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Chatterjea and Ćirič-Type Fixed-Point Theorems Using $(\alpha - \psi)$ Contraction on C^*-Algebra-Valued Metric Space

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Abstract: In the present paper, we provide and verify several results obtained by using the Chatterjea and Ćirič fixed-point theorems by using $(\alpha - \psi)$-contractive mapping in C^*-algebra-valued metric space. We provide some examples and an application to illustrate our results. Our study extends and generalizes the results of several studies in the literature.

Keywords: C^*-algebra-valued metric space; fixed-point theorem; $(\alpha - \psi)$-contractive mapping

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1. Introduction

The Banach contraction principle [1] is one of the most important tools of analysis and has many significant applications in various fields of science. It has been improved in many ways and generalized by many researchers. A map $T : \Omega \to \Omega$, where (Ω, d) is a complete metric space, is said to be a contraction map if there exists $\lambda \in (0,1)$, such that for all $\mu, \nu \in \Omega$

$$d(T\mu, T\nu) \leq \lambda d(\mu, \nu). \tag{1}$$

This result was introduced by Banach in 1922. Kannan [2] in 1968 proved that, if (Ω, d) is a complete metric space and $T : \Omega \to \Omega$ is a map satisfying

$$d(T\mu, T\nu) \leq \lambda(d(T\mu, \mu) + d(T\nu, \nu)), \tag{2}$$

where $\lambda \in (0, \frac{1}{2})$ for all $\mu, \nu \in \Omega$, then there is a unique fixed point on T. Later, in 1972, Chatterjea [3] proved that if (Ω, d) is a complete metric space and $T : \Omega \to \Omega$ is a mapping that exists $\lambda \in (0, \frac{1}{2})$, such that $\mu, \nu \in \Omega$, the inequality

$$d(T\mu, T\nu) \leq \lambda(d(T\mu, \nu) + d(T\nu, \mu)) \tag{3}$$

is satisfied; thus, T has a unique fixed point.

Ćirič [4] in 1974 introduced an interesting general contraction condition. If there exists $\lambda \in (0,1)$, such that for all $\mu, \nu \in \Omega$, and $T : \Omega \to \Omega$ is a map satisfying

$$d(T\mu, T\nu) \leq \lambda \cdot max\{d(\mu, \nu), d(T\mu, \mu)d(T\nu, \nu), d(T\mu, \nu), d(T\nu, \mu)\}, \tag{4}$$

then T has a unique fixed point.

On the other hand, Samet et al. [5,6] studied α-ψ-contractive mappings in metric spaces. Many researchers have established related studies to α-admissible and $\alpha - \psi$-contractive mappings and related fixed-point theorems (see [7–15]).

Recently, Ma et al. [10] introduced the more generalized notion 0f a C^*-algebra-valued metric space by replacing real numbers with the positive cone of C^*-algebra. This line of research was continued in [16–22], where several other fixed-point results were obtained in the framework of C^*-algebra-valued metric space.

Throughout this paper, we suppose that A is a unital C^*-algebra with a unit I_A. We mean that a unital C^*-algebra is a complex Banach algebra A with an involution map $* : A \to A, a \to a^*$, such that $(a^*)^* = a$, $(ab)^* = a^*b^*$, $(a+b)^* = a^* + b^*$ and $(\lambda a)^* = \overline{\lambda} a^*$ for $a, b, A, \lambda \in \mathbb{C}$, such that $\|a^*a\| = \|a\|^2$. Set $A_h = \{a \in A : a = a^*\}$. An element $a \in A$ is a positive element if $a = a^*$ and $\sigma(a) \subset \mathbb{R}^+$, where $\sigma(a)$ is the spectrum of a. We define a partial ordering $\preceq$ on A as $a \preceq b$ if $0_A \preceq b - a$, where 0_A means the zero element in A, and we let A^+ denote the $\{a \in A : a \succeq 0_A\}$ and $|a| = (a^*a)^{\frac{1}{2}}$.

The results described in this article extend some fixed-point theorems in C^*-algebra-valued metric spaces. C^*-algebras are considered typical examples of quantum spaces and non-commutative spaces. They play an important role in the non-commutative geometry project introduced by Alain Connes [23]. Thus, the theory of metric space-valued C^*-algebras should apply to many problems in quantum spaces, such as matrices and bounded linear operators on Hilbert spaces. Therefore, C^*-algebras and their metric provide a non-commutative version of ordinary metric spaces.

2. Preliminaries

In this section, we introduce some basic notions which will be used in the following work.

Lemma 1. *Suppose that A is a unital C^*-algebra with unit I_A. The following holds.*
(1) If $a \in A$, with $\|a\| < \frac{1}{2}$, then $1 - a$ is invertible and $\|a(1-a)^{-1}\| < 1$.
(2) If $a, b \in A^+$ and $ab = ba$, then $a.b \succeq 0_A$.
(3) Let $a \in A'$. If $b, c \in A$ with $b \succeq c \succeq 0_A$ and $1 - a \in (A')^+$ is an invertible element, then $(I_A - a)^{-1} b \succeq (I_A - a)^{-1} c$, where $A' = \{b \in A : ab = ba \ \ \forall a \in A\}$.
We refer to [24] for more C^*algebra details.

Definition 1. *[10] Let Ω be a non-empty set. Suppose the mapping $d_A : \Omega \times \Omega \to A$ satisfies:*
(1) $d_A(\mu, \nu) \succeq 0_A$ for all $\mu, \nu \in \Omega$ and $d_A(\mu, \nu) = 0_A \Leftrightarrow \mu = \nu$.
(2) $d_A(\mu, \nu) = d_A(\nu, \mu)$ for all $\mu, \nu \in \Omega$.
(3) $d_A(\mu, \xi) \preceq d_A(\mu, \nu) + d_A(\nu, \xi)$ for all $\mu, \nu, \xi \in \Omega$.
Then, d_A is called a C^*-algebra-valued metric on Ω and (Ω, A, d_A) is called C^*-algebra-valued metric space.

Example 1. *Let Ω be a Banach space and $d_A : \Omega \times \Omega \to A$ given by $d_A(\mu, \nu) = \|\mu - \nu\| \cdot a$, for all $\mu, \nu \in \Omega$, which should be where $a \in A^+$, $a \succeq 0$.*
It is easy to verify that (Ω, A, d_A) is a C^*-algebra-valued metric space.

Example 2. *Let $\Omega = \mathbb{C}$ and $A = M_n(\mathbb{C})$. It is obvious that A is a C^*-algebra with the matrix norm and the involution given by $* : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, $(Z_{ij})_{1 \le i,j \le n} \to (Z_{ij})^*_{1 \le i,j \le n} = (\overline{Z_{ji}})_{1 \le i,j \le n}$, where $\overline{Z_{ij}}$ is the conjugate of Z_{ij}, $Z_{ij} \in \mathbb{C}$. Define a mapping $d_A : \Omega \times \Omega \to A$, by:*

$$
\begin{aligned}
d(Z_1, Z_2) &= diag(e^{i\theta_1}|Z_1 - Z_2|, ..., e^{i\theta_k}|Z_1 - Z_2|, ...e^{i\theta_n}|Z_1 - Z_2|) \\
&= \begin{pmatrix}
e^{i\theta_1}|Z_1 - Z_2| & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & & \vdots \\
0 & \cdots & e^{i\theta_k}|Z_1 - Z_2| & \cdots & 0 \\
\vdots & & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & e^{i\theta_n}|Z_1 - Z_2|
\end{pmatrix},
\end{aligned}
$$

for all $Z_1, Z_2 \in \mathbb{C}$, $i = \sqrt{-1}$, $k = 1, ..., n$, $\theta_k \in [0, \frac{\pi}{2}]$. Then, (Ω, A, d_A) is a C^*-algebra-valued metric space. It is clear that it is a generalization of the complex-valued metric space given in [25], when $A = \mathbb{C}$.

Definition 2. *Let (Ω, A, d_A) be a C^*-algebra-valued metric space, $\mu \in \Omega$, and $\{\mu_n\}_{n=1}^{+\infty}$ be a sequence in Ω. Then,*
(i) $\{\mu_n\}_{n=1}^{+\infty}$ convergent to μ whenever, for every $\varepsilon \in A$ with $\varepsilon \succ 0_A$, there is a natural number $N \in \mathbb{N}$, such that

$$d_A(\mu_n, \mu) \prec \varepsilon,$$

for all $n > N$. We denote this by $\lim_{n\to\infty} \mu_n = \mu$ or $\mu_n \to \mu$ as $n \to +\infty$.
(ii) $\{\mu_n\}_{n=1}^{+\infty}$ is said to be a Cauchy sequence whenever, for every $\varepsilon \in A$ with $\varepsilon \succ 0_A$, there is a natural number $N \in \mathbb{N}$, such that

$$d_A(\mu_n, \mu_m) \prec \varepsilon,$$

for all $n, m > N$.

Lemma 2. *(i) $\{\mu_n\}_{n=1}^{+\infty}$ is convergent in Ω if, for any element $\epsilon > 0$, there is $N \in \mathbb{N}$, such that for all $n > N$, $\|d(\mu_n, \mu)\| \leq \epsilon$.*
(ii) $\{\mu_n\}_{n=1}^{+\infty}$ is a Cauchy sequence in Ω if, for any $\epsilon > 0$ there is $N \in \mathbb{N}$, such that
$\|d_A(\mu_n, \mu_m)\| \leq \epsilon$, for all $n, m > N$. We say that (Ω, A, d_A) is a complete C^*-algebra-valued metric space if every Cauchy sequence is convergent with respect to A.

Example 3. *Let Ω be a compact Hausdorff space. We denote by $C(\Omega)$ the algebra of all complex-valued continuous functions on Ω with pointwise addition and multiplication. The algebra $C(\Omega)$ with the involution defined by $f^*(\mu) = \overline{f(\mu)}$ for each $f \in C(\Omega), \mu \in \Omega$ and with the norm $\|f\|_\infty = \sup\{|f(\mu)|, \mu \in \Omega\}$ is a commutative C^*-algebra where unit $I_{C(\Omega)}$ is the constant function. Let $C^+(\Omega) = \{f \in C(\Omega) : \overline{f(\mu)} = f(\mu), f(\mu) \geq 0\}$ denote the positive cone of $C(\Omega)$, with partial order relation $f \leq g$ if and only if $f(\mu) \leq g(\mu)$. Put $d_{C(\Omega)} : C(\Omega) \times C(\Omega) \to C(\Omega)$ as $d_{C(\Omega)}(f, g) = \sup_{\mu \in \Omega}\{|f(\mu) - g(\mu)|\}.I_{C(\Omega)}$. It is clear that $(C(\Omega), C(\Omega), d_{C(\Omega)})$ is a complete C^*-algebra-valued metric space.*

Definition 3. *[6] Let $T : \Omega \to \Omega$ be a self map and $\alpha : \Omega \times \Omega \to [0, +\infty)$. Then, T is called α-admissible if for all $\mu, \nu \in \Omega$ and $\alpha(\mu, \nu) \geq 1$ implies $\alpha(T\mu, T\nu) \geq 1$.*

Definition 4. *Let Ω be a non-empty set and $\alpha_A : \Omega \times \Omega \to (A^+)'$ be a function. We say that the self map T is α_A -admissible if for all $(\mu, \nu) \in \Omega \times \Omega$, $\alpha_A(\mu, \nu) \succeq I_A \Rightarrow \alpha_A(T\mu, T\nu) \succeq I_A$, where I_A is the unit of A.*

Definition 5. *Let (Ω, A, d_A) be a C^*-algebra-valued metric space and $T : \Omega \to \Omega$ be a mapping. We say that T is an α_A-ψ_A-contractive mapping if there exist two functions $\alpha_A : \Omega \times \Omega \to A_+$ and $\psi_A \in \Psi_A$, such that*

$$\alpha_A(\mu, \nu) d_A(T\mu, T\nu) \preceq \psi_A(d_A(\mu, \nu)),$$

for all $\mu, \nu \in \Omega$.

Definition 6. *Suppose that A and B are C^*-algebras. A mapping $\psi : A \to B$ is said to be a C^*-homomorphism if :*
(a) $\psi(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_1 \psi(a_1) + \lambda_2 \psi(a_2)$ for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and $a_1, a_2 \in A$;
(b) $\psi(a_1 a_2) = \psi(a_1)\psi(a_2)$, $\forall a_1, a_2 \in A$;
(c) $\psi(a^*) = \psi(a)^*$, $\forall a \in A$; and
(d) ψ maps the unit in A to the unit in B.

Definition 7. *If $\psi : A \to B$ is a linear mapping in C^*-algebra, it is said to be positive if $\psi(A^+) \subseteq B^+$. In this case, $\psi(A_h) \subseteq B_h$, and the restriction map $\psi : A_h \to B_h$ increases. Every C^*-homomorphism is contractive and hence bounded and every $*$-homomorphism is positive.*

Definition 8. *Let Ψ_A be the set of positive functions $\psi_A : A^+ \to A^+$ satisfying the following conditions:*
(a) $\psi_A(a)$ is continuous and non-decreasing, $\psi_A(a) \prec a$;
(b) $\psi_A(a) = 0$ iff $a = 0$; and
(c) $\sum\limits_{n=1}^{\infty} \psi_A^n(a) < \infty$, $\lim\limits_{n\to\infty} \psi_A^n(a) = 0$ for each $a \succ 0$, where ψ_A^n is the nth-iterate of ψ_A.

3. Main Results

In this section, we give some types of Chatterjea and Cìriç fixed-point theorems in a C^*-algebra-valued metric space using $(\alpha - \psi)$-contraction.

Theorem 1. *(Chatterjea Type) Let (Ω, A, d_A) be a complete C^*-algebra-valued metric space and $T : \Omega \to \Omega$, be a mapping satisfying:*

$$\alpha_A(\mu,\nu) d_A(T\mu, T\nu) \preceq \psi_A\left(\frac{d_A(T\mu,\nu) + d_A(T\nu,\mu)}{2}\right), \tag{5}$$

for $\mu, \nu \in \Omega$, where

$$\alpha_A : \Omega \times \Omega \to A^+ \;\; and \;\; \psi_A \in \Psi_A \; , \; \psi_A \prec \frac{1}{2}.I_A$$

and the following conditions hold:
(a) T is α_A-admissible;
(b) There exists $\mu_0 \in \Omega$, such that $\alpha_A(\mu_0, T\mu_0) \succeq I_A$; and
(c) T is continuous.
Then, T has a fixed point in Ω.

Proof. Let $\mu_0 \in \Omega$, such that $\alpha_A(\mu_0, T\mu_0) \succeq I_A$, and define the sequence $\{\mu_n\}_{n=0}^{+\infty}$ in Ω, such that $\mu_{n+1} = T\mu_n$ for all $n \in \mathbb{N}$. If $\mu_n = \mu_{n+1}$ for some $n \in \mathbb{N}$, then μ_n is a fixed point for T.

Suppose that $\mu_n \neq \mu_{\mu+1}$ for all $n \in \mathbb{N}$. Because T is α_A-admissible, we obtain

$$\alpha_A(\mu_0, \mu_1) = \alpha_A(\mu_0, T\mu_0) \succeq I_A \Rightarrow$$

$$\alpha_A(T\mu_0, T^2\mu_0) = \alpha_A(\mu_1, \mu_2) \succeq I_A. \tag{6}$$

By induction, we have $\alpha_A(\mu_n, \mu_{n+1}) \succeq I_A$ for all $n \in \mathbb{N}$.
By using inequalities (5) and (6), we have

$$\begin{aligned} d_A(\mu_n, \mu_{n+1}) &= d_A(T\mu_{n-1}, T\mu_n) \\ &\preceq \alpha_A(\mu_{n-1}, \mu_n) d_A(T\mu_{n-1}, T\mu_n) \\ &\preceq \psi_A\left(\frac{d_A(T\mu_{n-1}, \mu_n) + d_A(T\mu_n, \mu_{n-1})}{2}\right) \\ &= \psi_A\left(\frac{d_A(\mu_n, \mu_n) + d_A(T\mu_n, \mu_{n-1})}{2}\right) \\ &= \psi_A\left(\left(\frac{d_A(\mu_n, \mu_n)) + \psi_A(d_A(\mu_{n+1}, \mu_{n-1})}{2}\right)\right). \end{aligned}$$

Because $\varphi_A(0) = 0$, we obtain

$$d_A(\mu_n, \mu_{n+1}) \preceq \psi_A\left(\frac{d_A(\mu_{n+1}, \mu_{n-1})}{2}\right). \tag{7}$$

Applying triangular inequality in (7), we have

$$d_A(\mu_n, \mu_{n+1}) \preceq \psi_A \frac{(d_A(\mu_{n+1}, \mu_n) + d_A(\mu_n, \mu_{n-1}))}{2}.$$

Because ψ_A is additive, we have

$$d_A(\mu_n, \mu_{n+1}) \quad \preceq \quad \frac{\psi_A(d_A(\mu_{n+1}, \mu_n)}{2} + \frac{\psi_A(d_A(\mu_n, \mu_{n-1}))}{2}.$$

Thus,

$$(\frac{1}{2} - \psi_A)(d_A(\mu_n, \mu_{n+1})) \quad \preceq \quad \frac{1}{2}\psi_A(d_A(\mu_n, \mu_{n-1})),$$

and we have

$$d_A(\mu_n, \mu_{n+1}) \quad \preceq \quad \frac{1}{2}(\psi_A(\frac{1}{2} - \psi_A)^{-1})(d_A(\mu_n, \mu_{n-1})).$$

Putting $\frac{1}{2}\psi_A(\frac{1}{2} - \psi_A)^{-1} = \phi_A$ by induction, we have

$$d_A(\mu_n, \mu_{n+1}) \preceq \phi_A^n(d_A(\mu_0, \mu_1)),$$

for all $n \in \mathbb{N}$. Let $n, m \in \mathbb{N}$ with $m > n$. We obtain

$$d_A(\mu_n, \mu_m) \preceq \sum_{k=n}^{m-1} \phi_A^k(d_A(\mu_0, \mu_1)) \quad \to \quad 0_A \ (as \ \ n \to +\infty).$$

Therefore, we can prove that $\{\mu_n\}$ is a Cauchy sequence in the C^*-algebra metric space (Ω, A, d_A).

Because (Ω, A, d_A) is complete, there exists $\mu \in \Omega$, such that $\mu_n \to \mu$ as $n \to +\infty$. From the continuity of T, it follows that $\mu_{n+1} = T\mu_n \to T\mu$ is as $n \to +\infty$.

By continuity of this limit, we have $T\mu = \mu$—that is, μ is a fixed point of T.

The proof of the uniqueness is as follows. If $\nu(\neq \mu)$ is another fixed point of T, then

$$\begin{aligned}
0_A \preceq d_A(\mu, \nu) \quad &= \quad d_A(T\mu, T\nu) \\
&\preceq \quad \alpha_A(\mu, \nu) d_A(T\mu, T\nu) \\
&\preceq \quad \psi_A \frac{(d_A(T\mu, \nu) + d_A(T\nu, \mu))}{2} \\
&= \quad \psi_A \frac{(d_A(\mu, \nu) + d_A(\mu, \nu))}{2} \\
&= \quad I_A \psi_A(d_A(\mu, \nu)), \ \psi_A(a) \prec a \ \ for \ \ any \ \ a \in A, .
\end{aligned}$$

This implies that

$$0_A \preceq d_A(\mu, \nu) \prec d_A(\mu, \nu),$$

which gives a contradiction, and we can obtain $\mu = \nu$. This completes the proof. □

Corollary 1. *Let (Ω, A, d) be a complete C^*-algebra-valued metric space. Suppose $T : \Omega \to \Omega$ satisfies for all $\mu, \nu \in \Omega$*

$$d_A(T\mu, T\nu) \leq \mathbb{A}(d_A(T\mu, \nu) + d_A(T\nu, \mu)),$$

where $\mathbb{A} \in (A')^+$ and $\|\mathbb{A}\| \leq \frac{1}{2}$. Then, there exists a unique fixed point T in Ω [10].

Proof. This is an immediate consequence of Theorem 1, with $\alpha_A(\mu, \nu) = Id$, $\psi_A(a) = \mathbb{A}a$,, where $a \in A$, $\mathbb{A} \in (A')^+$. □

Theorem 2. *(Banach-Chatterjea Type) Let (Ω, A, d_A) be a complete C^*-algebra-valued metric space and $T : \Omega \to \Omega$ be a mapping satisfying*

$$\alpha_A(\mu,\nu)d_A(T\mu,T\nu) \preceq \frac{\psi_A(d_A(\mu,\nu)+(d_A(T\mu,\nu)+d_A(T\nu,\mu)))}{3},\ \psi_A \prec \frac{1}{3}.I_A \tag{8}$$

$for\ \mu,\nu \in \Omega$, where the following conditions hold:
(i) T is α_A-admissible;
(ii) there exists $\mu_0 \in \Omega$, such that $\alpha_A(\mu_0, T\mu_0) \succeq I_A$; and
(iii) T is continuous.
Then, T has a fixed point in Ω.

Proof. Following the first part of the proof in the Theorem 1, we obtain

$$\alpha_A(\mu_n,\mu_{n+1}) \succeq I_A\ \ for\ \ all\ \ n\ \in\ \mathbb{N}. \tag{9}$$

By using inequalities (8) and (9), we have

$$\begin{aligned} d_A(\mu_n,\mu_{n+1}) &= d_A(T\mu_{n-1},T\mu_n) \\ &\preceq \alpha_A(\mu_{n-1},\mu_n)(d_A(T\mu_{n-1},T\mu_n)) \\ &\preceq \frac{1}{3}\psi_A(d_A(\mu_{n-1},\mu_n)+d_A(T\mu_{n-1},\mu_n)+d_A(T\mu_n,\mu_{n-1})) \\ &= \frac{1}{3}\psi_A(d_A(\mu_{n-1},\mu_n)+d_A(\mu_n,\mu_n)+d_A(\mu_{n+1},\mu_{n-1})) \\ &= \frac{1}{3}\psi_A(d_A(\mu_{n-1},\mu_n)+d_A(\mu_{n+1},\mu_{n-1})). \end{aligned}$$

By using triangular inequality, we obtain

$$\begin{aligned} d_A(\mu_n,\mu_{n+1}) &\preceq \frac{1}{3}\psi_A(d_A(\mu_{n-1},\mu_n)+d_A(\mu_{n-1},\mu_n)+d_A(\mu_n,\mu_{n+1})) \\ &= \frac{2}{3}\psi_A(d_A(\mu_{n-1},\mu_n))+\frac{1}{3}\psi_A(d_A(\mu_n,\mu_{n+1})). \end{aligned}$$

Thus, we have

$$(1-\frac{1}{3}\psi_A)(d_A(\mu_n,\mu_{n+1})) \preceq \frac{2}{3}\psi_A(d_A(\mu_{n-1},\mu_n)).$$

This implies that

$$d_A(\mu_n,\mu_{n+1}) \preceq \frac{2}{3}\psi_A(1-\frac{1}{3}\psi_A)^{-1}(d_A(\mu_{n-1},\mu_n)).$$

Putting $\phi_A = \frac{2}{3}\psi_A(1-\frac{1}{3}\psi_A)^{-1}$, we obtain

$$d_A(\mu_n,\mu_{n+1}) \preceq \phi_A^n(d_A(\mu_0,\mu_1))$$

for $m \geq n$. Thus, we obtain

$$\begin{aligned} d_A(\mu_n,\mu_m) &\preceq \sum_{k=n}^{m-1}\phi_A^k(d_A(\mu_0,\mu_1)) \\ &\to 0\ \ as\ \ (n\to+\infty). \end{aligned}$$

Thus, $\{\mu_n\}$ is a Cauchy sequence in Ω with respect to (Ω, A, d_A) .

Because (Ω, A, d_A) is a complete C^*-algebra-valued metric space, we conclude that $\{\mu_n\}$ is a convergence sequence, and so $\{\mu_n\} \to \mu$ as $n \to +\infty$ and $T\mu = \mu$ as $n \to +\infty$. Therefore, μ is a fixed point of T.

To prove the uniqueness, we suppose that $(\nu \neq \mu)$ is another fixed point of T. Thus,

$$\begin{aligned} 0_A \preceq d_A(\mu, \nu) &= d_A(T\mu, T\nu) \\ &\preceq \alpha_A(\mu, \nu)\psi_A(d_A(T\mu, T\nu)) \\ &\preceq \frac{1}{3}\psi_A(d_A(\mu, \nu) + d_A(T\mu, \nu) + d_A(T\nu, \mu)) \\ &\preceq \frac{1}{3}\psi_A(d_A(\mu, \nu) + d_A(\mu, \nu) + d_A(\mu, \nu)) \\ &\preceq \psi_A(d_A(\mu, \nu)) \prec d_A(\mu, \nu). \end{aligned}$$

This is a contradiction, so $d_A(\mu, \nu) = 0_A$ and $\mu = \nu$. □

Corollary 2. *Let (Ω, d) be a complete real-valued metric space. Suppose $T : \Omega \to \Omega$ satisfies for all $\mu, \nu \in \Omega$*

$$d(T\mu, T\nu) \leq k(d(\mu, \nu) + d(T\mu, \nu) + d(T\nu, \mu)),$$

where $k \in (0, \frac{1}{3})$. Then, T has a unique fixed point in Ω.

Proof. This is an immediate consequence of Theorem 2, with $\mathbb{A} = \mathbb{R}$ and $\alpha_A(\mu, \nu) = I$ and $\psi_A(t) = kt$, $t \in \mathbb{R}$. □

Theorem 3. *(Ćirić Contraction Type) Let (Ω, A, d_A) be a complete C^*-algebra-valued metric space and $T : \Omega \to \Omega$ be a mapping satisfying*

$$\alpha_A(\mu, \nu)d_A(T\mu, T\nu) \preceq \psi_A(M_A(\mu, \nu)) \tag{10}$$

$$M_A(\mu, \nu) = \frac{I_A}{3}[d_A(\mu, \nu) + (d_A(T\mu, \mu) + d_A(T\nu, \nu)) + (d_A(T\mu, \nu) + d_A(T\nu, \mu)], \ \psi_A \prec \frac{1}{2}.I_A$$

$for\ \mu, \nu \in \Omega$, where the following conditions hold:
(i) T is α_A-admissible;
(ii) there exists $\mu_0 \in \Omega$, such that $\alpha_A(\mu_0, T\mu_0) \succeq I_A$; and
(iii) T is continuous.
Then, T has a fixed point in Ω.

Proof. Following the first part of the proof in the Theorem 1, we obtain

$$\alpha_A(\mu_n, \mu_{n+1}) \succeq I_A \ for\ all\ n \ \in \ \mathbb{N}. \tag{11}$$

By using (10) and (11), we have

$$\begin{aligned} d_A(\mu_n, \mu_{n+1}) &= d_A(T\mu_{n-1}, T\mu_n) \\ &\preceq \alpha_A(\mu_{n-1}, \mu_n)d_A(T\mu_{n-1}, T\mu_n) \\ &\preceq \psi_A(M_A(\mu_{n-1}, \mu_n)). \end{aligned} \tag{12}$$

On the other hand, we have

$$
\begin{aligned}
M_A(\mu_{n-1},\mu_n) &= \frac{1}{3}(d_A(\mu_{n-1},\mu_n) + d_A(T\mu_{n-1},\mu_{n-1}) + d_A(T\mu_n,\mu_n) \\
&+ d_A(T\mu_{n-1},\mu_n) + d_A(T\mu_n,\mu_{n-1})).I_A \\
So,\ M_A(\mu_{n-1},\mu_n) &= \frac{1}{3}I_A(d_A(\mu_n,\mu_{n-1}) + d_A(\mu_n,\mu_{n-1}) + d_A(\mu_n,\mu_{n+1}) \\
&+ d_A(\mu_n,\mu_n) + d_A(\mu_{n+1},\mu_{n-1})).
\end{aligned}
$$

Because $d_A(\mu,\mu) = 0$, we obtain

$$
M_A(\mu_{n-1},\mu_n) \preceq \frac{1}{3}I_A(d_A(\mu_n,\mu_{n-1}) + d_A(\mu_n,\mu_{n-1}) + d_A(\mu_n,\mu_{n+1}) + d_A(\mu_{n+1},\mu_{n-1})).
$$

$$
So,\ M_A(\mu_{n-1},\mu_n) \preceq \frac{1}{3}I_A(2d_A(\mu_n,\mu_{n-1}) + d_A(\mu_n,\mu_{n+1}) + d_A(\mu_{n+1},\mu_{n-1}))).
$$

By using triangular inequality, we obtain

$$
\begin{aligned}
d_A(\mu_n,\mu_{n+1}) &\preceq \frac{1}{3}\psi_A I_A[2d_A(\mu_n,\mu_{n-1}) + d_A(\mu_n,\mu_{n+1}) + d_A(\mu_n,\mu_{n+1}) + d_A(\mu_n,\mu_{n-1})]. \\
d_A(\mu_n,\mu_{n+1}) &\preceq \frac{1}{3}\psi_A I_A[3d_A(\mu_{n-1},\mu_n) + 2d_A(\mu_n,\mu_{n+1})].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(1-\frac{2}{3}\psi_A)(d_A(\mu_n,\mu_{n+1})) &\preceq \psi_A I_A(d_A(\mu_{n-1},\mu_n)) \\
d_A(\mu_n,\mu_{n+1}) &\preceq \psi_A(1-\frac{2}{3}\psi_A)^{-1}I_A(d_A(\mu_{n-1},\mu_n)).
\end{aligned}
$$

Putting $\phi_A = \psi_A(1-\frac{2}{3}\psi_A)^{-1}$, $\|\psi_A\| < \frac{1}{2}$; then, we obtain

$$
d_A(\mu_n,\mu_{n+1}) \preceq \phi_A^n(d_A(\mu_0,\mu_1)). \tag{13}
$$

Let $n, m \in \mathbb{N}$, such that $m > n$. We thus obtain

$$
\begin{aligned}
d_A(\mu_n,\mu_m) &\preceq \sum_{k=n}^{m-1}\phi_A^k(d_A(\mu_0,\mu_1)) \\
&\to 0\ as\ (n \to +\infty).
\end{aligned}
$$

Thus, $\{\mu_n\}$ is a Cauchy sequence and $\mu_n \to \mu$ as $n \to +\infty$. Thus, we obtain $T\mu = \mu$ as a fixed point of T.

To prove the uniqueness, we suppose that $(\nu \neq \mu)$ is another fixed point of T. Thus,

$$
\begin{aligned}
0_A \preceq d_A(\mu,\nu) &= d_A(T\mu,T\nu) \\
&\preceq \alpha_A(\mu,\nu)d_A(T\mu,T\nu) \\
&\preceq \frac{1}{3}\psi_A(d_A(\mu,\nu) + d_A(T\mu,\nu) + d_A(T\nu,\mu) + d_A(T\mu,\mu) + d_A(T\nu,\nu)).I_A \\
&= \frac{1}{3}\psi_A(d_A(\mu,\nu) + d_A(\mu,\nu) + d_A(\nu,\mu) + d_A(\mu,\mu) + d_A(\nu,\nu)).I_A \\
&= \frac{1}{3}\psi_A(3d_A(\mu,\nu)).I_A, \\
so,\ 0_A \preceq d_A(\mu,\nu) &\preceq \psi_A(d_A(\mu,\nu)).
\end{aligned}
$$

Because $\psi_A(a) \prec a$, this implies that $0 \preceq d_A(\mu,\nu) \prec d_A(\mu,\nu)$,, which gives a contradiction. Then, we obtain $\mu = \nu$. $\square$

Example 4. *Let Ω be a Banach space and $d_A : \Omega \times \Omega \to A$ be defined as $d_A(\mu,\nu) = \|\mu - \nu\| \cdot I_A$ for all $\mu,\nu \in \Omega$. I_A is the unit of A because Ω is a Banach space. Then, (Ω, A, d_A) is a complete C^*-algebra-valued metric space. Define $T : \Omega \to \Omega$ as $T\mu = 2\mu$ and define $\psi_A : A^+ \to A^+$ as $\psi_A(a) = 3aI_A$ for all $a \in A^+$, where A^+ is the positive cone of A. Additionally, $\alpha_A : \Omega \times \Omega \to A^+$ is defined by $\alpha_A(\mu,\nu) = I_A$, where*

$$\alpha_A(T\mu, T\nu) = \alpha_A(2\mu, 2\nu) = 2\alpha_A(\mu,\nu) = 2I_A \succeq I_A.$$

Now,

$$\begin{aligned} d_A(T\mu, T\nu) &= \|T\mu - T\nu\| \cdot I_A = \|2\mu - 2\nu\| \cdot I_A \\ &= \|2\mu - 2\nu + \nu - \nu + \mu - \mu\| \cdot I_A \\ &= \|(2\mu - \nu) - (2\nu - \mu) - (\mu - \nu)\| \cdot I_A \\ &\preceq (\|2\mu - \nu\| + \|2\nu - \mu\| + \|\mu - \nu\|) \cdot I_A \\ &\preceq (\|T\mu - \nu\| + \|T\nu - \mu\| + \|\mu - \nu\|) \cdot I_A \\ &= (d_A(T\mu,\nu) + d_A(T\nu,\mu) + d_A(\mu,\nu)) \\ &\preceq \frac{1}{3}\psi_A(d_A(T\mu,\nu) + d_A(T\nu,\mu) + d_A(\mu,\nu)). \end{aligned}$$

Applying $\alpha_A(\mu,\nu)$, we obtain

$$\alpha_A(\mu,\nu) d_A(T\mu, T\nu) \preceq \frac{1}{3}\psi_A(d_A(T\mu,\nu) + d_A(T\nu,\mu) + d_A(\mu,\nu)).$$

This satisfies the conditions in Theorem 2. Then, T has a fixed point of Ω.

We introduce a numerical example, assuming that the metric space is valued-non-commutative C^*-algebra $M_2(\mathbb{R})$

Example 5. *Let $\Omega = \mathbb{R}$ and $A = M_2(\mathbb{R})$, where $M_2(\mathbb{R})$ is the set of all 2×2 matrices entries in $\mathbb{R}$. It is obvious that $M_2(\mathbb{R})$ is a C^*-algebra with matrix norm and involution $* : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by $* : a \to a^t$, where a^t is the transpose of a, $a \in M_2(\mathbb{R})$. Define*

$$d_A(\mu,\nu) = \begin{pmatrix} |\mu - \nu| & 0 \\ 0 & k|\mu - \nu| \end{pmatrix},$$

for all $\mu,\nu \in \Omega$, $k > 0$. It is clear that (Ω, A, d_A) is C^*-algebra-valued metric space. To verify the contraction conditions in Theorem 3, we take $\mu = 1$, $\nu = 2$, $k = 3$.

Additionally, we define $T : \Omega \to \Omega$ by $T(\mu) = 2\mu$ and $\alpha_A : \Omega \times \Omega \to M_2(\mathbb{R})^+$ by

$$\alpha_A(\mu,\nu) = 2\begin{pmatrix} |\mu - \nu| & 0 \\ 0 & |\mu - \nu| \end{pmatrix},$$

and $\psi_A : M_2(\mathbb{R})^+ \to M_2(\mathbb{R})^+$, by $\psi_A(a) = 3a$, for $a \in M_2(\mathbb{R})^+$, $\mu,\nu \in Z$, where $M_2(\mathbb{R})^+$ is the set of positive matrices of $M_2(\mathbb{R})$.

Now, by simple calculation, we obtain

$$d_A(\mu,\nu) = d_A(1,2) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix},$$

$$d_A(T\mu, T\nu) = d_A(2,4) = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix},$$

$$d_A(T\mu,\nu) = d_A(2,2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$d_A(T\nu,\mu) = d_A(4,1) = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix},$$

$$d_A(T\mu,\mu) = d_A(4,1) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix},$$

$$d_A(T\nu,\nu) = d_A(4,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

$$\alpha_A(\mu,\nu) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Thus, we calculate the right hand side of the inequality (10) in Theorem 3 as

$$\begin{aligned} M_A(\mu,\nu) &= \frac{1}{3}(d_A(\mu,\nu) + d_A(T\mu,\nu) + d_A(T\nu,\mu) + d_A(T\mu,\mu) + d_A(T\nu,\nu)) \\ &= \frac{1}{3}\begin{pmatrix} 7 & 0 \\ 0 & 21 \end{pmatrix}. \end{aligned}$$

Therefore, $\psi(M_A(\mu,\nu) = \begin{pmatrix} 7 & 0 \\ 0 & 21 \end{pmatrix}$.

On the other hand, the left hand side of the inequality (10) in Theorem 3 is given by $\alpha_A(\mu,\nu)d_A(T\mu,T\nu) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 12 \end{pmatrix}$.

Hence, it is obvious that T is $\alpha_A - \psi_A$*-admissible and, because* $\begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix} \leq \begin{pmatrix} 7 & 0 \\ 0 & 21 \end{pmatrix}$, *we can obtain*

$$\alpha_A(\mu,\nu)d_A(T\mu,T\nu) \leq \psi_A(M_2(\mu,\nu)).$$

Thus, all conditions of Theorem 3 are satisfied. Therefore, there exists a unique fixed point of T, and the zero matrix is the fixed point of $T \in \Omega$.

We discuss a numerical example that satisfies the conditions of Theorem 3, where the metric space in this example is valued-commutative C^*-algebra $\mathbb{C}^2$.

Example 6. *Let* $\Omega = [0,\infty)$ *and* $A = \mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$, *the set of direct sum of two copies of complex numbers.* $\mathbb{C}^2$ *with the vector addition and pointwise multiplication defined by* $(Z_1,Z_2) + (W_1,W_2) = (Z_1 + W_1, Z_2 + W_2)$, *and* $(Z_1,Z_2) \cdot (W_1,W_2) = (Z_1 \cdot W_1, Z_2 \cdot W_2)$, *for all* $Z_1,Z_2,W_1,W_2 \in \mathbb{C}$, *is a* C^**-algebra with the maximum norm given by* $\|(Z_1,Z_2)\| = max\{|Z_1|,|Z_2|\}$, *and involution* $* : \mathbb{C}^2 \to \mathbb{C}^2$ *given by* $(Z_1,Z_2)^* = (\overline{Z_1},\overline{Z_2})$, *for all* $Z_1,Z_2 \in \mathbb{C}$. *Define a partial order* $\preceq$ *on* $\mathbb{C}^2 : (Z_1,Z_2) \preceq (W_1,W_2)$ *if and only if*

(a) $Re(Z_1) \leq Re(W_1)$, $Im\, W_1 \leq Im\, W_1$, *and*

(b) $Re(Z_2) \leq Re(W_2)$, $Im\, W_2 \leq Im\, W_2$.

Thus, $(W_1,W_2) - (Z_1,Z_2) \succeq 0$ *iff* $(Z_1,Z_2) \preceq (W_1,W_2)$. *Additionally,* $(Z_1,Z_2) \succeq 0$ *if* $Z_1 \succeq 0$ *and* $Z_2 \succeq 0$. *In addition,* $Re(Z_1) \geq 0$, $ImZ_1 \geq 0$ *and* $Re(Z_2) \geq 0$, $ImZ_2 \geq 0$

Let $\mathbb{C}^2_+$ *be the set of all positive element in* $\mathbb{C}^2$. *Suppose* $\Omega = [0,\infty)$ *and* $d_A : \Omega \times \Omega \to \mathbb{C}^2$ *be a mapping defined by* $d_A(|\mu-\nu| + i|\mu-\nu|, |\mu-\nu| + 2i|\mu-\nu|)$ *for all* $\mu,\nu \in \Omega$ *and* $i = \sqrt{-1}$.

It is clear that (Ω, A, d_A) *is* C^**-algebra-valued metric space.*

Now, define $T : \Omega \to \Omega$ *by* $T\mu = e^\mu$ *and* $\alpha_A : \Omega \times \Omega \to \mathbb{C}^2_+$ *as* $\alpha_A(\mu,\nu) = I_A$. *In addition, assume* $\psi_A : \mathbb{C}^2_+ \to \mathbb{C}^2_+$ *defined by* $\psi_A(a) = 3a\ \forall\, a \in \mathbb{C}^2_+$.

To verify the contraction conditions in Theorem 3, we take $\mu = 1$, $\nu = 2$. By calculation, one can obtain the following:

$$\begin{aligned}
d_A(\mu,\nu) &= d(1,2) = (1+i, 1+2i), \\
d_A(T\mu,T\nu) &= d_A(e,e^2), \\
&\simeq (4.670+4.670i, 4.670+9.340i), \\
d_A(T\mu,\nu) &= d_A(e,2), \\
&\simeq (0.718+0.718i, 0.718+1.436i), \\
d_A(T\nu,\mu) &= d_A(e^2,1), \\
&\simeq (6.389+6.389i, 6.389+12.778i), \\
d_A(T\mu,\mu) &= d_A(e,1), \\
&\simeq (1.718+1.718i, 1.718+3.436i), \\
d_A(T\nu,\nu) &= d_A(e^2,2), \\
&\simeq (5.389+5.389i, 5.389+10.778i), \\
\alpha_A(\mu,\nu) &= \alpha_A(1,2) = (1,2).
\end{aligned}$$

We calculate the right-hand side of the inequality (10) in the Theorem 3 and obtain

$$\psi_A(M_A(\mu,\nu)) \simeq (15.214+15.214i, 15.214+30.428i).$$

On the other hand, the left-hand side of the inequality (10) in the Theorem 3 gives

$$\alpha_A(\mu,\nu)d_A(T\mu,T\nu) \simeq (4.670+4.670i, 4.670+18.680i).$$

It is clear that $\alpha_A(\mu,\nu)d_A(T\mu,T\nu) \leq \psi_A(M_A(\mu,\nu))$, and this satisfies all conditions of the Theorem 3.

In the following, we provide an application scenario with which to study the existence and uniqueness of the solution of a system of matrix equations. The existence and uniqueness of the solution of the linear matrix equations are very interesting and important in linear systems.

Here, we are interested in using C^*-algebra-valued metric spaces to find a positive definite hermitian solution for a system of matrix equations with complex entries.

The proof is based on the positive cones and the linear continuous operator mapping a cone into itself.

4. Application

Suppose that $M_n(\mathbb{C})$ is the set of all $n \times n$ matrices with complex entries. Additionally, $M_n(\mathbb{C})^+$ is the set of all positive definite matrices of $M_n(\mathbb{C})$. $M_n(\mathbb{C})$ is a Banach space with matrix norm and $M_n(\mathbb{C})$ is also a C^*-algebra with matrix norm and the involution $* : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, $Z \to Z^*$, where $Z = (Z_{ij})_{1\leq i,j\leq n} \in M_n(\mathbb{C})$ and $Z^* = (Z_{ij})^*_{1\leq i,j\leq n} = (\overline{Z_{ji}})_{1\leq i,j\leq n} \in M_n(\mathbb{C})$.

Let $A_1, A_2, ..., A_n \in M_n(\mathbb{C})$, $Z, W \in M_n(\mathbb{C})$. Additionally, $Q \in M_n(\mathbb{C})^+$. Then, the matrix equation

$$Z - \sum_{k=1}^{n} A_k^* Z A_k = Q \tag{14}$$

has a unique solution.

Proof. For $Z, W \in M_n(\mathbb{C})$, define $d_{M_n(\mathbb{C})} : M_n(\mathbb{C}) \times M_n(\mathbb{C}) \to M_n(\mathbb{C})$, as

$$d_{M_n(\mathbb{C})}(Z,W) = \|Z-W\| \cdot I_{d_{M_n(\mathbb{C})}}.$$

Then, $(M_n(\mathbb{C}), d_{M_n(\mathbb{C})})$ is a C^*-algebra-valued metric space and is complete, because the set $M_n(\mathbb{C})$ is complete. Consider $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, defined by $T(Z) = \sum_{k=1}^{n} A_k^* Z A_k + Q$. Additionally, $\psi_{M_n(\mathbb{C})}(Z) = 3Z \cdot I_{M_n(\mathbb{C})}$. Define

$$\alpha_{M_n(\mathbb{C})} : M_n(\mathbb{C}) \times M_n(\mathbb{C}) \to M_n(\mathbb{C})^+$$

$$\alpha_{M_n(\mathbb{C})}(Z, W) = I_{M_n(\mathbb{C})}.$$

It is clear that T is $\alpha_{M_n(\mathbb{C})} - \psi_{M_n(\mathbb{C})}$ admissible. Then,

$$\begin{aligned}
d_{M_n(\mathbb{C})}(TZ, TW) &= \|TZ - TW\| \cdot I_{M_n(\mathbb{C})} \\
&= \|(\sum_{k=1}^{n} A_k^* Z A_k + Q) - (\sum_{k=1}^{n} A_k^* W A_k + Q)\| \cdot I_{M_n(\mathbb{C})} \\
&= \|((\sum_{k=1}^{n} A_k^* Z A_k + Q) - W) - ((\sum_{k=1}^{n} A_k^* W A_k + Q) - Z) - (Z - W)\| \cdot I_{M_n(\mathbb{C})} \\
&\preceq \|((\sum_{k=1}^{n} A_k^* Z A_k + Q) - W)\| \cdot I_{M_n(\mathbb{C})} + ((\sum_{k=1}^{n} A_k^* W A_k + Q) - Z)\| \cdot \|I_{M_n(\mathbb{C})} \\
&+ \|(Z - W)\| \cdot I_{M_n(\mathbb{C})} \\
&= \|TZ - W\| \cdot I_{M_n(\mathbb{C})} + \|TW - Z\| \cdot I_{M_n(\mathbb{C})} + \|(Z - W)\| \cdot I_{M_n(\mathbb{C})} \\
&= d_{M_n(\mathbb{C})}(TZ, W) + d_{M_n(\mathbb{C})}(TW, Z) + d_{M_n(\mathbb{C})}(Z, W) \\
&\preceq \frac{1}{3}\psi_{M_n(\mathbb{C})}(d_{M_n(\mathbb{C})}(TZ, W) + d_{M_n(\mathbb{C})}(TW, Z) + d_{M_n(\mathbb{C})}(Z, W)).
\end{aligned}$$

Thus,

$$\alpha_A(Z, W) d_{M_n(\mathbb{C})}(TZ, TW) \preceq \frac{1}{3}\psi_{M_n(\mathbb{C})}(d_{M_n(\mathbb{C})}(TZ, W) + d_{M_n(\mathbb{C})}(TW, Z) + d_{M_n(\mathbb{C})}(Z, W)).$$

This satisfies the conditions of Theorem 2. Thus, the system of matrix Equation (14) has a unique hermitian matrix solution. □

5. Conclusions

In this paper, we provide some results obtained for the Chatterjea and Čirič fixed-point theorems by using α_A-ψ_A-contractive mapping in a C^*-algebra-valued metric space. Furthermore, illustrated examples and an application scenario are introduced. It is worth mentioning that these results generalize and extend some results described in [1–3,5,9,23,24,26–30].

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