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Graph Colorings and Labelings Having Multiple Restrictive Conditions in Topological Coding

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Abstract: With the fast development of networks, one has to focus on the security of information running in real networks. A technology that might be able to resist attacks equipped with AI techniques and quantum computers is the so-called topological graphic password of topological coding. In order to further study topological coding, we use the multiple constraints of graph colorings and labelings to propose 6C-labeling, 6C-complementary labeling, and its reciprocal-inverse labeling, since they can be applied to build up topological coding. We show some connections between 6C-labeling and other graph labelings/colorings and show graphs admitting twin-type 6C-labelings, as well as the construction of graphs admitting twin-type 6C-labelings.

Keywords: topological authentication; coloring; 6C-labeling; reciprocal-inverse matching; topological coding

MSC: 05C90

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1. Introduction

In addition to RSA, DSA, and ECDSA [1], there are many important cryptographic systems that are considered to be resistant to classical and quantum computers, such as hash-based cryptography, code-based cryptography [2,3], lattice-based cryptography [4], and key cryptography. Another technology that might be able to resist attacks equipped with AI techniques and quantum computers is the so-called topological graphic passwordof topological coding. The foundation of topological cryptography is based on topological graphic passwords consisting of topological structures and mathematical restrictions [5], and topological graphic passwords belong to a combined branch of topological coding, graph theory, and cryptography. Since topological graphic passwords are related to many mathematical conjectures and NP-hard problems [6,7], topological graphic passwords are computationally unbreakable or have provable security, and the investigation of topological graphic passwords was introduced in [8–10].

Topological authentication is a new technique based on topological coding, a mixed branch of discrete mathematics, number theory, algebraic groups, graph theory, and so on. Wang et al. in [8] designed a topological code consisting of a topological structure and graph colorings and labeling. Graph labelings were first introduced in the mid-1960s. In the intervening years, over 200 graph labeling techniques have been studied in over 3000 papers, which provides some technical support for topology coding. As is known, topological coding is related to many mathematical conjectures or NP-hardproblems in *graph theory* [11], and *operations research* [7], so topological coding is *computationally unbreakable* or *has provable security*.

1.1. An Example of Topological Cryptosystems

To understand topological graphic passwords, we provide an example as follows: Four colored graphs H, H_1, H_2, H_3 shown in Figure 1 are topological graphic passwords in topological coding, and they are used as public keys and private keys. The first graph H is a public key, and H_1, H_2, H_3 are private keys, respectively. We obtain a matrix for vertices, edges, and vertices in order, which we call a topological matrix; a topological matrix of order $3 \times q$ is a graph G with p vertices and q edges, that is

$$T_{code} = \begin{pmatrix} \alpha(x_1) & \alpha(x_2) & \cdots & \alpha(x_q) \\ \alpha(x_1y_1) & \alpha(x_2y_2) & \cdots & \alpha(x_qy_q) \\ \alpha(y_1) & \alpha(y_2) & \cdots & \alpha(y_q) \end{pmatrix}_{3\times q} \tag{1}$$

α represents a function to obtain the color of vertices and edges. The topological matrix of G can derive $(3q)!$ number strings, which can be used as digital-based passwords, and preset coloring may be needed during authentication, called *topological authentication*. We obtain the topological matrix $T_{code}(H)$ from Figure 1 and then convert it into a many-number-based string $N_i(H), i \in [1, n]$ as follows:

$$T_{code}(H) = \begin{pmatrix} 0 & 0 & 2 & 2 & 2 & 6 & 6 & 6 & 8 & 8 & 10 \\ 21 & 17 & 19 & 15 & 13 & 9 & 5 & 11 & 7 & 3 & 1 \\ 21 & 17 & 21 & 17 & 15 & 15 & 11 & 17 & 15 & 11 & 11 \end{pmatrix} \tag{2}$$

$$N_1(H) = 021210171721921215172131569156511611178715831110111$$

$$N_2(H) = 002226668810137115913151917212117211715151117151111$$

$$N_3(H) = 101000038871517116651115962131517152219211717002121$$

and so on. In the same way, the topological matrix and number-based string of the other three private keys can be obtained.

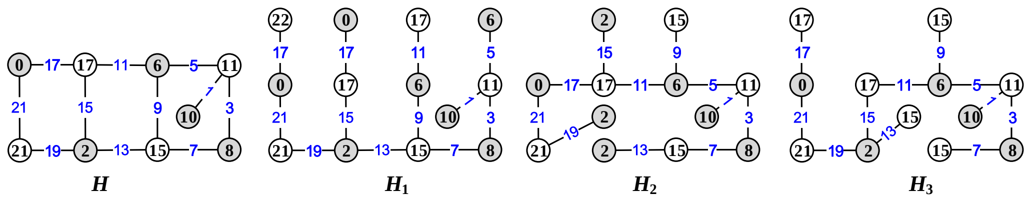

Figure 1. A set-ordered odd-graceful topological public key H and its own topological private keys H_1, H_2, H_3.

We use the multiple constraints of graph colorings and labelings to propose 6C-labeling, 6C-complementary labeling, and its reciprocal-inverse labeling, since they can be applied to building up topological coding. We show some connections between 6C-labeling and other graph labelings/colorings and show graphs admitting twin-type 6C-labelings, as well as the construction of graphs admitting twin-type 6C-labelings and constructing graph classes with new labels to provide a new technology for topological coding. See the examples in Figures 2 and 3.

1.2. Definitions

The standard terminology and notation of graph theory can be found in [12,13]. The following terminology, notation, labelings, particular graphs, and definitions will be used in later discussions:

- A symbol $[a, b]$ stands for a consecutive set $\{a, a+1, \dots, b\}$ with integers a, b holding $0 \le a < b$; $[a, b]^o$ denotes an *odd-set* $\{a, a+2, \dots, b\}$ with odd integers a, b with respect to $1 \le a < b$; $[\alpha, \beta]^e$ is an *even-set* $\{\alpha, \alpha+2, \dots, \beta\}$ with even integers α, β.
- The cardinality of a set X is denoted as $|X|$.
- The number $\deg_G(v) = |N(v)|$ is called the *degree* of the vertex v, where $N(v)$ is the set of neighbors of the vertex v. If $\deg_G(v) = 1$. we call the vertex v a *leaf*.
- G is a (p, q)-graph having p vertices and q edges.

Definition 1 ([13]). *Let a connected (p, q)-graph G with $1 \le p-1 \le q$ admit a mapping $\alpha : V(G) \to \{0, 1, 2, \dots\}$. For each $xy \in E(G)$, the labelings are defined as $\alpha(xy) = |\alpha(x) - \alpha(y)|$, and we write the vertex color set by $\alpha(V(G)) = \{\alpha(u) : u \in V(G)\}$ and the edge color set as $\alpha(E(G)) = \{\alpha(xy) : xy \in E(G)\}$. We have the following restrictions:*

(1) $|\alpha(V(G))| = p$;
(2) $\alpha(V(G)) \subseteq [0, q]$, $\min \alpha(V(G)) = 0$;
(3) $\alpha(V(G)) \subset [0, 2q-1]$, $\min \alpha(V(G)) = 0$;
(4) $\alpha(E(G)) = \{\alpha(xy) : xy \in E(G)\} = [1, q]$;
(5) $\alpha(E(G)) = \{\alpha(xy) : xy \in E(G)\} = [1, 2q-1]^o$;
(6) $\alpha(uv) = \alpha(u) + \alpha(v)\ (mod\ 2q)$;
(7) *G is a bipartite graph with the bipartition (X, Y) such that* $\max\{\alpha(x) : x \in X\} < \min\{\alpha(y) : y \in Y\}$ *($\alpha(X) < \alpha(Y)$ for short).*

We define: a graceful labeling *α satisfying the restriction (1), (2), and (4); a* set-ordered graceful labeling *α satisfies (1), (2), (4), and (7), simultaneously; an* odd-graceful labeling *α satisfies the restriction (1), (3), and (5); a* set-ordered odd-graceful labeling *θ satisfies the restriction (1), (3), (5), and (7); a* set-ordered odd-elegant labeling *α satisfies the restriction (1), (3), (5), (6), and (7).*

Definition 2 ([14]). *A total labeling $\alpha : V(G) \cup E(G) \to [1, p+q]$ for a bipartite (p, q)-graph G is a bijection holding:*

(i) $\alpha(uv) + |\alpha(u) - \alpha(v)| = k$;

(ii) Each edge uv corresponding to another edge xy holds $\alpha(uv) = |\alpha(x) - \alpha(y)|$ (or $\alpha(uv) = (p+q+1) - |\alpha(x) - \alpha(y)|$);

(iii) Let $s(uv) = |\alpha(u) - \alpha(v)| - \alpha(uv)$ for $uv \in E(G)$, then there exists a constant k' such that each edge uv corresponds to another edge $u'v'$ holding $s(uv) + s(u'v') = k'$ (or $(p+q+1) + s(uv) + s(u'v') = k'$) true;

(iv) $\min \alpha(V(G)) > \max \alpha(E(G))$ (or $\max \alpha(V(G)) < \min \alpha(E(G))$, or $\alpha(V(G)) \subseteq \alpha(E(G))$, or $\alpha(E(G)) \subseteq \alpha(V(G))$, or $\alpha(V(G))$ is an odd-set, and $\alpha(E(G))$ is an even-set;

(v) (ve-corresponding) each edge uv corresponds to one vertex w such that $\alpha(uv) + \alpha(w) = k''$, where k'' is a fixed constant, and each vertex z corresponds to one edge xy such that $\alpha(z) + \alpha(xy) = k''$, except the singularity $\alpha(x_0) = \lfloor \frac{p+q+1}{2} \rfloor$;

(vi) $\max \alpha(X) < \min \alpha(Y)$ (or $\min \alpha(X) > \max \alpha(Y)$) for the bipartition (X, Y) of $V(G)$.

Then, α is called a 6C-labeling *of the bipartite (p, q)-graph G.*

Definition 3. *Let (p, q)-tree G admit a 6C-labeling f and g be a 6C-labeling of (p, q)-tree H; if they hold, $f(V(G)) \setminus X^* = g(E(H))$, $f(E(G)) = g(V(H)) \setminus X^*$ and $f(V(G)) \cap g(V(H)) = X^* = \{z_0\}$ with $z_0 = \lfloor \frac{p+q+1}{2} \rfloor$, then f and g are pairwise reciprocal-inverse. The graph $\odot_1\langle G, H\rangle$ obtained by the coinciding of the vertex x_0 of G having $f(x_0) = z_0$ with the vertex w_0 of H having $g(w_0) = z_0$ is called a 6C-complementary labeling.*

Definition 4. *Let (p, q)-graph G admit a total labeling $f : V(G) \cup E(G) \to [1, p+q]$ and a (q, p)-graph H admit another total labeling $g : V(H) \cup E(H) \to [1, p+q]$. If $f(E(G)) = g(V(H)) \setminus X^*$ and $f(V(G)) \setminus X^* = g(E(H))$ for $X^* = f(V(G)) \cap g(V(H))$, then f and g are reciprocal-inverse (or reciprocal complementary) to each other, and H (or G) is an inverse labeling of G (or H).*

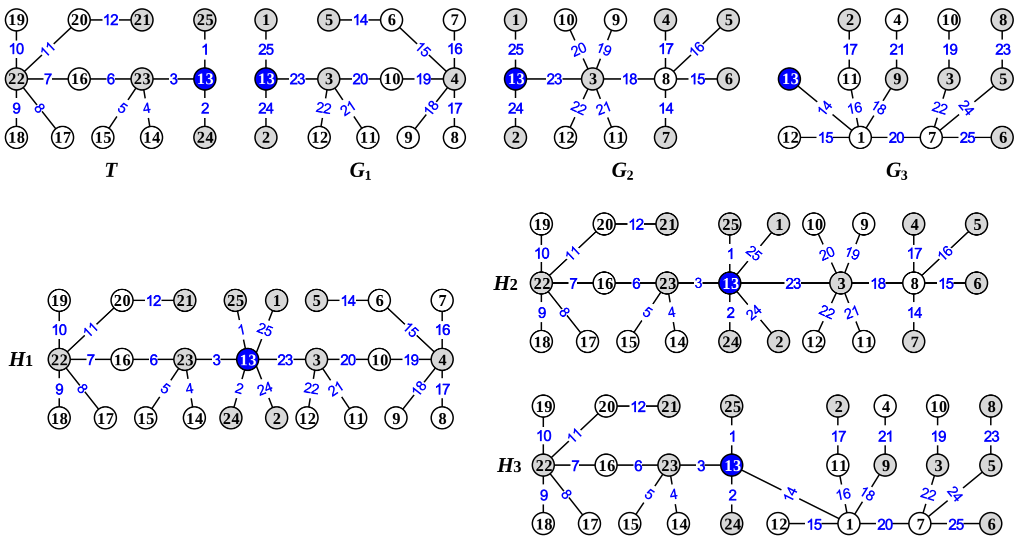

Figure 2. Examples The graphs T, G_1, G_2, G_3 for illustrating the 6C-labeling defined in Definition 2, the graphs TH_1, H_2, H_3 reciprocal-inverse labeling defined in Definition 3, and the 6C-complimentary matching defined in Definition 4.

Example 1. *In Figure 3, a tree T admits a 6C-labeling f_T, and other trees G_i admit a 6C-labeling f_i for $i = 1, 3$, where:*

(i) $f_T(uv) + |f_T(u) - f_T(v)| = 13$ for each edge $uv \in E(T)$;

(ii) $f_1(xy) + |f_1(x) - f_1(y)| = 13$ for each edge $xy \in E(G_1)$;

(iii) $f_3(xy) + |f_3(x) - f_3(y)| = 26$ for each edge $xy \in E(G_3)$;

The graph T and G_2 are as defined, then f_T and g are reciprocal-inverse (or reciprocal complementary) to each other, and T (or G_2) is an inverse labeling of G_2 (or T).

Moreover, each 6C-labeling f_i is a reciprocal-inverse labeling of the 6C-labeling f_T, that is $f_T(V(T)) \setminus \{13\} = f_i(E(G_i))$ and $f_T(E(T)) = f_i(V(G_i)) \setminus \{13\}$ for $i = [1, 3]$.

The 6C-labeling f_T and its reciprocal-inverse labeling f_i form a 6C-complimentary labeling $\langle f_T, f_i \rangle$ for $i = [1, 3]$. Each vertex-coinciding tree $H_i = \odot\langle T, G_i \rangle$ admits a 6C-complimentary labeling $g_i = \langle f_T, f_i \rangle$ for $i = [1, 3]$, where $\odot\langle T, G_1 \rangle$ is a self-isomorphic ve-image, since $T \cong G_1$;

(iv) $g_j(uv) + |g_j(u) - g_j(v)| = 13$ or $g_j(uv) - |g_j(u) - g_j(v)| = 13$ with $uv \in E(H_j)$ and $j = 1, 2$;

(v) $g_3(xy) + |g_3(x) - g_3(y)| = 13$ or $g_3(xy) + |g_3(x) - g_3(y)| = 26$ with $xy \in E(H_3)$.

If a (p, q)-graph T has two subgraphs T_1 and T_2 such that $V(T_1) \cap V(T_2) = \{w\}$ and $E(T) = E(T_1) \cup E(T_2)$, we denote T as $T = T_1 \diamond T_2$, called a *vertex-identified graph* (vi-graph for short). Moreover, we call T a uniformly vertex-identified graph (uniformly vi-(p, q)-graph) if $q = 2|E(T_1)| = 2|E(T_2)|$.

Definition 5. *Let a uniformly vi-(p, q)-graph $T = T_1 \diamond T_2$ have a mapping f: $V(T) \to [0, q]$ such that:*

(i) $f(x) \neq f(y)$ for any pair of vertices $x, y \in V(T)$;

(ii) f is an odd-graceful labeling (ogl) of T_1;

(iii) The edge label set $f(E(T_1)) = \{f(uv) = |f(u) - f(v)| : uv \in E(T_1)\} = [1, q-1]^o$.

Then, T is called a twin odd-graceful vi-(p, q)-graph*, f a* twin odd-graceful labeling (togl) *of T, T_1 a* source graph*, and T_2 an* associated graph *of T_1.*

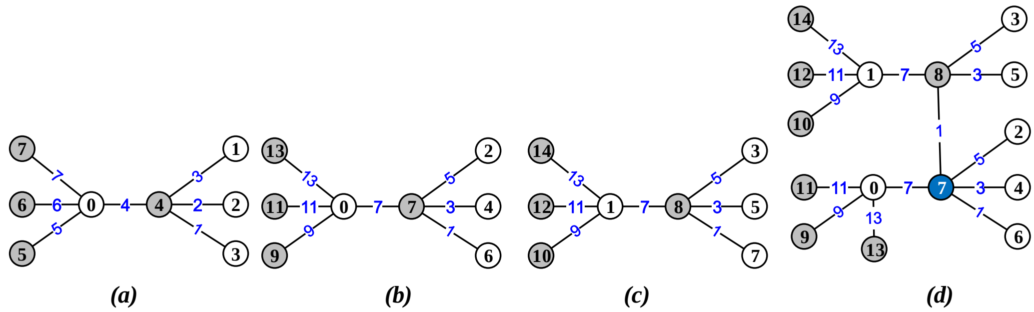

Figure 3. Examples for illustrating Corollary 2. (**a**–**d**) are T, T', T'', and $T' \diamond T''$, respectively.

2. Connections between 6C-Labeling and Other Labelings

Lemma 1. *A (p,q)-tree T admits a set-ordered graceful labeling if and only if it admits a 6C-labeling.*

Proof. Let $X = \{x_i : i \in [1,s]\}$ and $Y = \{y_j : j \in [1,t]\}$ be the bipartition of vertex set (X,Y) of a tree T, where $|V(T)| = p = s+t$ and $|E(T)| = s+t-1$.

First, notice that σ is a set-ordered graceful labeling of T, then $\sigma(x_i) = i-1$ for $i \in [1,s]$ and $\sigma(y_j) = s+j-1$ for $j \in [1,t]$, and each edge $x_iy_j \in E(T)$ has its label $\sigma(x_iy_j) = \sigma(y_j) - \sigma(x_i) = s+j-i$.

Another labeling Φ is defined for the tree T as: $\Phi(w) = p + \sigma(w)$ for $w \in V(T)$, and

$$\Phi(x_iy_j) = p - \sigma(x_iy_j) = p - |\sigma(x_i) - \sigma(y_j)| \tag{3}$$

for $x_iy_j \in E(T)$. The vertex set and edge set is

$$\Phi(V(T)) = [p, 2p-1], \quad \Phi(E(T)) = [1, p-1]. \tag{4}$$

then we obtain:

(i) $\Phi(x_iy_j) + |\Phi(x_i) - \Phi(y_j)| = p - \sigma(x_iy_j) + \sigma(x_iy_j) = p$.

(ii) Each edge $x_iy_j \in E(T)$ with another edge $x'_iy'_j \in E(T)$ corresponds and holds $p - \sigma(x_iy_j) = \sigma(x'_iy'_j)$ such that

$$\begin{aligned}\Phi(x_iy_j) &= p - \sigma(x_iy_j) = \sigma(x'_iy'_j) = |\sigma(x'_i) - \sigma(y'_j)| \\ &= |p + \sigma(x'_i) - [p + \sigma(y'_j)]| = |\Phi(x'_i) - \Phi(y'_j)|.\end{aligned} \tag{5}$$

(iii) Let $s(x_iy_j) = |\Phi(x_i) - \Phi(y_j)| - \Phi(x_iy_j)$ for $x_iy_j \in E(T)$, so

$$s(x_iy_j) = |\Phi(x_i) - \Phi(y_j)| - \Phi(x_iy_j) = |\sigma(x'_i) - \sigma(y'_j)| - p + \sigma(x_iy_j) = 2\sigma(x_iy_j) - p, \tag{6}$$

then we obtain a set $[2-p, p-2]^e$, where p is even, or a set $[2-p, p-2]^o$, where p is odd. Therefore, each edge $x_iy_j \in E(T)$ and another edge $x''_iy''_j \in E(T)$ correspond, so $s(x_iy_j) + s(x''_iy''_j) = 0$, except that edge e holds $s(e) = 0$ as p is even.

(iv) $\max \Phi(E(T)) < \min \Phi(V(T))$ from Equation (4).

(v) $\Phi(uv) + \Phi(w) = 2p$ for each edge uv corresponding to one vertex w, and $\Phi(z) + \Phi(xy) = 2p$ for each vertex z corresponding to one edge xy, except the *singularity* $f(w') = p$.

(vi) We obtain $\max f(X) < \min f(Y)$ for the bipartition (X,Y) of $V(G)$.

Therefore, Φ admits a 6C-labeling.

For the converse, let ψ be a 6C-labeling of T. According to the property (iv) and $\psi(V(T) \cup E(T)) = [1, 2p-1]$, we obtain the edge label and the vertices' label set $\psi(E(T)) = [1, p-1]$ and $\psi(V(T)) = [p, 2p-1]$, respectively. A labeling ψ^* is defined as: $\psi^*(w) = \psi(w) - p$ for $w \in V(T)$, which gives $\psi^*(V(T)) = [0, p-1]$; $\psi^*(x_iy_j) = p - \psi(x_iy_j)$ for each edge $x_iy_j \in E(T)$, so $\psi^*(E(T)) = [1, p-1]$. The property (i) enables us to compute

$$\begin{aligned}\psi^*(x_iy_j) &= p - \psi(x_iy_j) = p - [p - |\psi(x_i) - \psi(y_j)|] \\ &= |[\psi(x_i) - p] - [\psi(y_j) - p]| = |\psi^*(x_i) - \psi^*(y_j)|,\end{aligned} \tag{7}$$

that is ψ^* is graceful. The graceful labeling ψ^* is set-ordered according to the property (vi). □

Theorem 1. *If two (p,q)-trees admit set-ordered graceful labelings, then they are a 6C-complementary labeling.*

Proof. Let each tree T_i of p vertices admit a set-ordered graceful labeling f_i and (X_i, Y_i) be the bipartition of T_i with $i = 1,2$. Therefore, we have $\max f_i(X_i) < \min f_i(Y_i)$ where $X_i = \{x_{i,j} : j \in [1, s_i]\}$ and $Y_i = \{y_{i,j} : j \in [1, t_i]\}$ for $s_i + t_i = p$ with $i = 1,2$. Then, we can label the vertices' set as $f_i(x_{i,j}) = j - 1$ for $j \in [1, s_i]$, $f_i(y_{i,j}) = s_i + j - 1$ for $j \in [1, t_i]$ and label the edge set as $f_i(x_{i,s}y_{i,t}) = f_i(y_{i,t}) - f_i(x_{i,s}) = s_i + t - s$ for each edge $x_{i,s}y_{i,t} \in E(T_i)$, and $f_i(E(T_i)) = [1, p]$ for $i = 1,2$.

We define another labeling f_1^* of T_1 as: $f_1^*(w) = p + f_1(w)$ for $w \in V(T_1)$ and $f_1^*(x_{1,s}y_{1,t}) = p - f(x_{1,s}y_{1,t})$ for each edge $x_{1,s}y_{1,t} \in E(T_1)$. Therefore, we can compute $f_1^*(V(T_1)) = [p, 2p - 1]$ and $f_1^*(E(T_1)) = [1, p - 1]$.

Next, we define another labeling f_2^* of T_2 as: $f_2^*(w) = f_2(w) + 1$ for each vertex $w \in V(T_2)$ and $f_2^*(x_{2,i}y_{2,j}) = p + f_2(x_{2,i}y_{2,j})$ for each edge $x_{2,i}y_{2,j} \in E(T_2)$. Thereby, we obtain $f_2^*(V(T_2)) = [1, p]$, $f_2^*(E(T_2)) = [p + 1, 2p - 1]$.

Notice that $f_1^*(V(T_1)) \setminus \{p\} = f_2^*(E(T_2))$ and $f_1^*(E(T_1)) = f_2^*(V(T_2)) \setminus \{p\}$. By Lemma 1, we have proven the theorem. □

Theorem 2. *If a (p,q)-tree T admits a 6C-labeling such that $\odot_1\langle T, T'\rangle$ is a 6C-complementary matching, where the tree T' is a 6C-labeling tree.*

Proof. Let tree T admit a 6C-labeling f, another labeling g of T as $g(x) = 2(p + q + 1) - f(x)$ for each vertex $x \in V(T)$ and $g(uv) = (p + q + 1) - f(uv)$ for each edge $uv \in E(T)$. We obtain $E(T) = \{u_iv_i : i \in [1, q]\}$ with $g(u_jv_j) < g(u_{j+1}v_{j+1})$ true with $j \in [1, q - 1]$. Next, a labeling π is defined for a copy T' of T in this way: $\pi(x) = g(x)$ for $x \in V(T') = V(T)$, and $\pi(u_iv_i) = 2p + q + 1 - g(u_iv_i)$ for $u_iv_i \in E(T') = E(T)$. Notice that

$$\begin{aligned}\pi(u_iv_i) &= 2p + q + 1 - g(u_iv_i) = 2p + q + 1 - [(p + q + 1) - f(u_iv_i)] \\ &= f(u_iv_i) + p\end{aligned} \tag{8}$$

for $u_iv_i \in E(T') = E(T)$. We claim that π is a 6C-labeling of T' as well, which means that the proof of Theorem 2 is complete. □

Corollary 1. *Let two trees T and H with p vertices admit set-ordered graceful labelings. Then, $G^* = \odot_1\langle T, H\rangle$ admits a 6C-labeling τ with $\tau(xy) + |\tau(x) - \tau(y)| = p$ for each edge $xy \in E(T)$, and $\tau(uv) - |\tau(u) - \tau(v)| = p$ for each edge $xy \in E(H)$.*

Corollary 2. *Let a tree T admit a set-ordered odd-graceful labeling, then there exists H admitting a twin odd-graceful labeling, where H is a self-corresponding $H = T' \diamond T''$ graph.*

Proof. According to the supposition of the corollary, let a tree T have its own vertex bipartition (X, Y) with $X = \{x_1, x_2, \dots, x_s\}$ and $Y = \{y_1, y_2, \dots, y_t\}$ with $|V(T)| = p = s + t$ and $|E(T)| = s + t - 1 = p - 1$. For a set-ordered graceful labeling f of T, we obtain $\phi(x_i) = i - 1$ for $i \in [1, s]$ and $\phi(y_j) = s + j - 1$ for $j \in [1, t]$, and $\phi(x_iy_j) = \phi(y_j) - \phi(x_i) = s + j - i$ for each edge $x_iy_j \in E(T)$:

(1) The labeling ϕ^* is defined for a copy T' of T with $(X', Y') = (X, Y)$ as: $\phi^*(x_i') = 2f(x_i) = 2(i - 1)$ for $i \in [1, s]$ and $\phi^*(y_j') = 2f(y_j) - 1 = 2(s + j) - 3$ for $j \in [1, t]$; immediately,

$$\phi^*(x'_iy'_j) = |\phi^*(y'_j) - \phi^*(x'_i)| = |2(s+j) - 3 - 2(i-1)| = 2s + 2(j-i) - 1 \\ = 2(s+j-i) - 1 = 2\phi(x_iy_j) - 1. \tag{9}$$

Therefore, ϕ_1^* is an *ogl* of T', since $\phi_1^*(X') = [0, 2(s-1)]^e$ is an even-set, $\phi_1^*(Y') = [2s-1, 2p-3]^o$ is an odd-set, and $\phi_1^*(E(T)) = [1, 2p-3]^o$ is an odd-set as well. Next, we another another graph T'' obtained by copying T with $(X'', Y'') = (X, Y)$ and make a *complementary labeling* ϕ_2^* of the *ogl* ϕ_1^* by setting $\phi_2^*(w) = \phi_1^* + 1$ for $w \in V(T)$; clearly, $\phi_2^*(E(T)) = \phi_1^*(E(T))$. Moreover, $\phi_2^*(X) = [1, 2s-1]^o$, $\phi_2^*(Y) = [2s, 2p-2]^e$; we can see $\phi_2^*(V(T)) \cap \phi_1^*(V(T)) = \{2s-1\}$ and $\phi_2^*(V(T)) \cup \phi_1^*(V(T)) = [0, 2p-2]$. Therefore, T'' is the *complementary* of T'. Therefore, $T' \diamond T''$ admits a *togl*.

The proof is finished for the corollary. □

Theorem 3. *Let $T_1, T_2, \dots, T_{m-1}, T_m$ be disjoint trees, and each T_i admit a set-ordered graceful labeling for $i \in [1, m-1]$; T_m is a graceful tree. Then, there are vertices $u_i \in V(T_i)$ $(i \in [1, m])$ such that joining $u_j \in V(T_j)$ and $u_{j+1} \in V(T_{j+1})$ for $j \in [1, m-1]$ produces a new tree T admitting a 6C-labeling.*

Proof. For each $i \in [1, m]$, each tree T_i has n_i vertices and bipartition (X_i, Y_i), where $X_i = \{x_{i,r} : r \in [1, s_i]\}$ and $Y_i = \{y_{i,j} : j \in [1, t_i]\}$ with $s_i + t_i = |V(T_i)| = n_i$.

By the assumption of the theorem, each tree T_i for $i \in [1, m-1]$ has a set-ordered graceful coloring f_i holding $f_i(X_i) < f_i(Y_i)$, $f_i(x_{i,r}) = r - 1$, $f_i(y_{i,j}) = s_i - 1 + j$, as well as $f_i(x_{i,r}y_{i,j}) = |f_i(x_{i,r}) - f_i(y_{i,j})| = s_i + j - r$, which shows that $f_i(V(T_i)) = [0, n_i - 1]$ and $f_i(E(T_i)) = [1, n_i - 1]$. T_m has a graceful labeling f_m defined as $f_m(x_i) = i - 1$ for $x_i \in V(T_m)$ and $i \in [1, n_m]$ such that $f_m(V(T_m)) = [0, n_m - 1]$ and $f_m(E(T_m)) = [1, n_m - 1]$.

We join the vertex $y_{l,1} \in V(T_l)$ with the vertex $x_{l+1,1} \in V(T_{l+1})$ by an edge, $l \in [1, m-2]$; we join the vertex $y_{m-1,1} \in V(T_{m-1})$ with the vertex $x_1 \in V(T_m)$ by an edge, so the resulting tree is denoted as T. Next, we define a labeling g of T in the following steps. Let $A(a, b) = \sum_{l=a}^{b} n_l$ and $B(a, b) = \sum_{l=a}^{b} s_l$, $A(1, m) = \sum_{l=1}^{m} n_l = p$.

We define another labeling g as follows:

Step 1.For each $k \in [1, m-1]$, the vertices' color of $x_{i,r}$ as $g(x_{i,r}) = f_i(x_{r,i}) + B(1, i-1) + A(1, m)$, $i \in [1, s_i]$, where $x_{i,r} \in X_i \subset V(T_i)$.

Step 2. The vertices' color of x_i as $g(x_i) = f_m(x_i) + B(1, m-1) + A(1, m)$, $x_i \in V(T_m)$, where $i \in [1, n_m]$.

Step 3. The vertices' color of $y_{i,j}$ as $g(y_{i,j}) = f_i(y_{i,j}) + B(1, i-1) + A(i+1, m) + A(1, m)$ for each $i \in [1, m-1]$, $j \in [1, t_i]$, where $y_{i,j} \in Y_i \subset V(T_i)$.

Step 4. The edges $x_{i,r}y_{i,j}$ are colored as $g(x_{i,r}y_{i,j}) = A(1, m) - |g(x_{i,r}) - g(y_{i,j})|$ for $i \in [1, m-1]$, $g(x_ix_j) = A(1, m) - |g(x_i) - h(x_j)|$, $i, j \in [1, n_m]$.

Step 5. The edges $x_{l+1,1}y_{l,1}$ are colored as $h(x_{l+1,1}y_{l,1}) = A(1, m) - \{h(y_{l,1}) - h(x_{l+1,1})\} = A[l+1, m]$ for $l \in [1, m-2]$, $h(x_1y_{m-1,1}) = A(1, m) - \{h(y_{m-1,1}) - h(x_1)\} = n_m$.

We can verify the vertices' and edges' set as follows:

$h(x_{i,r}) \in [A(1, m), B(1, m-1) + A(1, m) - 1]$, $i \in [1, s_i]$, $i \in [1, m-1]$;

$h(x_i) \in [B(1, m-1) + A(1, m), B(1, m-1) + A(1, m) + n_m - 1]$, $i \in [1, n_m]$;

$h(y_{i,j}) \in [B(1, m-1) + A(1, m) + n_m, 2A(1, m) - 1]$, $j \in [1, t_i]$, $i \in [1, m-1]$;

$g(x_{i,r}y_{i,j}) = A(1, m) - |g(x_{i,r}) - g(y_{i,j})| = A(1, m) - \{f_i(y_{i,j}) - f_i(x_{i,r}) + A(i+1, m)\} = A(1, m) - f_i(x_{i,r}y_{i,j}) - A(i+1, m) \in [1 + A(i+1, m), A(i, m) - 1]$, $i \in [1, m-1]$;

$h(x_ix_j) \in [1, n_m - 1]$ for $x_ix_j \in E(T_m)$.

Let us continue to validate the restriction of labeling g as follows:

(i) Each edge $x_{i,r}y_{i,j} \in E(T)$:

$$g(x_{i,r}y_{i,j}) + |g(x_{i,r}) - g(y_{i,j})| = p - |f(x_{i,r}y_{i,j}) - f(x_{i,r}y_{i,j})| = A(1, m) \tag{10}$$

holds true.

(ii) Each edge $x_{i,r}y_{i,j} \in E(T)$ corresponding to another edge $x'_{i,r}y'_{i,j} \in E(T)$ holds $p - g(x_{i,r}y_{i,j}) = g(x'_{i,r}y'_{i,j})$ such that

$$\begin{aligned} f(x_{i,r}y_{i,j}) &= p - g(x_iy_j) = g(x'_{i,r}y'_{i,j}) = |g(x'_{i,r}) - g(y'_{i,j})| \\ &= |p + g(x'_{i,r}) - [p + g(y'_{i,j})]| = |f(x'_{i,r}) - f(y'_{i,j})|. \end{aligned} \tag{11}$$

(iii) Let $s(x_iy_j) = |f(x_i) - f(y_j)| - f(x_iy_j)$ for $x_iy_j \in E(T)$, so

$$\begin{aligned} s(x_iy_j) &= |f(x_i) - f(y_j)| - f(x_iy_j) = |g(x'_i) - g(y'_j)| - p + g(x_iy_j) \\ &= 2g(x_iy_j) - p, \end{aligned} \tag{12}$$

which distributes a set $[2-p, p-2]^e$ if p is even or a set $[2-p, p-2]^o$ if p is odd. Thereby, each edge $x_iy_j \in E(T)$ corresponds to another edge $x''_iy''_j \in E(T)$ such that $s(x_iy_j) + s(x''_iy''_j) = 0$, except that edge e holds$s(e) = 0$ as p is even.

The proof of (iv)–(vi) is the same as Lemma 1.

Hence, we claim that the labeling f admits really a 6C-labeling defined in Definition 2.

Thereby, we claim that $g(V(T)) = [A(1,m), 2A(1,m) - 1]$ and $g(E(T)) = [1, A(1,m) - 1]$, which shows that g is a 6C-labeling of T. See an example in Figures 4 and 5. □

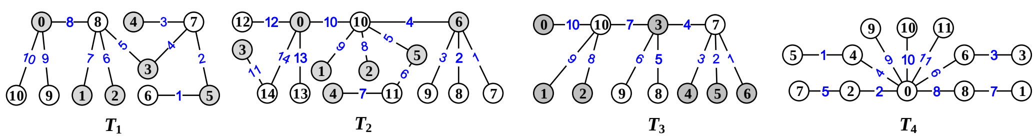

Figure 4. Four graphs T_i with $i \in [1,4]$ for understanding the proof of Theorem 3.

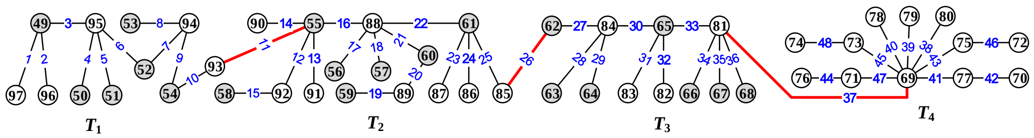

Figure 5. An example for illustrating the proof of Theorem 3.

3. Conclusions

This paper studied the 6C-labeling having multiple restrictive conditions and showed the graphic codes. We showed some connections between 6C-labeling and other graph labelings/colorings and showed graphs admitting twin-type 6C-labelings, as well as the construction of graphs admitting twin-type 6C-labelings and constructing graph classes with new labels to provide a new technology for topology coding. The result of Theorem 3 can be used to construct some graphic lattice [15]. If artificial intelligence technology and quantum computer attacks use labels and coloring-generated passwords, the decryption process will involve the determination of the graph isomorphism and some coloring conjectures, so it is not easy to crack. In addition, according to the method proposed in this paper, the topological coding has a variety of graph structures, topological matrices, and number-based strings of topological coding. We can use labeled graphs to design topology coding, so as to ensure information security. We can see that this is easy to obtain from the graph to the matrix and then to the string, and the reverse is difficult. Multiple restrictions also ensure the security of encryption.

Topological graphic passwords are based on the open structural cryptographic platform, that is this platform allows people to make themselves pan-topological graphic passwords by their remembered and favorite knowledge kept firmly in mind. We believe: "If a project has practical and effective applications and is supported by mathematics, it can go farther and farther. Practical application makes it live longer, and mathematics makes it stronger and faster. This project gives people feedback on material enjoyment and returns new objects and problems to mathematics".

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