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Extending the Applicability of Highly Efficient Iterative Methods for Nonlinear Equations and Their Applications

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Abstract: Numerous three-step methods of high convergence order have been developed to produce sequences approximating solutions of equations usually defined on the Euclidean space with a finite dimension. The local convergence order is determined by Taylor expansions requiring the existence of derivatives that are not present on the methods. The more interesting semi-local convergence analysis for these methods has not been considered before. The semi-local is also provided based on generalized ω-continuity conditions on the derivative of the operator involved and the majorizing sequences, thus limiting their usage to only solving equations with operators that are many times differentiable. However, these methods may convergence to a solution of the equation even if these high-order derivatives do not exist. That is why a methodology is utilized on two sixth convergence order methods and in the more general setting of a Banach space. This time, the convergence depends only on the operators and the first derivative on the method. Therefore, by this methodology the applicability of the methods is in the extended area. Although this methodology is demonstrated on two competing and efficient methods, it can also be utilized for the same reasons on other methods involving inverses of operators that are linear. This is the motivation and novelty of the paper. The numerical applications further validate the theoretical results both in the local as well as the semi-local convergence case.

Keywords: three-step method; banach space; convergence order

MSC: 49M15; 65G99; 65H10

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1. Introduction

In this paper, we are concerned with the problem of approximating a solution x_* of the nonlinear equation

$$F(x) = 0. \tag{1}$$

Here, $F : \Omega \subset \mathbb{B}_0 \to \mathbb{B}$ stands for a differentiable operator in the Fréchet sense, $\mathbb{B}_0$ and $\mathbb{B}$ denote Banach spaces, and $\Omega \neq \phi$ is a convex and open set. The analytical form of the solution x_* is possible only in some special cases. That is why researchers resort to the development of iterative methods generating a sequence convergent to x_* under some conditions on the initial data. A popular example of one such method is Newton's method [1–5]. However, the convergence order of this method is two. In order to increase the order of convergence of Newton's method, a plethora of single and multi-step methods have been developed [6–11].

In particular, we study the three-step methods of convergence order six proposed by Sharma and Parhi [9] and Behl et al. [12], which are given below, respectively:

$$
\begin{aligned}
y_n &= x_n - F'(x_n)^{-1}F(x_n)\\
z_n &= x_n - 2\Big(F'(x_n) + F'(y_n)\Big)^{-1}F(x_n)\\
x_{n+1} &= z_n - \left(\frac{7}{2}I - 4F'(x_n)^{-1}F'(y_n) + \frac{3}{2}\Big(F'(x_n)^{-1}F'(y_n)\Big)^2\right)F'(x_n)^{-1}F(z_n),
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n)\\
z_n &= x_n - \left(\frac{5}{8}I + \frac{3}{8}\Big(F'(y_n)^{-1}F'(x_n)\Big)^2\right)F'(x_n)^{-1}F(x_n),\\
x_{n+1} &= z_n - 2a_n^{-1}\Big(F'(x_n) + \alpha F'(y_n)\Big)F'(x_n)^{-1}F(z_n),
\end{aligned} \tag{3}
$$

where

$$a_n = -3(1+\alpha)F'(x_n) + (5\alpha+3)F'(y_n),\ \alpha \in \mathbb{R}.$$

Notice that both methods (2) and (3) are adopting the same number of functional evaluations, e.g. two functions, two first derivatives, and two linear operator inversions. **The motivation for writing this paper**: the convergence order was shown in [12] using the seventh-order derivatives that do not appear in the method, thus limiting the applicability in the special case when $\mathbb{B}_0 = \mathbb{B} = \mathbb{R}^m$.

As a motivational and simple example, define the function F on $\mathbb{B}_0 = \mathbb{B} = \mathbb{R}$, $\Omega = [-\frac{1}{\pi}, \frac{2}{\pi}]$ by

$$F(\tau) = \begin{cases} 2\tau^5 - 3\tau^4 + 6\tau^3 \log t, & \tau \neq 0 \\ 0, & x = 0 \end{cases}.$$

Then, the first three derivatives are

$$
\begin{aligned}
&F'(\tau) = 10\tau^4 - 12\tau^3 + 6\tau^2 + 18\tau^2\log(\tau),\\
&F''(\tau) = 40\tau^3 - 36\tau^2 + 30\tau + 36\tau\log(\tau),\\
&\text{and}\\
&F'''(\tau) = 120\tau^2 - 72\tau + 36\log(\tau) + 66.
\end{aligned}
$$

Then, one can easily find that the function $F'''(\tau)$ is unbounded on Ω at the point $\tau = 0$. Hence, the local convergence results in [12] cannot show the convergence of methods (2) and (3) or their special cases utilizing hypotheses on the seventh derivative of function F or higher. However, these derivatives are not on the methods (2) and (3). There are other problems with the study of these methods. As an example, there are no computable error estimates on the distances $\|x_* - x_n\|$ that can be determined. Moreover, there are no results concerning the uniqueness of the solution ball. Notice that, in-particular, there is a plethora of iterative methods for approximating the solutions of nonlinear equations [13–17], which cause the same concerns.

The novelty of the paper: we address these concerns in the more general setting of Banach spaces. In particular, the applicability of methods (2) and (3) is extended using only the first derivative, which appears on these methods. Moreover, the computational order of convergence (COC) [6] or approximate computational order of convergence (ACOC) [6] are used for the derivation of the convergence order. These computational orders are found using only the operator F', which only appears on the methods. Furthermore, the upper bounds on the distances are provided based on ω-continuity conditions. The uniqueness of the solution ball is also determined.

Our technique can be utilized to extend the usage of other methods of linear operators analogously [18–22]. That will be the topic of future research.

We present the local convergence analysis in Sections 2 and 3. The application in Section 4 validates the theoretical study. The conclusions can be seen in the concluding Section 5.

2. Local Analysis

In this section, the local convergence analysis utilizes real parameters and functions. Set $A = [0, \infty)$.

Suppose equation
$$\Delta_0(\tau) - 1 = 0 \tag{4}$$
has a minimal solution $R_0 \in A - \{0\}$ for some non-decreasing and continuous function $\Delta_0 : A \to A$. Set $A_0 = [0, R_0)$.

Suppose equation
$$g_1(\tau) - 1 = 0 \tag{5}$$
has a minimal solution $\rho_1 \in A_0 - \{0\}$ for some non-decreasing and continuous functions $\Delta : A_0 \to A$, where $g_1 : A_0 \to A$ is defined by
$$g_1(\tau) = \frac{\int_0^1 \Delta\Big((1-\theta)\tau\Big)d\theta}{1 - \Delta_0(\tau)}.$$

Suppose equation
$$p(\tau) - 1 = 0 \tag{6}$$
has a minimal solution $R_1 \in A_0 - \{0\}$, where $p : A_0 \to A$ is defined by
$$p(\tau) = \frac{1}{2}\Big[\Delta_0(\tau) + \Delta_0\Big(g_1(\tau)\tau\Big)\Big].$$

Set $R_2 = \min\{R_0, R_1\}$ and $A_1 = [0, R_2)$.

Suppose equation
$$g_2(\tau) - 1 = 0 \tag{7}$$
has a minimal solution $\rho_2 \in A_1 - \{0\}$, where $g_2 : A_1 \to A$ is defined by
$$g_2(\tau) = g_1(\tau) + \frac{\Bigg(\Delta_0(\tau) + \Delta_0\Big(g_1(\tau)\tau\Big)\Bigg)\int_0^1 \Delta_1(\theta\tau)d\theta}{2\Big(1 - \Delta_0(\tau)\Big)\Big(1 - p(\tau)\Big)}.$$

Suppose equation
$$\Delta_0\Big(g_2(\tau)\tau\Big) - 1 = 0 \tag{8}$$
has a minimal solution $R_3 \in A_1 - \{0\}$.

Set $R = \min\{R_2, R_3\}$ and $A_2 = [0, R)$.

Suppose equation
$$g_3(\tau) - 1 = 0 \tag{9}$$
has minimal solution $\rho_3 \in A_2 - \{0\}$, where

$$g_3(\tau) = \left[g_1(g_2(\tau)\tau) + \frac{\left(\Delta_0(\tau) + \Delta_0\left(g_2(\tau)\tau\right)\right)\int_0^1 \Delta_1(\theta g_2(\tau)\tau)d\theta}{\left(1-\Delta_0(\tau)\right)\left(1-\Delta_0(g_2(\tau)\tau)\right)} \right.$$
$$+ \frac{1}{2}\left(3\left(\frac{\Delta_0(\tau)+\Delta_0\left(g_1(\tau)\tau\right)}{1-\Delta_0(\tau)}\right)^2 + \left(\frac{\Delta_0(\tau)+\Delta_0\left(g_1(\tau)\tau\right)}{1-\Delta_0(\tau)}\right)\right)$$
$$\left. \times \frac{\int_0^1 \Delta_1\left(\theta g_2(\tau)\tau\right)d\theta}{1-\Delta_0(\tau)} \right] g_2(\tau).$$

Next, we prove

$$\rho = \min\{\rho_i\},\ i = 1,2,3, \tag{10}$$

is a possible convergence radius of method (2).

Set $T = [0,\rho)$. It follows by (10) that for each $\tau \in T$,

$$0 \le \Delta_0(\tau) < 1, \tag{11}$$

$$0 \le \Delta_0(g_1(\tau)\tau) < 1, \tag{12}$$

$$0 \le \Delta_0(g_2(\tau)\tau) < 1, \tag{13}$$

$$0 \le p(\tau) < 1 \tag{14}$$

and

$$0 \le g_i(\tau) < 1. \tag{15}$$

We shall use the notations $U(x_*,\delta), \bar{U}(x_*,\delta)$ for the open ball in and its closure, respectively.

The main local result uses conditions (H) with the "w" functions as previously defined. Assume:

(H_1) $F : \Omega \to \mathbb{B}$ is differentiable and x_* is a solution of the Equation (1), such that $F'(x_*)^{-1} \in \ell(\mathbb{B},\mathbb{B}_0)$.

(H_2) $\left\|F'(x_*)^{-1}\left(F'(x) - F'(x_*)\right)\right\| \le \Delta_0(\|x - x_*\|)$ for each $x \in \Omega$. Set $\Omega_0 = \Omega \cap U(x_*, R_0)$.

(H_3) $\left\|F'(x_*)^{-1}\left(F'(x) - F'(y)\right)\right\| \le \Delta(\|x - y\|)$ and $\|F'(x_*)^{-1}F'(x)\| \le \Delta_1(\|x - x_*\|)$ for each $x, y \in \Omega_0$.

(H_4) $\bar{U}(x_*,\rho) \subset \Omega$ for some ρ_* to be determined and

(H_5) $\beta \ge \rho$ exist, satisfying

$$\int_0^1 \Delta_0(\theta\beta)d\theta < 1.$$

Set $\Omega_1 = \Omega \cap \bar{U}(x_*,\beta)$.

The main local convergence result follows next using the preceding notations with the conditions (H).

Theorem 1. *Under the conditions (H_1)–(H_5) for $\rho_* = \rho$, further pick the starting point $x_0 \in U(x_*,\rho) - \{x_*\}$. Then, the following hold for method (2):*

$$\{x_n\} \subset U(x_*,\rho), \quad \lim_{n\to\infty} x_n = x_*, \tag{16}$$

$$\|y_n - x_*\| \le g_1(\|x_n - x_*\|)\|x_n - x_*\| \le \|x_n - x_*\| < \rho, \tag{17}$$

$$\|z_n - x_*\| \le g_2(\|x_n - x_*\|)\|x_n - x_*\| \le \|x_n - x_*\|, \tag{18}$$

$$\|x_{n+1} - x_*\| \le g_3(\|x_n - x_*\|)\|x_n - x_*\| \le \|x_n - x_*\|, \tag{19}$$

with the radius ρ defined by (10) and function g_i as defined previously. Moreover, the only solution of Equation (1) in the set Ω_1 is x_*.

Proof. Estimations (16)–(19) are shown by induction on integer k. By (H_1), (H_2), (10), (11), and $c \in U(x_*, \rho)$

$$\begin{aligned} \left\| F'(x_*)^{-1}\Big(F'(c) - F(x_*)\Big)\right\| \leq & \Delta_0(\|c - x_*\|) \\ \leq & \Delta_0(\rho) < 1. \end{aligned} \tag{20}$$

Using (20) and the lemma due to Banach on linear invertible operators [2,4,13], we deduce $F'(c)^{-1}F'(x_*) \in \ell(\mathbb{B}, \mathbb{B}_0)$, and

$$\left\| F'(c)^{-1}F(x_*)\right\| \leq \frac{1}{\Delta_0(\|c - x_*\|)}. \tag{21}$$

We also have by (21) (for $c = x_0$) and method (2) for $n = 0$ that y_0 exists, and we can write

$$\begin{aligned} y_0 - x_* =& x_0 - x_* - F'(x_0)^{-1}F(x_0) \\ =& \Big(F'(x_0)^{-1}F'(x_*)\Big)\left[\int_0^1 F'(x_0)^{-1}\Big(F'(x_* + \theta(x_0 - x_*)) - F'(x_0)\Big)d\theta(x_0 - x_*)\right] \end{aligned} \tag{22}$$

Using (10), (15) (for $i = 1$), (21) (for $c = x_0$), (H_3), and (22), we obtain

$$\begin{aligned} \|y_0 - x_*\| =& \frac{\int_0^1 \Delta\Big((1-\theta)\|x_0 - x_*\|\Big)d\theta\|x_0 - x_*\|}{1 - \Delta_0(\|x_0 - x_*\|)} \\ \leq & g_1(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < \rho, \end{aligned} \tag{23}$$

proving that the iterate $y_0 \in U(x_*, \rho)$ and (17) for $n = 0$. The linear operator $\Big(F'(x_0) + F'(y_0)\Big)^{-1} \in \ell(\mathbb{B}, \mathbb{B}_0)$.

Indeed, using (10), (14), (H_2), and (23), we have in turn that

$$\begin{aligned} & \left\|\Big(2F'(x_*)\Big)^{-1}\left[\Big(F'(x_0) - F'(x_*)\Big) + \Big(F'(y_0) - F'(x_*)\Big)\right]\right\| \\ & \leq \frac{1}{2}\left[\left\|F'(x_*)^{-1}\Big(F'(x_0) - F'(x_*)\Big)\right\| + \left\|F'(x_*)^{-1}\Big(F'(y_0) - F'(x_*)\Big)\right\|\right] \\ & \leq p(\|x_0 - x_*\|) \leq p(\rho) < 1. \end{aligned} \tag{24}$$

Thus, we obtain

$$\left\|\Big(F'(x_0) + F'(y_0)\Big)^{-1}F'(x_*)\right\| \leq \frac{1}{2\Big(1 - p(\|x_0 - x_*\|)\Big)}. \tag{25}$$

Then, the iterate z_0 exists, and we can write

$$\begin{aligned} z_0 - x_* =& x_0 - x_* - F'(x_0)^{-1}F(x_0) + \left[F'(x_0)^{-1} - 2\Big(F'(x_0) + F'(y_0)\Big)^{-1}\right]F(x_0) \\ &= x_0 - x_* - F'(x_0)^{-1}F(x_0) \\ &+ F'(x_0)^{-1}\Big(F'(y_0) - F'(x_0)\Big)\Big(F'(x_0) + F'(y_0)\Big)^{-1}F(x_0). \end{aligned} \tag{26}$$

Using (10), (15) (for $i = 2$), (21) (for $c = x_0$), (23), (25) and (26), we have

$$\begin{aligned}\|z_0 - x_*\| &\leq \left[g_1(\|x_0 - x_*\|) + \frac{\Big(\Delta_0(\|x_0 - x_*\|) + \Delta_0(\|y_0 - x_*\|)\Big)\int_0^1 \Delta_1\Big(\theta\|x_0 - x_*\|\Big)d\theta}{2\Big(1 - \Delta_0(\|x_0 - x_*\|)\Big)\Big(1 - p(\|x_0 - x_*\|)\Big)}\right]\|x_0 - x_*\| \\ &\leq g_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|,\end{aligned} \tag{27}$$

proving that the iterate $z_0 \in U(x_*, \rho)$ and (18) for $n = 0$.

Notice that x_1 is well defined by the third substep of the method (2), and $F'(z_0)^{-1} \in \ell(\mathbb{B}, \mathbb{B}_0)$ by (21) for $c = z_0$.

Moreover, the third substep of the method (2) gives

$$\begin{aligned}x_1 - x_* &= z_0 - x_* - F'(z_0)^{-1}F(z_0) + \Big(F'(z_0)^{-1} - F'(x_0)^{-1}\Big)F(z_0) \\ &\quad - \frac{1}{2}\left[5I - 8F'(x_0)^{-1}F'(y_0) + 3\Big(F'(x_0)^{-1}F'(y_0)\Big)^2\right]F'(x_0)^{-1}F(z_0) \\ &= z_0 - x_* - F'(z_0)^{-1}F(z_0) + F'(z_0)^{-1}\Big(F'(x_0) - F'(z_0)\Big)F'(x_0)^{-1}F(z_0) \\ &\quad - \frac{1}{2}\left[3\Big(F'(x_0)^{-1}\Big(F'(y_0) - F'(x_0)\Big)\Big)^2 - 2F'(x_0)^{-1}\Big(F'(y_0) - F'(x_0)\Big)\right]F'(x_0)^{-1}F(z_0).\end{aligned} \tag{28}$$

In view of (10), (15) (for $i = 3$), (21) (for $c = x_0, z_0$), (23), (27) and (28), we obtain

$$\begin{aligned}\|x_1 - x_*\| &= \left[g_1(\|z_0 - x_*\|) + \frac{\Big(\Delta_0(\|z_0 - x_*\|) + \Delta_0(\|x_0 - x_*\|)\Big)\int_0^1 \Delta_1(\theta\|z_0 - x_*\|)d\theta}{\Big(1 - \Delta_0(\|z_0 - x_*\|)\Big)\Big(1 - \Delta_0(\|x_0 - x_*\|)\Big)}\right. \\ &\quad + \frac{1}{2}\left(3\left(\frac{\Delta_0(\|x_0 - x_*\|) + \Delta_0(\|y_0 - x_*\|)}{1 - \Delta_0(\|x_0 - x_*\|)}\right)^2\right. \\ &\quad \left.\left. + 2\left(\frac{\Delta_0(\|x_0 - x_*\|) + \Delta_0(\|y_0 - x_*\|)}{1 - \Delta_0(\|x_0 - x_*\|)}\right)\right) \times \frac{\int_0^1 \Delta_1\Big(\theta\|z_0 - x_*\|\Big)d\theta}{1 - \Delta_0(\|x_0 - x_*\|)}\right]\|z_0 - x_*\| \\ &\leq g_3(\|x_0 - x_*\|)\|x_0 - x_*\|\|x_0 - x_*\| \leq \|x_0 - x_*\|,\end{aligned} \tag{29}$$

proving that the iterate $x_1 \in U(x_*, \rho)$ and (19) for $n = 0$.

Simply, switch x_k, y_k, z_k, x_{k+1} with x_0, y_0, z_0, x_1 in the preceding calculations; we terminate the induction for estimations (16)–(19). It then follows by the calculation

$$\|x_{k+1} - x_*\| \leq \xi\|x_k - x_*\| < \rho, \tag{30}$$

where $c = g_3(\|x_0 - x_*\|) \in [0, 1)$ that $x_{k+1} \in U(x_*, \rho)$, and $\lim_{k\to\infty} x_k = x_*$. Set $M = \int_0^1 F'\Big(x_* + \theta(v - x_*)\Big)d\theta$. Then, by (H_2) and (H_5)

$$\begin{aligned}\left\|F'(x_*)^{-1}\Big(M - F'(x_*)\Big)\right\| &\leq \int_0^1 \Delta_0\Big(\theta\|v - x_*\|\Big)d\theta \\ &\leq \int_0^1 \Delta_0(\theta\beta)d\theta < 1.\end{aligned}$$

Therefore, it follows that $v = x_*$ is concluded from the identity $0 = F(v) = F(x_*) = M(v - x_*)$ and the invertability of linear operator M. $\square$

Remark 1. *Next, the local convergence analysis is developed for method (3) in an analogous way. This time, the functions $\bar{g}_i$ are defined (for $\alpha \neq -1$), respectively, by*

$$
\begin{aligned}
\bar{g}_1(\tau) =& \frac{\int_0^1 \Delta\Big((1-\theta)\tau\Big)d\theta + \frac{1}{3}\int_0^1 \Delta_1(\theta\tau)d\theta}{1-\Delta_0(\tau)},\\
\bar{g}_2(\tau) =& g_1(\tau) + \frac{3}{8}\frac{\Big(\Delta_0(\bar{g}_1(\tau)\tau)+\Delta_0(\tau)\Big)\Big(2+\Delta_0(\tau)+\Delta_0(\bar{g}_1(\tau)\tau)\Big)\int_0^1 \Delta_1(\theta\tau)d\theta}{\Big(1-\Delta_0(\bar{g}_1(\tau)\tau)\Big)^2\Big(1-\Delta_0(\tau)\Big)},\\
\bar{g}_3(\tau) =& g_1\Big(\bar{g}_2(\tau)\Big) + \frac{\Big(\Delta_0(\tau)+\Delta_0(\bar{g}_2(\tau)\tau)\Big)\int_0^1 \Delta_1\Big(\theta\bar{g}_2(\tau)\tau\Big)d\theta}{\Big(1-\Delta_0(\tau)\Big)\Big(1-\Delta_0(\bar{g}_2(\tau)\tau)\Big)}\\
&+ \frac{3|1+\alpha|\left(\Delta_0(\tau)+\Delta_0\Big(\bar{g}_2(\tau)\tau\Big)\right)\int_0^1 \Delta_1\Big(\theta\bar{g}_2(\tau)\tau\Big)d\theta}{2|1+\alpha|\Big(1-q(\tau)\Big)\Big(1-\Delta_0(\tau)\Big)},
\end{aligned}
$$

where $q(\tau) = \frac{1}{2|1+\alpha|}\Big[|1+3\alpha|\Delta_0(\tau) + |5\alpha+3|\Delta_0\Big(\bar{g}_1(\tau)\tau\Big)\Big]$ and $\bar{\rho}_i$ are the least positive solutions of the equations (assuming that they exist),

$$\bar{g}_i(\tau) - 1 = 0.$$

Set $\rho = \min\{\bar{\rho}_i\}$. We need the estimates

$$y_0 - x_* = x_0 - x_* - F'(x_0)^{-1}F(x_0) + \frac{1}{3}F'(x_0)^{-1}F(x_0),$$

so

$$
\begin{aligned}
\|y_0 - x_*\| =& \frac{\Big[\int_0^1 \Delta\Big((1-\theta)\|x_0-x_*\|\Big)d\theta + \frac{1}{3}\int_0^1 \Delta_1(\theta\|x_0-x_*\|)d\theta\Big]\|x_0-x_*\|}{1-\Delta_0(\|x_0-x_*\|)}\\
\leq& \bar{g}_1(\|x_0-x_*\|)\|x_0-x_*\| \leq \bar{\rho},\\
z_0 - x_* =& x_0 - x_* - F'(x_0)^{-1}F(x_0) - \left[-\frac{3}{8}I + \Big(F'(y_0)^{-1}F'(x_0)\Big)^2\right]F'(x_0)^{-1}F(x_0)\\
&= x_0 - x_* - F'(x_0)^{-1}F(x_0)\\
&+ \frac{3}{8}\Big(I - F'(y_0)^{-1}F'(x_0)\Big)\Big(I + F'(y_0)^{-1}F'(x_0)\Big)F'(x_0)^{-1}F(x_0)\\
&= x_0 - x_* - F'(x_0)^{-1}F(x_0)\\
&+ \frac{3}{8}F'(y_0)^{-1}\Big(F'(y_0) - F'(x_0)\Big)F'(y_0)^{-1}\Bigg[\Big(F'(y_0) - F'(x_*)\Big)\\
&+ \Big(F'(x_0) - F'(x_*)\Big) + 2F'(x_*)\Bigg]F'(x_0)^{-1}F(x_0).
\end{aligned}
$$

Hence, we attain

$$
\begin{aligned}
\|z_0 - x_*\| =&\Bigg[g_1(\|x_0 - x_*\|) \\
&+\frac{3}{8}\frac{\Big(\Delta_0(\|y_0 - x_*\|) + \Delta_0(\|x_0 - x_*\|)\Big)\Big(2 + \Delta_0(\|x_0 - x_*\|) + \Delta_0(\|y_0 - x_*\|)\Big)}{\Big(1 - \Delta_0(\|y_0 - x_*\|)\Big)^2\Big(1 - \Delta_0(\|x_0 - x_*\|)\Big)} \\
&\times \int_0^1 \Delta_1(\theta\|x_0 - x_*\|)d\theta\Bigg]\|x_0 - x_*\|, \\
&\leq \bar{g}_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|,
\end{aligned}
$$

and

$$
\begin{aligned}
x_1 - x_* =& z_0 - x_* - F'(z_0)^{-1}F(z_0) + \Big(F'(z_0)^{-1} - F'(x_0)^{-1}\Big)F(z_0) \\
&+ \Big[I - 2a_0^{-1}\Big(F'(x_0) + \alpha F'(y_0)\Big)\Big]F'(x_0)^{-1}F(z_0) \\
=& \Big(z_0 - x_* - F'(z_0)^{-1}F(z_0)\Big) + F'(z_0)^{-1}\Big(F'(x_0) - F'(z_0)\Big)F'(x_0)^{-1}F(z_0) \\
&+ a_0^{-1}\Big[a_0 - 2\Big(F'(x_0) + \alpha F'(y_0)\Big)\Big]F'(x_0)^{-1}F(z_0),
\end{aligned}
$$

which further yields

$$
\begin{aligned}
\|x_1 - x_*\| =&\Bigg[g_1(\|z_0 - x_*\|) \\
&+ \frac{\Big(\Delta_0(\|x_0 - x_*\|) + \Delta_0(\|z_0 - x_*\|)\Big)\int_0^1 \Delta_1\Big(\theta\|z_0 - x_*\|\Big)d\theta}{\Big(1 - \Delta_0(\|x_0 - x_*\|)\Big)\Big(1 - \Delta_0(\|z_0 - x_*\|)\Big)} \\
&+ \frac{\Big(|4 + 3\alpha|\Delta_0(\|x_0 - x_*\|) + |4\alpha + 3|\Delta_0(\|y_0 - x_*\|) + 7|1 + \alpha|\Big)}{2|\alpha|\Big(1 - q(\|x_0 - x_*\|)\Big)\Big(1 - \Delta_0(\|x_0 - x_*\|)\Big)} \\
&\times \int_0^1 \Delta_1\Big(\theta\|z_0 - x_*\|\Big)d\theta\Bigg]\|z_0 - x_*\|, \\
&\leq \bar{g}_3(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|,
\end{aligned}
$$

where we also used

$$
\begin{aligned}
\left\|\Big(2(1+\alpha)F'(x_*)\Big)^{-1}\Big(a_0 - 2(1+\alpha)F'(x_*)\Big)\right\| =& \frac{1}{2|1+\alpha|}\Bigg[|1 + 3\alpha|\left\|F'(x_*)^{-1}\Big(F'(x_0) - F'(x_*)\Big)\right\| \\
&+ |5\alpha + 3|\left\|F'(x_*)^{-1}\Big(F'(y_0) - F'(x_*)\Big)\right\|\Bigg] \\
\leq& q(\|x_0 - x_*\|) < q(\bar{\rho}) < 1,
\end{aligned}
$$

Therefore, we obtain

$$
\left\|a_0^{-1}F'(x_*)\right\| \leq \frac{1}{2|1+\alpha|\Big(1 - q(\|x_0 - x_*\|)\Big)}.
$$

Hence, we arrived at the corresponding local result for method (3).

Theorem 2. *Under the conditions* (H) *for* $\rho_* = \bar{\rho}$*, further pick the starting point* $x_0 \in U(x_*, \rho) - \{x_*\}$*. Then, the conclusions of Theorem 1 hold with* $\bar{\rho}$ *and* $\bar{g}_i$ *being replaced by* ρ *and* g_i*, respectively.*

3. Semi-Local Analysis

The idea of a majorizing sequence is applied to first show the convergence of the method (2). Let $d \geq 0$ be a given parameter. Define the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ as follows for each $n = 0, 1, 2, \ldots$, $a_0, b_0 = d$,

$$\begin{aligned} c_n =& b_n + \frac{\bar{\Delta}_n (b_n - a_n)}{2(1 - p(a_n))}, \\ a_{n+1} =& c_n + \frac{1}{2}\left[6\left(\frac{\bar{\Delta}_n}{1 - \Delta_0(a_n)}\right)^2 + \frac{4\bar{\Delta}_n}{1 - \Delta_0(a_n)} + 5\right]\frac{\delta_n}{1 - \Delta_0(a_n)}, \\ b_{n+1} =& a_{n+1} + \frac{\delta^1_{n+1}}{1 - \Delta_0(a_{n+1})}, \end{aligned} \tag{31}$$

where $p(a_n) = \frac{1}{2}\Big(\Delta_0(a_n) + \Delta_0(b_n)\Big)$,

$$\bar{\Delta}_n = \begin{cases} \Delta_0(a_n) + \Delta_0(b_n) \\ \text{or} \\ \Delta(b_n - a_n), \end{cases}$$

$$\delta_n = \left(1 + \int_0^1 \Delta_0\Big(\theta(c_n - a_n)\Big)\right)(c_n - a_n) + \Big(1 + \Delta_0(a_n)\Big)(b_n - a_n)$$

and

$$\delta^1_{n+1} = \int_0^1 \Delta_0\Big((1 - \theta)(a_{n+1} - a_)\Big)(a_{n+1} - a_n) + \Big(1 + \Delta_0(a_n)\Big)(a_{n+1} - b_n),$$

where the functions "w" have the same properties as the "Δ" functions in the semi-local. A general sufficient convergence result is useful.

Lemma 1. *Suppose that for each $n = 0, 1, 2, \ldots$,*

$$p(a_n) < 1, \quad \Delta_0(a_n) < 1 \quad \text{and} \quad a_n \leq \beta, \tag{32}$$

for some parameter $\beta \geq 0$. Then, the sequence $\{a_n\}$ produced by the formula (31) is non-decreasingly convergent to some $a_* \in [0, \beta]$.

Proof. The definition (31) and the conditions (32) imply the conclusion. In particular, a_* is the least upper bound of the sequence $\{a_n\}$, which is unique. $\square$

A relationship is provided between the scalar function and operator F'.
Suppose:

- (E_1) That an element $x_0 \in \Omega$ and a parameter $d \geq$ exist so that $F'(x_0)^{-1} \in \ell(\mathbb{B}, \mathbb{B}_0)$ and $\left\|F'(x_0)^{-1}F(x_0)\right\| \leq d$.
- (E_2) $\left\|F'(x_0)^{-1}\Big(F'(y) - F'(x_0)\Big)\right\| \leq \Delta_0(\|y - x_0\|)$ for each $y \in \Omega$.
- (E_3) The equation $\Delta_0(t) - 1 = 0$ has a minimal positive solution denoted by h. Set $\Omega_2 = U(x_0, h) \cap \Omega$.
- (E_4) $\left\|F'(x_0)^{-1}\Big(F'(y) - F'(\bar{y})\Big)\right\| \leq \Delta(\|y - \bar{y}\|)$ for each $y, \bar{y} \in \Omega_2$.
- (E_5) Condition (32) holds,
  and
- (E_6) $U(x_0, a_*) \subset \Omega$ for some parameter $\bar{t}_*$ with $0 \leq \bar{t}_* \leq h$.

Some Ostrowski-like representations for method (2) are useful.

Lemma 2. *Suppose that the iterates of the method (2) exist for each $n = 0, 1, 2, \cdots$. Then, the following items hold:*

$$z_n - y_n = \Big(F'(x_n) + F'(y_n)\Big)^{-1}\Big(F'(y_n) - F'(x_n)\Big)(y_n - x_n), \tag{33}$$

$$x_{n+1} - z_n = \frac{1}{2}\left[\Big(I - F'(x_n)^{-1}F'(y_n)\Big)^2 - 4(I - F'(x_n)^{-1}F'(y_n)\Big) + 5I\right]F'(x_n)^{-1}F(z_n), \tag{34}$$

$$F(z_n) = \int_0^1 F'\Big(x_n + \theta(z_n - x_n)\Big)d\theta(z_n - x_n) - F'(x_n)(y_n - x_n) \tag{35}$$

and

$$F(z_n) = \int_0^1 \left(F'\Big(x_n + \theta(x_{n+1} - x_n)\Big) - F'(x_n)\right)d\theta(x_{n+1} - x_n) + F'(x_n)(x_{n+1} - y_n). \tag{36}$$

Proof. In view of the first two substeps of the method (2), we have in turn that

$$\begin{aligned} z_n - y_n =& \left(F'(x_n)^{-1} - 2\Big(F'(x_n) + F'(y_n)\Big)^{-1}\right)F(x_n) \\ =& \Big(F'(x_n) + F'(y_n)\Big)^{-1}\left(2F'(x_n) - \Big(F'(x_n) + F'(y_n)\Big)\right)F'(x_n)^{-1}F(x_n) \\ =& \Big(F'(x_n) + F'(y_n)\Big)^{-1}\Big(F'(y_n) - F'(x_n)\Big)(y_n - x_n), \end{aligned}$$

proving identity (33). Moreover, item (34) follows from the third substep of method (2). Furthermore, we can write

$$\begin{aligned} F(z_n) =& F(z_n) - F(x_n) + F(x_n) \\ =& \int_0^1 F'\Big(x_n + \theta(z_n - x_n)\Big)d\theta(z_n - x_n) - F'(x_n)(y_n - x_n). \end{aligned}$$

Finally, from the first substep of the method (2), we have

$$\begin{aligned} F(x_{n+1}) =& F(x_{n+1}) - F(x_n) - F'(x_n)(y_n - x_n) \\ =& F(x_{n+1}) - F(x_n) - F'(x_n)(x_{n+1} - x_n) + F'(x_n)(x_{n+1} - y_n) \\ =& \int_0^1 \left(F'\Big(x_n + \theta(x_{n+1} - x_n)\Big)d\theta - F'(x_n)\right)(x_{n+1} - x_n) + F'(x_n)(x_{n+1} - x_n). \end{aligned}$$

This ends the proof. □

We can prove the semi-local convergence for method (2), with the assistance of conditions (E_1)–(E_5) and Lemma 2.

Theorem 3. *Suppose that the conditions (E_1)–(E_5) hold for $\bar{t}_* = a_*$. Then, there exists $x_* \in U[x_0, a_*]$, which solves the equation $F(x) = 0$.*

Proof. Condition (E_1) and (31) give

$$\|y_0 - x_0\| = \left\|F'(x_0)^{-1}F(x_0)\right\| \leq d = b_0 - a_0 = b_0 \leq a_*,$$

proving that $y_0 \in U[x_0, a_*]$. As in Theorem 1, we obtain by (E_2), (E_3), and (33) that

$$\begin{aligned}\|z_n - y_n\| \leq & \left\|\left(F'(x_n) + F'(y_n)\right)^{-1}F'(x_0)\right\|\left\|F'(x_0)^{-1}\left(F'(y_n) - F'(x_n)\right)\right\|\|y_n - x_n\| \\ = & \frac{\bar{\Delta}_n\|y_n - x_n\|}{2\left(1 - p(a_n)\right)} \leq \frac{\bar{\Delta}_n(b_n - a_n)}{2\left(1 - p(a_n)\right)} = c_n - b_n,\end{aligned}$$

and

$$\|z_n - x_0\| \leq \|z_n - y_n\| + \|y_n - x_0\| \leq c_n - b_n + b_n - a_0 = c_n \leq a_*.$$

Then, by the third substep of the method (2), (E_2), (34), and (35), we have in turn that

$$\begin{aligned}\|x_{n+1} - z_n\| = & \frac{1}{2}\left\|\left[6\left(I - F'(x_n)^{-1}F'(y_n)\right)^2 - 4\left(I - F'(x_n)^{-1}F'(y_n)\right) + 5I\right]\right\| \\ & \times \left\|F'(x_n)^{-1}F(z_n)\right\| \\ \leq & \frac{1}{2}\left[6\left(\frac{\bar{\Delta}_n}{1 - \Delta_0(a_n)}\right)^2 + 4\left(\frac{\bar{\Delta}_n}{1 - \Delta_0(a_n)}\right) + 5\right]\frac{\delta_n}{1 - \Delta_0(a_n)} \\ = & a_{n+1} - c_n,\end{aligned}$$

and

$$\|x_{n+1} - x_0\| = \|x_{n+1} - z_n\| + \|z_n - x_0\| \leq a_{n+1} - c_n + c_n - a_0 = a_{n+1} \leq a_*,$$

where

$$\bar{\Delta}_n = \begin{cases}\Delta_0(a_n) + \Delta_0(b_n) \\ w(b_n - a_n).\end{cases}$$

Then, by (36) and the first substep of the method (2), we obtain

$$\begin{aligned}\|y_{n+1} - x_{n+1}\| = & \left\|F'(x_{n+1})^{-1}F'(x_0)\right\|\left\|F'(x_0)^{-1}F'(x_{n+1})\right\| \\ \leq & \frac{\int_0^1 \Delta\left((1-\theta)\|x_{n+1} - x_n\|\right)d\theta\|x_{n+1} - x_n\| + \left(1 + \Delta_0(\|x_n - x_0\|)\right)\|x_{n+1} - y_n\|}{1 - \Delta_0(\|x_{n+1} - x_0\|)} \\ \leq & \frac{\int_0^1 \Delta\left((1-\theta)(a_{n+1} - a_n)\right)d\theta(a_{n+1} - a_n) + \left(1 + \Delta_0(a_n)\right)(a_{n+1} - b_n)}{1 - \Delta_0(a_{n+1})} \\ = & b_{n+1} - a_{n+1},\end{aligned}$$

and

$$\begin{aligned}\|y_{n+1} - x_0\| = & \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_0\| \\ \leq & b_{n+1} - a_{n+1} + a_{n+1} - a_0 \\ = & b_{n+1} \leq a_*.\end{aligned}$$

Thus, the iterates $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ belong in $U[x_0, a_*]$ and are fundamental since $\{a_n\}$ is also fundamental as convergent. It follows that $x_* \in U[x_0, a_*]$ exists such that $\lim_{n\to\infty} x_n = x_*$. Finally, if $n \to \infty$ the calculation $\left\|F'(x_0)^{-1}F(x_{n+1})\right\| \leq \delta^1_{n+1}$ gives $F(x_*) = 0$ (by the continuity of F). □

A uniqueness domain for the solution results follows.

Proposition 1. *Suppose:*

(i) $\rho_3 \geq 0$ *and* $y_* \in U[x_0, \rho_3)$ *exist such that* $F(y_*) = 0$.
(ii) *Condition* (E_2) *holds on the ball* $U[x_0, \rho_3)$,
and
(iii) $\rho_4 \geq \rho_3$ *exists such that*

$$\int_0^1 \Delta_0\Big((1-\theta)\rho_3 + \theta\rho_4\Big)d\theta < 1.$$

Set $\Omega_3 = U[x_0, \rho_4] \cap \Omega$. *Then, the equation* $F(x) = 0$ *is uniquely solvable by* y_* *in the region* Ω_3.

Proof. Let $y_*^0 \in \Omega_3$ with $F(y_*^0) = 0$. Define the linear operator L by $L = \int_0^1 F'\Big(y_* + \theta(y_*^0 - y_*)\Big)d\theta$. Then, we obtain by (ii) and (iii)

$$\begin{aligned}
\Big\|F'(x_0)^{-1}\Big(L - F'(x_0)\Big)\Big\| \leq& \Big\|\int_0^1 \Delta_0\Big((1-\theta)\|\|y_* - x_0\| + \theta\|\|y_*^0 - x_0\|\Big)d\theta\Big\| \\
\leq& \int_0^1 \Delta_0\Big((1-\theta)\|\rho_3 + \theta\rho_4\Big)d\theta < 1
\end{aligned}$$

So, $L^{-1} \in \ell(\mathbb{B}, \mathbb{B}_0)$ and consequently $y_*^0 - y_*$. □

Remark 2. *The conditions* (E_1)–(E_2) *are not used in Proposition 1. However, if all of the conditions* (E_1)–(E_2) *are used, one can set* $y_* = x_*$ *and* $\rho_3 = a_*$.

The corresponding majorizing sequence for method (3) is defined for each $n = 0, 1, 2, \cdots$, $a_0 = 0$, $b_0 = d$ as

$$\begin{aligned}
c_n =& b_n + \frac{1}{24}\left[8 + 9\left(\frac{\bar{\Delta}_n}{1 - \Delta_0(b_n)}\right)\left(2 + \frac{\bar{\Delta}_n}{1 - \Delta_0(b_n)}\right)\right](b_n - a_n), \\
\delta_n^2 =& \left(1 + \int_0^1 \Delta_0\Big(\theta(c_n - a_n)\Big)d\theta\right)(c_n - a_n) + \Big(1 + \Delta_0(a_n)\Big)(b_n - a_n), \\
a_{n+1} =& c_n + \frac{\Big(1 + |\alpha|\Delta_0(a_n) + |\alpha|\Delta_0(b_n)\Big)\delta_n^2}{\Big(1 - \Delta_0(a_n)\Big)\Big(1 - \bar{q}(a_n)\Big)}, \\
\bar{q}_n =& \frac{1}{2|\alpha + 1|}\Big(|1 + 3\alpha|\Delta_0(a_n) + |5\alpha + 3|\Delta_0(b_n)\Big), \\
\delta_n^3 =& \left(1 + \int_0^1 \Delta_0\Big(a_n + \theta(a_{n+1} - a_n)\Big)d\theta\right)(a_{n+1} - a_n) + \frac{3}{2}\Big(1 + \Delta_0(a_n)\Big)(b_n - a_n)
\end{aligned}$$

and

$$b_{n+1} = a_{n+1} + \frac{\delta_{n+1}^3}{1 - \Delta_0(a_{n+1})}.$$

Lemma 3. *Suppose that for each* $n = 0, 1, 2, \cdots$, $q(a_n) < 1$, $\Delta_0(a_n) < 1$ *and* $a_n \leq \beta$ *for some* $\beta > 0$. *Then, the sequence* $\{a_n\}$ *is non-decreasingly convergent to its unique least upper bound* $a_*^1 \in [0, \beta]$.

Proof. It follows immediately as in Lemma 1. □

As in the proof of Theorem 1, Lemma 2, and the Theorem 3 by assuming that the iterates $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ exist, we have in turn by the substeps two, three, and one that

$$z_n - y_n = \frac{1}{24}\left[9\Big(I - F'(y_n)^{-1}F'(x_n)\Big)^2 - 18\Big(F'(y_n)^{-1}F'(x_n) - I\Big) - 8I\right]F'(x_n)^{-1}F(x_n)$$
$$x_{n+1} - y_n = -2\Big(-3(1+\alpha)F'(x_n) + (5\alpha+3)F'(y_n)\Big)^{-1}\Big(F'(x_n) + \alpha F'(y_n)\Big)F'(x_n)^{-1}F(z_n)$$

and

$$\begin{aligned} F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - \frac{3}{2}F'(x_n)(y_n - x_n) \\ &= \int_0^1 F'\Big(x_n + \theta(x_{n+1} - x_n)\Big)d\theta(x_{n+1} - x_n) - \frac{3}{2}F'(x_n)(y_n - x_n). \end{aligned}$$

Thus, we can prove the corresponding semi-local convergence result for method (3).

Theorem 4. *Under the conditions* (E_1)–(E_5) *for* $\bar{t}_* = a_*^1$, *the conclusions of the Theorem 3 hold but for the method* (3).

Proof. It follows by the preceding identities and the proofs of the method (2) and Theorem 3 that

$$\begin{aligned} \|z_n - y_n\| &= c_n - b_n, \\ \|x_{n+1} - y_n\| &= a_{n+1} - c_n, \\ \|y_{n+1} - x_{n+1}\| &= b_{n+1} - a_{n+1} \end{aligned}$$

and that all of the iterates belong in the ball $U[x_0, a_*^1]$.

The rest follows as in the proof of the Theorem 3. □

The uniqueness of the solution x_* is already given in the Proposition 1.

4. Numerical Examples

Computational results are developed based on the suggested theoretical results in this work. We select three applied science problems 2 and 4 for the computational results. The corresponding results are listed in the Tables. Additionally, we obtain the *COC* approximated by means of

$$\lambda = \frac{\ln \frac{\|x_{m+1} - x_*\|}{|x_m - x_*\|}}{\ln \frac{\|x_m - x_*\|}{\|x_{m-1} - x_*\|}}, \quad \text{for } m = 1, 2, \ldots \tag{37}$$

or *ACOC* [6,7] by:

$$\lambda^* = \frac{\ln \frac{\|x_{m+1} - x_m\|}{\|x_m - x_{m-1}\|}}{\ln \frac{\|x_m - x_{m-1}\|}{\|x_{m-1} - x_{m-2}\|}}, \quad \text{for } m = 2, 3, \ldots \tag{38}$$

In addition, we adopt $\epsilon = 10^{-100}$ as the error tolerance and the terminating criteria to the solve nonlinear system or the scalar equations (i) $\|x_{m+1} - x_m\| < \epsilon$, and (ii) $\|F(x_m)\| < \epsilon$.

The computations are performed with the package of *Mathematica* 11 and multiple precision arithmetic.

Example 1. *Let* $\Omega = U[0,1]$ *and* $\mathbb{B}_0 = \mathbb{B} = C[0,1]$. *Consider the nonlinear integral equation of the first kind of Hammerstein operator* H, *which is defined by*

$$H(v)(x) = v(x) - 7\int_0^1 x\lambda v(\lambda)^3 d\lambda.$$

The calculation for the derivative gives

$$H'\Big(v(q)\Big)(x) = q(x) - 21\int_0^1 x\lambda v(\lambda)^2 q(\lambda)d\lambda,$$

for $q \in C[0,1]$. *By this value of the operator* H', *the conditions* (E_1)–(E_4) *are verified so that we choose*

$$\Delta_0(\tau) = 10.5\tau, \quad \Delta_1(\tau) = 1 + 10.5\tau \text{ and } \quad \Delta(\tau) = 21\tau.$$

In Tables 1 and 2, we present radii for methods (2) and (3), respectively, for example (1).

Table 1. Radii of method (2) for example (1).

R_0	R_1	R_3	R	ρ_1	ρ_2	ρ_3	ρ
0.095238	0.063492	0.042254	0.042254	0.047619	0.030544	0.023638	0.023638

Table 2. Radii of method (3) for example (1).

α	$\bar{\rho}_1$	$\bar{\rho}_2$	$\bar{\rho}_3$	ρ
0.5	0.029304	0.015727	0.018479	0.015727
1	0.029304	0.015727	0.018117	0.015727
3	0.029304	0.015727	0.017548	0.015727

Example 2. *Let* $\Omega = U[0,1]$ *and* $\mathbb{B}_0 = \mathbb{B} = \mathbb{R}^3$. *Then, for* $w = (w_1,\ w_2,\ w_3)^T$ *as*

$$T(w) == \left(w_1,\ e^{w_2} - 1,\ \frac{e-1}{2}w_3^2 + w_3\right)^T. \tag{39}$$

It follows by this definition that the derivative T' *is*

$$F'(u) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{w_2} & 0 \\ 0 & 0 & (e-1)w_3 + 1 \end{bmatrix}.$$

Notice also that $x_* = (0,\ 0,\ 0)^T$. *Consequently,* $F'(x_*) = F'(x_*)^{-1} = diag\{1,\ 1,\ 1\} = I$. *By plugging the values of* T' *in the conditions* (E_1)–(E_4), *we see that*

$$\Delta_0(\tau) = (e-1)\tau, \quad h = \frac{1}{e-1}, \quad \Omega_2 = U(0,h) \cap U(0,1) = U(0,h),$$
$$\Delta(\tau) = e^{\frac{1}{e-1}}\tau, \quad \text{and} \quad \Delta_1(\tau) = e^{\frac{1}{e-1}}.$$

In Tables 3 and 4, we present radii for methods (2) and (3), respectively, for example (2). Further, in Table 5, we present a number of iterations and the convergence order of example (1).

Table 3. Radii of method (2) for example (2).

R_0	R_1	R_3	R	ρ_1	ρ_2	ρ_3	ρ
0.58198	0.44149	0.270369	0.270369	0.38269	0.19655	0.13665	0.13665

Table 4. Radii of method (3) for example (2).

α	$\bar{\rho}_1$	$\bar{\rho}_2$	$\bar{\rho}_3$	ρ
0.5	0.15441	0.053865	0.082356	0.053865
1	0.15441	0.053865	0.081149	0.053865
3	0.15441	0.053865	0.079279	0.053865

Table 5. Number of iterations and convergence order of example (2).

Cases	α	x_0	itr	λ
Method (2)	-	$(0.12, 0.12, 0.12)^T$	3	5.1238
Method (3)	0.5	$(0.05)^T$	3	6.1292
Method (3)	1	$(0.05, 0.05, 0.05)^T$	3	6.1251
Method (3)	3	$(0.05, 0.05, 0.05)^T$	3	6.0011

Example 3. *The kinematic synthesis problem for steering [8] is given as*

$$[E_i(\nu_2\sin(\eta_i)-\nu_3)-H_i(\nu_2\sin(\varphi_i)-\nu_3)]^2+[H_i(\nu_2\cos(\varphi_i)+1)-H_i(\nu_2\cos(\eta_i)-1)]^2$$
$$-[\nu_1(\nu_2\sin(\eta_i)-\nu_3)(\nu_2\cos(\varphi_i)+1)-\nu_1(\nu_2\cos(\eta_i)-\nu_3)(\nu_2\sin(\varphi_i)-\nu_3)]^2=0,\ \textit{for } i=1,2,3,$$

where

$$E_i=-\nu_3\nu_2(\sin(\varphi_i)-\sin(\varphi_0))-\nu_1(\nu_2\sin(\varphi_i)-\nu_3)+\nu_2(\cos(\varphi_i)-\cos(\varphi_0)),\ i=1,2,3$$

and

$$H_i=-\nu_3\nu_2\sin(\eta_i)+(-\nu_2)\cos(\eta_i)+(\nu_3-\nu_1)\nu_2\sin(\eta_0)+\nu_2\cos(\eta_0)+\nu_1\nu_3,\ i=1,2,3.$$

In Table 6, we present the values of η_i and φ_i (in radians).

Table 6. Values of η_i and φ_i (in radians) for example (3).

i	η_i	φ_i
0	1.3954170041747090114	1.7461756494150842271
1	1.7444828545735749268	2.0364691127919609051
2	2.0656234369405315689	2.2390977868265978920
3	2.4600678478912500533	2.4600678409809344550

In Table 7, we present the number of iterations and the convergence order of example (3).

Table 7. Number of iterations and convergence order of example (3).

Cases	α	x_0	itr	λ
Method (2)	-	$(0.89, 0.68, 0.64)^T$	4	5.1006
Method (3)	0.5	$(0.89, 0.68, 0.64)^T$	3	6.0535
Method (3)	1	$(0.89, 0.68, 0.64)^T$	3	6.0091
Method (3)	3	$(0.89, 0.68, 0.64)^T$	3	6.1154

Methods (2) and (3) converge to the approximated root $(0.9051\ldots, 0.6977\ldots, 0.6508\ldots)^T$.

Example 4. *Let us consider the Van der Pol equation [23], which is defined as follows:*

$$y''-\mu(y^2-1)y'+y=0,\ \mu>0, \tag{40}$$

which governs the flow of the current in a vacuum tube, with the boundary conditions y(0) = 0, y(2) = 1. Further, we consider the partition of the given interval $[0,\ 2]$, *which is given by*

$$x_0=0<x_1<x_2<x_3<\cdots<x_n,\ \textit{where } x_i=x_0+ih,\ h=\frac{2}{n}.$$

Moreover, we assume that

$$y_0=y(x_0)=0,\ y_1=y(x_1),\ \ldots,\ y_{n-1}=y(x_{n-1}),\ y_n=y(x_n)=1.$$

If we discretize the above problem (40) by using the second-order-divided difference for the first and second derivatives, which are given by

$$y'_k = \frac{y_{k+1} - y_{k-1}}{2h}, \; y''_k = \frac{y_{k-1} - 2y_k + y_{k+1}}{h^2}, \; k = 1,\, 2,\, \ldots,\, n-1,$$

then, we obtain a $(n-1) \times (n-1)$ *system of nonlinear equations*

$$2h^2 x_k - h\mu\left(x_k^2 - 1\right)(x_{k+1} - x_{k-1}) + 2(x_{k-1} + x_{k+1} - 2x_k) = 0,\; k = 1, 2, \ldots, n-1.$$

Let us consider $\mu = \frac{1}{2}$ and $n = 6$ so that we can obtain a 5×5 system of nonlinear equations. The obtained results are depicted in Table 8.

Table 8. Number of iterations and convergence order of example (4).

Cases	α	x_0	itr	λ
Method (2)	-	$(0.83, 1.5, 2.2, 3.1, 7.9)^T$	3	5.1267
Method (3)	0.5	$(0.83, 1.5, 2.2, 3.1, 7.9)^T$	3	6.0281
Method (3)	1	$(0.83, 1.5, 2.2, 3.1, 7.9)^T$	3	6.0404
Method (3)	3	$(0.83, 1.5, 2.2, 3.1, 7.9)^T$	3	5.9843

Methods (2) and (3) converge to the approximated root $(0.8243131\ldots,\ 1.516531\ldots,\ 2.187958\ldots,\ 3.123402\ldots,\ 7.824242\ldots)^T$.

Example 5. *Let us consider the following nonlinear system of nonlinear equation [19]:*

$$\begin{cases} x_j^2 x_{j+1} - 1 = 0,\; 1 \le j \le n-1, \\ x_n^2 x_1 - 1 = 0. \end{cases} \tag{41}$$

In Table 9, we present the number of iterations and the convergence order of example (5) for $n = 100$.

Table 9. Number of iterations and convergence order of example (5).

Cases	α	x_0	itr	λ
Method (2)	-	$(1.1, 1.1, 1.1, \overset{100}{\cdots}, 1.1)^T$	3	6.2193
Method (3)	0.5	$(1.1, 1.1, 1.1, \overset{100}{\cdots}, 1.1)^T$	3	6.1665
Method (3)	1	$(1.1, 1.1, 1.1, \overset{100}{\cdots}, 1.1)^T$	3	6.1567
Method (3)	3	$(1.1, 1.1, 11, \overset{100}{\cdots}, 1.1)^T$	3	6.1514

Methods (2) and (3) converge to the approximated root $\xi = (1,\ 1,\ 1,\ \overset{100}{\cdots},\ 1)^T$.

5. Conclusions

At the beginning of this paper, we provided the motivation for writing this paper by looking at the problems that exist with the application of method (2) and method (3) and consequently of other high convergence-order methods [24–29]. In view of these concerns, a general methodology is introduced to extend the usage for these two efficient sixth-order methods and in the more general setting of Banach-space-valued nonlinear equations. The local convergence is shown under weak w–continuity conditions on the operator F'. This is in contrast to earlier local convergence results based on at least the seventh-order assumptions of the operator F. The more interesting semi-local convergence is also given and based on the concept of a majorizing sequence. Such a result was not presented in [12]. The convergence order six is recovered using the formula COC or the formula $ACOC$. The developed methodology does not depend on the studied methods (2) and (3). Therefore, it can also be employed [24–29] on other single, two-step, or multi-step methods in order to provide the same benefits. Hence, we revealed the direction of our future research topics.

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