

$$N(x)(y) = \int_0^c h(s) \ln\big(1 + |x(s)(y)|^{\frac{1}{2}}\big) ds.$$

Then, $B : U \to Z$ is a bounded linear operator and N is the compact operator satisfied (H_4). For $y \in (0, 1)$, the operator W is defined as

$$(Wu)(y) = \int_0^1 (1 - s)^{\frac{-1}{3}} \mathcal{Q}_{\frac{2}{3}}(1 - s)\beta\omega(s, y) ds.$$

For $t \in [0, 1]$,

$$\mathcal{Q}_{\frac{2}{3}}(x(s)) = \frac{2}{3} \int_0^\infty \theta M_{\frac{2}{3}}(\theta) x(t^{\frac{2}{3}} + s) d\theta,$$

where the Wright-type function, $M_{\frac{2}{3}}(\theta) = \sum_{k=1}^{\infty} \frac{(-\theta)^{k-1}}{\Gamma(1-\frac{2}{3}k)(k-1)!}$. Suppose that W satisfies (H_3); then, all of the statements of Theorem 4 are accomplished. Therefore, the HF stochastic differential Equations (1) and (2) are nonlocal controllable on V.

4.2. Example-II

Digital filters perform an incredibly considerable aspect in the field of digital signal processing (DSP). The execution of digital filters is extraordinary, and each of the essential factors of DSP has grown in acclaim. Commonly, we classify filters with two main usages: one is signal separation, and the other is signal restoration. Digital filters are an important entity in various fields of signal processing and have unexceptionally received high-level prominence in the field of biomedical signal processing too. As most biomedical signals are low-frequency by nature, the general problem in processing those signals is the small amplitude of the signal, which is naturally dominated by artefacts and various other noises. The efficiency of the system lies in retrieving those signals in a substantial amount of time so as not to generate a high level of delay in processing and analyzing those signals. In general, biomedical systems consist of complex cascaded blocks such that the delay in producing the output of each block cumulatively affects the generation of the final output. The speed, as well as the accuracy, of processing is highly essential in modelling any biomedical system to retrieve such error-free processed signals.

Further, the biological signals are the ones with minimal amplitudes and lower frequencies (a few brain signals range from 1–4 Hz). The entire requirement of error-free signal generation shall be attributed to the proper controllability of the modelled system obtained through the solution. The signal processing unit is implemented through a block diagram as depicted in Figure 1 using MATLAB-Simulink software with the aid of the proposed system of Equations (1) and (2) that shall be suitable for a few biomedical applications. The magnitude response of the digital filter implemented using a fractional differentiation model, as shown in Figure 2, exhibits a sharp cut-off frequency with better signal-restoration capability [50,51]. Therefore, the output is obtained, which is bounded and controllable with better stability in the amplitude in the low-frequency corner.

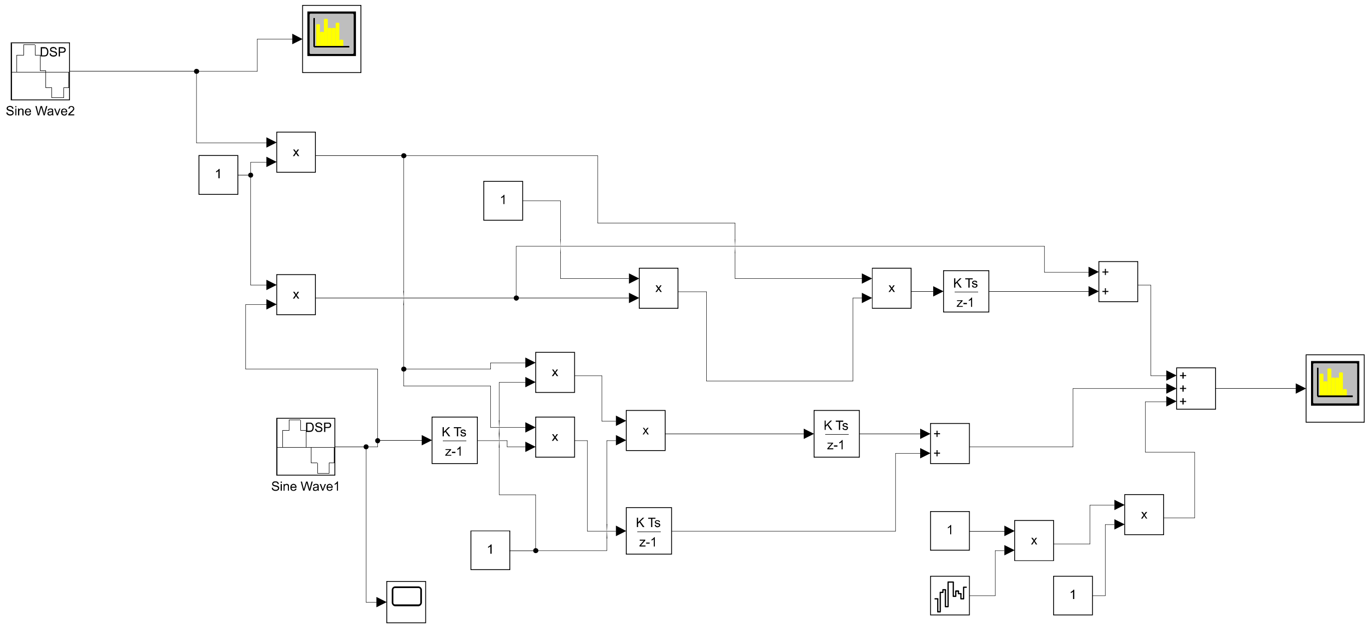

Figure 1. Digital filter model.

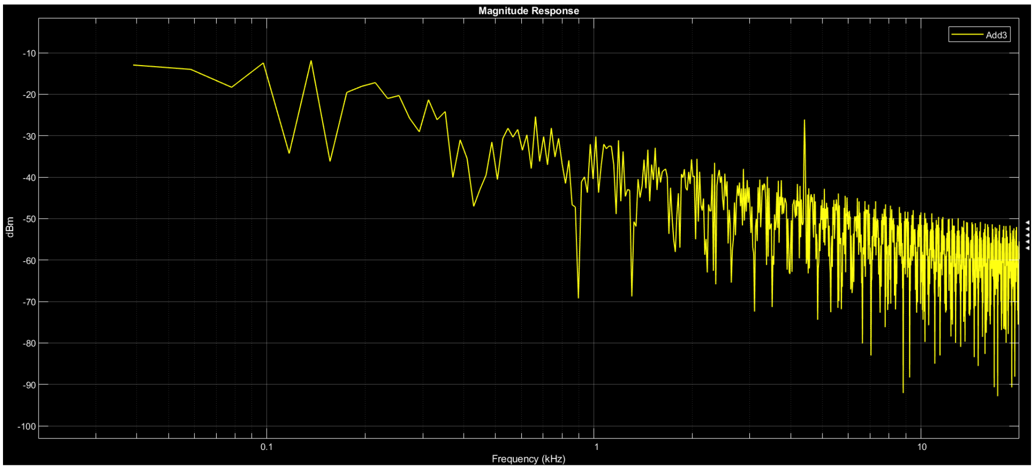

Figure 2. Magnitude response.

Motivated by the filter systems defined in [52–54], we present our filter system in Figures 1 and 2. Figure 1 describes the rough pattern of the digital filter model, and Figure 2 describes the frequency response of the digital filter, which aids in enhancing the usefulness of the solution with a minimum measure of inputs.

1. Product modulator 1 receives the input $x(s)$, and G produces $G(s, x(s))$.
2. Product modulator 2 receives the input $u(s)$, and B produces $B(u(s))$.
3. Product modulator 3, receives the input $[x(0) - N(x)]$, and $S_{\gamma,\delta}$ at time $t = 0$ produces $S_{\gamma,\delta}(t)[x(0) - N(x)]$.
4. The integrator performs the integral of

$$\mathcal{Q}_\gamma(t-s)[G(s, x(s)) + Bu(s)],$$

over the period s.

Furthermore,

(i) Inputs $\mathcal{Q}_\gamma(t-s)$, $G(s,x(s))$ are combined and multiplied with an output of the integrator over $(0,t)$.
(ii) Inputs $\mathcal{Q}_\gamma(t-s)$, $B(u(s))$ are combined and multiplied with an output of the integrator over $(0,t)$.

Finally, we move all of the outputs from the integrators to the summer network. Therefore, the output of $x(t)$ is attained; it is bounded and controllable.

5. Conclusions

This manuscript concentrated on the almost sectorial operator-based nonlocal controllability of HF stochastic differential equations. By applying findings and concepts from fractional calculus, almost sectorial operators, MNC, and the fixed-point method, the primary outcomes are obtained. Researchers established the necessary criteria for the mild solution's existence and the system's nonlocal controllability. Finally, we offer a tool for putting theoretical findings into practice. Future research will focus on almost sectorial operators' approximate controllability of HF stochastic differential systems with infinite and finite delay.

These are the contributions we made: (1) The authors present a number of requirements for the nonlocal controllability of HF stochastic differential equations via almost sectorial operators. (2) As far as we are aware, there has not been an investigation into the existence and nonlocal controllability of the HF stochastic differential system using almost sectorial operators. (3) To wrap up, we offer an example of the results.

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