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Remarks on Nonlocal Dirichlet Problems

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Abstract: We study a nonlocal Dirichlet problem with the $(p(b(u)), q(b(u)))$-Laplacian operator and integrable data on a bounded domain with smooth boundary. We establish the existence of at least one weak solution in the case the variable exponents of the leading operator depend on the solution u, without assuming any growth conditions on g. The proof is based on the characterization of the energy functional associated to the problem, using the methods of the calculus of variations.

Keywords: $(p(b(u)), q(b(u)))$-Laplacian operator; integrable data; nonlocal problem; nontrivial weak solution

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1. Introduction

We consider a nonlocal Dirichlet boundary value problem of the following form:

$$\begin{cases} -\mathrm{div}\,(a(u, \nabla u)) = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$. Here, g is a suitable given data, and the leading operator is defined by

$$a(u, \nabla u) = (|\nabla u|^{p(b(u))-2} + |\nabla u|^{q(b(u))-2})\nabla u,$$

which is the sum of a $p(\cdot)$-Laplacian operator and of a $q(\cdot)$-Laplacian operator. The variable exponents are bounded and bounded away by 1, and they are precisely defined by referring to the following maps:

$$p, q : \mathbb{R} \to [1, +\infty),$$
$$b : W_0^{1,\alpha}(\Omega) \to \mathbb{R}.$$

By $W_0^{1,\alpha}(\Omega)$, we mean the Dirichlet–Sobolev space with constant exponent α satisfying $1 < \alpha < +\infty$ (that is, $W_0^{1,\alpha}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,\alpha}(\Omega)$) (for a better understanding of the role of constant exponent α, see Lemma 2 and subsequent discussion). To underline the degree of generality in defining exponents p, q, we recall two typical examples of maps b of the following form:

$$b(u) = \|\nabla u\|_{L^\alpha(\Omega)}, \quad b(u) = \|u\|_{L^s(\Omega)}, \; s \le \alpha^*, \tag{2}$$

namely, we may link $b(\cdot)$ to two norm definitions that are relevant from a mathematical point of view. Here, α^* denotes the critical Sobolev exponent of α (namely, $\alpha^* = \dfrac{N\alpha}{N-\alpha}$

if $\alpha < N$ and $\alpha^* = +\infty$ otherwise); see also (7) for the precise definition in the case of variable exponents.

The differential operator $a(u, \nabla u)$ is a nonhomogeneous operator that is usually known as $(p(\cdot), q(\cdot))$-Laplacian operator. Differential problems involving such operator play a crucial role in modeling various physical phenomena and life science dynamics. For some references, we note the works of Ružička [1], Shi et al. [2] and Zhang and Rădulescu [3] (electrorheological fluid). Some other applications to model porous media and viscous flows can be found in Antontsev and Shmarev [4], where the authors consider various evolution equations and discuss existence, uniqueness, localization and asymptotic behavior of solutions under appropriate growth conditions. Briefly, the analysis of variational problems with the $p(\cdot)$-Laplacian operator arises from the nonlinear elasticity theory, with the works of Marcellini [5] and Zhikov [6]. In [5,6], the authors present a study of energy integral functionals under suitable growth conditions for the integrand. Namely, they consider an energy functional of the following form:

$$u \to \int_\Omega f(z, x) dz,$$

under a $p\&q$-growth condition ($q < p$) given by

$$|x|^q \le f(z, x) \le c_0[|x|^p + |x|^q] \text{ for a.e. } z \in \Omega, \text{ all } x \in \mathbb{R}^N, \text{ some } c_0 > 0.$$

Clearly, in the case of problem (1), we have the following energy functional

$$u \to \int_\Omega \left[|\nabla u|^{p(b(u))} + |\nabla u|^{q(b(u))} \right] dz.$$

The interest for such variational problems was recently revived by Mingione and co-authors, who obtained significant regularity results for local minimizers of functionals (see, for example, Baroni et al. [7] and the references therein). The case where variable exponents depend on the unknown solution u is not largely investigated; see, for example, the recent works of Chipot and de Oliveira [8] and Vetro [9]. This situation is relevant in the context of variational image denoising methods, where certain numerical approaches estimate the orientations of image structures from the data and, hence, use this information in building an energy functional to minimize. The performance of this minimization process benefits from using explicitly u-dependence or ∇u-dependence (recall (2)), for image regularization (see Tiirola [10] and the references therein). Moreover, for more information about nonlocal problems, the interested reader can refer to the works of Chipot and de Oliveira [8] and Corvellec and Hantoute [11] and the book of Diening et al. [12].

Our approach involves the energy functional associated to problem (1) using certain estimates of some integral terms together with the properties of auxiliary operators. We establish the existence of at least one weak solution in a suitable anisotropic Sobolev space without assuming any restriction on the growth of data g.

The manuscript is organized as follows. In Section 2, we present some notations and results used in the paper; in Section 3, we provide the main theorems with complete proofs. Section 4 concludes the manuscript.

2. Notation and Materials

In this section we introduce our notation and collect some useful materials. In the context of Banach spaces, if we denote by X a Banach space, then its topological dual will be given as X^*. Now, we focus on the setting of Lebesgue and Sobolev spaces with variable exponents, but we also link these spaces to their counterparts with constant exponents. For a more complete view on framework structures, we suggest the recent monographies of Antontsev and Shmarev [4] and Zhikov [6,13] (about differential problems subject to nonstandard growth conditions), Cruz-Uribe and Fiorenza [14] and Diening et al. [12] (about variable Lebesgue and Sobolev spaces). The exponent functions in our finding here are introduced as elements of the set, namely $\mathcal{M}(\Omega)$, of all Lebesgue-measurable functions $p : \Omega \to [1, +\infty)$ for which their essential infimum and essential supremum are given as follows:

$$p^- = \operatorname*{ess\,inf}_{z\in\Omega} p(z),\ p^+ = \operatorname*{ess\,sup}_{z\in\Omega} p(z).$$

For $p \in \mathcal{M}(\Omega) \cap L^\infty(\Omega)$, we introduce variable exponent Lebesgue space $L^{p(z)}(\Omega)$ defined by the following:

$$L^{p(z)}(\Omega) = \{u : \Omega \to \mathbb{R} \ :\ u \text{ is measurable and } \rho_p(u) < +\infty\},$$

where

$$\rho_p(u) = \int_\Omega |u(z)|^{p(z)} dz < +\infty \quad \text{(that is, we assume } \rho_p(\cdot) \text{ is finite).} \tag{3}$$

The space $(L^{p(z)}(\Omega), \|u\|_{L^{p(z)}(\Omega)})$ is a Banach space, where

$$\|u\|_{L^{p(z)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_p\left(\frac{u}{\lambda}\right) \le 1\right\}$$

is the well-known Luxembourg norm. From Fan and Zhao [15], we recall some results involving $(L^{p(z)}(\Omega), \|u\|_{L^{p(z)}(\Omega)})$. For their proofs, we refer to Theorems 1.6 and 1.8 of [15], for which their conclusions are summarized in the following result.

Theorem 1. *The following facts hold:*

(i) $(L^{p(z)}(\Omega), \|\cdot\|_{L^{p(z)}(\Omega)})$ *is a separable Banach space;*

(ii) $C_0^\infty(\Omega)$ *is dense in* $(L^{p(z)}(\Omega), \|\cdot\|_{L^{p(z)}(\Omega)})$.

Additionally, looking at Theorem 1.10 of [15], we note the following result.

Theorem 2. $(L^{p(z)}(\Omega), \|\cdot\|_{L^{p(z)}(\Omega)})$ *is an uniform convex space (thus, reflexive too), provided that* $1 < p^- \le p^+ < +\infty$.

To increase the discussion about norm properties, we recall Theorems 1.2 and 1.3 of [15] in the following result involving also (3).

Theorem 3. *If* $u \in L^{p(z)}(\Omega)$, *then we have the following:*

(i) $\|u\|_{L^{p(z)}(\Omega)} < 1\ (= 1, > 1)$ *if and only if* $\rho_p(u) < 1\ (= 1, > 1)$;

(ii) *If* $\|u\|_{L^{p(z)}(\Omega)} > 1$, *then* $\|u\|^{p^-}_{L^{p(z)}(\Omega)} \le \rho_p(u) \le \|u\|^{p^+}_{L^{p(z)}(\Omega)}$;

(iii) *If* $\|u\|_{L^{p(z)}(\Omega)} < 1$, *then* $\|u\|^{p^+}_{L^{p(z)}(\Omega)} \le \rho_p(u) \le \|u\|^{p^-}_{L^{p(z)}(\Omega)}$;

(iv) *In the case* $u \not\equiv 0$, *then* $\|u\|_{L^{p(z)}(\Omega)} = a$ *if and only if* $\rho_p(u/a) = 1$.

According to the classical notation in variable exponents Lebesgue and Sobolev spaces, we will denote the dual space of $L^{p(z)}(\Omega)$ by $L^{p'(z)}(\Omega)$ (instead than $(L^{p(z)}(\Omega))^*$). We point out that $p'(\cdot)$ means the Hölder conjugate of $p(\cdot)$ in the sense that the following equality holds true:

$$\frac{1}{p'(\cdot)} + \frac{1}{p(\cdot)} = 1.$$

The similar concept of Hölder conjugate applies in the case of constant exponents (for example, α' will denote the Hölder conjugate of α). From the context, it is clear that

$$1 < (p^+)' \le \operatorname*{ess\,inf}_{z\in\Omega} p'(z) \le \operatorname*{ess\,sup}_{z\in\Omega} p'(z) \le (p^-)' < +\infty.$$

Additionally, in the case $1 < p^-$, we consider the Hölder's inequality given as

$$\int_\Omega uvdz \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|u\|_{L^{p(z)}(\Omega)}\|v\|_{L^{p'(z)}(\Omega)} \le 2\|u\|_{L^{p(z)}(\Omega)}\|v\|_{L^{p'(z)}(\Omega)},$$

for all $u \in L^{p(z)}(\Omega)$ and $v \in L^{p'(z)}(\Omega)$. The above inequalities are also useful in establishing relevant embedding results. For example, Theorem 1.11 of [15] uses the Hölder inequality to ensure the continuity of embedding in a bounded domain Ω. Namely, we have the following result.

Theorem 4. *Assume that the Lebesgue measure of Ω is finite (namely, $|\Omega| < +\infty$) and $p_1, p_2 \in \mathcal{M}(\Omega) \cap L^\infty(\Omega)$. Then, we have*

$$L^{p_2(z)}(\Omega) \hookrightarrow L^{p_1(z)}(\Omega) \quad \Longleftrightarrow \quad p_1(z) \le p_2(z) \text{ for a.e. } z \in \Omega.$$

Moreover, the embedding is continuous.

We complete the discussion about the spaces involved in our study by introducing the variable exponent Sobolev space $W^{1,p(z)}(\Omega)$ defined by

$$W^{1,p(z)}(\Omega) = \{u \in L^{p(z)}(\Omega) : |\nabla u| \in L^{p(z)}(\Omega)\}.$$

On this space, we consider the norm given as

$$\|u\|_{W^{1,p(z)}(\Omega)} = \|u\|_{L^{p(z)}(\Omega)} + \|\nabla u\|_{L^{p(z)}(\Omega)} \quad (\text{where } \|\nabla u\|_{L^{p(z)}(\Omega)} = \||\nabla u|\|_{L^{p(z)}(\Omega)}).$$

Similarly to the variable exponent Lebesgue space $L^{p(z)}(\Omega)$, we note that $W^{1,p(z)}(\Omega)$ is separable when $1 \le p^- \le p^+ < +\infty$. Moreover, $W^{1,p(z)}(\Omega)$ is reflexive if $1 < p^- \le p^+ < +\infty$. About the embedding properties, we remark that

$$W^{1,p_2(z)}(\Omega) \hookrightarrow W^{1,p_1(z)}(\Omega) \text{ if } p_1(z) \le p_2(z) \text{ for a.e. } z \in \Omega. \tag{4}$$

For our analysis, it is useful to mention the anisotropic Dirichlet variable exponent space (see, for example, [12]) defined by

$$W_0^{1,p(z)}(\Omega) = \{u \in W_0^{1,1}(\Omega) : |\nabla u| \in L^{p(z)}(\Omega)\},$$

where we consider the norm

$$\|u\|_{W_0^{1,p(z)}(\Omega)} = \|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^{p(z)}(\Omega)}. \tag{5}$$

If $p \in C(\overline{\Omega}) \cap \mathcal{M}(\Omega)$, we can find some constant $c = c(p, \Omega, N)$ such that

$$\|u\|_{L^{p(z)}(\Omega)} \le c\|\nabla u\|_{L^{p(z)}(\Omega)} \text{ for all } u \in W_0^{1,p(z)}(\Omega).$$

For more information we refer again to [12]. Indeed, it is relevant to note that the norms $\|u\|_{W^{1,p(z)}(\Omega)}$ and $\|\nabla u\|_{L^{p(z)}(\Omega)}$ are equivalent to each other on $W_0^{1,p(z)}(\Omega)$. Consequently, we can use $\|\nabla u\|_{L^{p(z)}(\Omega)}$ instead of $\|u\|_{W^{1,p(z)}(\Omega)}$. With the abuse of notation, we write

$$\|u\| = \|\nabla u\|_{L^{p(z)}(\Omega)} \text{ in } W_0^{1,p(z)}(\Omega).$$

Even if the variable exponent space originates as natural extension of the corresponding constant exponent space, there are some source of difficulties in developing the theory of $L^{p(z)}(\Omega)$, $W^{1,p(z)}(\Omega)$ and $W_0^{1,p(z)}(\Omega)$. For example, smooth functions are not necessarily dense in $W_0^{1,p(z)}(\Omega)$. By letting the following:

$$H_0^{1,p(z)}(\Omega) = \text{the closure of } C_0^\infty \text{ with respect to } \|\cdot\|_{W^{1,p(z)}(\Omega)},$$

we note that generally

$$H_0^{1,p(z)}(\Omega) \subsetneqq W_0^{1,p(z)}(\Omega).$$

However, in the case where Ω is a bounded domain with Lipschitz-continuous boundary $\partial\Omega$ and $p(\cdot)$ is log-Hölder continuous, the density of $C_0^\infty(\Omega)$ in $W_0^{1,p(z)}(\Omega)$ holds true (for additional details, look at Theorem 2.6 of [15]). For the reader's convenience, we remark that $p(\cdot)$ is a log-Hölder continuous function if the following condition is satisfied:

$$\exists\, C > 0 : -|p(z) - p(y)| \ln |z - y| \leq C \text{ for all } z, y \in \Omega,\ 2|z - y| < 1. \tag{6}$$

Summing up, (6) gives us the equality

$$H_0^{1,p(z)}(\Omega) = W_0^{1,p(z)}(\Omega).$$

The log-Hölder continuity of $p(\cdot)$ is ensured whenever

$$p \in C^{0,\lambda}(\Omega) \text{ for some } \lambda \in (0,1).$$

For a given Sobolev space $W^{1,p(z)}(\Omega)$, we define the critical Sobolev exponent of $p(\cdot)$ by

$$p^*(z) = \begin{cases} \dfrac{N\,p(z)}{N - p(z)} & \text{if } p(z) < N, \\ +\infty & \text{if } p(z) \geq N. \end{cases} \tag{7}$$

Referring to this notion, Fan and Zhao [15] established the following result.

Proposition 1. *Let $p \in C(\overline{\Omega})$ satisfying $p^- > 1$. If $\alpha \in C(\overline{\Omega})$ and $1 < \alpha(z) < p^*(z)$ for all $z \in \Omega$, then there is a continuous and compact embedding $W^{1,p(z)}(\Omega) \hookrightarrow L^{\alpha(z)}(\Omega)$.*

We conclude this section by recalling some results related to classical properties of operators (see also Chipot [16]). These results will be used in concluding the proof of our main theorem.

Lemma 1. *For all $\xi, \eta \in \mathbb{R}^N$, we have the following implications:*

$$2 \leq p < +\infty \Rightarrow 2^{1-p}|\xi - \eta|^p \leq (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta); \tag{8}$$

$$1 < p < 2 \Rightarrow (p-1)|\xi - \eta|^2 \leq (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \cdot (|\xi|^p + |\eta|^p)^{\frac{2-p}{p}}. \tag{9}$$

The above lemma is linked to monotonicity, and the next theorem is linked to subjectivity.

Theorem 5 (Minty-Browder)**.** *Assume that $\mathcal{L} : X \to X^*$ is a bounded, continuous, coercive and a monotone operator, where X is a real, reflexive Banach space. Then, for each data $g \in X^*$, there is an element u satisfying the equation:*

$$\mathcal{L}(u) = g,$$

that is, $\mathcal{L}(X) = X^*$.

The following lemma is a consequence of similar one established by Chipot and de Oliveira.

Lemma 2 ([8], Lemma 3.1). *Let $\{r_n\}, \{s_n\} \subset \mathbb{R}$ be two sequences. Assume that there exist $\alpha, \beta \in (1, +\infty)$ such that the following is the case:*

(i) $\alpha \le r_n, s_n \le \beta$ *for all* $n \in \mathbb{N}$;

(ii) $r_n \to r, s_n \to s$ *as* $n \to +\infty$;

(iii) $\nabla u_n \xrightarrow{w} \nabla u$ *in* $L^1(\Omega)^N$, *as* $n \to +\infty$;

(iv) $\||\nabla u_n|^{r_n}\|_{L^1(\Omega)}, \||\nabla u_n|^{s_n}\|_{L^1(\Omega)} \le C$, *for some constant* $C > 0$ *not depending on* n.

Then, $|\nabla u|^r, |\nabla u|^s \in L^1(\Omega)$ *and*

$$\liminf_{n\to+\infty} \int_\Omega |\nabla u_n|^{r_n} dz \ge \int_\Omega |\nabla u|^r dz \text{ and } \liminf_{n\to+\infty} \int_\Omega |\nabla u_n|^{s_n} dz \ge \int_\Omega |\nabla u|^s dz. \tag{10}$$

Proof. From Lemma 3.1 of [8], we obtain $|\nabla u|^r, |\nabla u|^s \in L^1(\Omega)$ and

$$\liminf_{n\to+\infty} \int_\Omega |\nabla u_n|^{r_n} dz \ge \int_\Omega |\nabla u|^r dz, \quad \liminf_{n\to+\infty} \int_\Omega |\nabla u_n|^{s_n} dz \ge \int_\Omega |\nabla u|^s dz.$$

Thus, (10) holds. □

We define the set where we are going to look for the solutions to problem (1) as follows:

$$W_0^{1,p(b(u))}(\Omega) = \left\{ u \in W_0^{1,1}(\Omega) : \int_\Omega |\nabla u|^{p(b(u))} dz < +\infty \right\}.$$

If $1 < p(b(u)) < +\infty$ for all $u \in \mathbb{R}$, this set is a Banach space for norm $\|u\|_{W_0^{1,p(\cdot)}(\Omega)}$ defined in (5), which is equivalent to $\|\nabla u\|_{L^{p(b(u))}(\Omega)}$ in the case of $p(b(u)) \in C(\overline{\Omega})$. If, for some constant α, $p \ge \alpha > 1$, p and b are continuous, then $W_0^{1,p(b(u))}(\Omega)$ is a closed subspace of $W_0^{1,\alpha}(\Omega)$ in view of (4); therefore, it is separable and reflexive. In what follows, $W^{-1,\alpha'}(\Omega) = W_0^{1,\alpha}(\Omega)^*$, with $1 < \alpha < +\infty$, denotes as usual the dual space of $W_0^{1,\alpha}(\Omega)$.

3. Main Results

We prove an existence theorem of at least one weak solution to the nonlocal Dirichlet problem (1). Thus, we place some restrictions to the exponents and assume that $p(\cdot)$ and $q(\cdot)$ are real functions satisfying the following:

$$p, q \text{ are continuous}, \quad 1 < \alpha < q \le p \le \beta, \tag{11}$$

for some constants α and β. With respect to constant α, we define domain $W_0^{1,\alpha}(\Omega)$ of the real map $b(\cdot)$, and additionally we impose the following:

$$b \text{ is continuous}, \quad b \text{ is bounded}, \tag{12}$$

that is, $b(\cdot)$ sends bounded sets of $W_0^{1,\alpha}(\Omega)$ into bounded sets of $\mathbb{R}$. Since we are interested in solutions in a weak sense, we recall the following definition.

Definition 1. *For a weak solution to the problem* (1) *we mean a function* $u \in W_0^{1,p(b(u))}(\Omega)$ *such that the following is the case:*

$$\int_\Omega \left[|\nabla u|^{p(b(u))-2} + |\nabla u|^{q(b(u))-2} \right] \nabla u \nabla v dz = \langle g, v \rangle \text{ for all } v \in W_0^{1,p(b(u))}(\Omega), \tag{13}$$

with $\langle \cdot, \cdot \rangle$ *being the duality pairing of* $(W_0^{1,p(b(u))}(\Omega)^*, W_0^{1,p(b(u))}(\Omega))$.

We remark that quantities $p(b(u))$ and $q(b(u))$ reduce to real numbers and not functions. Consequently, we can treat variable exponent Sobolev spaces in Definition 1 as constant exponent Sobolev spaces.

As ingredients of the existence theorem (namely Theorem 6), we consider certain classes of approximating problems (for short $(1)_n$, $n \in \mathbb{N}$) obtained from problem (1) by assuming the following one as the leading operator:

$$a_n(u, \nabla u) = (|\nabla u|^{p_n - 2} + |\nabla u|^{q_n - 2})\nabla u, \quad n \in \mathbb{N},$$

where p_n and q_n with $q_n \le p_n$ are constant exponents. For $n \in \mathbb{N}$, as an approximating solution of problem $(1)_n$, we mean function $u_n \in W_0^{1,p_n}(\Omega)$ such that the following is the case:

$$\int_\Omega \Big[|\nabla u_n|^{p_n - 2} + |\nabla u_n|^{q_n - 2}\Big]\nabla u_n \nabla v dz = \langle g, v\rangle \text{ for all } v \in W_0^{1,p_n}(\Omega), \tag{14}$$

with $\langle \cdot, \cdot \rangle$ being the duality pairing of $(W_0^{1,p_n}(\Omega)^*, W_0^{1,p_n}(\Omega))$. Clearly, (14) is (13) in special cases $p(b(u)) = p_n$ and $q(b(u)) = q_n$.

Remark 1. *We recall that the operator $A : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)^*$ is defined by*

$$\langle A(u), v\rangle = \int_\Omega \Big[|\nabla u|^{p-2} + |\nabla u|^{q-2}\Big]\nabla u \nabla v dz$$

for all $v \in W_0^{1,p}(\Omega)$ is bounded, continuous, strictly monotone and coercive. Therefore, weak solution $u_n \in W_0^{1,p_n}(\Omega)$ to problem $(1)_n$ exists by Theorem 5 and is unique by the strict monotonicity; that is, there is a unique $u_n \in W_0^{1,p_n}(\Omega)$ that satisfies (14).

In this context, we establish the following convergence result.

Lemma 3. *For $n \in \mathbb{N}$, let $u_n \in W_0^{1,p_n}(\Omega)$ be the solution to (14). If sequences $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ and the given data g satisfy the following conditions:*

$$p_n \to p,\ q_n \to q \text{ as } n \to +\infty, \text{ where } p, q \in (1, +\infty), \tag{15}$$

$$g \in W^{-1,s'}(\Omega) \text{ for some constant } s \in (1, +\infty) \text{ such that } 1 < s < q \le p, \tag{16}$$

then the sequence of approximating solutions $\{u_n\}_{n\in\mathbb{N}}$ converges as follows:

$$u_n \to u \text{ in } W_0^{1,s}(\Omega) \text{ as } n \to +\infty, \tag{17}$$

where $u \in W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega)$ is the solution to the equation

$$\int_\Omega \Big[|\nabla u|^{p-2} + |\nabla u|^{q-2}\Big]\nabla u \nabla v dz = \langle g, v\rangle \text{ for all } v \in W_0^{1,p}(\Omega). \tag{18}$$

Proof. The convergence in Equation (17) is strong in $W_0^{1,s}(\Omega)$. However, we construct this result over an auxiliary weak convergence result for the gradient of u_n, $n \in \mathbb{N}$, in the constant exponent Lebesgue space $L^s(\Omega)^N$. To obtain this goal, by combining the convergences in (15) with the relations among the involved exponents p, q, s (that is, $s < q \le p$), we suppose without any loss of generality that the following is the case:

$$p + 1 > p_n \ge q_n > s \text{ for all } n \in \mathbb{N}. \tag{19}$$

From (14) with test function $v = u_n$, we derive the following inequality linking the integrand in (18) to suitable norms of data g and the gradient of approximating solutions. Namely, we have

$$\int_\Omega [|\nabla u_n|^{p_n} + |\nabla u_n|^{q_n}]dz \le \|g\|_{W^{-1,s'}(\Omega)}\|u_n\|_{W_0^{1,s}(\Omega)} = \|g\|_{W^{-1,s'}(\Omega)}\|\nabla u_n\|_{L^s(\Omega)}. \tag{20}$$

We remark that $\|g\|_{W^{-1,s'}(\Omega)}$ means the operator norm in Sobolev space $W^{-1,s'}(\Omega)$ associated to the norm of gradient in Lebesgue space $L^s(\Omega)$. The relations in (19) and an application of Hölder's inequality lead to the following:

$$\left(\int_\Omega |\nabla u_n|^s dz\right)^{\frac{1}{s}} \leq \left(\int_\Omega |\nabla u_n|^{p_n} dz\right)^{\frac{1}{p_n}} |\Omega|^{\frac{1}{s}-\frac{1}{p_n}} \tag{21}$$

and

$$\left(\int_\Omega |\nabla u_n|^s dz\right)^{\frac{1}{s}} \leq \left(\int_\Omega |\nabla u_n|^{q_n} dz\right)^{\frac{1}{q_n}} |\Omega|^{\frac{1}{s}-\frac{1}{q_n}}, \tag{22}$$

where by $|\Omega|$, we mean again the Lebesgue measure of Ω. From (20) and (21), we obtain the following:

$$\begin{aligned}
&\|\nabla u_n\|^{p_n}_{L^{p_n}(\Omega)} \leq \|g\|_{W^{-1,s'}(\Omega)} \|\nabla u_n\|_{L^s(\Omega)} \\
&\leq \|g\|_{W^{-1,s'}(\Omega)} \|\nabla u_n\|_{L^{p_n}(\Omega)} |\Omega|^{\frac{1}{s}-\frac{1}{p_n}}, \\
\Longrightarrow \quad &\|\nabla u_n\|_{L^{p_n}(\Omega)} \leq \|g\|^{\frac{1}{p_n-1}}_{W^{-1,s'}(\Omega)} |\Omega|^{\left(\frac{1}{s}-\frac{1}{p_n}\right)\frac{1}{p_n-1}}, \\
\Longrightarrow \quad &\|\nabla u_n\|_{L^{p_n}(\Omega)} \leq \max_{t\in[s,p+1]} \left\{ \|g\|^{\frac{1}{t-1}}_{W^{-1,s'}(\Omega)} |\Omega|^{\left(\frac{1}{s}-\frac{1}{t}\right)\frac{1}{t-1}} \right\} = C,
\end{aligned} \tag{23}$$

where $C = C(p,s,\Omega,g) > 0$. Similarly, from (20) and (22), we obtain the following:

$$\|\nabla u_n\|_{L^{q_n}(\Omega)} \leq \max_{t\in[s,p+1]} \left\{ \|g\|^{\frac{1}{t-1}}_{W^{-1,s'}(\Omega)} |\Omega|^{\left(\frac{1}{s}-\frac{1}{t}\right)\frac{1}{t-1}} \right\} = C, \tag{24}$$

where $C = C(p,q,s,\Omega,g) > 0$. Therefore, from (21)–(24), we obtain the following upper bound:

$$\|\nabla u_n\|_{L^s(\Omega)} \leq C \min\{\|\nabla u_n\|_{L^{p_n}(\Omega)}, \|\nabla u_n\|_{L^{q_n}(\Omega)}\} \leq C, \tag{25}$$

for some $C = C(p,q,s,\Omega,g) > 0$, where we point out that constant $C > 0$ is independent of the index n. Moreover, the upper bound (25) ensures that we can find some subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (with abuse of notation, we use $\{u_n\}_{n\in\mathbb{N}}$ to denote also this subsequence) and some $u \in W^{1,s}(\Omega)$ (namely, u is a weak limit of the sequence $\{u_n\}_{n\in\mathbb{N}}$ in the Sobolev space $W^{1,s}(\Omega)$) such that

$$\nabla u_n \xrightarrow{w} \nabla u \text{ in } L^s(\Omega)^N \text{ as } n \to +\infty. \tag{26}$$

The last convergence is the auxiliary weak result that we mentioned at the beginning of the proof. Now, combining the convergences in (15) and (26), with the relations among the involved exponents in (19), with the upper bounds in (23) and (24), we can apply Lemma 2 to conclude that

$$\liminf_{n\to+\infty} \int_\Omega |\nabla u_n|^{p_n} dz \geq \int_\Omega |\nabla u|^p dz \text{ and } \liminf_{n\to+\infty} \int_\Omega |\nabla u_n|^{q_n} dz \geq \int_\Omega |\nabla u|^q dz.$$

By these inferior limits, we deduce that

$$u \in W_0^{1,p}(\Omega) \text{ and } u \in W_0^{1,q}(\Omega). \tag{27}$$

We note that the equality in (14) is equivalent to the following inequality:

$$\int_\Omega \left[|\nabla u_n|^{p_n-2} + |\nabla u_n|^{q_n-2}\right] \nabla u_n \nabla (v - u_n) dz \geq \langle g, v - u_n\rangle \text{ for all } v \in W_0^{1,p_n}(\Omega).$$

Referring to the Minty's lemma, we obtain the following inequality:

$$\int_\Omega \left[|\nabla v|^{p_n-2} + |\nabla v|^{q_n-2}\right]\nabla v \nabla(v-u_n)dz \geq \langle g, v-u_n\rangle \quad \text{for all } v \in W_0^{1,p_n}(\Omega). \tag{28}$$

Then, for $v \in C_0^\infty(\Omega)$ in (28), the convergences in (15) and (26) lead to the following inequality:

$$\int_\Omega \left[|\nabla v|^{p-2} + |\nabla v|^{q-2}\right]\nabla v \nabla(v-u)dz \geq \langle g, v-u\rangle \text{ for all } v \in C_0^\infty(\Omega), \tag{29}$$

as n goes to $+\infty$ in (28). On the other hand, we note that space $C_0^\infty(\Omega)$ is dense into $W_0^{1,p}(\Omega)$; hence, the inequality of (29) remains true for every $v \in W_0^{1,p}(\Omega)$. Therefore, we can consider test functions $v = u \pm \delta y$ in (29), where $y \in W_0^{1,p}(\Omega)$ and $\delta > 0$, and we obtain the following:

$$\int_\Omega \left[|\nabla(u \pm \delta y)|^{p-2} + |\nabla(u \pm \delta y)|^{q-2}\right]\nabla(u \pm \delta y)\nabla y dz = \langle g, y\rangle. \tag{30}$$

Clearly, if we take the limit as $\delta \to 0$ in (30), then we obtain the following:

$$\int_\Omega \left[|\nabla u|^{p-2} + |\nabla u|^{q-2}\right]\nabla u \nabla y dz = \langle g, y\rangle \text{ for all } y \in W_0^{1,p}(\Omega),$$

which implies that u solves (18). What remains is to show that the convergence in (17) holds true. To this aim, we first prove that the convergence of gradient terms in (26) is in fact strong. Referring to the right hand side of (14) (with test function $v = u_n$), we remark that

$$\langle g, u_n\rangle \to \langle g, u\rangle \text{ as } n \to +\infty, \text{ since } u_n \xrightarrow{w} u. \tag{31}$$

Choosing again as test function $v = u_n$ in (14) and using the above convergence (that is, (31)), we deduce that

$$\lim_{n\to+\infty}\int_\Omega [|\nabla u_n|^{p_n} + |\nabla u_n|^{q_n}]dz = \int_\Omega [|\nabla u|^p + |\nabla u|^q]dz. \tag{32}$$

In addition, we refine the discussion about exponents p_n and q_n starting from the situation where

$$p_n \geq p,\ q_n \geq q \text{ for all } n \in \mathbb{N}.$$

In this case, we use the Hölder's inequality to obtain the following:

$$\int_\Omega |\nabla u_n|^p dz \leq \left(\int_\Omega |\nabla u_n|^{p_n}dz\right)^{\frac{p}{p_n}}|\Omega|^{1-\frac{p}{p_n}}$$

and

$$\int_\Omega |\nabla u_n|^q dz \leq \left(\int_\Omega |\nabla u_n|^{q_n}dz\right)^{\frac{q}{q_n}}|\Omega|^{1-\frac{q}{q_n}},$$

where by $|\Omega|$ we mean again the Lebesgue measure of Ω. Referring to the first part of the proof of the present lemma, we note that the sequences

$$\left\{\int_\Omega |\nabla u_n|^{p_n}dz\right\}_{n\in\mathbb{N}} \text{ and } \left\{\int_\Omega |\nabla u_n|^{q_n}dz\right\}_{n\in\mathbb{N}}$$

are bounded. This implies that we can find some subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (with the abuse of notation, we use $\{u_n\}_{n\in\mathbb{N}}$ to denote also this subsequence) such that the following limits exist, namely

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{p_n}dz \text{ and } \lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{q_n}dz.$$

We can deduce that

$$\lim_{n\to+\infty}\left(\int_\Omega |\nabla u_n|^{p_n}dz\right)^{\frac{p}{p_n}}|\Omega|^{1-\frac{p}{p_n}} = \lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{p_n}dz$$

and

$$\lim_{n\to+\infty}\left(\int_\Omega |\nabla u_n|^{q_n}dz\right)^{\frac{q}{q_n}}|\Omega|^{1-\frac{q}{q_n}} = \lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{q_n}dz.$$

Referring to the limit in (32), we obtain the following inequality

$$\begin{aligned}
&\limsup_{n\to+\infty}\int_\Omega |\nabla u_n|^{p}dz + \limsup_{n\to+\infty}\int_\Omega |\nabla u_n|^{q}dz \\
&\leq \lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{p_n}dz + \lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{q_n}dz \\
&= \int_\Omega |\nabla u|^{p}dz + \int_\Omega |\nabla u|^{q}dz.
\end{aligned}$$

On the other hand, we say that

$$\int_\Omega |\nabla u|^{p}dz \leq \liminf_{n\to+\infty}\int_\Omega |\nabla u_n|^{p}dz \text{ and } \int_\Omega |\nabla u|^{q}dz \leq \liminf_{n\to+\infty}\int_\Omega |\nabla u_n|^{q}dz.$$

Thus, we obtain the following

$$\int_\Omega |\nabla u|^{p}dz = \lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{p}dz \text{ or } \int_\Omega |\nabla u|^{q}dz = \lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{q}dz.$$

This implies that

$$u_n \to u \text{ (strongly) in } W_0^{1,p}(\Omega) \text{ or in } W_0^{1,q}(\Omega), \text{ as } n\to+\infty. \tag{33}$$

Thus, we obtain $u_n \to u$ (strongly) in $W_0^{1,p}(\Omega)$ and in $W_0^{1,q}(\Omega)$, as $n\to+\infty$. Finally, since $W_0^{1,p}(\Omega), W_0^{1,q}(\Omega) \subset W_0^{1,s}(\Omega)$, we conclude the convergence in (17) as a byproduct of (33).

Next, we develop similar arguments in the situation where the following is the case:

$$s < q_n \leq p_n < p \text{ for all } n\in\mathbb{N}. \tag{34}$$

We introduce the non-negative monotone operator A_n defined by

$$\begin{aligned}
A_n = \int_\Omega \Big(\Big[|\nabla u_n|^{p_n-2} + |\nabla u_n|^{q_n-2}\Big]\nabla u_n - \Big[|\nabla u|^{p_n-2} + |\nabla u|^{q_n-2}\Big]\nabla u \Big) \\
\times(\nabla u_n - \nabla u)dz.
\end{aligned} \tag{35}$$

From (14),we deduce the equivalent form of A_n as follows

$$A_n = \langle g, u_n - u\rangle - \int_\Omega \Big[|\nabla u|^{p_n-2} + |\nabla u|^{q_n-2}\Big]\nabla u \nabla(u_n - u)dz.$$

The hypothesis on the data (see (16)) and the weak convergence in (26) imply that

$$\langle g, u_n - u\rangle \to 0 \text{ as } n\to+\infty. \tag{36}$$

In addition, by using (27), we deduce the following bounds:

$$\Big||\nabla u|^{p_n-2}\nabla u\Big| \leq \max\{1, |\nabla u|\}^{p-1} \in L^{p'}(\Omega) \tag{37}$$

and
$$\left||\nabla u|^{q_n-2}\nabla u\right| \le \max\{1,|\nabla u|\}^{p-1} \in L^{p'}(\Omega). \tag{38}$$

We combine the information in (34), (36)–(38) to conclude that
$$A_n \to 0 \text{ as } n \to +\infty. \tag{39}$$

We involve in the proof in Lemma 1. Precisely, if we suppose that
$$p_n \ge 2,$$
then we refer to the first implication of Lemma 1; that is, we use (8) in (35) and deduce the following:
$$\begin{aligned} A_n &\ge \int_\Omega \Big(|\nabla u_n|^{p_n-2}\nabla u_n - |\nabla u|^{p_n-2}\nabla u\Big)(\nabla u_n - \nabla u)dz \\ &\ge \frac{1}{2^{p_n-1}}\int_\Omega |\nabla(u_n-u)|^{p_n}dz. \end{aligned} \tag{40}$$

On the other hand, $p_n > s$ (recall (34)); hence, an application of Hölder's inequality, together with (39) and (40), gives us the following:
$$\int_\Omega |\nabla(u_n-u)|^s dz \le \left(\int_\Omega |\nabla(u_n-u)|^{p_n}dz\right)^{\frac{s}{p_n}}|\Omega|^{1-\frac{s}{p_n}} \to 0,$$
as $n \to +\infty$. Consequently, the convergence in (17) holds true. The other situation to consider is the second implication in Lemma 1. Namely, starting from the following:
$$p_n < 2 \quad\Longrightarrow\quad q_n < 2,$$
we apply Hölder's inequality to obtain that
$$\begin{aligned} &\int_\Omega |\nabla(u_n-u)|^{q_n}dz \\ &= \int_\Omega |\nabla(u_n-u)|^{q_n}(|\nabla u_n|^{q_n}+|\nabla u|^{q_n})^{\frac{q_n-2}{2}}(|\nabla u_n|^{q_n}+|\nabla u|^{q_n})^{\frac{2-q_n}{2}}dz \\ &\le \left[\int_\Omega |\nabla(u_n-u)|^2(|\nabla u_n|^{q_n}+|\nabla u|^{q_n})^{\frac{q_n-2}{q_n}}dz\right]^{\frac{q_n}{2}}\left[\int_\Omega (|\nabla u_n|^{q_n}+|\nabla u|^{q_n})dz\right]^{1-\frac{q_n}{2}}. \end{aligned} \tag{41}$$

Thus, from (9), we deduce the following:
$$\begin{aligned} A_n &\ge \int_\Omega \Big(|\nabla u_n|^{q_n-2}\nabla u_n - |\nabla u|^{q_n-2}\nabla u\Big)(\nabla u_n - \nabla u)dz \\ &\ge C\int_\Omega |\nabla(u_n-u)|^2(|\nabla u_n|^{q_n}+|\nabla u|^{q_n})^{\frac{q_n-2}{q_n}}dz, \end{aligned} \tag{42}$$
for some constant depending on q_n; that is, $C = C(q_n) > 0$. Now, (41) and (42) together with the bounds in (24) lead to the following limit
$$\lim_{n\to+\infty}\int_\Omega |\nabla(u_n-u)|^{q_n}dz = 0.$$

This completes the proof of the convergence in (17). □

We are ready to establish the main result of the paper in the form of the following existence theorem. Lemma 3 is the key tool of the proof.

Theorem 6. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain. If assumptions (11) and (12) hold and $g \in W^{-1,\alpha'}(\Omega)$, then problem (1) admits at least one weak solution.*

Proof. Starting from the assumption on data g, we point out that

$$g \in (W^{-1,\alpha'}(\Omega))^* \subset (W^{-1,\delta'}(\Omega))^* \text{ for any } \delta > \alpha.$$

Now, for each $\lambda \in \mathbb{R}$, we can find a unique solution $u = u_\lambda \in W^{1,p(\lambda)}(\Omega)$ (see Remark 1) relative to the auxiliary $p(\lambda)$-Laplacian problem

$$\int_\Omega \left[|\nabla u|^{p(\lambda)-2} + |\nabla u|^{q(\lambda)-2} \right] \nabla u \nabla v dz = \langle g, v \rangle \text{ for all } v \in W_0^{1,p(\lambda)}(\Omega). \tag{43}$$

For choice $v = u = u_\lambda$, from (43), we deduce the inequality

$$\int_\Omega \left[|\nabla u_\lambda|^{p(\lambda)} + |\nabla u_\lambda|^{q(\lambda)} \right] dz \le \|g\|_{W^{-1,\alpha'}(\Omega)} \|\nabla u_\lambda\|_{L^\alpha(\Omega)}. \tag{44}$$

For the second norm term in (44), the Hölder's inequality leads to

$$\|\nabla u_\lambda\|_{L^\alpha(\Omega)} \le \min\left\{ \|\nabla u_\lambda\|_{L^{p(\lambda)}(\Omega)} |\Omega|^{\frac{1}{\alpha} - \frac{1}{p(\lambda)}}, \|\nabla u_\lambda\|_{L^{q(\lambda)}(\Omega)} |\Omega|^{\frac{1}{\alpha} - \frac{1}{q(\lambda)}} \right\} \tag{45}$$

and, hence, inequality (44) gives us

$$\|\nabla u_\lambda\|^{p(\lambda)-1}_{L^{p(\lambda)}(\Omega)} \le \|g\|_{W^{-1,\alpha'}(\Omega)} |\Omega|^{\frac{1}{\alpha} - \frac{1}{p(\lambda)}}, \quad \|\nabla u_\lambda\|^{q(\lambda)-1}_{L^{q(\lambda)}(\Omega)} \le \|g\|_{W^{-1,\alpha'}(\Omega)} |\Omega|^{\frac{1}{\alpha} - \frac{1}{q(\lambda)}}. \tag{46}$$

From inequalities (45) and (46), keeping in mind assumption (11), we deduce the following:

$$\begin{aligned} \|\nabla u_\lambda\|_{L^\alpha(\Omega)} &\le \min\left\{ \|g\|^{\frac{1}{p(\lambda)-1}}_{W^{-1,\alpha'}(\Omega)} |\Omega|^{\left(\frac{1}{\alpha} - \frac{1}{p(\lambda)}\right)\frac{p(\lambda)}{p(\lambda)-1}}, \|g\|^{\frac{1}{q(\lambda)-1}}_{W^{-1,\alpha'}(\Omega)} |\Omega|^{\left(\frac{1}{\alpha} - \frac{1}{q(\lambda)}\right)\frac{q(\lambda)}{q(\lambda)-1}} \right\} \\ &\le \max_{p,q \in [\alpha,\beta]} \left\{ \|g\|^{\frac{1}{p-1}}_{W^{-1,\alpha'}(\Omega)} |\Omega|^{\left(\frac{1}{\alpha} - \frac{1}{p}\right)\frac{p}{p-1}}, \|g\|^{\frac{1}{q-1}}_{W^{-1,\alpha'}(\Omega)} |\Omega|^{\left(\frac{1}{\alpha} - \frac{1}{q}\right)\frac{q}{q-1}} \right\} = C, \end{aligned} \tag{47}$$

where $C = C(\alpha, \beta, \Omega, g) > 0$. Next, the second part of assumption (12) says that $b(\cdot)$ is a bounded function. This fact and the inequality (47) imply that we can find $L \in \mathbb{R}$ satisfying the range constraint for $b(\cdot)$ given as

$$b(u_\lambda) \in [-L, L] \text{ for all } \lambda \in \mathbb{R}.$$

In view of the λ-dependence herein, we look at function $B : [-L, L] \to [-L, L]$ given by

$$B(\lambda) = b(u_\lambda) \text{ for all } \lambda \in [-L, L].$$

We note that B is a continuous function. Indeed, $\lambda_n \to \lambda$, as $n \to +\infty$, implies $p(\lambda_n) \to p(\lambda)$ and $q(\lambda_n) \to q(\lambda)$ (recall the first part of assumption (11)). We set $p_n = p(\lambda_n)$ and $q_n = q(\lambda_n)$, we use Lemma 3 to conclude that

$$u_{\lambda_n} \to u_\lambda \text{ in } W_0^{1,\alpha}(\Omega) \text{ as } n \to +\infty.$$

From the first part of assumption (12), we note the continuity of $b(\cdot)$. Thus, the following is obtained:

$$B(u_{\lambda_n}) \to B(u_\lambda) \text{ as } n \to +\infty,$$

and, hence, the continuity of $B(\cdot)$ is established. Consequently, $B(\cdot)$ has a fixed point λ_0 and u_{λ_0} solution of (43) (for $\lambda = \lambda_0$) is clearly a solution to (13). We conclude that function u_{λ_0} corresponding to the fixed point λ_0 of $B(\cdot)$ is a weak solution to (1). □

4. Conclusions

Nonhomogeneous differential operators with exponents depending on solution u may represent useful models in the understanding of the dynamic process for applications. As mentioned in the Introduction (recall the work of Tiirola [10]), these models together with numerical techniques may improve certain sub-fields of image processing, as restoration, noising and denoising and registration. In this manuscript, combining the tools of the calculus of variations, with suitable classes of approximating problems and certain a priori estimates, we study the weak solutions of nonlocal Dirichlet problems. Further studies will be needed to explore the existence and multiplicity of ground state solutions (that is, minimizers of the energy functional associated to the problem, among all nontrivial solutions), also with a nonlinearity in the critical growth range. It is well-known that the presence of a critical term (consider, for example, the critical power term $|u|^{p^*-2}u$) is a source of difficulties, as the energy functional loses its regularity. For more information, we refer to Papageorgiou et al. [17], where suitable cut-off techniques are involved to bypass the critical term.

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