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Remark on a Fixed-Point Theorem in the Lebesgue Spaces of Variable Integrability $L^{p(\cdot)}$

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Abstract: In a personal communication, Prof. Domínguez Benavides noted that a fixed-point theorem for modular nonexpansive mappings in $L^{p(\cdot)}(\Omega)$ obtained under the assumptions $p_+ < \infty$ and the property (R) satisfied by ρ will force $p_- > 1$. Therefore, the conclusion is well known. In this note, we establish said conclusion without the assumption $p_+ < \infty$.

Keywords: electrorheological fluid; fixed point; modular strict convexity; modular vector space; modular uniform convexity; Nakano modular

MSC: 47H09; 46B20; 47H10; 47E10

1. Introduction

In 1965, W. Kirk [1] established his celebrated fixed-point theorem for nonexpansive mappings. Specifically, he proved that any nonexpansive map:

$$T : C \to C$$

on a non-empty, closed, convex subset of a reflexive Banach space, which has the normal structure (see below), has a fixed point.

It is worthwhile to mention that in Kirk's proof, the reflexivity of a Banach space X is used in the following equivalent form (established by Smulian): Every decreasing sequence of non-empty, bounded, closed, and convex subsets of X has a non-empty intersection.

In [2], a modular version of Kirk's theorem was utilized in order to show a fixed-point property of variable-exponent Lebesgue spaces. Specifically, Theorem 5 in [2] reads as follows (we refer the reader to the body of the paper for the relevant terminology):

Theorem 1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and let $p \in \mathcal{P}(\Omega)$; assume that $|\Omega_1| = 0$, $p_+ < \infty$ and that ρ has property (R). Let C be a non-empty, ρ-bounded, ρ-closed, and convex subset $L^{p(\cdot)}(\Omega)$. If a map $T : C \to C$ is ρ nonexpansive, then it has a fixed point.*

It was rightly observed by Prof. Domínguez Benavídes that the assumption $p_+ < \infty$ is equivalent to the Δ_2 condition, which in turn, implies that the norm topology and the modular topology on $L^{p(\cdot)}(\Omega)$ coincide, from which it follows that the intersection property (R), alluded to in the statement of Theorem 1 is just the intersection property (R) for the norm. However, the latter implies the reflexivity of $L^{p(\cdot)}(\Omega)$, which is equivalent to $1 < p_- \leq p_+ < \infty$. Under these conditions, the conclusion of Theorem 1 is already known [3].

In this note, we prove the conclusion of Theorem 1, without the condition $p_+ < \infty$, to reveal the original modular nature of the result.

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2. The Modular Geometry of the Variable-Exponent Lebesgue Spaces

In the interest of clarity, the definition of the variable-exponent Lebesgue spaces is recalled [4].

Definition 1. *Let $\Omega \subset \mathbb{R}^n$ be a domain. As usual, $\mathcal{M}(\Omega)$ will stand for the vector space of all real-valued Borel-measurable functions defined on Ω, and the Lebesgue measure of a subset $A \subset \mathbb{R}^n$ will be denoted by $|A|$. Let $\mathcal{P}(\Omega)$ be the subset of $\mathcal{M}$ consisting of functions $p : \Omega \longrightarrow [1, \infty]$. For each such p, define the sets:*

$$\Omega_1 := \{t \in \Omega : \ p(t) = 1\} \quad \text{and} \quad \Omega_\infty := \{x \in \Omega : p(x) = \infty\}.$$

The function $\rho : \mathcal{M}(\Omega) \longrightarrow [0, \infty]$, defined by

$$\rho(u) = \int\limits_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} d\mu + \sup_{x \in \Omega_\infty} |u(x)|,$$

is a convex and continuous modular on $\mathcal{M}(\Omega)$ in the sense of Nakano [2]. The associated modular vector space is denoted by $L^{p(\cdot)}(\Omega)$.

The first systematic treatment of the variable-exponent Lebesgue class is the work [4]; we refer the reader to [5] for a more recent survey on this topic. These spaces are by no means artificial constructions: their study has intensified due to their fairly recent applications to the hydrodynamics of electrorheological fluids and refined mathematical models used for image restoration, to name only two; see [5] and the references therein.

It has been recently observed that, under very mild conditions on the variable exponent $p(\cdot)$, the modular ρ_p is uniformly convex in every direction. More precisely, the following theorem holds:

Theorem 2 ([2,6])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and let $p \in \mathcal{P}(\Omega)$. Then the following properties are equivalent:*

(a) $|\Omega_1| = |\Omega_\infty| = 0$,

(b) *The modular ρ is uniformly convex in every direction in the following sense: for any z_1 and z_2 in $L^{p(\cdot)}(\Omega)$ such that $z_1 \neq z_2$ and $R > 0$, there exists $\Delta(R, z_1, z_2) > 0$ such that for any $u \in L^{p(\cdot)}(\Omega)$, we have*

$$\rho\left(u - \frac{z_1 + z_2}{2}\right) \leq R\ (1 - \Delta(z_1, z_2, R)),$$

provided $\rho(u - z_1) \leq R$ and $\rho(u - z_2) \leq R$.

As in the case of Banach spaces, the modular uniform convexity as stated in Theorem 2 implies the modular normal structure property [7]:

Proposition 1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and let $p \in \mathcal{P}(\Omega)$. Assume $|\Omega_1| = |\Omega_\infty| = 0$. Then, for any non-empty ρ-bounded, ρ-closed, and convex subset C of $L^{p(\cdot)}(\Omega)$ not reduced to one point, there exists $f \in C$ such that*

$$\sup_{g \in C} \rho(f - g) < \delta_\rho(C) = \sup\{\rho(a - b)\,, a \in C\,, b \in C\}.$$

This is known as the ρ-normal structure property.

For $p \in \mathcal{P}(\Omega)$, set

$$p_- := \operatorname*{ess\,inf}_{t \in \Omega} p(t) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{t \in \Omega} p(t).$$

Clearly, in our setting, $1 \leq p_- \leq p_+ \leq \infty$. In particular, if $p_- > 1$, a rather stronger form of modular uniform convexity holds for $L^{p(\cdot)}(\Omega)$.

Theorem 3 ([8])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open and $p \in \mathcal{P}(\Omega)$. If $|\Omega_\infty| = 0$ and $p_- > 1$, then ρ satisfies the following modular uniform convexity property. Set*

$$D(r,\varepsilon) = \left\{ (u,v) \in L^{p(\cdot)}(\Omega) \times L^{p(\cdot)}(\Omega) : \rho(u) \leq r\,,\, \rho(v) \leq r\,,\, \rho\left(\frac{u-v}{2}\right) \geq \varepsilon r \right\}$$

and

$$\delta(r,\varepsilon) = \inf\left\{ 1 - \frac{1}{r}\rho\left(\frac{u+v}{2}\right) : (u,v) \in D(r,\varepsilon) \right\}.$$

(If $D(r,\varepsilon) = \emptyset$, we define $\delta(r,\varepsilon) = 1$.) Then, for each $s \geq 0$, $\varepsilon > 0$, there exists $\eta(s,\varepsilon) > 0$ such that, for arbitrary $r > s > 0$,

$$\delta(r,\varepsilon) \geq \eta(s,\varepsilon).$$

For further reference, the following standard definition is recalled:

Definition 2. *A family $(C_i)_{i \in I}$ of sets is said to have the finite intersection property if, for every finite subset $\{i_1, \ldots i_k\} \subset I$, it holds that $\bigcap\limits_{j=1}^{k} C_{i_j} \neq \emptyset$.*

In this regard, the following theorem was proven in [8]:

Theorem 4. *Assume that $p_- > 1$. Then, ρ satisfies the strong-(R) property, i.e., for any $C \subset L^{p(\cdot)}(\Omega)$ ρ-closed, ρ-bounded, and convex non-empty subset, then if $(C_i)_{i \in I} \subset 2^C$ is a family of ρ-closed convex subsets of C having the finite intersection property, it necessarily holds that $\bigcap\limits_{i \in I} C_i \neq \emptyset$.*

We will say that ρ satisfies the property (R) if the conclusion of Theorem 4 holds for countable families, i.e., for any $(C_n)_{n \in \mathbb{N}}$ decreasing sequence of ρ-closed, ρ-bounded, and convex non-empty subsets, it necessarily holds that $\bigcap\limits_{n \in \mathbb{N}} C_n \neq \emptyset$.

3. Fixed-Point Theorems for $L^{p(\cdot)}(\Omega)$

In this section, the main fixed-point result of this work will be addressed. The following definition is a prerequisite:

Definition 3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and let $p \in \mathcal{P}(\Omega)$. Let $\emptyset \neq C \subset L^{p(\cdot)}(\Omega)$ and $T : C \to C$ be a mapping. T is said to be ρ-nonexpansive if*

$$\rho(T(x) - T(y)) \leq \rho(x - y), \quad \textit{for any } x, y \in C.$$

A point $x \in C$ that satisfies $T(x) = x$ is said to be a fixed point of T.

The field of the fixed-point theory of maps acting on modular function spaces is vast and deep; the interested reader is referred to [7] for a comprehensive treatment of the subject.

The next result is the modular version of Kirk's celebrated fixed-point theorem [1]. The proof is constructive and was first used in the Banach-space setting by Kirk [9] and relaxes the compactness assumption in the above theorem. The main ingredient in Kirk's constructive proof is a technical lemma due to Gillespie and Williams [10].

Theorem 5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and let $p \in \mathcal{P}(\Omega)$; assume that $|\Omega_1| = 0$, $|\Omega_\infty| = 0$ and that ρ has property (R). Let $\emptyset \neq C \subset L^{p(\cdot)}(\Omega)$ be ρ-bounded, ρ-closed, and convex. If a map $T : C \to C$ is ρ nonexpansive, then it has a fixed point.*

Proof. Let $\mathcal{F}$ be the family of non-empty ρ-closed and convex subsets of C, which are T-invariant. The family $\mathcal{F}$ is not empty since $C \in \mathcal{F}$. Define $\tilde{\delta} : \mathcal{F} \to [0, +\infty)$ by

$$\tilde{\delta}(D) = \inf\{\delta_\rho(B) : \ B \in \mathcal{F} \text{ and } B \subset D\}.$$

Set $D_1 = C$. By the definition of $\tilde{\delta}(D_1)$, there exists $D_2 \in \mathcal{F}$ such that $D_2 \subset D_1$ and $\delta_\rho(D_2) < \tilde{\delta}(D_1) + 1$. Assume $D_1, D_2, \cdots, D_n$, for $n \geq 1$, are constructed. Again by the definition of $\tilde{\delta}(D_n)$, there exists $D_{n+1} \in \mathcal{F}$ such that $\delta_\rho(D_{n+1}) < \tilde{\delta}(D_n) + \frac{1}{n}$ and $D_{n+1} \subset D_n$. The property (R) implies $D_\infty = \bigcap\limits_{n\geq 1} D_n$ is not empty. Clearly, it holds that $D_\infty \in \mathcal{F}$. Assume that D_∞ contains more than one point. Using Proposition 1, one derives the existence of $f_0 \in D_\infty$ such that

$$r = \sup_{g \in D_\infty} \rho(f_0 - g) < \delta_\rho(D_\infty) = \sup\{\rho(a-b)\,, a \in D_\infty\,, b \in D_\infty\}.$$

Hence, the set

$$D = \bigcap_{g \in D_\infty} B_\rho(g, r) \cap D_\infty$$

is a non-empty, ρ-closed and convex subset of D_∞. Note that there is no reason for D to be T-invariant, i.e., $T(D) \subset D$. Consider the family $\mathcal{F}^* = \{M \in \mathcal{F} : \ D \subset M\}$. Obviously, $\mathcal{F}^*$ is not an empty since $C \in \mathcal{F}^*$. Set $L = \bigcap\limits_{M \in \mathcal{F}^*} M$. The set L is a non-empty, ρ-closed, and convex subset of C, which is T-invariant. Consider $B = D \cup T(L)$, and observe that $\overline{conv}_\rho(B) = L$ (where $\overline{conv}_\rho(B)$ is the intersection of all ρ-closed, convex subsets, which contain B). Indeed, since L contains D and is T-invariant, it is readily concluded that $B \subset L$. Since L is ρ-closed and convex, it follows that $\overline{conv}_\rho(B) \subset L$, whence

$$T(\overline{conv}_\rho(B)) \subset T(L) \subset B \subset \overline{conv}_\rho(B).$$

Hence $\overline{conv}_\rho(B) \in \mathcal{F}^*$ and $L \subset \overline{conv}_\rho(B)$. This implies the desired equality $L = \overline{conv}_\rho(B)$. Define $D^* = \bigcap\limits_{g \in L} B_\rho(g, r) \cap L$. Observe that D^* is non-empty since it contains D (by the definition of D and $D_\infty \in \mathcal{F}^*$) and is a ρ-closed, convex subset of C. On the other hand, it is clear that $\delta_\rho(D^*) \leq r$. Note that D^* is T-invariant. Indeed, let $f \in D^*$. It is clear by the definition of D^* that $L \subset B_\rho(f, r)$. Since T is ρ-nonexpansive, one has $T(L) \subset B_\rho(T(f), r)$. For any $g \in D$, it holds $L \subset B_\rho(g, r)$. However, $T(f) \in L$, so $T(f) \in B_\rho(g, r)$, which implies $g \in B_\rho(T(f), r)$. Hence, $D \subset B_\rho(T(f), r)$ holds. Since $B = D \cup T(L)$, it follows that $B \subset B_\rho(T(f), r)$. Therefore, one must have

$$\overline{conv}_\rho(B) = L \subset B_\rho(T(f), r).$$

By the definition of D^*, it follows that $T(f) \in D^*$. In other words, D^* is T-invariant. Since $L \subset D_\infty$ one has $D^* \subset D_\infty$. Therefore, the above construction yields $D^* \in \mathcal{F}$ and $D^* \subset D_\infty$ such that $\delta_\rho(D^*) \leq r$. Since $D^* \subset D_n$, it is clear that

$$\delta_\rho(D^*) \leq \delta_\rho(D_\infty) \leq \delta_\rho(D_{n+1}) \leq \tilde{\delta}(D_n) + \frac{1}{n} \leq \delta_\rho(D^*) + \frac{1}{n},$$

for any $n \geq 1$. Letting now $n \to \infty$, it is readily seen that $\delta_\rho(D^*) = \delta_\rho(D_\infty)$, which implies $\delta_\rho(D_\infty) \leq r$. This is in contradiction with the inequality $r < \delta_\rho(D_\infty)$. Hence, D_∞ must consist of exactly one point, which is a fixed point of T since D_∞ is T-invariant. $\square$

Note that the conclusion of Theorem 5 requires only the validity of the intersection property (R) for ρ on C and not on the entire space $L^{p(\cdot)}(\Omega)$.

Remark 1. *The proof of Theorem 5 can be significantly simplified in case ρ satisfies the so-called strong-(R) property alluded to in Theorem 4.*

Indeed, consider the family of non-empty, ρ-closed, and convex subsets of C, which are T-invariant $\mathcal{F}$, introduced in the previous proof. It is clear that $\mathcal{F} \neq \emptyset$, since $C \in \mathcal{F}$. The strong-(R) property combined with Zorn's lemma immediately yields the existence of a minimal element in $\mathcal{F}$. Let K be one such minimal element. It will be shown that K consists of exactly one point. First, notice that, since $T(K) \subset K$, it necessarily holds that $T\left(\overline{conv}_\rho(T(K))\right) \subset T(K) \subset \overline{conv}_\rho(T(K))$. The minimality of K forces $\overline{conv}_\rho(T(K)) = K$. Fix $f_0 \in K$; set $r = \sup\limits_{g \in K} \rho(f_0 - g)$; define

$$K_r = \left\{ f \in K;\ r_\rho(f) = \sup_{g \in K} \rho(f - g) \leq r \right\}.$$

Note that K_r is a non-empty, ρ-closed, and convex subset of K ($f_0 \in K_r$). K_r is T-invariant. To see this, let $f \in K_r$, and observe that $K \subset B_\rho(f, r)$. Since T is ρ-nonexpansive, one must have $T(K) \subset B_\rho(T(f), r)$, which implies

$$K = \overline{conv}_\rho(T(K)) \subset B_\rho(T(f), r).$$

Hence, $T(f) \in K_r$. Since K_r is a T-invariant subset of K, it follows that $K = K_r$. This clearly implies $r = \sup\limits_{g \in K} \rho(f_0 - g) = \delta_\rho(K)$, which only holds for subsets that consist of exactly one point, on account of the ρ-normal structure property. In other words, T has a fixed point, as claimed.

Corollary 1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and let $p \in \mathcal{P}(\Omega)$. Assume that $|\Omega_1| = 0$, $|\Omega_\infty| = 0$ and $p_- > 1$. Let $C \subset L^{p(\cdot)}(\Omega)$ be a non-empty ρ-bounded, ρ-closed, and convex subset. If a map $T : C \to C$ is ρ-nonexpansive, then it has a fixed point.*

Proof. If $p_- > 1$, Theorem 4 asserts that ρ satisfies the strong-(R) property. The proof follows directly from the above remark. □

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	linear dichroism

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