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On the Regularity of Weak Solutions to Time-Periodic Navier–Stokes Equations in Exterior Domains

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Abstract: Consider the time-periodic viscous incompressible fluid flow past a body with non-zero velocity at infinity. This article gives sufficient conditions such that weak solutions to this problem are smooth. Since time-periodic solutions do not have finite kinetic energy in general, the well-known regularity results for weak solutions to the corresponding initial-value problem cannot be transferred directly. The established regularity criterion demands a certain integrability of the purely periodic part of the velocity field or its gradient, but it does not concern the time mean of these quantities.

Keywords: time-periodic solutions; weak solutions; exterior domain; regularity criterion; Serrin condition; Oseen problem

MSC: 35B10; 35B65; 35Q30; 76D03; 76D05; 76D07

1. Introduction

We consider the time-periodic flow of a viscous incompressible fluid past a three-dimensional body that translates with constant non-zero velocity v_∞. We assume v_∞ to be directed along the x_1-axis such that $v_\infty = \tau\, e_1$ with $\tau > 0$. In a frame attached to the body, the fluid motion is then governed by the Navier–Stokes equations:

$$\partial_t u - \Delta u - \tau\partial_1 u + u\cdot\nabla u + \nabla\mathfrak{p} = f \qquad \text{in } \mathbb{T}\times\Omega, \tag{1a}$$

$$\operatorname{div} u = 0 \qquad \text{in } \mathbb{T}\times\Omega, \tag{1b}$$

$$u = u_* \qquad \text{on } \mathbb{T}\times\partial\Omega, \tag{1c}$$

$$\lim_{|x|\to\infty} u(t,x) = 0 \qquad \text{for } t\in\mathbb{T}, \tag{1d}$$

where $\Omega\subset\mathbb{R}^3$ is the exterior domain occupied by the fluid.

The functions $u\colon \mathbb{T}\times\Omega\to\mathbb{R}^3$ and $\mathfrak{p}\colon \mathbb{T}\times\Omega\to\mathbb{R}$ are velocity and pressure of the fluid flow, $f\colon \mathbb{T}\times\Omega\to\mathbb{R}^3$ is an external body force, and $u_*\colon \mathbb{T}\times\partial\Omega\to\mathbb{R}^3$ denotes the velocity field at the boundary. The time axis is given by the torus group $\mathbb{T} := \mathbb{R}/\mathcal{T}\mathbb{Z}$, which ensures that all functions appearing in Equation (1) are time periodic with a prescribed period $\mathcal{T} > 0$.

In this article, we study weak solutions to the problem in Equation (1), and we provide sufficient conditions such that these weak solutions possess more regularity and are actually smooth solutions. In the context of the initial-value problem for the Navier–Stokes equations, these criteria have been studied extensively. Existence of weak solutions was shown several decades ago in the seminal works by Leray [1] and Hopf [2] together with a corresponding energy inequality, but it remained unclear for many decades whether solutions in this *Leray–Hopf class* are unique, even when the external forcing is smooth (or even 0). Note that Albritton, Brué and Colombo [3] recently showed that there are forcing terms such that multiple Leray–Hopf solutions to the initial-value problem exist, so that uniqueness fails for general forcing terms. However, Leray–Hopf solutions come along with

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a weak-strong uniqueness principle that states that weak solutions coincide with strong solutions if the latter exist. This also motivated the development of criteria that ensured higher regularity of weak solutions. The first results in this direction are due to Leray [1] and Serrin [4], who showed that, if a weak solution is an element of $\mathrm{L}^{\rho}(0,T;\mathrm{L}^{\kappa}(\Omega)^3)$ for some $\kappa,\rho \in (1,\infty)$ such that $\frac{2}{\rho}+\frac{3}{\kappa}<1$, then it is a strong solution and smooth with respect to the spatial variables. Since then, there appeared many other regularity criteria that ensured higher-order regularity of a weak solution to the initial-value problem; see [5–10] and the references therein.

To obtain similar regularity results for weak solutions to the time-periodic problem in Equation (1), the first idea might be to identify these with weak solutions to the initial-value problem for a suitable initial value. However, this procedure is not successful in the considered framework of an exterior domain Ω since regularity of weak solutions to the initial-value problem is usually investigated within the class $\mathrm{L}^{\infty}(0,T;\mathrm{L}^2(\Omega)^3)$, but weak solutions u to the time-periodic problem are merely elements of $\mathrm{L}^2(\mathbb{T};\mathrm{L}^6(\Omega)^3)$ at the outset; see Definition 1 below. To see that we cannot expect the same integrability as for the initial-value problem, observe that every weak solution to the steady-state problem is also a time-periodic solution. In general, these steady-state solutions do not have finite kinetic energy but only belong to $\mathrm{L}^s(\Omega)^3$ for $s>2$; see Theorem 4 below. Therefore, one cannot reduce the time-periodic situation to that of the initial-value problem.

For the formulation of suitable regularity criteria for time-periodic weak solutions, we decompose functions into a time-independent part, given by the time mean over one period, and a time-periodic remainder part. To this decomposition, we associate a pair of complementary projections $\mathcal{P}$ and $\mathcal{P}_\perp$ such that

$$\mathcal{P}u := \int_{\mathbb{T}} u(t,\cdot)\,\mathrm{d}t, \qquad \mathcal{P}_\perp u := u - \mathcal{P}u.$$

Then, $\mathcal{P}u$ is called the *steady-state part* of u, and $\mathcal{P}_\perp u$ denotes the *purely periodic part* of u.

In this article, we consider weak solutions to (1) in the following sense.

Definition 1. *Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}\times\Omega)^3$ and $u_* \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}\times\partial\Omega)^3$. A function $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}\times\Omega)^3$ is called* weak solution *to (1) if it satisfies the following properties:*

i. $\nabla u \in \mathrm{L}^2(\mathbb{T}\times\Omega)^{3\times 3}$, $u \in \mathrm{L}^2(\mathbb{T};\mathrm{L}^6(\Omega)^3)$, $\operatorname{div} u = 0$ *in* $\mathbb{T}\times\Omega$, $u = u_*$ *on* $\mathbb{T}\times\partial\Omega$,

ii. $\mathcal{P}_\perp u \in \mathrm{L}^{\infty}(\mathbb{T};\mathrm{L}^2(\Omega))^3$,

iii. *the identity*

$$\int_{\mathbb{T}}\int_{\Omega}\big[-u\cdot\partial_t\varphi + \nabla u : \nabla\varphi - \tau\partial_1 u\cdot\varphi + (u\cdot\nabla u)\cdot\varphi\big]\,\mathrm{d}x\mathrm{d}t = \int_{\mathbb{T}}\int_{\Omega} f\cdot\varphi\,\mathrm{d}x\mathrm{d}t$$

holds for all test functions $\varphi \in \mathrm{C}^{\infty}_{0,\sigma}(\mathbb{T}\times\Omega)$.

The existence of weak solutions in the sense of Definition 1 satisfying an associated energy inequality was shown in [11] for $\Omega=\mathbb{R}^3$. Their asymptotic properties as $|x|\to\infty$ were investigated in [12–14]. For these results, it was necessary to ensure higher regularity of the solution u, which was achieved by assuming that

$$\mathcal{P}_\perp u \in \mathrm{L}^{\rho}(\mathbb{T};\mathrm{L}^{\kappa}(\Omega)^3) \tag{2}$$

holds for some $\kappa=\rho\in(5,\infty)$. Moreover, it was shown in [15] that u satisfies an energy equality if Equation (2) holds for some $\kappa\in[4,\infty]$ and $\rho\in[2,4]$ with $\frac{2}{\rho}+\frac{2}{\kappa}\leq 1$. It is remarkable that, in both cases, the additional integrability is only assumed for the purely periodic part $\mathcal{P}_\perp u$, but not for the whole weak solution u as is achieved for the initial-value problem. The main result of this article is in the same spirit and can be seen as an extension of the regularity results used in [12–14]. More precisely, we consider the criteria

$$\exists\, \kappa, \rho \in (1,\infty) \text{ with } \frac{2}{\rho} + \frac{3}{\kappa} < 1: \qquad \mathcal{P}_\perp u \in \mathrm{L}^\rho(\mathbb{T}; \mathrm{L}^\kappa(\Omega)^3), \tag{3}$$

$$\exists\, \kappa, \rho \in (1,\infty) \text{ with } \frac{2}{\rho} + \frac{3}{\kappa} < 2: \quad \nabla\mathcal{P}_\perp u \in \mathrm{L}^\rho(\mathbb{T}; \mathrm{L}^\kappa(\Omega)^{3\times 3}). \tag{4}$$

If the domain has a smooth boundary and the data are smooth, then both lead to smooth solutions.

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with a boundary of class C^∞, and let $\tau > 0$. Let $f \in \mathrm{C}_0^\infty(\mathbb{T}\times\Omega)$ and $u_* \in \mathrm{C}^\infty(\mathbb{T}\times\partial\Omega)$, and let u be a weak time-periodic solution to (1) in the sense of Definition 1 such that (3) or (4) is satisfied. Then, there exists a corresponding pressure field $\mathfrak{p}$ such that $(u,\mathfrak{p})$ is a smooth solution to Equation (1) and*

$$u \in \mathrm{C}^\infty(\mathbb{T}\times\overline{\Omega})^3, \qquad \mathfrak{p} \in \mathrm{C}^\infty(\mathbb{T}\times\overline{\Omega}).$$

As an intermediate step, we show the following result that assumes less smooth data.

Theorem 2. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with boundary of class C^2, and let $\tau > 0$. Let f and u_* be such that*

$$\forall q, r \in (1,\infty): \; f \in \mathrm{L}^r(\mathbb{T}; \mathrm{L}^q(\Omega)^3), \tag{5a}$$

$$u_* \in \mathrm{C}(\mathbb{T}; \mathrm{C}^2(\partial\Omega)^3) \cap \mathrm{C}^1(\mathbb{T}; \mathrm{C}(\partial\Omega)^3). \tag{5b}$$

Let u be a weak time-periodic solution to Equation (1) in the sense of Definition 1 such that Equation (3) or Equation (4) is satisfied. Then, $v := \mathcal{P}u$ and $w := \mathcal{P}_\perp u$ satisfies

$$\forall s_2 \in (1, \frac{3}{2}]: \quad v \in \mathrm{D}^{2,s_2}(\Omega)^3, \tag{6}$$

$$\forall s_1 \in (\frac{4}{3}, \infty]: \quad v \in \mathrm{D}^{1,s_1}(\Omega)^3, \tag{7}$$

$$\forall s_0 \in (2, \infty]: \quad v \in \mathrm{L}^{s_0}(\Omega)^3, \tag{8}$$

$$\forall q, r \in (1,\infty): \quad w \in \mathrm{W}^{1,r}(\mathbb{T}; \mathrm{L}^q(\Omega)^3) \cap \mathrm{L}^r(\mathbb{T}; \mathrm{W}^{2,q}(\Omega)^3), \tag{9}$$

and there exists a pressure field $\mathfrak{p} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}\times\Omega)$ with $p := \mathcal{P}\mathfrak{p}$ and $\mathfrak{q} := \mathcal{P}_\perp\mathfrak{p}$ such that

$$\forall s_2 \in (1, \frac{3}{2}]: \; p \in \mathrm{D}^{2,s_2}(\Omega)^3, \qquad \forall q, r \in (1,\infty): \; \nabla\mathfrak{q} \in \mathrm{L}^r(\mathbb{T}; \mathrm{L}^q(\Omega)^3) \tag{10}$$

and the identities in Equation (1) are satisfied in the strong sense.

Additionally, if Ω has a C^3-boundary, and if $\mathcal{P}f \in \mathrm{W}^{1,q}(\Omega)^3$ and $\mathcal{P}u_* \in \mathrm{W}^{3-1/q_1,q_1}(\partial\Omega)^3$ for some $q_1 \in (3,\infty)$, then

$$\forall s_2 \in (1,\infty): \quad v \in \mathrm{D}^{2,s_2}(\Omega)^3, \quad p \in \mathrm{D}^{1,s_2}(\Omega). \tag{11}$$

Theorems 1 and 2 are the main results of this article and will be proved in Section 5.

Comparing the regularity criteria of Theorems 1 and 2 with those used in [12–14], we see that the present article extends them in two directions. Firstly, by Equation (3), we extend the range of admissible parameters ρ, κ in the sufficient condition (2) by also allowing the mixed case $\rho \neq \kappa$. Secondly, Equation (4) is an alternative condition on certain integrability of the purely periodic part of the gradient ∇u. In particular, we can replace the assumption in Equation (2) for some $\kappa = \rho \in (5,\infty)$ with one of the assumptions in Equation (3) or Equation (4) in the main results of [12–14], and the results on the spatially asymptotic behavior of the velocity and the vorticity field derived there are also valid under the alternative regularity criteria from Equation (3) or Equation (4).

In Section 2, we next introduce the general notation used in this article. In Section 3, we recall the notion of Fourier multipliers in spaces with mixed Lebesgue norms and introduce a corresponding transference principle, from which we derive an embedding theorem. Section 4 recalls a well-known regularity result for the steady-state Navier–Stokes equations, and it contains a similar result for the time-periodic Oseen problem, which is a linearized version of Equation (1). Finally, Theorems 1 and 2 will be proved in Section 5, and we conclude the paper by a short outlook in Section 6.

2. Notation

For the whole article, the time period $\mathcal{T} > 0$ is a fixed constant, and $\mathbb{T} := \mathbb{R}/\mathcal{T}\mathbb{Z}$ denotes the corresponding torus group, which serves as the time axis. The spatial domain is usually given by a three-dimensional exterior domain $\Omega \subset \mathbb{R}^3$, that is, the domain Ω is the complement of a compact connected set. We write $\partial_t u$ and $\partial_j u := \partial_{x_j} u$ for partial derivatives with respect to time and space, and we set $\Delta u := \partial_j \partial_j u$ and $\operatorname{div} u := \partial_j u_j$, where we used Einstein's summation convention.

We equip the compact abelian group $\mathbb{T}$ with the normalized Lebesgue measure given by

$$\forall f \in \mathrm{C}(\mathbb{T}): \qquad \int_{\mathbb{T}} f(t)\,\mathrm{d}t = \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} f(t)\,\mathrm{d}t,$$

and the group $\mathbb{Z}$, which can be identified with the dual group of $\mathbb{T}$, with the counting measure. The Fourier transform $\mathscr{F}_G$ on the locally compact group $G\colon \mathbb{T} \times \mathbb{R}^n$, $n \in \mathbb{N}_0$, and its inverse $\mathscr{F}_G^{-1}$ are formally given by

$$\mathscr{F}_G[f](k,\xi) := \int_{\mathbb{T}} \int_{\mathbb{R}^n} f(t,x)\,\mathrm{e}^{-i\frac{2\pi}{\mathcal{T}}kt - ix\cdot\xi}\,\mathrm{d}x\mathrm{d}t,$$

$$\mathscr{F}_G^{-1}[f](t,x) := \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} f(k,\xi)\,\mathrm{e}^{i\frac{2\pi}{\mathcal{T}}kt + ix\cdot\xi}\,\mathrm{d}\xi,$$

where the Lebesgue measure $\mathrm{d}\xi$ is normalized appropriately such that $\mathscr{F}_G\colon \mathscr{S}(G) \to \mathscr{S}(\widehat{G})$ defines an isomorphism with inverse $\mathscr{F}_G^{-1}$. Here, $\mathscr{S}(G)$ is the so-called Schwartz–Bruhat space, which is a generalization of the classical Schwartz space in the Euclidean setting; see [16,17]. By duality, this induces an isomorphism $\mathscr{F}_G\colon \mathscr{S}'(G) \to \mathscr{S}'(\widehat{G})$ of the dual spaces $\mathscr{S}'(G)$ and $\mathscr{S}'(\widehat{G})$, the corresponding spaces of tempered distributions.

By $\mathrm{L}^q(\Omega)$ and $\mathrm{W}^{m,q}(\Omega)$ as well as $\mathrm{L}^q(\mathbb{T}\times\Omega)$ and $\mathrm{W}^{m,q}(\mathbb{T}\times\Omega)$, we denote the classical Lebesgue and Sobolev spaces, and $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}\times\Omega)$ denote the respective classes of locally integrable functions. We define homogeneous Sobolev spaces by

$$\mathrm{D}^{m,q}(\Omega) := \left\{ u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega) \;\middle|\; \nabla^m u \in \mathrm{L}^q(\Omega) \right\},$$

where $\nabla^m u$ denotes the collection of all (spatial) weak derivatives of the u of m-th order. We further set

$$\mathrm{C}^\infty_{0,\sigma}(\Omega) := \{\varphi \in \mathrm{C}^\infty_0(\Omega)^3 \mid \operatorname{div}\varphi = 0\},$$

where $\mathrm{C}^\infty_0(\Omega)$ is the class of compactly supported smooth functions on Ω. For $q \in [1,\infty]$ and a (semi-)normed vector space X, $\mathrm{L}^q(\mathbb{T};X)$ denotes the corresponding Bochner–Lebesgue space on $\mathbb{T}$, and

$$\mathrm{W}^{1,q}(\mathbb{T};X) := \left\{ u \in \mathrm{L}^q(\mathbb{T};X) \;\middle|\; \partial_t u \in \mathrm{L}^q(\mathbb{T};X) \right\}.$$

The projections

$$\mathcal{P}f := \int_{\mathbb{T}} f(t)\,\mathrm{d}t, \qquad \mathcal{P}_\perp f := f - \mathcal{P}f$$

decompose $f \in \mathrm{L}^1(\mathbb{T};X)$ into a time-independent *steady-state* part $\mathcal{P}f$ and a *purely periodic* part $\mathcal{P}_\perp f$.

We further study the fractional time derivative D_t^α for $\alpha \in (0,\infty)$, which is defined by

$$D_t^\alpha u(t) := \mathscr{F}_{\mathbb{T}}^{-1}\big[\big|\tfrac{2\pi}{\mathcal{T}}k\big|^\alpha \mathscr{F}_{\mathbb{T}}[u]\big](t) = \sum_{k\in\mathbb{Z}} \big|\tfrac{2\pi}{\mathcal{T}}k\big|^\alpha u_k \,\mathrm{e}^{i\frac{2\pi}{\mathcal{T}}kt}$$

for $u \in \mathscr{S}(\mathbb{T})$. By Plancherel's theorem (see [18][Prop. 3.1.16] for example), one readily verifies the integration-by-parts formula

$$\int_{\mathbb{T}} D_t^\alpha u\, v \,\mathrm{d}x = \int_{\mathbb{T}} u\, D_t^\alpha v \,\mathrm{d}x \tag{12}$$

for all $u, v \in \mathscr{S}(\mathbb{T})$. By duality, D_t^α extends to an operator on the distributions $\mathscr{S}'(\mathbb{T})$. Note that in general we have $D_t^\alpha u \neq \partial_t^\alpha u$ for $\alpha \in \mathbb{N}$, but

$$D_t^\alpha u \in \mathrm{L}^p(\mathbb{T}) \iff \partial_t^\alpha u \in \mathrm{L}^p(\mathbb{T})$$

holds for $\alpha \in \mathbb{N}$ and $p \in (1,\infty)$. If $\alpha = j/2$ for some $j \in \mathbb{N}$, we usually write $\sqrt{D_t^j}u := D_t^{j/2}u$.

3. Transference Principle and Embedding Theorem

To analyze mapping properties of the fractional derivative and other operators, we need the notion of Fourier multipliers on the locally compact abelian group $G = \mathbb{T} \times \mathbb{R}^n$ for $n \in \mathbb{N}_0$. We are interested in multipliers that induce bounded operators between mixed-norm spaces of the form $\mathrm{L}^p(\mathbb{T};\mathrm{L}^q(\mathbb{R}^n))$ for $p, q \in (1,\infty)$. We call $M \in \mathrm{L}^\infty(\mathbb{Z}\times\mathbb{R}^n)$ an $\mathrm{L}^p(\mathbb{T};\mathrm{L}^q(\mathbb{R}^n))$-*multiplier* if there is $C > 0$ such that

$$\forall u \in \mathscr{S}(\mathbb{T}\times\mathbb{R}^n): \quad \big\|\mathscr{F}_{\mathbb{T}\times\mathbb{R}^n}^{-1}\big[M\,\mathscr{F}_{\mathbb{T}\times\mathbb{R}^n}[u]\big]\big\|_{\mathrm{L}^p(\mathbb{T};\mathrm{L}^q(\mathbb{R}^n))} \leq C\|u\|_{\mathrm{L}^p(\mathbb{T};\mathrm{L}^q(\mathbb{R}^n))},$$

and we call $m \in \mathrm{L}^\infty(\mathbb{R}\times\mathbb{R}^n)$ an $\mathrm{L}^p(\mathbb{R};\mathrm{L}^q(\mathbb{R}^n))$-*multiplier* if there is $C > 0$ such that

$$\forall u \in \mathscr{S}(\mathbb{R}\times\mathbb{R}^n): \quad \big\|\mathscr{F}_{\mathbb{R}\times\mathbb{R}^n}^{-1}\big[m\,\mathscr{F}_{\mathbb{R}\times\mathbb{R}^n}[u]\big]\big\|_{\mathrm{L}^p(\mathbb{R};\mathrm{L}^q(\mathbb{R}^n))} \leq C\|u\|_{\mathrm{L}^p(\mathbb{R};\mathrm{L}^q(\mathbb{R}^n))}.$$

The smallest such constant C is denoted by $\|M\|_{\mathcal{M}_{p,q}(\mathbb{T}\times\mathbb{R}^n)}$ and $\|m\|_{\mathcal{M}_{p,q}(\mathbb{R}\times\mathbb{R}^n)}$ and called the multiplier norm of M and m, respectively. The following transference principle enables us to reduce multipliers on $\mathbb{T}\times\mathbb{R}^n$ to multipliers on $\mathbb{R}\times\mathbb{R}^n$.

Proposition 1. *Let $p, q \in (1,\infty)$, and let $m \in \mathrm{C}(\mathbb{T}\times\mathbb{R}^n)$ be an $\mathrm{L}^p(\mathbb{R};\mathrm{L}^q(\mathbb{R}^n))$-multiplier. Then, $M := m|_{\mathbb{Z}\times\mathbb{R}^n}$ is an $\mathrm{L}^p(\mathbb{T};\mathrm{L}^q(\mathbb{R}^n))$-multiplier with norm*

$$\|M\|_{\mathcal{M}_{p,q}(\mathbb{T}\times\mathbb{R}^n)} \leq \|m\|_{\mathcal{M}_{p,q}(\mathbb{R}\times\mathbb{R}^n)}$$

Proof. The statement can be shown as in [19], where a transference principle from scalar-valued $\mathrm{L}^p(\mathbb{R})$-multipliers to $\mathrm{L}^p(\mathbb{T})$-multipliers was shown. For a more direct and modern approach, one may also follow the proof of ([20] [Proposition 5.7.1]), where an operator-valued version of the result from [19] was established. □

We now apply this transference principle to show the following result, which is an extension of ([21] [Theorem 4.1]) to the case of mixed norms. Moreover, we also take fractional time derivatives into account.

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded or exterior domain with Lipschitz boundary, and let $q, r \in (1,\infty)$. For $\alpha \in [0,2]$, let*

$$r_0 \in \begin{cases} [1, \frac{2r}{2-\alpha r}] & \text{if } \alpha r < 2, \\ [1,\infty) & \text{if } \alpha r = 2, \\ [1,\infty] & \text{if } \alpha r > 2, \end{cases} \qquad q_0 \in \begin{cases} [q, \frac{nq}{n-(2-\alpha)q}] & \text{if } (2-\alpha)q < n, \\ [q,\infty) & \text{if } (2-\alpha)q = n, \\ [q,\infty] & \text{if } (2-\alpha)q > n, \end{cases}$$

and, for $\beta \in [0,1]$, *let*

$$r_1 \in \begin{cases} \left[1, \frac{2r}{2-\beta r}\right] & \text{if } \beta r < 2, \\ [1,\infty) & \text{if } \beta r = 2, \\ [1,\infty] & \text{if } \beta r > 2, \end{cases} \qquad q_1 \in \begin{cases} \left[q, \frac{nq}{n-(1-\beta)q}\right] & \text{if } (1-\beta)q < n, \\ [q,\infty) & \text{if } (1-\beta)q = n, \\ [q,\infty] & \text{if } (1-\beta)q > n. \end{cases}$$

Then, there is $C = C(n,q,r,\alpha,\beta) > 0$ *such that all* $u \in \mathrm{W}^{1,r}(\mathbb{T};\mathrm{L}^q(\Omega)) \cap \mathrm{L}^r(\mathbb{T};\mathrm{W}^{2,q}(\Omega))$ *satisfy the inequality*

$$\begin{aligned} \|u\|_{\mathrm{L}^{r_0}(\mathbb{T};\mathrm{L}^{q_0}(\Omega))} + \|\nabla u\|_{\mathrm{L}^{r_1}(\mathbb{T};\mathrm{L}^{q_1}(\Omega))} + \|\sqrt{D}_t u\|_{\mathrm{L}^{r_1}(\mathbb{T};\mathrm{L}^{q_1}(\Omega))} + \|\sqrt{D}_t \nabla u\|_{\mathrm{L}^{r}(\mathbb{T};\mathrm{L}^{q}(\Omega))} \\ \leq C\big(\|u\|_{\mathrm{W}^{1,r}(\mathbb{T};\mathrm{L}^q(\Omega))} + \|u\|_{\mathrm{L}^r(\mathbb{T};\mathrm{W}^{2,q}(\Omega))}\big). \end{aligned} \tag{13}$$

Proof. For the proof, we proceed analogously to ([21] [Theorem 4.1]). However, we have to modify some arguments in the case $p \neq q$, and we also derive estimates for the fractional time derivative, which is why we give some details here. Using Sobolev extension operators and the density properties of $\mathscr{S}(\mathbb{T} \times \mathbb{R}^n)$, it suffices to show the estimate from Equation (13) for $\Omega = \mathbb{R}^n$ and $u \in \mathscr{S}(G)$ with $G = \mathbb{T} \times \mathbb{R}^n$.

We begin with the estimate of u. By means of the Fourier transform, we obtain

$$u = \mathscr{F}_G^{-1}\left[\frac{1}{1+|\xi|^2 + i\frac{2\pi}{\mathcal{T}}k}\mathscr{F}_G\left[u + \partial_t u - \Delta u\right]\right] = \left(\gamma_{\alpha/2} \otimes \Gamma_{2-\alpha}\right) * F, \tag{14}$$

where

$$\gamma_\mu := \mathscr{F}_{\mathbb{T}}^{-1}\left[\left(1-\delta_{\mathbb{Z}}(k)\right)\left|\tfrac{2\pi}{\mathcal{T}}k\right|^{-\mu}\right], \qquad \Gamma_\nu := \mathscr{F}_{\mathbb{R}^n}^{-1}\left[(1+|\xi|^2)^{-\nu/2}\right],$$

$$F := \mathscr{F}_G^{-1}\left[M\,\mathscr{F}_G[u + \partial_t u - \Delta u]\right], \qquad M(k,\xi) := \frac{(1+|\xi|^2)^{1-\alpha/2}\left|\frac{2\pi}{\mathcal{T}}k\right|^{\alpha/2}}{1+|\xi|^2 + i\frac{2\pi}{\mathcal{T}}k}.$$

Here, $\delta_{\mathbb{Z}}$ is the delta distribution on $\mathbb{Z}$, that is, $\delta_{\mathbb{Z}} \colon \mathbb{Z} \to \{0,1\}$ with $\delta_{\mathbb{Z}}(k) = 1$ if and only if $k = 0$. We can extend $M \colon \mathbb{Z} \times \mathbb{R}^n \to \mathbb{A}$ to a continuous function $m \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{A}$ in a trivial way such that $M = m|_{\mathbb{Z}\times\mathbb{R}^n}$. One readily shows that m satisfies the Lizorkin multiplier theorem ([22] [Corollary 1]), so that the function m is an $\mathrm{L}^r(\mathbb{R};\mathrm{L}^q(\mathbb{R}^n))$-multiplier. Due to the transference principle from Proposition 1, this implies that M is an $\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\mathbb{R}^n))$-multiplier, and we have

$$\begin{aligned} \|F\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\mathbb{R}^n))} &\leq C\|u + \partial_t u - \Delta u\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\mathbb{R}^n))} \\ &\leq C\big(\|u\|_{\mathrm{W}^{1,r}(\mathbb{T};\mathrm{L}^q(\mathbb{R}^n))} + \|u\|_{\mathrm{L}^r(\mathbb{T};\mathrm{W}^{2,q}(\mathbb{R}^n))}\big). \end{aligned} \tag{15}$$

Moreover, from ([18] [Example 3.1.19]) and ([23] [Proposition 6.1.5]), we conclude

$$\gamma_\mu \in \mathrm{L}^{\frac{1}{1-\mu},\infty}(\mathbb{T}), \qquad \forall s \in \left[1, \frac{1}{1-\mu}\right) : \ \gamma_\mu \in \mathrm{L}^s(\mathbb{T}),$$

$$\Gamma_\nu \in \mathrm{L}^{\frac{n}{n-\nu},\infty}(\mathbb{R}^n), \qquad \forall s \in \left[1, \frac{n}{n-\nu}\right) : \ \Gamma_\nu \in \mathrm{L}^s(\mathbb{R}^n)$$

for $\mu \in (0,1)$ and $\nu \in (0,n)$. Young's inequality thus implies that $\varphi \mapsto \gamma_{\alpha/2} * \varphi$ defines a continuous linear operator $\mathrm{L}^r(\mathbb{T}) \to \mathrm{L}^{r_0}(\mathbb{T})$ if $r_0 \geq r$, and $\psi \mapsto \Gamma_{2-\alpha} * \psi$ defines a continuous linear operator $\mathrm{L}^q(\mathbb{T}) \to \mathrm{L}^{q_0}(\mathbb{T})$. Therefore, Equation (14) yields

$$\begin{aligned}
\|u\|_{\mathrm{L}^{r_0}(\mathbb{T};\mathrm{L}^{q_0}(\mathbb{R}^n))} &= \left(\int_{\mathbb{T}} \left\| \int_{\mathbb{T}} \gamma_\alpha(t-s) \Gamma_{2-\alpha} *_{\mathbb{R}^n} F(s,\cdot)\,\mathrm{d}s \right\|_{q_0}^{r_0} \mathrm{d}t \right)^{\frac{1}{r_0}} \\
&\leq \left(\int_{\mathbb{T}} \left(\int_{\mathbb{T}} |\gamma_\alpha(t-s)| \, \|\Gamma_{2-\alpha} *_{\mathbb{R}^n} F(s,\cdot)\|_{q_0} \,\mathrm{d}s \right)^{r_0} \mathrm{d}t \right)^{\frac{1}{r_0}} \\
&\leq C \left(\int_{\mathbb{T}} \|\Gamma_{2-\alpha} *_{\mathbb{R}^n} F(t,\cdot)\|_{q_0}^r \,\mathrm{d}t \right)^{\frac{1}{r}} \\
&\leq C \|F\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\mathbb{R}^n))}.
\end{aligned}$$

Invoking now Equation (15), we arrive at the desired estimate for u if $r_0 \geq r$. Since $\mathbb{T}$ is compact, the estimate for $r_0 < r$ follows immediately.

The remaining estimates of ∇u, $\sqrt{D}_t u$ and $\sqrt{D}_t \nabla u$ can be shown in the same way as those for u. Note that, for the estimates of $\sqrt{D}_t u$ and $\sqrt{D}_t \nabla u$, the procedure has to be slightly modified since the trivial extension of the corresponding multipliers to $\mathbb{R} \times \mathbb{R}^n$ is not continuous. To demonstrate this, we focus on the estimate for $\sqrt{D}_t \nabla u$, which means nothing else than the boundedness of the linear operator

$$\sqrt{D}_t \nabla \colon \mathrm{W}^{1,r}(\mathbb{T};\mathrm{L}^q(\mathbb{R}^n)) \cap \mathrm{L}^r(\mathbb{T};\mathrm{W}^{2,q}(\mathbb{R}^n)) \to \mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\mathbb{R}^n)).$$

Similarly to the above, this boundedness follows if the function

$$M \colon \mathbb{Z} \times \mathbb{R}^n \to \mathbb{A}, \qquad M(k,\xi) = \frac{|\frac{2\pi}{\mathcal{T}} k|^{\frac{1}{2}} \xi_j}{|\xi|^2 + i \frac{2\pi}{\mathcal{T}} k}$$

is an $\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\mathbb{R}^n))$-multiplier for $j = 1, \dots, n$. Note that its trivial extension is not a continuous function in $(0,0) \in \mathbb{R} \times \mathbb{R}^n$, which is necessary for application of the transference principle from Proposition 1. However, since $M(0,\xi) = 0$, we can introduce a smooth cut-off function $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ with $\operatorname{supp} \chi \subset (-1,1)$ and such that $\chi(\eta) = 1$ for $|\eta| \leq \frac{1}{2}$. We define

$$m \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{A}, \qquad m(\eta,\xi) = \frac{(1-\chi(\eta)) |\frac{2\pi}{\mathcal{T}} \eta|^{\frac{1}{2}} \xi_j}{|\xi|^2 + i \frac{2\pi}{\mathcal{T}} \eta}.$$

Then, m is a smooth function with $m|_{\mathbb{Z} \times \mathbb{R}^n} = M$, and one readily verifies that m satisfies the multiplier theorem by Lizorkin ([22] [Corollary 1]). Finally, Proposition 1 shows that M is an $\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\mathbb{R}^n))$-multiplier, which implies the estimate for $\sqrt{D}_t \nabla u$. □

As mentioned in the proof, the lower bound 1 for r_0 and r_1 is valid since the torus $\mathbb{T}$ has finite measure. In the same manner, the lower bound for q_0 and q_1 can be replaced with 1 if Ω is a bounded domain.

In ([24] [Theorem 4.1]), a homogeneous version of Theorem 3 was shown, but only in the case $q = r$. Modifying the proof in [24] and using similar arguments as above, this result is easily extended to the case $q \neq r$.

We might also formulate the assumptions on the integrability exponents in Theorem 3 as follows: Let $r_0, p_0, r_1, p_1 \in [1, \infty]$ such that

$$\begin{aligned}
\frac{2}{r} - \alpha &< \frac{2}{r_0} \leq 2, & \qquad \frac{n}{q} - (2-\alpha) &< \frac{n}{q_0} \leq \frac{n}{q}, \\
\frac{2}{r} - \beta &< \frac{2}{r_1} \leq 2, & \qquad \frac{n}{q} - (1-\beta) &< \frac{n}{q_1} \leq \frac{n}{q},
\end{aligned}$$

where in each of the four conditions the left $<$ can be replaced with $\leq$ if the respective lower bound is different from 0.

4. Preliminary Regularity Results

As a preparation for the proof of the main theorems, we first consider the steady-state Navier–Stokes equations

$$-\Delta v - \tau\partial_1 v + v\cdot\nabla v + \nabla p = F \quad \text{in } \Omega, \tag{16a}$$
$$\operatorname{div} v = 0 \quad \text{in } \Omega, \tag{16b}$$
$$v = v_* \quad \text{on } \partial\Omega \tag{16c}$$

and recall the following result on the regularity of weak solutions.

Theorem 4. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with a C^2-boundary. Let $q_0 \in (1,2)$ such that*

$$F \in \mathrm{L}^q(\Omega)^3, \qquad v_* \in \mathrm{W}^{2-\frac{1}{q},q}(\partial\Omega)^3 \tag{17}$$

for $q = q_0$ and for $q = \frac{3}{2}$. If v is a weak solution to Equation (16), then there exists an associated pressure field p such that

$$v \in \mathrm{D}^{2,q_0}(\Omega)^3 \cap \mathrm{D}^{1,4q_0/(4-q_0)}(\Omega)^3 \cap \mathrm{L}^{2q_0/(2-q_0)}(\Omega)^3, \qquad p \in \mathrm{D}^{1,q_0}(\Omega),$$

and Equation (16) is satisfied in the strong sense. Additionally, if there exists $q_1 \in (3,\infty)$ such that Equation (17) holds for all $q \in (1,q_1]$, then v satisfies Equations (7) and (8). Moreover, if Ω has C^3-boundary and $F \in \mathrm{W}^{1,q_1}(\Omega)^3$ and $v_* \in \mathrm{W}^{3-1/q_1,q_1}(\partial\Omega)^3$, then Equation (6) holds and $p \in \mathrm{D}^{1,q}(\Omega)$ for all $q \in (1,\infty]$.

Proof. See ([25] [Lemma X.6.1 and Theorem X.6.4]). □

We further derive a similar regularity result for weak solutions to the time-periodic Oseen problem, which is the linearization of Equation (1) given by

$$\partial_t u - \Delta u - \tau\partial_1 u + \nabla\mathfrak{p} = f \quad \text{in } \mathbb{T}\times\Omega, \tag{18a}$$
$$\operatorname{div} u = 0 \quad \text{in } \mathbb{T}\times\Omega, \tag{18b}$$
$$u = u_* \quad \text{on } \mathbb{T}\times\partial\Omega. \tag{18c}$$

Here, we focus on the case of purely oscillatory data. To shorten the notation, we denote the mixed-norm parabolic space by

$$\mathcal{W}_{q,r} := \mathrm{W}^{1,r}(\mathbb{T};\mathrm{L}^q(\Omega)^3) \cap \mathrm{L}^r(\mathbb{T};\mathrm{W}^{2,q}(\Omega)^3).$$

Lemma 1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class C^2, and let u_* be as in Equation (5b), and let $f \in \mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega)^3)$ for some $r,q \in (1,\infty)$ such that $\mathcal{P}f = 0$ and $\mathcal{P}u_* = 0$. Let $u \in \mathrm{L}^\infty(\mathbb{T};\mathrm{L}^2(\Omega)^3)$ with $\nabla u \in \mathrm{L}^2(\mathbb{T}\times\Omega)^{3\times 3}$ and $\mathcal{P}u = 0$ be a weak solution to Equation (18), that is, $u = u_*$ on $\mathbb{T}\times\partial\Omega$, $\operatorname{div} u = 0$ and*

$$\int_{\mathbb{T}}\int_{\Omega}\big[-u\cdot\partial_t\varphi + \nabla u : \nabla\varphi - \tau\partial_1 u\cdot\varphi\big]\,\mathrm{d}x\mathrm{d}t = \int_{\mathbb{T}}\int_{\Omega} f\cdot\varphi\,\mathrm{d}x\mathrm{d}t \tag{19}$$

for all $\varphi \in \mathrm{C}^\infty_{0,\sigma}(\mathbb{T}\times\Omega)$. Then, $u \in \mathcal{W}_{q,r}$, and there exists $\mathfrak{p} \in \mathrm{L}^q(\mathbb{T};\mathrm{D}^{1,q}(\Omega))$ such that $(u,\mathfrak{p})$ is a strong solution to Equation (18).

Proof. For $q = r$, the result was shown in ([14] [Lemma 5.1]). Arguing in the same way, we can show that it suffices to treat the case $u_* = 0$. In this case, first consider a solution of the time-periodic Stokes problem, that is, the system

$$\begin{aligned} \partial_t U - \Delta U + \nabla \mathfrak{P} &= f && \text{in } \mathbb{T} \times \Omega, && (20a) \\ \operatorname{div} U &= 0 && \text{in } \mathbb{T} \times \Omega, && (20b) \\ U &= 0 && \text{on } \mathbb{T} \times \partial\Omega. && (20c) \end{aligned}$$

We now use the result from ([26] [Theorem 5.5]) on maximal regularity for this system for right-hand sides in $\mathrm{L}^r(\mathbb{T}; \mathrm{L}^q(\Omega)^3)$. From this, we conclude the existence of a unique solution $(U, \mathfrak{P})$ with $\mathcal{P}U = 0$ and $U \in \mathcal{W}_{q,r}$. The embedding Theorem 3 implies that $\partial_1 U \in \mathrm{L}^{\tilde{r}}(\mathbb{T}; \mathrm{L}^q(\Omega))$ for $\tilde{r} \in (1, \infty)$ with $\frac{1}{\tilde{r}} \in (\frac{1}{r} - \frac{1}{2}, 1]$. We again employ the the maximal regularity result from ([26] [Theorem 5.5]) to obtain the existence of a unique solution (V, P) to

$$\begin{aligned} \partial_t V - \Delta V + \nabla P &= \tau \partial_1 U && \text{in } \mathbb{T} \times \Omega, \\ \operatorname{div} V &= 0 && \text{in } \mathbb{T} \times \Omega, \\ V &= 0 && \text{on } \mathbb{T} \times \partial\Omega, \end{aligned}$$

such that $\mathcal{P}V = 0$ and $V \in \mathcal{W}_{q,\tilde{r}}$ for all $\tilde{r}$ as above. Employing Theorem 3 once more, we see that $\partial_1 V \in \mathrm{L}^{\hat{r}}(\mathbb{T}; \mathrm{L}^q(\Omega))$ for any $\hat{r} \in (1, \infty)$. In particular, we can choose $\hat{r} = q$, that is, we have $\partial_1 V \in \mathrm{L}^q(\mathbb{T} \times \Omega)$. Now we can use the maximal regularity result ([21] [Theorem 5.1]) for the Oseen system for the right-hand sides in $\mathrm{L}^q(\mathbb{T} \times \Omega)$ to find a solution $(W, \mathfrak{Q})$ to

$$\begin{aligned} \partial_t W - \Delta W - \tau \partial_1 W + \nabla \mathfrak{Q} &= \tau \partial_1 V && \text{in } \mathbb{T} \times \Omega, \\ \operatorname{div} W &= 0 && \text{in } \mathbb{T} \times \Omega, \\ W &= 0 && \text{on } \mathbb{T} \times \partial\Omega, \end{aligned}$$

such that $W \in \mathcal{W}_{q,q}$. Theorem 3 further implies $\partial_1 W \in \mathrm{L}^{\bar{r}}(\mathbb{T}; \mathrm{L}^q(\Omega))$ for $\frac{1}{\bar{r}} \in (\frac{1}{q} - \frac{1}{2}, 1]$ such that $W \in \mathcal{W}_{\bar{r},q}$ by ([26] [Theorem 5.5]). Repeating this argument once again, we obtain $W \in \mathcal{W}_{q,\bar{r}}$ for $\bar{r} \in (1, \infty)$. In total, we see that $\tilde{u} := U + V + W$ and $\tilde{\mathfrak{p}} := \mathfrak{P} + P + \mathfrak{Q}$ satisfy the Oseen system from Equation (18) and $\tilde{u} \in \mathcal{W}_{q,r}$. To conclude that $u = \tilde{u}$, one can now proceed as in the proof of ([14] [Lemma 5.1]). The regularity of the pressure $\mathfrak{p}$ follows immediately. □

Observe that, for the proof, we combined two results on maximal regularity: one for the Stokes problem in Equation (20) for right-hand sides in $\mathrm{L}^r(\mathbb{T}; \mathrm{L}^q(\Omega)^3)$, and one for the Oseen problem in Equation (18) for right-hand sides in $\mathrm{L}^q(\mathbb{T} \times \Omega)^3$. The argument could be shortened severely if such a result would be available for the Oseen problem in Equation (18) for right-hand sides in $\mathrm{L}^r(\mathbb{T}; \mathrm{L}^q(\Omega)^3)$. For a proof, one can use the approach developed in ([26] [Theorem 5.5]), which would also give corresponding a priori estimates.

5. Regularity of Time-Periodic Weak Solutions

Now, we begin with the proof of Theorem 2, for which we proceed by a bootstrap argument that increases the range of admissible integrability exponents step by step. To shorten the notation, we introduce the *Serrin number*

$$s_{q,r} := \frac{2}{r} + \frac{3}{q}.$$

For the whole section, let f and u_* satisfy Equation (5), and let u be a weak solution in the sense of Definition 1. We decompose u and set $v := \mathcal{P}u$ and $w := \mathcal{P}_\perp u$.

We first show that the definition of weak solutions already implies some degree of increased regularity and that there exists a pressure such that the Navier–Stokes equations are satisfied in the strong sense.

Lemma 2. *There exists a pressure field* $\mathfrak{p} = p + \mathfrak{q}$ *such that*

$$\forall s_2 \in (1, \frac{3}{2}] : \ v \in \mathrm{D}^{2,s_2}(\Omega)^3, \ \ p \in \mathrm{D}^{1,s_2}(\Omega)^3, \tag{21}$$

$$\forall s_1 \in (\frac{4}{3}, 3] : \ v \in \mathrm{D}^{1,s_1}(\Omega)^3, \qquad \forall s_0 \in (2, \infty) : \ v \in \mathrm{L}^{s_0}(\Omega)^3, \tag{22}$$

$$\forall r, q \in (1, \infty) \text{ with } s_{q,r} = 4 : \quad w \in \mathcal{W}_{q,r}, \quad \mathfrak{q} \in \mathrm{L}^r(\mathbb{T}; \mathrm{D}^{1,q}(\Omega)), \tag{23}$$

and the Navier–Stokes equations in Equation (1) *are satisfied in the strong sense. More precisely, it holds that*

$$-\Delta v - \tau \partial_1 v + v \cdot \nabla v + \nabla p = \mathcal{P} f - \mathcal{P}[w \cdot \nabla w] \qquad \text{in } \Omega, \tag{24a}$$

$$\operatorname{div} v = 0 \qquad \text{in } \Omega, \tag{24b}$$

$$v = \mathcal{P} u_* \qquad \text{on } \partial\Omega. \tag{24c}$$

and

$$\partial_t w - \Delta w - \tau \partial_1 w + \nabla \mathfrak{q} = \mathcal{P}_\perp f - v \cdot \nabla w - w \cdot \nabla v - \mathcal{P}_\perp (w \cdot \nabla w) \text{ in } \mathbb{T} \times \Omega, \tag{25a}$$

$$\operatorname{div} w = 0 \qquad \text{in } \mathbb{T} \times \Omega, \tag{25b}$$

$$w = \mathcal{P}_\perp u_* \qquad \text{on } \mathbb{T} \times \partial\Omega. \tag{25c}$$

Proof. From the integrability of w, we conclude with Hölder's inequality that $w \cdot \nabla w \in \mathrm{L}^1(\mathbb{T}; \mathrm{L}^{3/2}(\Omega)) \cap \mathrm{L}^2(\mathbb{T}; \mathrm{L}^1(\Omega))$. We thus have $\mathcal{P} f - \mathcal{P}(w \cdot \nabla w) \in \mathrm{L}^1(\Omega) \cap \mathrm{L}^{3/2}(\Omega)$, and Theorem 4 yields the existence of p such that Equation (24) and Equations (21) and (22) also hold.

To obtain the regularity statement for w, note that Equation (22) implies $v \cdot \nabla w \in \mathrm{L}^2(\mathbb{T}; \mathrm{L}^q(\Omega)^3)$ for all $q \in (1, 2)$. Moreover, we have $w \in \mathrm{L}^2(\mathbb{T}; \mathrm{L}^6(\Omega)^3) \cap \mathrm{L}^\infty(\mathbb{T}; \mathrm{L}^2(\Omega)^3) \hookrightarrow \mathrm{L}^r(\mathbb{T}; \mathrm{L}^q(\Omega)^3)$ for all $r \in [2, \infty]$ and $q \in [2, 6]$ with $s_{q,r} = \frac{3}{2}$ by the Sobolev inequality and interpolation. By virtue of Equation (22) and Hölder's inequality, we conclude that $w \cdot \nabla(v + w) \in \mathrm{L}^r(\mathbb{T}; \mathrm{L}^q(\Omega)^3)$ for all $q \in (1, \frac{3}{2}]$ and $r \in [1, 2)$ with $s_{q,r} = 4$. In consequence, we obtain

$$\mathcal{P}_\perp f - v \cdot \nabla w - w \cdot \nabla v - \mathcal{P}_\perp (w \cdot \nabla w) \in \mathrm{L}^r(\mathbb{T}; \mathrm{L}^q(\Omega)^3)$$

for all such q and r. Now, Lemma 1 yields the existence of a pressure $\mathfrak{q}$ such that Equation (25) is satisfied in the strong sense and Equation (23) holds. □

In the following lemmas, we always assume that $w \in \mathcal{W}_{q,r}$ for some given $q, r \in (1, \infty)$, and the goal is to extend the range of one of the parameters q or r while the other one remains fixed. We use the assumption on additional regularity from Equation (3) or Equation (4), or the embedding properties from Theorem 3 to conclude that

$$w \in \mathrm{L}^{r_0}(\mathbb{T}; \mathrm{L}^{q_0}(\Omega)^3) \tag{26}$$

for a class of parameters $q_0, r_0 \in [1, \infty]$, and

$$\nabla w \in \mathrm{L}^{r_1}(\mathbb{T}; \mathrm{L}^{q_1}(\Omega)^{3\times 3}) \tag{27}$$

for a class of parameters $q_1, r_1 \in [1, \infty]$, and we use Lemma 2 or Theorem 4 to deduce

$$v \in \mathrm{L}^{s_0}(\Omega)^3 \tag{28}$$

for certain $s_0 \in [1, \infty]$ and

$$\nabla v \in \mathrm{L}^{s_1}(\Omega)^{3\times 3} \tag{29}$$

for certain $s_1 \in [1, \infty]$. Then, Hölder's inequality yields suitable estimates of the nonlinear terms and of the total right-hand side

$$\mathcal{P}_\perp f - v \cdot \nabla w - w \cdot \nabla v - \mathcal{P}_\perp(w \cdot \nabla w) \in \mathrm{L}^{r_5}(\mathbb{T}; \mathrm{L}^{q_5}(\Omega)^3) \tag{30}$$

for a certain class of parameters $q_5, r_5 \in (1, \infty)$. Invoking now the regularity result from Lemma 1, we conclude $w \in \mathcal{W}_{q_5,r_5}$.

As a preparation, we first derive suitable estimates of the nonlinear terms if we have $w \in \mathcal{W}_{q,r}$. In the next lemma, we start with the nonlinear term

$$w \cdot \nabla w \in \mathrm{L}^{r_2}(\mathbb{T}; \mathrm{L}^{q_2}(\Omega)^3), \tag{31}$$

and we show better integrability for v and ∇v for sufficiently large q.

Lemma 3. *Let $w \in \mathcal{W}_{q,r}$ for some $q, r \in (1, \infty)$. Then, Equation (31) holds for*

i. $\frac{3}{q_2} \in \big(\max\{0, s_{q,r} - 1, \frac{3}{q} + s_{q,r} - 2\}, \min\{3, \frac{6}{q}\}\big]$ *and $r_2 = r$, and*

ii. $q_2 = q$ *and* $\frac{2}{r_2} \in \big(\max\{0, s_{q,r} - 1\}, 2\big]$.

Moreover, if $s_{q,r} < \frac{3}{2} + \frac{1}{\max\{2,r\}}$ or $q > 3$, then the steady-state part v satisfies Equations (7) and (8).

Proof. At first, Theorem 3 yields Equation (26) for $r_0 = \infty$ and $\frac{3}{q_0} \in \big(\max\{0, s_{q,r} - 1\}, \frac{3}{q}\big]$, and Equation (27) for $r_1 = r$ and $\frac{3}{q_1} \in \big(\max\{0, \frac{3}{q} - 1\}, \frac{3}{q}\big]$, so that we deduce Equation (31) for $r_2 = r$ and q_2 as asserted in i. Moreover, Theorem 3 yields $w \in \mathrm{L}^{r_0}(\mathbb{T}; \mathrm{L}^q(\Omega)^3)$ for $r_0 \in [1, \infty]$ as well as $\nabla w \in \mathrm{L}^{r_1}(\mathbb{T}; \mathrm{L}^\infty(\Omega)^{3\times 3})$ for $\frac{2}{r_1} \in \big(\max\{0, s_{q,r} - 1\}, 2\big]$. Now, Hölder's inequality implies the integrability of $w \cdot \nabla w$ asserted in ii.

If, additionally, $s_{q,r} < \frac{3}{2} + \frac{1}{\max\{2,r\}}$, then the lower bound in i. is smaller than 1, so that $\mathcal{P}f - \mathcal{P}(w \cdot \nabla w) \in \mathrm{L}^{q_2}(\Omega)^3$ for some $q_2 \in (3, \infty)$. The same follows from ii. for $q_2 = q > 3$. Now, Theorem 4 yields Equations (7) and (8). □

Next, we treat the nonlinear terms that involve v and ∇v, namely we show that $w \in \mathcal{W}_{q,r}$ implies

$$v \cdot \nabla w \in \mathrm{L}^{r_3}(\mathbb{T}; \mathrm{L}^{q_3}(\Omega)^3) \tag{32}$$

and

$$w \cdot \nabla v \in \mathrm{L}^{r_4}(\mathbb{T}; \mathrm{L}^{q_4}(\Omega)^3) \tag{33}$$

for suitable parameters $q_3, r_3, q_4, r_4 \in [1, \infty]$.

Lemma 4. *Let $w \in \mathcal{W}_{q,r}$ for some $q, r \in (1, \infty)$. Then, Equation (32) holds for*

i. $\frac{3}{q_3} \in \big(\max\{0, \frac{3}{q} - 1\}, \min\{3, \frac{3}{q} + \frac{3}{2}\}\big)$ *and $r_3 = r$, and*

ii. $q_3 = q$ *and* $\frac{2}{r_3} \in \big(\max\{0, \frac{2}{r} - 1\}, 2\big]$,

and Equation (33) holds for

iii. $\frac{3}{q_4} \in \big(\max\{\frac{3}{4}, \frac{3}{q} - \frac{3}{4}, \frac{6}{q} - \frac{11}{4}\}, \min\{3, \frac{3}{q} + \frac{9}{4}\}\big)$ *and $r_4 = r$, and*

iv. $\frac{3}{q_4} \in (0, \min\{3, \frac{3}{q} + \frac{9}{4}\})$ *and $r_4 = r$ if $q > 3$, and*

v. $q_4 = q$ *and* $\frac{2}{r_4} \in \big(\max\{0, \frac{2}{r} - 1\}, 2\big]$.

Proof. Theorem 3 implies Equation (27) for $\frac{3}{q_1} \in \big(\max\{0, \frac{3}{q} - 1\}, \frac{3}{q}\big]$ and $r_1 = r$ as well as for $\frac{3}{q_1} = \frac{3}{q} - \delta$ and $\frac{2}{r_1} \in \big(\max\{0, \frac{2}{r} - (1 - \delta)\}, \frac{2}{r}\big]$ for $\delta > 0$ small. Moreover, by Lemma 2, we have Equation (28) for $s_0 \in (2, \infty)$, and Hölder's inequality implies Equation (32) for q_3 and r_3 as in i. or ii.

Theorem 4 and Lemma 2 yield Equation (29) for all $\frac{1}{s_1} \in \big(\max\{\frac{1}{4}, \frac{1}{q} - \frac{1}{4}\}, \frac{3}{4}\big)$. Theorem 3 implies Equation (26) for $r_0 = r$ and $\frac{3}{q_0} \in \big(\max\{0, \frac{3}{q} - 2\}, \frac{3}{q}\big]$. Hölder's inequality now yields Equation (33) for q_4 and r_4 as in iii. Additionally, if $q > 3$, then we obtain Equation (7) by Lemma 3. Theorem 3 further implies Equation (26) for $r_0 = r$ and $q_0 \in [q, \infty]$, so that Hölder's inequality yields Equation (33) for q_4 and r_4 as in iv.

For v., we distinguish two cases. Firstly, if $q \leq 3$, then Theorem 3 implies Equation (26) for $\frac{3}{q_0} = \frac{3}{q} - 1 - \delta$ and $\frac{2}{r_0} \in \big(\max\{0, \frac{2}{r} - 1 + \delta\}, 2\big]$ for $\delta \in (0,1)$, and Lemma 2 yields Equation (29) for all $s_1 = \frac{3}{1+\delta}$, so that Hölder's inequality implies Equation (33) for $q_4 = q_0$ and $r_4 = r_0$. Secondly, if $q > 3$, then we use Lemma 3 again to conclude Equation (7). Moreover, Theorem 3 yields Equation (26) for $q_0 = q$ and $r_0 \in [1, \infty]$, and we conclude Equation (33) for $q_4 = q$ and $r_4 = r_0 \in [1, \infty]$. Combining both cases, we obtain v. □

The results from Lemmas 3 and 4 are not sufficient to conclude the proof, and we need to invoke the additional regularity assumptions from Equation (3) or Equation (4) to obtain Equation (31) for other parameters q_2 and r_2. We define $\delta_{\kappa,\rho} > 0$ by

$$\delta_{\kappa,\rho} := \begin{cases} 1 - s_{\kappa,\rho} & \text{if Equation (3) is assumed,} \\ 2 - s_{\kappa,\rho} & \text{if Equation (4) is assumed.} \end{cases}$$

Lemma 5. *Assume either Equation (3) or Equation (4), and let $w \in \mathcal{W}_{q,r}$ for some $q, r \in (1, \infty)$. Then, Equation (31) holds for*

i. $\frac{3}{q_2} \in \big(\max\{\frac{3}{\kappa}, \frac{3}{q} - \delta_{\kappa,\rho}\}, \min\{3, \frac{3}{q} + \frac{3}{\kappa}\}\big]$ *and* $r_2 = r$, *and*

ii. $q_2 = q$ *and* $\frac{2}{r_2} \in \big(\max\{\frac{2}{\rho}, \frac{2}{r} - \delta_{\kappa,\rho}\}, 2\big]$.

Proof. At first, let us assume Equation (3). Then, $\rho > 2$ and Theorem 3 with $\beta = \frac{2}{\rho} \in (0,1)$ yields Equation (27) for $\frac{3}{q_1} \in \big(\max\{0, \frac{3}{q} + s_{\kappa,\rho} - 1 - \frac{3}{\kappa}\}, \frac{3}{q}\big]$ and $\frac{2}{r_1} = \frac{2}{r} - \frac{2}{\rho}$. Combining this with Equation (3) and using Hölder's inequality, we obtain Equation (31) for q, r as in i. Moreover, we have $\kappa > 3$, and Theorem 3 with $\beta = 1 - \frac{3}{\kappa} \in (0,1)$ yields Equation (27) for $\frac{2}{r_1} \in \big(\max\{0, \frac{2}{r} + s_{\kappa,\rho} - 1 - \frac{2}{\rho}\}, 2\big]$ and $\frac{3}{q_1} = \frac{3}{q} - \frac{3}{\kappa}$. Combining this with Equation (3) and using Hölder's inequality, we obtain Equation (31) for q, r as in ii.

Now, let us assume Equation (4). From Theorem 3 with $\alpha = \frac{2}{\rho} \in (0,2)$, we deduce Equation (26) for $\frac{3}{q_0} \in \big(\max\{0, \frac{3}{q} + s_{\kappa,\rho} - 2 - \frac{3}{\kappa}\}, \frac{3}{q}\big]$ and $\frac{2}{r_0} = \frac{2}{r} - \frac{2}{\rho}$. Combining this with Equation (4) and using Hölder's inequality, we also obtain Equation (31) in this case for q, r as in i. Moreover, we have $\kappa > \frac{3}{2}$, and Theorem 3 with $\alpha = 2 - \frac{3}{\kappa} \in (0,2)$ yields Equation (26) for $\frac{2}{r_0} \in \big(\max\{0, \frac{2}{r} + s_{\kappa,\rho} - 2 - \frac{2}{\rho}\}, 2\big]$ and $\frac{3}{q_0} = \frac{3}{q} - \frac{3}{\kappa}$. Combining this with Equation (4) and using Hölder's inequality, we also obtain Equation (31) in this case for q, r as claimed in ii. □

Now, we have prepared everything to iteratively increase the range of parameters q, r such that $w \in \mathcal{W}_{q,r}$. By Lemma 2, we start with q, r such that $s_{q,r} = 4$. In particular, both parameters cannot be chosen as large, and we use Lemmas 4 and 5 to extend the range of admissible parameters. An iteration leads to sufficiently large parameters such that Lemma 3 can be invoked to further iterate until the full range $(1, \infty)$ is admissible for both parameters, which proves the regularity result from Theorem 2.

Proof of Theorem 2. As a first step, we show that $w \in \mathcal{W}_{q,r}$ for all $q \in (3, \infty)$ and all $r \in (2, \infty)$. To do so, observe that both Equations (3) and (4) imply that $\kappa > 3$ or $\rho > 2$. In what follows, we distinguish these two cases:

Consider the case $\kappa > 3$ at first. We show that $w \in \mathcal{W}_{\tilde{q},r}$ for all $\tilde{q} \in (1, \kappa)$ and $r \in (1,2)$. Let $q \in (1, \frac{3}{2})$ and $r \in (1,2)$ with $s_{q,r} = 4$, so that $w \in \mathcal{W}_{q,r}$ by Lemma 2. Then, we have Equation (31) for q_2, r_2 as in i. of Lemma 5, we have Equation (32) for q_3, r_3 as in i. of Lemma 4, and we have Equation (33) for q_4, r_4 as in iii. of Lemma 4. We thus obtain Equation (30) for $\frac{3}{q_5} \in \big(\max\{\frac{3}{\kappa}, \frac{3}{q} - \delta_{\kappa,\rho}, \frac{3}{q} - \frac{3}{4}, \frac{3}{4}, \frac{6}{q} - \frac{11}{4}\}, \min\{3, \frac{3}{q} + \frac{3}{\kappa}, \frac{3}{q} + \frac{3}{2}\}\big)$ and $r_5 = r$. Since $\kappa > \frac{3}{2}$, this interval is non-empty, and by the regularity result from Lemma 1, we conclude $w \in \mathcal{W}_{\tilde{q},r}$ for $\frac{3}{\tilde{q}} \in \big(\max\{\frac{3}{\kappa}, \frac{3}{q} - \delta_{\kappa,\rho}, \frac{3}{q} - \frac{3}{4}, \frac{3}{4}, \frac{6}{q} - \frac{11}{4}\}, \min\{3, \frac{3}{q} + \frac{3}{\kappa}, \frac{3}{q} + \frac{3}{2}\}\big)$. Repeating this argument iteratively with q replaced with a suitable $\tilde{q}$ within this range, we obtain $w \in \mathcal{W}_{\tilde{q},r}$ for $\tilde{q} \in (1, \min\{4,\kappa\})$. If $\kappa \leq 4$, this completes the first step. If this is not the case, we repeat the above argument for $q \in (3,4)$, but we use iv. of Lemma 4 instead of iii., which leads to Equation (30) for $\frac{3}{q_5} \in \big(\max\{\frac{3}{\kappa}, \frac{3}{q} - \delta_{\kappa,\rho}\}, \min\{3, \frac{3}{q} + \frac{3}{\kappa}, \frac{3}{q} + \frac{3}{2}\}\big)$ and $r_5 = r$, and thus $w \in \mathcal{W}_{\tilde{q},r}$ for for $\frac{3}{\tilde{q}} \in \big(\max\{\frac{3}{\kappa}, \frac{3}{q} - \delta_{\kappa,\rho}\}, \min\{3, \frac{3}{q} + \frac{3}{\kappa}, \frac{3}{q} + \frac{3}{2}\}\big)$. Repeating now this argument a sufficient number of times for admissible $\tilde{q} > q$ instead of q, we finally arrive at $w \in \mathcal{W}_{\tilde{q},r}$ for all $\tilde{q} \in (1,\kappa)$ and $r \in (1,2)$.

Since we assume $\kappa > 3$, we can now choose $q \in (3,\kappa)$. Let $r \in (1,2)$ such that $s_{q,r} < 2$. The previous step implies $w \in \mathcal{W}_{q,r}$, and we conclude Equation (31) for q_2, r_2 as in ii. of Lemma 3, we have Equation (32) for q_3, r_3 as in ii. of Lemma 4, and we have Equation (33) for q_4, r_4 as in v. of Lemma 4. We thus obtain Equation (30) for $q_5 = q$ and $\frac{2}{r_5} \in \big(\max\{0, s_{q,r} - 1\}, 2\big]$. Invoking Lemma 1, we obtain $w \in \mathcal{W}_{q,\tilde{r}}$ for $\frac{2}{\tilde{r}} \in \big(\max\{0, \frac{2}{r} + \frac{3}{q} - 1\}, 2\big)$, and an iteration as above yields $w \in \mathcal{W}_{\tilde{q},\tilde{r}}$ for all $\tilde{q} \in (3,\kappa)$ and all $\tilde{r} \in (1,\infty)$.

Now, let $q \in (3,\kappa)$ and $r \in (2,\infty)$. Then, $s_{q,r} < 2$ and, since $w \in \mathcal{W}_{q,r}$, we have Equation (31) for q_2, r_2 as in i. of Lemma 3, we have Equation (32) for q_3, r_3 as in i. of Lemma 4, and we have Equation (33) for q_4, r_4 as in iv. of Lemma 4. We thus obtain Equation (30) for $\frac{3}{q_5} \in \big(\max\{0, s_{q,r} - 1, \frac{3}{q} + s_{q,r} - 2\}, \frac{6}{q}\big)$ and $r_5 = r$, and Lemma 1 yields $w \in \mathcal{W}_{\tilde{q},r}$ for $\frac{3}{\tilde{q}} \in \big(\max\{0, \frac{3}{q} + \frac{2}{r} - 1, \frac{3}{q} + s_{q,r} - 2\}, \frac{6}{q}\big)$. An iteration of this argument leads to $w \in \mathcal{W}_{\tilde{q},\tilde{r}}$ for all $\tilde{q} \in (\frac{3}{2},\infty)$ and all $\tilde{r} \in (2,\infty)$.

Now consider the case $\rho > 2$. We first extend the range for r and show that $w \in \mathcal{W}_{\tilde{q},\tilde{r}}$ for all $\tilde{q} \in (1,\frac{3}{2})$ and $\tilde{r} \in (1,\rho)$. For this, fix $q \in (1,\frac{3}{2})$. Lemma 2 yields $w \in \mathcal{W}_{q,r}$ for some $r \in (1,2)$ such that $s_{q,r} = 4$. Then, we have Equation (31) for q_2, r_2 as in ii. of Lemma 5, we have Equation (32) for q_3, r_3 as in ii. of Lemma 4, and we have Equation (33) for q_4, r_4 as in v. of Lemma 4. We thus obtain Equation (30) for $q_5 = q$ and $\frac{2}{r_5} \in \big(\max\{\frac{2}{\rho}, \frac{2}{r} - \delta_{\kappa,\rho}, \frac{2}{r} - 1\}, 2\big]$. Using the regularity result from Lemma 1, we now obtain $w \in \mathcal{W}_{q,\tilde{r}}$ for $\frac{2}{\tilde{r}} \in \big(\max\{\frac{2}{\rho}, \frac{2}{r} - \delta_{\kappa,\rho}, \frac{2}{r} - 1\}, 2\big)$. Repeating this argument with r replaced with some $\tilde{r} > r$ in this range, we can successively increase the admissible range for $\tilde{r}$ until we obtain $w \in \mathcal{W}_{\tilde{q},\tilde{r}}$ for all $\tilde{q} \in (1,\frac{3}{2})$ and $\tilde{r} \in (1,\rho)$.

Since $\rho > 2$, we can choose $r \in (2,\rho)$, and from $w \in \mathcal{W}_{q,r}$ for $q \in (1,\frac{3}{2})$ and we conclude Equation (31) for q_2, r_2 as in i. of Lemma 3, we have Equation (32) for q_3, r_3 as in i. of Lemma 4, and we have Equation (33) for q_4, r_4 as in iii. of Lemma 4. We thus obtain Equation (30) for $r_5 = r$ and $\frac{3}{q_5} \in \big(\max\{\frac{3}{4}, \frac{3}{q} - \frac{3}{4}, \frac{6}{q} - \frac{11}{4}, s_{q,r} - 1, \frac{3}{q} + s_{q,r} - 2\}, \min\{3, \frac{6}{q}\}\big]$. Invoking Lemma 1, we obtain $w \in \mathcal{W}_{\tilde{q},r}$ for $\frac{3}{\tilde{q}} \in \big(\max\{\frac{3}{4}, \frac{3}{q} - \frac{3}{4}, \frac{6}{q} - \frac{11}{4}, s_{q,r} - 1, \frac{3}{q} + s_{q,r} - 2\}, \min\{3, \frac{6}{q}\}\big)$, and an iteration as above yields $w \in \mathcal{W}_{\tilde{q},r}$ for all $\tilde{q} \in (1,4)$. Now, we can choose $q = \tilde{q} > 3$, and repeating the argument with iv. of Lemma 4 instead of iii., we obtain $w \in \mathcal{W}_{\tilde{q},r}$ for $\frac{3}{\tilde{q}} \in \big(\max\{0, s_{q,r} - 1, \frac{3}{q} + s_{q,r} - 2\}, \min\{3, \frac{6}{q}\}\big)$. Another iteration now leads to $w \in \mathcal{W}_{\tilde{q},\tilde{r}}$ for all $\tilde{q} \in (1,\infty)$ and all $\tilde{r} \in (2,\rho)$ if $\rho > 2$.

Now, let $q \in (3,\infty)$ and $r \in (2,\rho)$. Then, $s_{q,r} < 2$ and, since $w \in \mathcal{W}_{q,r}$, we have Equation (31) for q_2, r_2 as in ii. of Lemma 3, we have Equation (32) for q_3, r_3 as in ii. of Lemma 4, and we have Equation (33) for q_4, r_4 as in v. of Lemma 4. We thus obtain Equation (30) for $q_5 = q$ and $\frac{2}{r_5} \in \big(\max\{0, s_{q,r} - 1\}, 2\big]$. Invoking Lemma 1, we obtain $w \in \mathcal{W}_{q,\tilde{r}}$ for $\frac{2}{\tilde{r}} \in \big(\max\{0, \frac{2}{r} + \frac{3}{q} - 1\}, 2\big)$, and an iteration as above yields $w \in \mathcal{W}_{\tilde{q},\tilde{r}}$ for all $\tilde{q} \in (3,\infty)$ and all $\tilde{r} \in (1,\infty)$.

Combining these two cases and using that $\mathbb{T}$ is compact, we have shown that $w \in \mathcal{W}_{q,r}$ for all $q \in (3,\infty)$ and $r \in (1,\infty)$. In particular, v satisfies Equations (8) and (7) by Lemma 3, and we have Equation (6) by Lemma 2. To conclude Equation (9), note that Theorem 3 implies Equations (26) and (27) for $q_0, q_1 \in (3,\infty]$ and $r_0, r_1 \in [1,\infty)$, so that Equation (31) holds for $q_2 \in (\frac{3}{2},\infty]$ and $r_2 \in (1,\infty)$, and i. and iv. of Lemma 4 yield Equation (32) for $q_3 \in (\frac{6}{5},\infty)$ and $r_3 \in (1,\infty)$, and Equation (33) for $q_4 \in (1,\infty)$ and $r_4 \in (1,\infty)$. We thus have obtained Equation (30) for $q_5 \in (\frac{3}{2},\infty)$ and $r_5 \in (1,\infty)$, and, from Lemma 1, we conclude $w \in \mathcal{W}_{q,r}$ for all $q \in (\frac{3}{2},\infty)$ and $r \in (1,\infty)$. Repeating the argument once more, we finally obtain Equation (9). Moreover, Equation (10) is a direct consequence of Equations (6)–(9) by virtue of Equations (24) and (25).

Finally, Equation (11) follows from Theorem 4 and the additional assumptions on $\partial\Omega$, f and u_* since Equation (9) implies that $\mathcal{P}(w \cdot \nabla w) \in \mathrm{W}^{1,q}(\Omega)$ for any $q \in (1,\infty)$. $\square$

Proof of Theorem 1. At first, we increase the time regularity of the solution inductively in steps of half a derivative. For $j \in \mathbb{N}$, let $\tilde{u}_j := \sqrt{D_t}^j u$ and $\tilde{w}_j := \sqrt{D_t}^j w$. We show that, for every $j \in \mathbb{N}$, we have

$$\forall q,r \in (1,\infty): \quad \tilde{w}_j \in \mathrm{W}^{1,r}(\mathbb{T};\mathrm{L}^q(\Omega)^3) \cap \mathrm{L}^r(\mathbb{T};\mathrm{W}^{2,q}(\Omega)^3). \tag{34}$$

By Theorem 2, there exists a pressure field $\mathfrak{p}$ such that $(u,\mathfrak{p})$ is a strong solution to Equation (1) with the regularity stated in Equations (6)–(11). In particular, this shows Equation (34) for $j = 0$. Now, assume that $\tilde{w}_j$ has the asserted regularity stated in Equation (34) for all $j \in \{0,\dots,k\}$. Then, Theorem 3 implies

$$\forall q,r \in (1,\infty): \quad \tilde{w}_{j+1} = \sqrt{D_t}\tilde{w}_j \in \mathrm{L}^r(\mathbb{T};\mathrm{W}^{1,q}(\Omega)^3) \cap \mathrm{L}^\infty(\mathbb{T};\mathrm{L}^\infty(\Omega)^3) \tag{35}$$

for $j = 0,\dots,k$. Let $\varphi \in \mathrm{C}^\infty_{0,\sigma}(\mathbb{T}\times\Omega)$ and multiply Equation (1a) by $\sqrt{D_t}^{k+1}\varphi$. Since $\tilde{w}_k = \tilde{u}_k$ for $k \geq 1$, after integrating by parts in space and time as well as by means of Equation (12), we obtain

$$\begin{aligned}\int_{\mathbb{T}}\int_\Omega \big[-\tilde{w}_{k+1}\cdot\partial_t\varphi + \nabla\tilde{w}_{k+1} : \nabla\varphi - \tau\partial_1\tilde{w}_{k+1}\cdot\varphi + (\tilde{w}_{k+1}\cdot\nabla\tilde{w}_{k+1})\cdot\varphi\big]\,\mathrm{d}x\mathrm{d}t \\ = \int_{\mathbb{T}}\int_\Omega f_{k+1}\cdot\varphi\,\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{36}$$

where

$$f_{k+1} := \sqrt{D_t}^{k+1} f + \tilde{w}_{k+1}\cdot\nabla\tilde{w}_{k+1} - \sqrt{D_t}^{k+1}\operatorname{div}(u\otimes u).$$

In virtue of the smoothness of the boundary data and the regularity of $\tilde{w}$, we see that $\tilde{w}$ is a weak solution to the Navier–Stokes equations from Equation (1) for the right-hand side f_{k+1}, which is an element of $\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega)^3)$ for all $q,r \in (1,\infty)$. For the first two terms in the definition of f_{k+1}, this follows from the assumptions and from Equation (35). For the term $\sqrt{D_t}^{k+1}\operatorname{div}(u\otimes u)$, we distinguish two cases.

If $k = 2N-1$ is an odd number, then this term is an element of $\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))$ if and only if $\partial_i\partial_t^N(u\otimes u)$ is an element of $\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))$ for $i = 1,2,3$. We write

$$\partial_i\partial_t^N(u\otimes u) = \sum_{\ell=0}^N \partial_i\partial_t^\ell u\otimes\partial_t^{N-\ell}u.$$

We can estimate the terms of this sum as

$$\begin{aligned}\|\partial_i u\otimes\partial_t^N u\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))} &\leq \|\nabla u\|_{\mathrm{L}^\infty(\mathbb{T};\mathrm{L}^\infty(\Omega))}\|\tilde{w}_{k+1}\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))}, \\ \|\partial_i\partial_t^N u\otimes u\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))} &\leq \|\nabla\tilde{w}_{k+1}\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))}\|u\|_{\mathrm{L}^\infty(\mathbb{T};\mathrm{L}^\infty(\Omega))}, \\ \|\partial_i\partial_t^\ell u\otimes\partial_t^{N-\ell}u\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))} &\leq \|\nabla\tilde{w}_{2\ell}\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))}\|\tilde{w}_{k+1-2\ell}\|_{\mathrm{L}^\infty(\mathbb{T};\mathrm{L}^\infty(\Omega))},\end{aligned}$$

for $\ell = 1,\dots,N-1$, where the respective right-hand side is finite due to Equations (7)–(9) and the embedding Theorem 3 as well as Equation (35) for $j \leq k$. If $k = 2N$ is an even number, then $\sqrt{D_t}^{k+1}\operatorname{div}(u\otimes u) \in \mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega)^3)$ if and only if this is true for

$$\sqrt{D_t}\partial_t^N\operatorname{div}(u\otimes u) = \sum_{\ell=0}^N \sqrt{D_t}\operatorname{div}\big(\partial_t^\ell u\otimes\partial_t^{N-\ell}u\big).$$

By Theorem 3, this is the case if $\partial_t^\ell u \otimes \partial_t^{N-\ell} u \in \mathrm{W}^{1,r}(\mathbb{T};\mathrm{L}^q(\Omega)) \cap \mathrm{L}^r(\mathbb{T};\mathrm{W}^{2,q}(\Omega))$ for $\ell = 0,\dots,N$. For example, for the terms with derivatives of highest order we obtain

$$\begin{aligned}
\|u \otimes \partial_t^{N+1} u\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))} &\leq \|u\|_{\mathrm{L}^\infty(\mathbb{T};\mathrm{L}^\infty(\Omega))} \|\partial_t \tilde{w}_k\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))}, \\
\|\partial_t^\ell u \otimes \partial_t^{N+1-\ell} u\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))} &\leq \|\tilde{w}_{2\ell}\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))} \|\partial_t \tilde{w}_{k-2\ell}\|_{\mathrm{L}^\infty(\mathbb{T};\mathrm{L}^\infty(\Omega))}, \\
\|u \otimes \partial_t^N \nabla^2 u\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))} &\leq \|u\|_{\mathrm{L}^\infty(\mathbb{T};\mathrm{L}^\infty(\Omega))} \|\nabla^2 \tilde{w}_k\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))}, \\
\|\partial_t^N u \otimes \nabla^2 u\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))} &\leq \|\tilde{w}_k\|_{\mathrm{L}^\infty(\mathbb{T};\mathrm{L}^\infty(\Omega))} \|\nabla^2 u\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))}, \\
\|\partial_t^\ell u \otimes \partial_t^{N-\ell} \nabla^2 u\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))} &\leq \|\tilde{w}_{2\ell}\|_{\mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega))} \|\nabla^2 \tilde{w}_{k-2\ell}\|_{\mathrm{L}^\infty(\mathbb{T};\mathrm{L}^\infty(\Omega))},
\end{aligned}$$

which are all finite by the same argument as above. Similarly, this follows for the lower-order terms.

In summary, we obtain $f_{k+1} \in \mathrm{L}^r(\mathbb{T};\mathrm{L}^q(\Omega)^3)$ for all $q,r \in (1,\infty)$ in both cases. By Equation (35), the function $\tilde{w}_{k+1}$ is subject to both regularity assumptions from Equations (3) and (4), and Theorem 2 implies that $\tilde{w}_{k+1} = \mathcal{P}_\perp \tilde{w}_{k+1}$ satisfies Equation (34) for $j = k+1$. We thus have shown Equation (34) for all $j \in \mathbb{N}_0$.

To increase the spatial regularity, we recall that $(u,\mathfrak{p})$ is a strong solution by Theorem 2, so that the N-th time derivative, $N \in \mathbb{N}_0$, satisfies the Stokes system

$$\begin{aligned}
-\Delta \partial_t^N u + \nabla \partial_t^N \mathfrak{p} &= F_N && \text{in } \Omega, \\
\operatorname{div} \partial_t^N u &= 0 && \text{in } \Omega, \\
\partial_t^N u &= \partial_t^N u_* && \text{on } \partial\Omega
\end{aligned}$$

a.e. in $\mathbb{T}$, where

$$F_N := \partial_t^N f - \partial_t^{N+1} u + \tau \partial_1 \partial_t^N u - \partial_t^N (u \cdot \nabla u).$$

Since $\partial_t^\ell u = \partial_t^\ell w$ for $\ell \geq 1$, Theorem 2 and Equation (34) imply $F_N \in \mathrm{L}^r(\mathbb{T};\mathrm{W}^{1,q}(\Omega_R)^3)$ for all $q,r \in (1,\infty)$ and all $R > 0$ such that $\partial\Omega \subset B_R$, where we define $\Omega_R := \Omega \cap B_R$, and $B_R \subset \mathbb{R}^3$ is the ball with radius R and centered at $0 \in \mathbb{R}^3$. By a classical regularity result for the steady-state Stokes problem (see [25] [Theorem IV.5.1] for example), we obtain $\partial_t^N u \in \mathrm{L}^r(\mathbb{T};\mathrm{W}^{3,q}(\Omega_R))$ for all $R > 0$ sufficiently large and all $N \in \mathbb{N}_0$. This implies $F_N \in \mathrm{L}^r(\mathbb{T};\mathrm{W}^{2,q}(\Omega_R))$, and can again apply ([25] [Theorem IV.5.1]) to deduce $\partial_t^N u \in \mathrm{L}^r(\mathbb{T};\mathrm{W}^{4,q}(\Omega_R))$. Iterating this argument, we finally obtain

$$u \in \mathrm{W}^{N,r}(\mathbb{T};\mathrm{W}^{M,q}(\Omega_R))$$

for all $N,M \in \mathbb{N}_0$, all $q,r \in (1,\infty)$ and all $R > 0$ such that $\partial\Omega \subset B_R$. This completes the proof. $\square$

6. Conclusions and Outlook

As the main result, this paper contains new regularity criteria for time-periodic weak solutions to the Navier–Stokes equations with a non-zero drift term in exterior domains. These criteria are given in the form of a Serrin-type condition on the purely periodic part $\mathcal{P}_\perp u$ of the velocity field u or its gradient, but they do not involve the steady-state part $\mathcal{P}u$. This is a severe difference to known regularity results for the initial-value problem. Moreover, this article generalizes the regularity criterion used in [12–14], so that the results on the asymptotic behavior of time-periodic solutions also hold under more general assumptions.

A natural question for further research would be whether the conditions Equations (3) and (4) can be extended to the critical case, that is, if one can still obtain smoothness of weak solutions if the strict inequalities in Equations (3) and (4) are replaced with equalities. In this case, the presented proof is not applicable, but analogous results are well known for the initial-value problem.

Moreover, the present article focuses on the case $\tau > 0$, that is, the flow around a translating body. In the case $\tau = 0$, corresponding to the flow around a body at rest, the above bootstrap argument cannot be employed since the decay properties of the velocity field are worse, similarly to the properties of the time-independent problem (see [25] [Ch. X]). Moreover, as mentioned above, time-periodic solutions cannot be expected to have finite kinetics and thus cannot be identified with Leray–Hopf weak solutions to the initial-value problem. Therefore, it remains an open question how to establish regularity criteria such as Equation (3) or Equation (4) in the case $\tau = 0$.

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