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Left-Invariant Einstein-like Metrics on Compact Lie Groups

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Abstract: In this paper, we study left-invariant Einstein-like metrics on the compact Lie group G. Assume that there exist two subgroups, $H \subset K \subset G$, such that G/K is a compact, connected, irreducible, symmetric space, and the isotropy representation of G/H has exactly two inequivalent, irreducible summands. We prove that the left metric $\langle \cdot, \cdot \rangle_{t_1,t_2}$ on G defined by the first equation, must be an $\mathcal{A}$-metric. Moreover, we prove that compact Lie groups do not admit non-naturally reductive left-invariant $\mathcal{B}$-metrics, such as $\langle \cdot, \cdot \rangle_{t_1,t_2}$.

Keywords: homogeneous space; compact Lie groups; Einstein-like metric; $\mathcal{A}$-metric; $\mathcal{B}$-metric

MSC: 53C25; 53C30

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1. Introduction

The paper aims to discuss generalizations of Einstein metrics. Recall that the Einstein manifold is a Riemannian manifold (M, g), satisfying $\mathrm{Ric} = \lambda g$, where λ is a constant and Ric is the Ricci tensor. The metric g is called an Einstein metric. The Einstein metric is very important both in mathematics and physics, but the existence of Einstein metrics in general cases is still an open problem [1]. However, there has been some interesting progress for homogeneous Einstein metrics [2]. For instance, Wolf classified non-symmetric homogeneous spaces G/H where H acts irreducibly [3]. Moreover, Wolf proved that there is a unique G-invariant Riemannian metric which is an Einstein metric on G/H.

Let $M = G/H$ be a simply connected homogeneous space, where G is a simple, connected Lie group and H is a connected, closed subgroup. The homogeneous space M, whose isotropy representation decomposed exactly into two irreducible summands, was studied by Dickinson and Kerr [4]. Based on works of Dynkin, Wolf, and Krämer [3,5–7], Dickinson and Kerr found a complete list of G/H and completely determined whether there is a G-invariant metric on the homogeneous space G/H. Assuming there is an intermediate subgroup $H \subset K \subset G$, they classified all the G-invariant Einstein metrics on G/H.

In general, Böhm and Kerr proved that there is a G-invariant Einstein metric on each simply connected homogeneous space G/H, whose dimension is no more than 11 [8]. In addition, Wang and Ziller proved this result is optimal. They found a 12-dimensional homogeneous space $\mathrm{SU}(4)/\mathrm{SU}(2)$, which does not admit any homogeneous Einstein metrics.

As generalizations of the Einstein metric, Gray [9] proposed two classes of Riemannian metrics, as follows:

- A Riemannian metric g is called an $\mathcal{A}$-metric if $(\nabla_X \mathrm{Ric})(X, X) = 0$ for any tangent field $X \in TM$;
- A Riemannian metric g is called a $\mathcal{B}$-metric if $(\nabla_X \mathrm{Ric})(Y, Z) = (\nabla_Y \mathrm{Ric})(X, Z)$ for any tangent fields $X, Y, Z \in TM$;

where ∇ denotes the covariant derivative of (M, g). Let $\mathcal{E}$ and $\mathcal{P}$ be sets of all Einstein manifolds and Ricci-parallel manifolds (i.e., $(\nabla_X \mathrm{Ric})(Y, Z) = 0$ for all $X, Y, Z \in TM$), respectively. Then, Gray gave the following inclusions between various classes:

$$\mathcal{E} \subset \mathcal{P} = \mathcal{A} \cap \mathcal{B} \subset A \text{ (or } \mathcal{B}).$$

Hence, Einstein metrics must be $\mathcal{A}$-metrics as well as $\mathcal{B}$-metrics. On the contrary, there are some examples of $\mathcal{A}$-metrics and $\mathcal{B}$-metrics, which are not Einstein metrics [9]. It is worth mentioning that Tang and Yan [10] constructed compact, simply connected manifolds with $\mathcal{A}$-metrics, which are neither locally homogeneous nor locally isometric to Riemannian products and have non-parallel Ricci tensor, to the Besse's problem [1].

On the other hand, the classifications of $\mathcal{A}$-metrics and $\mathcal{B}$-metrics are also unsolved, even for homogeneous spaces. There exists a homogeneous space G/H, on which G-invariant metrics are not always $\mathcal{A}$-metrics [9,11]. For the $\mathcal{B}$-metrics, there is no example of a non-Ricci-parallel Riemannian manifold endowed with a $\mathcal{B}$-metric by now, which supports the following conjecture [12]:

Conjecture 1. *Any homogeneous Riemannian manifold with a $\mathcal{B}$-metric is Ricci-parallel.*

This conjecture is true in several special cases [11–13], and one of our main results in the present paper proving this conjecture is also true for some compact Lie groups with a class of left-invariant metrics.

Let G be a compact simple Lie group. Yan and Deng studied left-invariant Einstein metrics on G [14]. Based on the works of Dickinson and Kerr, Wolf, D'Atri, and Ziller [3,4,15], Yan and Deng found a method to construct non-naturally reductive left-invariant Einstein metrics on G. By this method, they found some new non-naturally reductive Einstein metrics on compact simple Lie group. For examples, $\mathrm{SO}(2n)$ and $\mathrm{Sp}(2n)$ all admit non-naturally reductive left-invariant Einstein metrics. Inspired by these results, we will study the $\mathcal{A}$-metrics and $\mathcal{B}$-metrics on compact Lie groups.

Let G be a compact Lie group with subgroups $H \subset K \subset G$. In this paper, we assume that G/K is a compact, connected, irreducible, symmetric space and the isotropy representation $\mathfrak{m}$ of G/H decomposes exactly into two summands denoted by $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Let B be the negative of the Killing form on the Lie algebra $\mathfrak{g}$ of G, then we have a decomposition of the Lie algebra with respect to B,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}_2 = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where $\mathfrak{h}$ and $\mathfrak{k}$ are Lie algebras of H and K, respectively. Since it is well known that there is a one-to-one correspondence between left-invariant metrics on G and inner products on $\mathfrak{g}$, we consider the following left-invariant metric on G,

$$\langle \cdot, \cdot \rangle_{t_1,t_2} = B|_{\mathfrak{h}} \oplus t_1 B|_{\mathfrak{m}_1} \oplus t_2 B|_{\mathfrak{m}_2}, \text{ where } t_1, t_2 \in \mathbb{R}^+. \tag{1}$$

For $\mathcal{A}$-metrics, we prove that

Theorem 1. *With notations above, left-invariant metrics determined by Equation (1) on the compact Lie group G must be $\mathcal{A}$-metrics.*

If G is simple and the left-invariant metric $\langle \cdot, \cdot \rangle_{t_1,t_2}$ is naturally reductive, then $\langle \cdot, \cdot \rangle_{t_1,t_2}$ is always holonomy irreducible [15]. Hence, $\langle \cdot, \cdot \rangle_{t_1,t_2}$ is a $\mathcal{B}$-metric if—and only if—it is an Einstein metric. For $\mathcal{B}$-metrics, we consider them to be non-naturally reductive left-invariant metrics, and we can prove this, as follows:

Theorem 2. *Let (G, K, H) be a triple of Lie groups in Table 1. Then the compact Lie group G admits no non-naturally reductive left-invariant $\mathcal{B}$-metrics, such as Equation (1).*

Remark 1. *For each case in Table 1, the compact Lie group G admits no non-naturally reductive left-invariant Einstein metrics, such as Equation (1) [14].*

Table 1. No non-naturally reductive left-invariant $\mathcal{B}$-metrics, such as Equation (1) on G.

G	K	H	Index
$\mathrm{SU}(4)$	$\mathrm{Sp}(2)$	$\mathrm{SU}(2)$	
$\mathrm{SU}(14)$	$\mathrm{Sp}(7)$	$\mathrm{Sp}(3)$	
$\mathrm{SO}(\frac{l(l-1)}{2}+1)$	$\mathrm{SO}(\frac{l(l-1)}{2})$	$\mathrm{SO}(l)$	$7 \leq l$
$\mathrm{SO}(m^2)$	$\mathrm{SO}(m^2-1)$	$\mathrm{SU}(m)$	$3 \leq m$
$\mathrm{SO}(2n^2+n+1)$	$\mathrm{SO}(2n^2+n)$	$\mathrm{Sp}(n)$	$2 \leq n$
$\mathrm{SO}(15)$	$\mathrm{SO}(14)$	G_2	
$\mathrm{SO}(17)$	$\mathrm{SO}(16)$	$\mathrm{Spin}(9)$	
$\mathrm{SO}(43)$	$\mathrm{SO}(42)$	$\mathrm{Sp}(4)$	
$\mathrm{SO}(129)$	$\mathrm{SO}(128)$	$\mathrm{Spin}(16)$	
$\mathrm{Spin}(9)$	$\mathrm{Spin}(8)$	$\mathrm{Spin}(7)$	

The rest of this paper is organized as follows. In Section 2, we will introduce geometries of the compact Lie group and prove Theorem 1. A useful proposition will be proposed in the end of Section 2. In Section 3, we will prove Theorem 2.

2. Geometries of the Compact Lie Group

Let G be a compact Lie group endowed with a left-invariant metric $\langle \cdot, \cdot \rangle$. Then, for any left-invariant fields $X, Y \in \mathfrak{g}$, the function $\langle X, Y \rangle : G \to \mathbb{R}$ is constant. In fact, since $\langle \cdot, \cdot \rangle$, X and Y are all left-invariant, for any $g \in G$, we have

$$\langle X, Y \rangle_g = \langle X_g, Y_g \rangle = \langle (dL_g)_e X_e, (dL_g)_e Y_e \rangle = \langle X_e, Y_e \rangle = \langle X, Y \rangle_e.$$

On the contrary, fixing an inner product on $\mathfrak{g}$, we can define a left-invariant metric on G by transformation. Moreover, the constant function $\langle X, Y \rangle$ on G implies $X\langle Y, Z \rangle = 0$, where $X, Y, Z \in \mathfrak{g}$. Then, by straightforward application of Koszul's formula [16], we obtain

$$2\langle \nabla_X Y, Z \rangle = -\langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle = 2\langle U(X, Y), Z \rangle + \langle Z, [X, Y] \rangle.$$

Assume there are subgroups $H \subset K \subset G$, such that G/K is a compact, connected, irreducible, symmetric space and the isotropy representation $\mathfrak{m}$ of G/H decomposes exactly into two summands $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Let $\mathfrak{h}, \mathfrak{k}$ and $\mathfrak{g}$ be Lie algebras of H, K, and G, respectively. We use $B_{\mathfrak{h}}, B_{\mathfrak{k}}$ and B to denote negatives of Killing forms on corresponding Lie algebras. Then, we have a decomposition of Lie algebras with respect to B, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}_2 = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$. It is direct to check that Lie brackets have the following relations:

$$[\mathfrak{h}, \mathfrak{m}_1] \subset \mathfrak{m}_1,\ [\mathfrak{k}, \mathfrak{m}_2] \subset \mathfrak{m}_2,\ [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{k},\ [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h},\ [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{k}.$$

Furthermore, there is a one-to-one correspondence between inner products $\langle \cdot, \cdot \rangle_e$ on $\mathfrak{g}$ and symmetric, positive, defined, linear maps P which is defined by $\langle X, Y \rangle_e := B(PX, Y)$ for any $X, Y \in \mathfrak{g}$. We consider the left-invariant metric on G defined by Equation (1), which corresponds to $P = \mathrm{Id}|_{\mathfrak{h}} \oplus t_1 \mathrm{Id}|_{\mathfrak{m}_1} \oplus t_2 \mathrm{Id}|_{\mathfrak{m}_2}$ on $\mathfrak{g}$.

For convenience, define $B_\pm : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by $B_\pm(X, Y) := \frac{1}{2}([X, PY] \mp [PX, Y])$ for any $X, Y \in \mathfrak{g}$ [17] and recall that $U : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is defined by $2(U(X, Y), Z) := ([Z, X], Y) + (X, [Z, Y])$ for all $X, Y, Z \in \mathfrak{g}$ [1]. Then, $U(X, Y) = P^{-1} B_+(X, Y)$, and by direct calculations, we obtain the following:

- for $X \in \mathfrak{h}$ and $Y \in \mathfrak{h}$: $B_-(X, Y) = [X, Y]$, $B_+(X, Y) = 0$, $U(X, Y) = 0$;
- for $X \in \mathfrak{h}$ and $Y \in \mathfrak{m}_1$: $B_-(X, Y) = \frac{t_1+1}{2}[X, Y]$, $B_+(X, Y) = \frac{t_1-1}{2}[X, Y]$, $U(X, Y) = \frac{t_1-1}{2t_1}[X, Y]$;

- for $X \in \mathfrak{h}$ and $Y \in \mathfrak{m}_2$: $B_-(X,Y) = \frac{t_2+1}{2}[X,Y]$, $B_+(X,Y) = \frac{t_2-1}{2}[X,Y]$, $U(X,Y) = \frac{t_2-1}{2t_2}[X,Y]$;
- for $X \in \mathfrak{m}_1$ and $Y \in \mathfrak{m}_1$: $B_-(X,Y) = t_1[X,Y]$, $B_+(X,Y) = 0$, $U(X,Y) = 0$;
- for $X \in \mathfrak{m}_1$ and $Y \in \mathfrak{m}_2$: $B_-(X,Y) = \frac{t_2+t_1}{2}[X,Y]$, $B_+(X,Y) = \frac{t_2-t_1}{2}[X,Y]$, $U(X,Y) = \frac{t_2-t_1}{2t_2}[X,Y]$;
- for $X \in \mathfrak{m}_2$ and $Y \in \mathfrak{m}_2$: $B_-(X,Y) = t_2[X,Y]$, $B_+(X,Y) = 0$, $U(X,Y) = 0$.

Hence, the covariant derivative is $\nabla_X Y = \frac{1}{2}[X,Y] + U(X,Y) = c[X,Y]$, where the value of c is showed in Table 2.

Table 2. $\nabla_X Y = c[X,Y]$.

$X \in$	$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{m}_1$	$\mathfrak{m}_1$	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{m}_2$	$\mathfrak{m}_2$
$Y \in$	$\mathfrak{h}$	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{h}$	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{h}$	$\mathfrak{m}_1$	$\mathfrak{m}_2$
c	$\frac{1}{2}$	$1-\frac{1}{2t_1}$	$1-\frac{1}{2t_2}$	$\frac{1}{2t_1}$	$\frac{1}{2}$	$1-\frac{t_1}{t_2}$	$\frac{1}{2t_2}$	$\frac{t_1}{2t_2}$	$\frac{1}{2}$

Next, we summarize formulae of the curvature tensor R and the Ricci curvature tensor Ric on the compact Lie group G.

Lemma 1 ([1]). *Let G be a compact Lie group endowed with a left-invariant metric $\langle\cdot,\cdot\rangle$. Then, for any $X, Y \in \mathfrak{g}$, we have*

- $\langle R(X,Y)X,Y\rangle = |U(X,Y)|^2 - \langle U(X,X), U(Y,Y)\rangle - \frac{3}{4}|[X,Y]|^2 + \frac{\langle [[X,Y],X],Y\rangle - \langle X,[[X,Y],Y]\rangle}{2}$,
- $\mathrm{Ric}(X,X) = -\frac{1}{2}\sum_i |[X,X_i]|^2 + \frac{1}{2}\sum_i \langle [[X,X_i],X],X_i\rangle + \frac{1}{4}\sum_{ij}\langle X,[X_j,X_i]\rangle^2$,

where $\{X_i\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle$.

For $p = 1$ or 2, define $C_{\mathfrak{m}_p} := -\sum_i (\mathrm{ad}_{h_i} \circ \mathrm{ad}_{h_i})|_{\mathfrak{m}_p}$ called Casimir operators [14], where $\{h_i\}$ is a B-orthonormal basis of $\mathfrak{h}$. It is not difficult to check the action of Casimir operator $C_{\mathfrak{m}_p}$ on $\mathfrak{m}_p$ is $\lambda_p \mathrm{Id}$, where λ_p is a constant number. Then, formulae of the Ricci curvature tensor can be rewritten as follows:

Proposition 1 ([14]). *Let G be a compact Lie group and $\langle\cdot,\cdot\rangle_{t_1,t_2}$ be the left-invariant metric on G defined by Equation (1). Assume $h \in \mathfrak{h}, u \in \mathfrak{m}_1$ and $p \in \mathfrak{m}_2$, then*

- $\mathrm{Ric}(h,h) = (\frac{1}{4} - \frac{1}{4t_1^2})B_{\mathfrak{h}}(h,h) + (\frac{1}{4t_1^2} - \frac{1}{4t_2^2})B_{\mathfrak{k}}(h,h) + \frac{1}{4t_2^2}B(h,h)$;
- $\mathrm{Ric}(u,u) = \frac{t_1^2}{4t_2^2}B(u,u) + (\frac{1}{4} - \frac{t_1^2}{4t_2^2})B_{\mathfrak{k}}(u,u) + (\frac{1}{2} - \frac{1}{2t_1})B(C_{\mathfrak{m}_1}(u),u)$;
- $\mathrm{Ric}(p,p) = (\frac{1}{2} - \frac{t_1}{4t_2})B(p,p) + (\frac{t_1}{2t_2} - \frac{1}{2t_2})B(C_{\mathfrak{m}_2}(p),p)$;
- $\mathrm{Ric}(h,u) = \mathrm{Ric}(h,p) = \mathrm{Ric}(u,p) = 0$.

If there are constants $c_1, c_2 > 0$, such that $B_{\mathfrak{k}} = c_1 B|_{\mathfrak{k}}$ and $B_{\mathfrak{h}} = c_2 B|_{\mathfrak{h}}$, then the formulae above are reduced to

- $\mathrm{Ric}(h,h) = (\frac{c_2}{4} + \frac{c_1-c_2}{4t_1^2} + \frac{1-c_1}{4t_2^2})B(h,h)$;
- $\mathrm{Ric}(u,u) = (\frac{(1-c_1)t_1^2}{4t_2^2} - \frac{1}{2t_1}\lambda_1 + \frac{c_1}{4} + \frac{1}{2}\lambda_1)B(u,u)$;
- $\mathrm{Ric}(p,p) = ((\frac{1}{2}\lambda_2 - \frac{1}{4})\frac{t_1}{t_2} - \frac{1}{2t_2}\lambda_2 + \frac{1}{2})B(p,p)$.

By the proposition, it follows a corollary directly.

Corollary 1 ([14]). *Let G be a compact Lie group. If $\langle\cdot,\cdot\rangle_{t_1,t_2}$ is the left-invariant metric on G defined by Equation (1), then $\langle\cdot,\cdot\rangle_{t_1,t_2}$ is an Einstein metric if—and only if—the following system of equations has a real solution.*

$$\begin{cases} \frac{c_2}{4} + \frac{c_1 - c_2}{4t_1^2} + \frac{1 - c_1}{4t_2^2} = \lambda, \\ \frac{(1-c_1)t_1^2}{4t_2^2} - \frac{1}{2t_1}\lambda_1 + \frac{c_1}{4} + \frac{1}{2}\lambda_1 = \lambda t_1, \\ (\frac{1}{2}\lambda_2 - \frac{1}{4})\frac{t_1}{t_2} - \frac{1}{2t_2}\lambda_2 + \frac{1}{2} = \lambda t_2, \end{cases}$$

where λ is a constant number. Moreover, if $t_1 = 1$ or $t_1 = t_2$ or $\mathfrak{h}$ is a ideal of $\mathfrak{k}$, then $\langle\cdot,\cdot\rangle_{t_1,t_2}$ is naturally reductive.

Now, we can introduce algebraic characterizations of $\mathcal{A}$-metric and $\mathcal{B}$-metric. Firstly, there are equivalent descriptions of $\mathcal{A}$-metrics and $\mathcal{B}$-metrics.

Proposition 2 ([1,11])**.** *Assume the notations as above.*

(1) *A left-invariant metric $\langle\cdot,\cdot\rangle$ on G is an $\mathcal{A}$-metric if—and only if—the following stands:* $\mathrm{Ric}(U(X,X),X) = 0$ *for each $X \in \mathfrak{g}$.*
(2) *A left-invariant metric $\langle\cdot,\cdot\rangle$ on G is a $\mathcal{B}$-metric if—and only if—the following stands:*

$$\begin{aligned} \frac{1}{2}\mathrm{Ric}([Z,X],Y) - \frac{1}{2}\mathrm{Ric}([Y,X],Z) + \mathrm{Ric}(X,[Z,Y]) & \\ +\mathrm{Ric}(U(Z,X),Y) - \mathrm{Ric}(U(Y,X),Z) &= 0, \end{aligned}$$

for any $X,\ Y,\ Z \in \mathfrak{g}$.

Because in this paper, the Lie algebra $\mathfrak{g}$ can be decomposed as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Assume dimensions of $\mathfrak{g}$ and summands on the right hand are $\dim \mathfrak{g} = l$, $\dim \mathfrak{h} = m$, $\dim \mathfrak{m}_1 = n$ and $\dim \mathfrak{m}_2 = l - m - n$, respectively. Then, let $\{e_a | 1 \le a \le l\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to B, such that $\mathfrak{h} = \mathrm{span}_{\mathbb{R}}\{e_i | 1 \le i \le m\}$, $\mathfrak{m}_1 = \mathrm{span}_{\mathbb{R}}\{e_p | m+1 \le p \le m+n\}$ and $\mathfrak{m}_2 = \mathrm{span}_{\mathbb{R}}\{e_\alpha | m+n+1 \le \alpha \le l\}$. Hence, the Lie bracket has three parts,

$$[e_a, e_b] = \sum_{i=1}^{m} c_{ab}^i e_i + \sum_{p=m+1}^{m+n} c_{ab}^p e_p + \sum_{\alpha=m+n+1}^{l} c_{ab}^\alpha e_\alpha,$$

where c_{ab}^d are called structure constants satisfied $c_{ab}^d = -c_{ba}^d$ with $1 \le a,b,d \le l$. Since B is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$, we have $B(\mathrm{ad}(Z)X, Y) = -B(X, \mathrm{ad}(Z)Y)$ for any $X, Y, Z \in \mathfrak{g}$, i.e., $c_{ab}^d = -c_{ad}^b$ for any $1 \le a,b,d \le l$.

With notations above, we can prove Theorem 1.

Proof of Theorem 1. By Proposition 2(1), the metric $\langle\cdot,\cdot\rangle_{t_1,t_2}$ on G is an $\mathcal{A}$-metric if—and only if—the following stands:

$$\mathrm{Ric}(U(E_1,E_2),E_3) + \mathrm{Ric}(U(E_2,E_3),E_1) + \mathrm{Ric}(U(E_3,E_1),E_2) = 0$$

for any E_1, E_2, E_3 belonging to the orthonormal basis,

$$\{e_i, \tfrac{1}{\sqrt{t_1}}e_p, \tfrac{1}{\sqrt{t_2}}e_\alpha | 1 \le i \le m, m+1 \le p \le m+n, m+n+1 \le \alpha \le l\},$$

of $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle_{t_1,t_2}$. For convenience, set $A_1 = \{e_i | 1 \le i \le m\}$, $A_2 = \{\frac{1}{\sqrt{t_1}}e_p | m+1 \le p \le m+n\}$, $A_3 = \{\frac{1}{\sqrt{t_2}}e_\alpha | m+n+1 \le \alpha \le l\}$ and

$$\mathcal{W}(E_1,E_2,E_3) = \mathrm{Ric}(U(E_1,E_2),E_3) + \mathrm{Ric}(U(E_2,E_3),E_1) + \mathrm{Ric}(U(E_3,E_1),E_2).$$

Then, we need to check whether $\mathcal{W}(E_1,E_2,E_3) = 0$ case by case, and by symmetries there are just 3 cases.

- for $E_1, E_2, E_3 \in A_1$ or A_2 or A_3, since $U(\mathfrak{h},\mathfrak{h}), U(\mathfrak{m}_1,\mathfrak{m}_1)$ and $U(\mathfrak{m}_2,\mathfrak{m}_2)$ are all 0, we obtain $\mathcal{W}(E_1,E_2,E_3)=0$.
- for $E_1, E_2 \in A_r$ and $E_3 \in A_s$ $(1 \le r \ne s \le 3)$, it is direct to check by formulae of U and Proposition 1 that $\mathcal{W}(E_1,E_2,E_3)=0$. For example, take $E_1,E_2 \in A_1$ and $E_3 \in A_2$. Because structure constants satisfy $c^a_{bd}+c^b_{ad}=0$, then $\mathcal{W}(E_1,E_2,E_3) = \mathrm{Ric}(\frac{t_1-1}{2t_1}[E_2,E_3],E_1)+\mathrm{Ric}(\frac{t_1-1}{2t_1}[E_1,E_3],E_2)=0$.
- for $E_r \in A_r (r=1,2,3)$, by formulae of U, we obtain

$$\mathcal{W}(E_1,E_2,E_3)=\mathrm{Ric}(\tfrac{t_1-1}{2t_1}[E_1,E_2],E_3)+\mathrm{Ric}(\tfrac{t_2-t_1}{2t_2}[E_2,E_3],E_1)+\mathrm{Ric}(\tfrac{t_2-1}{2t_2}[E_3,E_1],E_2).$$

Recall Lie bracket relations $[\mathfrak{h},\mathfrak{m}_1]\subset\mathfrak{m}_1$ and $[\mathfrak{k},\mathfrak{m}_2]\subset\mathfrak{m}_2$. Then, by Proposition 1, we have $\mathcal{W}(E_1,E_2,E_3)=0$.

□

Considering $\mathcal{B}$-metrics, we have the following result:

Proposition 3. *Let G be a compact Lie group, H be a closed subgroup, and K be an intermediate subgroup. Assume the triple (G,K,H) satisfied G/K is a connected, compact, irreducible, symmetric space and the isotropy representation $\mathfrak{m}$ of G/H decomposes exactly into two inequivalent irreducible summands $\mathfrak{m}_1$ and $\mathfrak{m}_2$. If the decomposition of the Lie algebra, $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}_1\oplus\mathfrak{m}_2$, satisfies that*

- *there are $u_1,v_1\in\mathfrak{m}_2$ and $u_2,v_2\in\mathfrak{m}_2$ such that $[u_1,v_1]_{\mathfrak{m}_1}\ne 0$ and $[u_2,v_2]_{\mathfrak{h}}\ne 0$,*
- *or there are $u_1,v_1\in\mathfrak{m}_1$ and $u_2,v_2\in\mathfrak{m}_2$ such that $[u_1,v_1]_{\mathfrak{h}}\ne 0$ and $[u_2,v_2]_{\mathfrak{h}}\ne 0$,*
- *or there are $u_1,v_1\in\mathfrak{m}_1$ and $u_2,v_2\in\mathfrak{m}_2$ such that $[u_1,v_1]_{\mathfrak{h}}\ne 0$ and $[u_2,v_2]_{\mathfrak{m}_1}\ne 0$,*

then the left-invariant metric $\langle\cdot,\cdot\rangle_{t_1,t_2}$ on G is a $\mathcal{B}$-metric if—and only if—there is an Einstein metric.

Proof. It is sufficient to prove the metric $\langle\cdot,\cdot\rangle_{t_1,t_2}$ is a $\mathcal{B}$-metric, then it must be an Einstein metric. Recall that the Lie algebra $\mathfrak{g}$ has an orthonormal basis

$$\{e_i, \tfrac{1}{\sqrt{t_1}}e_p, \tfrac{1}{\sqrt{t_2}}e_\alpha | 1\le i\le m, m+1\le p\le m+n, m+n+1\le\alpha\le l\}$$

with respect to $\langle\cdot,\cdot\rangle_{t_1,t_2}$. Then, by Proposition 2(2), the condition of being a $\mathcal{B}$-metric implies for any $X,Y,Z\in\mathfrak{g}$ there must be

$$\tfrac{1}{2}\mathrm{Ric}([Z,X],Y)-\tfrac{1}{2}\mathrm{Ric}([Y,X],Z)+\mathrm{Ric}(X,[Z,Y])+\mathrm{Ric}(U(Z,X),Y)-\mathrm{Ric}(U(Y,X),Z)=0.$$

Limited by space, here we just check the first case and other cases are similar. If $u_1,v_1\in\mathfrak{m}_2$ satisfy $[u_1,v_1]_{\mathfrak{m}_1}\ne 0$, then there must be a structure constant $c^p_{\alpha\beta}$, such that $c^p_{\alpha\beta}\ne 0$ for $m+1\le p\le m+n$ and $m+n+1\le\alpha,\beta\le l$. Hence, we can take $Z=e_\alpha$, $X=e_\beta$ and $Y=e_p$, and directly

$$\frac{1}{2}\mathrm{Ric}(\sum_{i=1}^{m}c^i_{\alpha\beta}e_i+\sum_{q=m+1}^{m+n}c^q_{\alpha\beta}e_q,e_p)+\sum_{\gamma=m+n+1}^{l}\{-\frac{1}{2}\mathrm{Ric}(c^\gamma_{p\beta}e_\gamma,e_\alpha)$$
$$+\mathrm{Ric}(e_\beta,c^\gamma_{\alpha p}e_\gamma)-\mathrm{Ric}(\frac{t_2-t_1}{2t_2}c^\gamma_{p\beta}e_\gamma,e_\alpha)\}=0,$$

which can be reduced to

$$\mathrm{Ric}(e_p,e_p)=\frac{t_1}{t_2}\mathrm{Ric}(e_\alpha,e_\alpha).$$

Similarly, if $u_2,v_2\in\mathfrak{m}_2$ satisfy $[u_2,v_2]_{\mathfrak{h}}\ne 0$, then there must be a structure constant $c^i_{\alpha\beta}$, such that $c^i_{\alpha\beta}\ne 0$ for $1\le i\le m$ and $m+n+1\le\alpha,\beta\le l$. Taking $Z=e_\alpha$, $X=e_\beta$ and $Y=e_i$, we obtain

$$\frac{1}{2}\mathrm{Ric}(\sum_{j=1}^{m} c_{\alpha\beta}^{j} e_j + \sum_{p=m+1}^{m+n} c_{\alpha\beta}^{p} e_p, e_i) + \sum_{\gamma=m+n+1}^{l} \{-\frac{1}{2}\mathrm{Ric}(c_{p\beta}^{\gamma} e_\gamma, e_\alpha) + \mathrm{Ric}(e_\beta, c_{\alpha p}^{\gamma} e_\gamma) - \mathrm{Ric}(\frac{t_2-1}{2t_2} c_{p\beta}^{\gamma} e_\gamma, e_\alpha)\} = 0,$$

which can be reduced to

$$\mathrm{Ric}(e_i, e_i) = \frac{1}{t_2}\mathrm{Ric}(e_\alpha, e_\alpha).$$

In conclusion, the left-invariant metric $\langle\cdot,\cdot\rangle_{t_1,t_2}$ is an Einstein metric. □

3. No Non-Naturally Reductive Left-Invariant $\mathcal{B}$-Metrics

In this section, we will prove Theorem 2. For each triple (G, K, H) in Table 1, there is a Lie algebra decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}_2 = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$, where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are inequivalent and irreducible. In each case, the compact Lie group G carries no non-naturally reductive left-invariant Einstein metrics [14], and each G/K is a connected, compact, irreducible, symmetric space. The proof of Theorem 2 is based on Proposition 3. To be precise, we will find vectors u_1, v_1, u_2, v_2 which satisfy the condition of Proposition 3 and every specific example in Table 1 will be discussed case by case. In this section, E_{ij} denotes the matrix whose (i,j)-entry is 1 and others are 0, and we define $G_{ij} := E_{ij} - E_{ji}$ and $F_{ij} := E_{ij} + E_{ji}$.

3.1. The $(\mathrm{SU}(4), \mathrm{Sp}(2), \mathrm{SU}(2))$ *Case*

The current case corresponds to a 12-dimensional homogeneous space $\mathrm{SU}(4)/\mathrm{SU}(2)$ which is proposed firstly by Wang and Ziller in [18] as the lowest dimensional example without $\mathrm{SU}(4)$-invariant homogenous Einstein metrics. In this case, $\mathrm{SU}(4)/\mathrm{Sp}(2)$ is a compact, connected, irreducible, symmetric space and the isotropy representation of $\mathrm{Sp}(2)/\mathrm{SU}(2)$ is irreducible. Moreover, we embed $\mathrm{SU}(2)$ into $\mathrm{SU}(4)$ by an irreducible complex representation of $\mathrm{SU}(2)$ with the highest weight $\lambda = 3$ [3,4,18]. It is well known [19] that the highest weight $\lambda \in \mathbb{N}^+$ determines the finite dimensional irreducible complex representation of $\mathrm{SU}(2)$. Then, we will construct the representation with respect to $\lambda = 3$ precisely.

Let $V = \mathbb{C}^2$ be the standard complex representation of $\mathrm{SU}(2)$ with a standard complex basis $\{e_1, e_2\}$. The nth symmetric power of V and the symmetrization of the n-tensor $(\otimes^i e_1) \otimes (\otimes^{n-i} e_2)$ $(1 \le i \le n)$ are denoted by $\mathrm{Sym}^n(V)$ and $e_1^i e_2^{n-i}$, respectively. For instance, take $n = 2$ and $i = 1$, then $e_1 e_2 = \frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)$. For the highest weight $\lambda = 3$, the corresponding irreducible complex representation of $\mathrm{SU}(2)$ is $\mathrm{Sym}^3(V) = \mathrm{span}_{\mathbb{C}}\{e_1^3, e_1^2 e_2, \ldots, e_1 e_2^2, e_2^3\}$ [19]. Denoting the standard Hermitian inner product on V by $\langle\cdot,\cdot\rangle_V$, we define a Hermitian inner product on the tensor space $\bigotimes^3 V$ by $\langle x_1 \otimes \cdots \otimes x_3, y_1 \otimes \cdots \otimes y_3\rangle_{\bigotimes^3 V} = \langle x_1, y_1\rangle_V \cdots \langle x_3, y_3\rangle_V$, which induces a Hermitian inner product on $\mathrm{Sym}^3(V)$. Hence, there is a unitary basis of $\mathrm{Sym}^3(V)$ with respect to the induced Hermitian inner product, i.e., $\{e_1^3, \sqrt{3}e_1^2 e_2, \sqrt{3}e_1 e_2^2, e_2^3\}$.

Recall that the Lie algebra

$$\mathfrak{su}(2) = \left\{ \left(\begin{array}{cc} a & b \\ -\bar{b} & \bar{a} \end{array} \right) \middle| a \in \mathrm{Im}(\mathbb{C}), b \in \mathbb{C} \right\},$$

where $\mathrm{Im}(\mathbb{C})$ is the set of all purely imaginary numbers. According to the construction above, the irreducible complex representation $\mathrm{Sym}^3(V)$ also induces an irreducible complex representation of Lie algebra $\mathfrak{su}(2)$, which is also denoted by $\mathrm{Sym}^3(V)$. Up to a conjugation in $\mathfrak{su}(4)$, the embedding ϕ of $\mathfrak{su}(2)$ into $\mathfrak{su}(4)$ is given as follows:

$$\phi : \left(\begin{array}{cc} a & b \\ -\bar{b} & \bar{a} \end{array} \right) \mapsto \left(\begin{array}{cccc} 3a & 0 & 0 & \sqrt{3}b \\ 0 & -a & \sqrt{3}b & -2\bar{b} \\ 0 & -\sqrt{3}\bar{b} & -3a & 0 \\ -\sqrt{3}\bar{b} & 2b & 0 & a \end{array} \right).$$

Recall that quaternions form a 4-dimensional algebra over the field of real numbers with basis $1, i, j, k$, and any element X of Lie algebra $\mathfrak{sp}(2)$ can be decomposed uniquely as $X = C + jD$, where $C, D \in M_{2,2}(\mathbb{C})$ satisfied $\bar{C}^t = -C$ and $D^t = D$, which induces an embedding ψ of $\mathfrak{sp}(2)$ into $\mathfrak{su}(4)$ by $\psi(X) = \begin{pmatrix} C & -\bar{D} \\ D & \bar{C} \end{pmatrix}$ [20]. Furthermore, it is direct to check $\phi(\mathfrak{su}(2)) \subset \psi(\mathfrak{sp}(2))$.

In this paper, we use $\mathrm{tr}(A)$ to denote the trace of the matrix A. Then, the negative of the Killing form of $\mathfrak{su}(4)$ is $B(X,Y) = -8\mathrm{tr}(XY)$ with $X, Y \in \mathfrak{su}(4)$. Hence, there is an orthonormal basis of $\mathfrak{su}(4)$ with respect to $B(\cdot,\cdot)$,

$h_1 = \frac{\sqrt{-10}}{40}(3E_{11} - E_{22} - 3E_{33} + E_{44})$, $h_2 = \frac{\sqrt{10}}{40}(\sqrt{3}G_{14} + \sqrt{3}G_{23} - 2G_{24})$,
$h_3 = \frac{\sqrt{-10}}{40}(\sqrt{3}F_{14} + \sqrt{3}F_{23} + 2F_{24})$, $u_1 = \frac{\sqrt{-10}}{40}(E_{11} + 3E_{22} - E_{33} - 3E_{44})$,
$u_2 = \frac{\sqrt{2}}{8}(G_{12} + G_{34})$, $u_3 = \frac{\sqrt{-2}}{8}(F_{12} - F_{34})$, $u_4 = \frac{1}{4}G_{13}$, $u_5 = \frac{\sqrt{-1}}{4}F_{13}$,
$u_6 = \frac{3\sqrt{15}}{60}(\frac{\sqrt{3}}{3}G_{14} + \frac{\sqrt{3}}{3}G_{23} + G_{24})$, $u_7 = \frac{3\sqrt{-15}}{60}(\frac{\sqrt{3}}{3}F_{14} + \frac{\sqrt{3}}{3}F_{23} - F_{24})$,
$p_1 = \frac{\sqrt{2}}{8}(G_{12} - G_{34})$, $p_2 = \frac{\sqrt{-2}}{8}(F_{12} + F_{34})$, $p_3 = \frac{\sqrt{2}}{8}(G_{14} - G_{23})$,
$p_4 = \frac{\sqrt{-2}}{8}(F_{14} - F_{23})$, $p_5 = \frac{\sqrt{-2}}{8}(-E_{11} + E_{22} - E_{33} + E_{44})$.

Then, the decomposition of Lie algebra is $\mathfrak{su}(4) = \mathfrak{su}(2) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$, where $\mathfrak{su}(2) = \mathrm{span}_{\mathbb{R}}\{h_i | 1 \le i \le 3\}$, $\mathfrak{m}_1 = \mathrm{span}_{\mathbb{R}}\{u_l | 1 \le q \le 7\}$ and $\mathfrak{m}_2 = \mathrm{span}_{\mathbb{R}}\{p_\alpha | 1 \le \alpha \le 5\}$. Lie bracket relations can be calculated directly,

- $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_1 \oplus \mathfrak{h}$:
$[u_1, u_2] = -\frac{\sqrt{10}}{20}u_3$, $[u_1, u_3] = \frac{\sqrt{10}}{20}u_2$, $[u_1, u_4] = \frac{\sqrt{10}}{20}u_5$, $[u_1, u_5] = -\frac{\sqrt{10}}{20}u_4$,
$[u_1, u_6] = -\frac{\sqrt{10}}{20}u_7 + \frac{\sqrt{15}}{10}h_3$, $[u_1, u_7] = \frac{\sqrt{10}}{20}u_6 - \frac{\sqrt{15}}{10}h_2$, $[u_2, u_3] = -\frac{\sqrt{10}}{20}u_1 + \frac{\sqrt{10}}{10}h_1$,
$[u_2, u_4] = -\frac{\sqrt{10}}{20}u_6 - \frac{\sqrt{15}}{20}h_2$, $[u_2, u_5] = -\frac{\sqrt{10}}{20}u_7 - \frac{\sqrt{15}}{20}h_3$, $[u_2, u_6] = \frac{\sqrt{10}}{20}u_4 + \frac{1}{4}h_2$,
$[u_2, u_7] = \frac{\sqrt{10}}{20}u_5 - \frac{1}{4}h_3$, $[u_3, u_4] = \frac{\sqrt{10}}{20}u_7 + \frac{\sqrt{15}}{20}h_3$, $[u_3, u_5] = -\frac{\sqrt{10}}{20}u_6 - \frac{\sqrt{15}}{20}h_2$,
$[u_3, u_6] = \frac{\sqrt{10}}{20}u_5 + \frac{1}{4}h_3$, $[u_3, u_7] = -\frac{\sqrt{10}}{20}u_4 + \frac{1}{4}h_2$, $[u_4, u_5] = \frac{\sqrt{10}}{20}u_1 + \frac{3\sqrt{10}}{20}h_1$,
$[u_4, u_6] = -\frac{\sqrt{10}}{20}u_2$, $[u_4, u_7] = \frac{\sqrt{10}}{20}u_3$, $[u_5, u_6] = -\frac{\sqrt{10}}{20}u_3$, $[u_5, u_7] = -\frac{\sqrt{10}}{20}u_2$,
$[u_6, u_7] = -\frac{\sqrt{10}}{20}u_1 + \frac{\sqrt{10}}{20}h_1$;
- $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_2$:
$[u_1, p_1] = -\frac{\sqrt{10}}{20}p_2$, $[u_1, p_2] = \frac{\sqrt{10}}{20}p_1$, $[u_1, p_3] = \frac{\sqrt{10}}{10}p_4$, $[u_1, p_4] = -\frac{\sqrt{10}}{10}p_3$,
$[u_1, p_5] = 0$, $[u_2, p_1] = 0$, $[u_2, p_2] = -\frac{\sqrt{2}}{4}p_5$, $[u_2, p_3] = 0$, $[u_2, p_4] = 0$,
$[u_2, p_5] = \frac{\sqrt{2}}{4}p_2$, $[u_3, p_1] = \frac{\sqrt{2}}{4}p_5$, $[u_3, p_2] = 0$, $[u_3, p_3] = 0$, $[u_3, p_4] = 0$,
$[u_3, p_5] = -\frac{\sqrt{2}}{4}p_1$, $[u_4, p_1] = -\frac{1}{4}p_3$, $[u_4, p_2] = \frac{1}{4}p_4$, $[u_4, p_3] = \frac{1}{4}p_1$,
$[u_4, p_4] = -\frac{1}{4}p_2$, $[u_4, p_5] = 0$, $[u_5, p_1] = -\frac{1}{4}p_4$, $[u_5, p_2] = -\frac{1}{4}p_3$, $[u_5, p_3] = \frac{1}{4}p_2$,
$[u_5, p_4] = \frac{1}{4}p_1$, $[u_5, p_5] = 0$, $[u_6, p_1] = -\frac{\sqrt{15}}{20}p_3$, $[u_6, p_2] = -\frac{\sqrt{15}}{20}p_4$, $[u_6, p_3] = \frac{\sqrt{15}}{20}p_1$,
$[u_6, p_4] = \frac{\sqrt{15}}{20}p_2 - \frac{\sqrt{5}}{10}p_5$, $[u_6, p_5] = \frac{\sqrt{5}}{10}p_4$, $[u_7, p_1] = \frac{\sqrt{15}}{20}p_4$, $[u_7, p_2] = -\frac{\sqrt{15}}{20}p_3$,
$[u_7, p_3] = \frac{\sqrt{15}}{20}p_2 + \frac{\sqrt{5}}{10}p_5$, $[u_7, p_4] = -\frac{\sqrt{15}}{20}p_1$, $[u_7, p_5] = -\frac{\sqrt{5}}{10}p_3$;
- $[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_1 \oplus \mathfrak{h}$:
$[p_1, p_2] = -\frac{\sqrt{10}}{20}u_1 + \frac{\sqrt{10}}{10}h_1$, $[p_1, p_3] = -\frac{1}{4}u_4 - \frac{\sqrt{15}}{20}u_6 + \frac{\sqrt{10}}{20}h_2$,
$[p_1, p_4] = -\frac{1}{4}u_5 + \frac{\sqrt{15}}{20}u_7 - \frac{\sqrt{10}}{20}h_3$, $[p_1, p_5] = \frac{\sqrt{2}}{4}u_3$, $[p_2, p_3] = -\frac{1}{4}u_5 - \frac{\sqrt{15}}{20}u_7 + \frac{\sqrt{10}}{20}h_3$,
$[p_2, p_4] = \frac{1}{4}u_4 - \frac{\sqrt{15}}{20}u_6 + \frac{\sqrt{10}}{20}h_2$, $[p_2, p_5] = -\frac{\sqrt{2}}{4}u_2$, $[p_3, p_4] = \frac{\sqrt{10}}{10}u_1 + \frac{\sqrt{10}}{20}h_1$,
$[p_3, p_5] = \frac{\sqrt{5}}{10}u_7 + \frac{\sqrt{30}}{20}h_3$, $[p_4, p_5] = -\frac{\sqrt{5}}{10}u_6 - \frac{\sqrt{30}}{20}h_2$,

Proof of Theorem 2 for the (SU(4), Sp(2), SU(2)) Case. By calculations above, we find $[p_1, p_2] = -\frac{\sqrt{10}}{20}u_1 + \frac{\sqrt{10}}{10}h_1$. Take $u_1 = u_2 = p_1$ and $v_1 = v_2 = p_2$. Obviously, we have $[u_1, v_1]_{\mathfrak{m}_1} \neq 0$ and $[u_2, v_2]_{\mathfrak{h}} \neq 0$. Then, by Proposition 3, the left-invariant metric $\langle\cdot,\cdot\rangle_{t_1,t_2}$ on compact Lie group SU(4) is a $\mathcal{B}$-metric if—and only if—it is an Einstein metric. Since there is no non-naturally reductive left-invariant Einstein metric such as Equation (1) on SU(4) [14], compact Lie group SU(4) admits no non-naturally reductive left-invariant $\mathcal{B}$-metric, such as Equation (1). □

3.2. The $(\mathrm{SU}(14), \mathrm{Sp}(7), \mathrm{Sp}(3))$ *Case*

Recall that the Lie algebra of $\mathrm{Sp}(3)$ is

$$\mathfrak{sp}(n) = \left\{ \begin{pmatrix} C & -\bar{D} \\ D & \bar{C} \end{pmatrix} \middle| \bar{C}^t = -C, D^t = D \text{ for } C, D \in M_{n,n}(\mathbb{C}) \right\}.$$

Let $\mathfrak{t}_n$ be a Cartan subalgebra of $\mathfrak{sp}(n)$ defined by $\mathfrak{t}_n := \{\mathrm{diag}(\theta_1, \cdots, \theta_n, -\theta_1, \cdots, -\theta_n)$ $|\theta_s \in \mathrm{Im}(\mathbb{C}), 1 \leq s \leq n\}$. Then, define a complex linear functional L_s on $\mathfrak{t}_n$ by

$$L_s(\mathrm{diag}(\theta_1, \cdots, \theta_n, -\theta_1, \cdots, -\theta_n)) = \theta_s,$$

where $1 \leq s \leq n$. In the present case, the Lie algebra $\mathfrak{sp}(3)$ is embedded into $\mathfrak{su}(14)$ by the irreducible complex representation of $\mathfrak{sp}(3)$ with the highest weight $L_1 + L_2 + L_3$ [3,4]. Next, we will introduce how to construct this representation [19,21].

In fact, consider the standard representation $\rho : \mathfrak{sp}(3) \times \mathbb{C}^6 \to \mathbb{C}^6$, which induces a representation $\bigwedge^3 \rho$ on $\bigwedge^3 \mathbb{C}^6$, i.e., the 3th exterior power of $\mathbb{C}^6$. Then the highest weight of $\bigwedge^3 \rho$ is $L_1 + L_2 + L_3$ [19,21]. However, $\bigwedge^3 \rho$ is not an irreducible representation. Actually, define a contraction map $\varphi_3 : \bigwedge^3 \mathbb{C}^6 \to \mathbb{C}^6$ by

$$\varphi_3(v_1 \wedge v_2 \wedge v_3) = b_0(v_1, v_2)v_3 - b_0(v_1, v_3)v_2 + b_0(v_2, v_3)v_1,$$

where $b_0(\cdot, \cdot)$ is a bilinear form on $\mathbb{C}^6$ that corresponds to $J = \begin{pmatrix} 0 & -\mathrm{Id}_3 \\ \mathrm{Id}_3 & 0 \end{pmatrix}$. It follows that the kernel W of the map φ_3 is exactly the irreducible complex representation of $\mathfrak{sp}(3)$ with the highest weight $L_1 + L_2 + L_3$ [19,21]. To give a unitary basis of the representation space W, let $\{e_1, \cdots, e_6\}$ be a standard basis of $\mathbb{C}^6$ and set

$$\begin{aligned}
&N_1 = e_1e_2e_3, N_2 = e_2e_3e_4, N_3 = -e_1e_3e_5, N_4 = -e_1e_2e_6, \\
&N_5 = -\tfrac{1}{2}(e_1e_2e_4 + e_2e_3e_6), N_6 = \tfrac{1}{2}(-e_1e_4e_6 + e_2e_5e_6), \\
&N_7 = \tfrac{1}{2}(e_1e_2e_5 - e_1e_3e_6), N_8 = e_4e_5e_6, N_9 = -e_1e_5e_6, \\
&N_{10} = e_2e_4e_6, N_{11} = e_3e_4e_5, N_{12} = \tfrac{1}{2}(e_1e_4e_5 + e_3e_5e_6), \\
&N_{13} = \tfrac{1}{2}(-e_1e_3e_4 + e_2e_3e_5), N_{14} = \tfrac{1}{2}(-e_2e_4e_5 + e_3e_4e_6),
\end{aligned}$$

where we omit the symbol $\wedge$ for convenience. Then, the representation space is given by $W = \mathrm{span}_{\mathbb{C}}\{N_a | 1 \leq a \leq 14\}$.

On the other side, the irreducible complex representation W is a symplectic type [21]. More precisely, define a complex linear map $\psi : W \to W$ by

- for $1 \leq k \leq 7$, then $\psi(N_k) = -N_{k+7}$,
- for $8 \leq k \leq 14$, then $\psi(N_k) = N_{k-7}$.

Moreover, define the conjugate map γ on W by $\gamma(\sum_{a=1}^{14} \lambda_a N_a) = \sum_{a=1}^{14} \bar{\lambda}_a N_a$, where $\lambda_a \in \mathbb{C}$ and $\bar{\lambda}_a$ is the conjugate of λ_a. Since $\psi^2 = -1$, the structure map $\mathcal{F} = -\psi \circ \gamma$ of W satisfies $\mathcal{F}^2 = -1$.

There is a natural Hermitian inner product $\langle \cdot, \cdot \rangle_W$ on W induced by the standard Hermitian inner product on $\bigotimes^3 \mathbb{C}^6$. Hence, we can find a unitary basis

$$\{\sqrt{6}N_m, 2\sqrt{3}N_p, \sqrt{6}\mathcal{F}(N_m), 2\sqrt{3}\mathcal{F}(N_p) | 1 \leq m \leq 4, 5 \leq p \leq 7\}$$

of W with respect to $\langle \cdot, \cdot \rangle_W$. In fact, by the representation W, we identify $\mathfrak{sp}(3)$ with a Lie subalgebra, which is also denoted by $\mathfrak{sp}(3)$, in $\mathfrak{sp}(7)$. Particularly, the vector $G_{14} \in \mathfrak{sp}(3)$ corresponds to

$$w_1 = -(G_{12} + G_{89}) - (G_{3,11} + G_{4,10}) - G_{7,14} \in \mathfrak{sp}(7).$$

Recall that the negative of the Killing form of $\mathfrak{su}(14)$ is $B(X,Y) = -28\mathrm{tr}(XY)$, where $X, Y \in \mathfrak{su}(14)$. Then, we obtain the Lie algebra decomposition

$$\mathfrak{su}(14) = \mathfrak{sp}(7) \oplus \mathfrak{m}_2 = \mathfrak{sp}(3) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where $\mathfrak{m}_1$ is the orthogonal complement of $\mathfrak{sp}(3)$ in $\mathfrak{sp}(7)$ and $\mathfrak{m}_2$ has a basis

$$\{G_{ij} - G_{7+i,7+j}, \sqrt{-1}(F_{ij} + F_{7+i,7+j}), G_{i,7+j} - G_{j,7+i}, \sqrt{-1}(F_{i,7+j} - F_{j,7+i}),$$
$$\sqrt{-1}(E_{11} - E_{qq} + E_{88} - E_{7+q,7+q}) | 1 \le i < j \le 7, 2 \le q \le 7\}.$$

Proof of Theorem 2 for the (SU(14), Sp(7), Sp(3)) Case. Take $u_1 = u_2 = G_{13} - G_{8,10} \in \mathfrak{m}_2$, $v_1 = v_2 = G_{23} - G_{9,10} \in \mathfrak{m}_2$ and $w_2 = (G_{12} + G_{89}) - (G_{3,11} + G_{4,10}) \in \mathfrak{m}_1$. It is not difficult to check $B([u_1, v_1], w_2) \neq 0$ and $B([u_1, v_1], w_1) \neq 0$. Hence, by Proposition 3, the left-invariant metric $\langle \cdot, \cdot \rangle_{t_1,t_2}$ on $\mathrm{SU}(14)$ is a $\mathcal{B}$-metric if—and only if—it is an Einstein metric. Since $\mathrm{SU}(14)$ admits no non-naturally reductive left-invariant Einstein-like metric such as Equation (1) [4], there is no non-naturally reductive left-invariant $\mathcal{B}$-metric, such as Equation (1) on $\mathrm{SU}(14)$. □

3.3. *The* $(\mathrm{SO}(\frac{l(l-1)}{2}+1), \mathrm{SO}(\frac{l(l-1)}{2}), \mathrm{SO}(l))$ *Case*

For $7 \le l$, the Lie algebra $\mathfrak{so}(l)$ is embedded into $\mathfrak{so}(\frac{l(l-1)}{2})$ by its adjoint representation. In fact, according to the adjoint representation of $\mathfrak{so}(l)$, we identify $\mathfrak{so}(l)$ with a Lie subalgebra, which is also denoted by $\mathfrak{so}(l)$, in $\mathrm{SO}(\frac{l(l-1)}{2})$. Particularly, the vector $G_{23} \in \mathfrak{so}(l)$ corresponds to

$$w_1 = G_{12} + \sum_{p=4}^{l} G_{l+p-3,2l+p-6} \in \mathfrak{so}(\frac{l(l-1)}{2}).$$

On the other hand, the negative of the Killing form of $\mathfrak{so}(\frac{l(l-1)}{2}+1)$ is $B(X,Y) = -(\frac{l(l-1)}{2} - 1)\mathrm{tr}(XY)$, where $X, Y \in \mathfrak{so}(\frac{l(l-1)}{2}+1)$. Then, there is a Lie algebra decomposition with respect to $B(\cdot,\cdot)$,

$$\mathfrak{so}(\frac{l(l-1)}{2}+1) = \mathfrak{so}(\frac{l(l-1)}{2}) \oplus \mathfrak{m}_2 = \mathfrak{so}(l) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where $\mathfrak{m}_1$ is the orthogonal complement of $\mathfrak{so}(l)$ in $\mathfrak{so}(\frac{l(l-1)}{2})$ and $\mathfrak{m}_2 = \mathrm{span}_{\mathbb{R}}\{G_{a,\frac{l(l-1)}{2}+1} | 1 \le a \le \frac{l(l-1)}{2}\}$.

Proof of Theorem 2 for the $(\mathrm{SO}(\frac{l(l-1)}{2}+1), \mathrm{SO}(\frac{l(l-1)}{2}), \mathrm{SO}(l))$ Case. Take $u_1 = u_2 = G_{1,\frac{l(l-1)}{2}+1} \in \mathfrak{m}_2$, $v_1 = v_2 = G_{2,\frac{l(l-1)}{2}+1} \in \mathfrak{m}_2$ and $w_2 = G_{12} - G_{l+1,2l-2} \in \mathfrak{m}_1$. It is not difficult to check that $B([u_1, v_1], w_2) \neq 0$ and $B([u_2, v_2], w_1) \neq 0$. Hence, by Proposition 3, the left-invariant metric $\langle \cdot, \cdot \rangle_{t_1,t_2}$ on $\mathrm{SO}(\frac{l(l-1)}{2}+1)$ is a $\mathcal{B}$-metric if—and only if—it is an Einstein metric. Since $\mathrm{SO}(\frac{l(l-1)}{2}+1)$ admits no non-naturally reductive left-invariant Einstein-like metric such as Equation (1) [14], there is no non-naturally reductive left-invariant $\mathcal{B}$-metric, such as Equation (1) on $\mathrm{SO}(\frac{l(l-1)}{2}+1)$. □

3.4. *The* $(\mathrm{SO}(m^2), \mathrm{SO}(m^2-1), \mathrm{SU}(m))$ *Case*

For $3 \le m$, the Lie algebra $\mathfrak{su}(m)$ is embedded into $\mathfrak{so}(m^2-1)$ by its adjoint representation. In fact, according to the adjoint representation of $\mathfrak{su}(m)$, we identify $\mathfrak{su}(m)$ with a Lie subalgebra, which is also denoted by $\mathfrak{su}(m)$, in $\mathfrak{so}(m^2-1)$. Particularly, the vector $G_{23} \in \mathfrak{su}(m)$ corresponds to

$$\begin{aligned} w_1 =& G_{12} + \sum_{p=4}^{m} G_{m+p-3,2m+p-6} + G_{\frac{m(m-1)}{2}+1,\frac{m(m-1)}{2}+2} + \sum_{p=4}^{m} G_{\frac{m^2+m}{2}+p-3,\frac{m^2+3m}{2}+p-6} \\ &+ G_{\frac{m^2+m}{2},m^2-m+1} - \sqrt{3}G_{\frac{m^2+m}{2},m^2-m+2} \in \mathfrak{so}(m^2-1). \end{aligned}$$

On the other hand, the negative of the Killing form of $\mathfrak{so}(m^2)$ is $B(X,Y) = -(m^2-2)\mathrm{tr}(XY)$, where $X, Y \in \mathfrak{so}(m^2)$. Then, we have a Lie algebra decomposition with respect to $B(\cdot,\cdot)$,

$$\mathfrak{so}(m^2) = \mathfrak{so}(m^2-1) \oplus \mathfrak{m}_2 = \mathfrak{su}(m) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where $\mathfrak{m}_1$ is the orthogonal complement of $\mathfrak{su}(m)$ in $\mathfrak{so}(m^2-1)$ and $\mathfrak{m}_2 = \mathrm{span}_{\mathbb{R}}\{G_{a,m^2} | 1 \leq a \leq m^2-1\}$.

Proof of Theorem 2 for the $(\mathbf{SO}(m^2), \mathbf{SO}(m^2-1), \mathbf{SU}(m))$ case. Take $u_1 = u_2 = G_{1,m^2} \in \mathfrak{m}_2$, $v_1 = v_2 = G_{2,m^2} \in \mathfrak{m}_2$ and $w_2 = G_{12} - G_{m+1,2m-2} \in \mathfrak{m}_1$. It is not difficult to check $B([u_1,v_1],w_2) \neq 0$ and $B([u_2,v_2],w_1) \neq 0$. Hence, by Proposition 3, the left-invariant metric $\langle\cdot,\cdot\rangle_{t_1,t_2}$ on compact Lie group $\mathrm{SO}(m^2)$ is a $\mathcal{B}$-metric if—and only if—it is an Einstein metric. Since $\mathrm{SO}(m^2)$ admits no non-naturally reductive left-invariant Einstein-like metric such as Equation (1) [14], there is no non-naturally reductive left-invariant $\mathcal{B}$-metric, such as Equation (1) on $\mathrm{SO}(m^2)$. □

3.5. *The* $(\mathrm{SO}(2n^2+n+1), \mathrm{SO}(2n^2+n), \mathrm{Sp}(n))$ *Case*

For $2 \leq n$, the Lie algebra $\mathfrak{sp}(n)$ is embedded into $\mathfrak{so}(2n^2+n)$ by its adjoint representation. In fact, according to the adjoint representation of $\mathfrak{sp}(n)$, we identify $\mathfrak{sp}(n)$ with a Lie subalgebra, which is also denoted by $\mathfrak{sp}(n)$, in $\mathfrak{so}(2n^2+n)$. Particularly, the vector $G_{23} + G_{n+2,n+3} \in \mathfrak{sp}(n)$ corresponds to

$$\begin{aligned} w_1 =& -G_{12} - \sum_{p=4}^{n} G_{n+p-3,2n+p-6} - G_{\frac{n(n-1)}{2}+1,\frac{n(n-1)}{2}+2} + \sqrt{2}G_{\frac{n(n+1)}{2},2n(n-1)+1} \\ &- \sqrt{2}G_{\frac{n(n+1)}{2},2n(n-1)+2} - \sum_{p=4}^{n} G_{\frac{n(n+1)}{2}+p-3,\frac{n^2+3n}{2}+p-6} - G_{n(n-1)+1,n(n-1)+2} \\ &+ \sqrt{2}G_{n^2,2n^2-n+2} - \sqrt{2}G_{n^2,2n^2-n+3} - \sum_{p=4}^{n} G_{n^2+p-3,n^2+n+p-6} - G_{\frac{3n(n-1)}{2}+1,\frac{3n(n-1)}{2}+2} \\ &+ \sqrt{2}G_{\frac{3n^2-n}{2},2n^2+2} - \sqrt{2}G_{\frac{3n^2-n}{2},2n^2+3} - G_{\frac{3n^2-n}{2}+p-3,\frac{3n^2+n}{2}+p-6} \in \mathfrak{so}(2n^2+n). \end{aligned}$$

On the other hand, the negative of the Killing form of $\mathfrak{so}(2n^2+n+1)$ is $B(X,Y) = -(2n^2+n-1)\mathrm{tr}(XY)$, where $X, Y \in \mathfrak{so}(2n^2+n+1)$. Then, we have a Lie algebra decomposition with respect to $B(\cdot,\cdot)$,

$$\mathfrak{so}(2n^2+n+1) = \mathfrak{so}(2n^2+n) \oplus \mathfrak{m}_2 = \mathfrak{sp}(n) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where $\mathfrak{m}_1$ is the orthogonal complement of $\mathfrak{sp}(n)$ in $\mathfrak{so}(2n^2+n)$ and $\mathfrak{m}_2 = \mathrm{span}_{\mathbb{R}}\{G_{a,2n^2+n+1} | 1 \leq a \leq 2n^2+n\}$.

Proof of Theorem 2 for the $(\mathbf{SO}(2n^2+n+1), \mathbf{SO}(2n^2+n), \mathbf{Sp}(n))$ Case. Take $u_1 = u_2 = G_{1,2n^2+n+1} \in \mathfrak{m}_2$, $v_1 = v_2 = G_{2,2n^2+n+1} \in \mathfrak{m}_2$ and $w_2 = G_{12} - G_{n+1,2n-2} \in \mathfrak{m}_1$. It is not difficult to check $B([u_1,v_1],w_2) \neq 0$ and $B([u_2,v_2],w_1) \neq 0$. Hence, by Proposition 3, the left-invariant metric $\langle\cdot,\cdot\rangle_{t_1,t_2}$ on $\mathrm{SO}(2n^2+n+1)$ is a $\mathcal{B}$-metric if—and only if—it is an Einstein metric. Since $\mathrm{SO}(2n^2+n+1)$ admits no non-naturally reductive left-invariant Einstein-like metric, such as Equation (1) [14], there is no non-naturally reductive left-invariant $\mathcal{B}$-metric, such as Equation (1) on $\mathrm{SO}(2n^2+n+1)$. □

3.6. The $(\mathrm{SO}(15), \mathrm{SO}(14), \mathrm{G}_2)$ Case

It is well known that the Lie algebra $\mathfrak{g}_2$ can be embedded into $\mathfrak{so}(7)$ [13,22]. In fact, any $X \in \mathfrak{g}_2$ is a linear combination of the following elements:

$$\alpha_1 G_{23} + \beta_1 G_{45} + \gamma_1 G_{76},\ \alpha_2 G_{31} + \beta_2 G_{46} + \gamma_2 G_{57},\ \alpha_3 G_{12} + \beta_3 G_{47} + \gamma_3 G_{65},$$
$$\alpha_4 G_{51} + \beta_4 G_{62} + \gamma_4 G_{73},\ \alpha_5 G_{14} + \beta_5 G_{72} + \gamma_5 G_{36},\ \alpha_6 G_{71} + \beta_6 G_{42} + \gamma_6 G_{35},$$
$$\alpha_7 G_{61} + \beta_7 G_{25} + \gamma_7 G_{34},\ \text{where } \alpha_p + \beta_p + \gamma_p = 0 \text{ for } 1 \le p \le 7.$$

Let $Q(X,Y) = -\frac{1}{2}\mathrm{tr}(XY)$, where $X, Y \in \mathfrak{so}(7)$. Then, we can take an orthonormal basis $\{N_a | 1 \le a \le 14\}$ of $\mathfrak{g}_2$ with respect to Q. Precisely, let $(\alpha_a, \beta_a, \gamma_a) = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0)$ for $1 \le a \le 7$ and $(\alpha_{a-7}, \beta_{a-7}, \gamma_{a-7}) = (\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3})$ for $8 \le a \le 14$. Moreover, by the adjoint representation of $\mathfrak{g}_2$, we identify $\mathfrak{g}_2$ with a Lie subalgebra, which is also denoted by $\mathfrak{g}_2$, in $\mathfrak{so}(14)$. Particularly, vectors $N_3, N_{10} \in \mathfrak{g}_2$ correspond to vectors in $\mathfrak{so}(14)$ as follows, respectively,

$$N_3 \mapsto w_1 = -\frac{\sqrt{2}}{4}G_{12} - \frac{\sqrt{6}}{4}G_{19} + \frac{\sqrt{6}}{4}G_{28} + \frac{\sqrt{2}}{4}G_{89} - \frac{\sqrt{2}}{2}G_{47} + \frac{\sqrt{2}}{2}G_{11,14} - \sqrt{2}G_{56};$$
$$N_{10} \mapsto w_2 - \frac{\sqrt{6}}{4}G_{12} + \frac{\sqrt{2}}{4}G_{19} - \frac{\sqrt{2}}{4}G_{28} - \frac{5\sqrt{6}}{12}G_{89} - \frac{\sqrt{6}}{2}G_{47} - \frac{\sqrt{6}}{6}G_{11,14} + \frac{\sqrt{6}}{3}G_{12,13}.$$

Recall that the negative of the Killing form of $\mathfrak{so}(15)$ is $B(X,Y) = -13\mathrm{tr}(XY)$, where $X, Y \in \mathfrak{g}_2$. Then, there is a Lie algebra decomposition with respect to $B(\cdot,\cdot)$,

$$\mathfrak{so}(15) = \mathfrak{so}(14) \oplus \mathfrak{m}_2 = \mathfrak{g}_2 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where $\mathfrak{m}_1$ is the orthogonal complement of $\mathfrak{g}_2$ in $\mathfrak{so}(14)$ and $\mathfrak{m}_2 = \mathrm{span}_{\mathbb{R}}\{G_{a,15} | 1 \le a \le 14\}$.

Proof of Theorem 2 for the $(\mathbf{SO(15), SO(14), G_2})$ Case. Take $u_1 = u_2 = G_{1,15}, v_1 = v_2 = G_{2,15}$ and $w_3 = -G_{12} + \frac{2\sqrt{3}}{3}G_{19} + G_{89} \in \mathfrak{m}_1$. It is not difficult to check $B([u_1, v_1], w_3) \ne 0$ and $B([u_2, v_2], w_1) \ne 0$. Hence, by Proposition 2, the left-invariant metric $\langle \cdot,\cdot \rangle_{t_1,t_2}$ on $\mathrm{SO}(15)$ is a $\mathcal{B}$-metric if—and only if—it is an Einstein metric. Since $\mathrm{SO}(15)$ admits no non-naturally reductive left-invariant Einstein-like metric such as Equation (1) [4], there is no non-naturally reductive left-invariant $\mathcal{B}$-metric, such as Equation (1) on $\mathrm{SO}(15)$. □

3.7. The $(\mathrm{SO}(17), \mathrm{SO}(16), \mathrm{Spin}(9))$ Case

Let $V = \mathbb{R}^9$ with inner product $\langle \cdot,\cdot \rangle_V$ and $\{e_1, \cdots, e_9\}$ be an orthonormal basis of V with respect to $\langle \cdot,\cdot \rangle_V$. Denote the Clifford algebra over V with $\langle \cdot,\cdot \rangle_V$ by $Cl(\mathbb{R}^9)$ and recall that the products in $Cl(\mathbb{R}^9)$ are determined by $e_r^2 = -1$ and $e_r e_s = -e_s e_r$ for $1 \le r \ne s \le 9$ [23]. Then, it is well known that $\mathfrak{spin}(9) = \mathrm{span}_{\mathbb{R}}\{e_r e_s | 1 \le r < s \le 9\}$. We must point out that a Clifford algebra $Cl(U, g)$ of a vector space U over a general field F with a quadratic form g is also called geometric algebra [24]. Moreover, for arbitrary vectors x, y the product in the geometric algebra can be calculated by $xy = x \cdot y + x \wedge y$, where $\cdot$ is the dot product and $\wedge$ is the outer product. For more results of the geometric algebra and its application, one can refer to [24–26]. In this case, the Lie algebra $\mathfrak{spin}(9)$ is embedded into $\mathfrak{so}(16)$ by a special observation [11].

Let $\mathbb{C}\mathrm{a}$ be the Cayley algebra which is isomorphic to $\mathbb{H} \oplus \mathbb{H}$ as a vector space, i.e., any Cayley number can be expressed as an ordered pair of quaternions. Then, the multiplication and the conjugate of Cayley numbers are defined by $(q_1, q_2)(q_1', q_2') = (q_1 q_1' - \bar{q}_2' q_2, q_2' q_1 + q_2 \bar{q}_1')$ and $\overline{(q_1, q_2)} = (\bar{q}_1, -q_2)$, respectively, where $\bar{q}_1, \bar{q}_1'$ and $\bar{q}_2'$ are conjugates of quaternions. Consider a vector space $\mathbb{R}^8$ endowed with an inner product $\langle \cdot,\cdot \rangle_{\mathbb{R}^8}$ and take the standard basis of $\mathbb{R}^8$ denoted by $\varepsilon_1, \cdots, \varepsilon_8$. Naturally, there is an isomorphism between $\mathbb{C}\mathrm{a}$ and $\mathbb{R}^8$, defined by mapping $(1,0), (i,0), \cdots, (0,k)$ to $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_8$, respectively. Consequently, $\mathbb{C}\mathrm{a} \oplus \mathbb{C}\mathrm{a}$ is isomorphic to $\mathbb{R}^{16}$ as a vector space and the inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}_{16}}$ on $\mathbb{R}_{16}$ is induced by $\langle\cdot,\cdot\rangle_{\mathbb{R}^8}$.

There is an action of the Cayley algebra on $\mathbb{C}\mathrm{a} \oplus \mathbb{C}\mathrm{a}$,

$$\rho : \mathbb{C}\mathrm{a} \times (\mathbb{C}\mathrm{a} \oplus \mathbb{C}\mathrm{a}) \to \mathbb{C}\mathrm{a} \oplus \mathbb{C}\mathrm{a},\ \rho(z)(x,y) = (\bar{y}z, -z\bar{x}),$$

and it is not difficult to prove that $\rho(z)^2 = -1$. Let $Cl(\mathbb{R}^8)$ be the Clifford algebra over $\mathbb{R}^8$ with $\langle\cdot,\cdot\rangle_V|_{\mathbb{R}^8}$, where $\mathbb{R}^8$ is a subspace of V spanned by $\{e_1,\cdots,e_8\}$. Then, ρ induces a representation of $Cl(\mathbb{R}^8)$,

$$\rho' : Cl(\mathbb{R}^8) \times (\mathbb{C}\mathrm{a} \oplus \mathbb{C}\mathrm{a}) \to (\mathbb{C}\mathrm{a} \oplus \mathbb{C}\mathrm{a}),\ \rho'(e_{\alpha_1}\dots e_{\alpha_t})(x,y) = \rho(\varepsilon_{\alpha_1})\dots\rho(\varepsilon_{\alpha_t})(x,y),$$

where $1 \leq \alpha_1 < \cdots < \alpha_t \leq 8$. Consider a subalgebra of Clifford algebra $Cl(\mathbb{R}^9)$ defined by $Cl(\mathbb{R}^9)_0 = \mathrm{span}_{\mathbb{R}}\{\prod_{s=1}^{9} e_s^{r_s} | r_s = 0 \text{ or } 1, \prod_{s=1}^{9} r_s \text{ is even}\}$ and observe that $Cl(\mathbb{R}^8)$ is isomorphic to $Cl(\mathbb{R}^9)_0$ with the isomorphic map $\phi : Cl(\mathbb{R}^8) \to Cl(\mathbb{R}^9)_0$ which is defined by $\phi(e_{\alpha_1}\dots e_{\alpha_t}) = e_{\alpha_1}\dots e_{\alpha_t}$ for t is even and $\phi(e_{\alpha_1}\dots e_{\alpha_t}) = e_{\alpha_1}\dots e_{\alpha_t}e_9$ for t is odd. Directly, there is a representation of $Cl(\mathbb{R}^9)_0$ on the vector space $\mathbb{C}\mathrm{a} \oplus \mathbb{C}\mathrm{a}$,

$$\begin{aligned}\rho'' : Cl(\mathbb{R}^9)_0 \times (\mathbb{C}\mathrm{a} \oplus \mathbb{C}\mathrm{a}) &\to \mathbb{C}\mathrm{a} \oplus \mathbb{C}\mathrm{a},\\ \rho''(e_{\alpha_1}\dots e_{\alpha_t}) = \rho'(e_{\alpha_1}\dots e_{\alpha_t}) &\text{ and } \rho''(e_{\alpha_1}\dots e_{\alpha_t}e_9) = \rho'(e_{\alpha_1}\dots e_{\alpha_t}).\end{aligned}$$

Recall the definition of Spin(9) [23], and the restriction of ρ'' to Spin(9) is a 16-dimensional representation of Spin(9) which induces a 16-dimensional representation $\tilde{\rho}$ of $\mathfrak{spin}(9)$ on $\mathbb{C}\mathrm{a} \oplus \mathbb{C}\mathrm{a}$. The representation $\tilde{\rho}$ is determined by $\tilde{\rho}(e_\alpha e_\beta)(x,y) = \rho(\varepsilon_\alpha)\rho(\varepsilon_\beta)(x,y)$ and $\tilde{\rho}(e_\alpha e_9)(x,y) = \rho(\varepsilon_\alpha)(x,y)$, where $1 \leq \alpha < \beta \leq 8$. In fact,

- $\tilde{\rho}(e_\alpha e_\beta)(\varepsilon_m,\varepsilon_n) = (-(\varepsilon_m\bar{\varepsilon}_\beta)\varepsilon_\alpha, -\varepsilon_\alpha(\bar{\varepsilon}_\beta\varepsilon_n))$, where $1 \leq \alpha < \beta \leq 8$ and $1 \leq m,n \leq 8$;
- $\tilde{\rho}(e_\alpha e_9)(\varepsilon_m,\varepsilon_n) = (\bar{\varepsilon}_n\varepsilon_\alpha, -\varepsilon_\alpha\bar{\varepsilon}_m)$, where $1 \leq \alpha \leq 8$ and $1 \leq m,n \leq 8$.

Denote $N_a = (\varepsilon_a, 0)$ for $1 \leq a \leq 8$ and $N_a = (0, \varepsilon_{a-8})$ for $9 \leq a \leq 16$, then $\{N_a | 1 \leq a \leq 16\}$ is an orthonormal basis of $\mathbb{R}^{16}$ with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}^{16}}$. Under the representation $\tilde{\rho}$ we identify the Lie algebra $\mathfrak{spin}(9)$ with a Lie subalgebra, which is also denoted by $\mathfrak{spin}(9)$, in $\mathfrak{so}(16)$. Particularly, vectors $e_1e_2, e_3e_4, e_5e_6, e_7e_8 \in \mathfrak{spin}(9)$ correspond to the vectors in $\mathfrak{so}(16)$ as follows, respectively,

$$\begin{aligned}
e_1e_2 &\mapsto w_1 = G_{12} - G_{34} - G_{56} + G_{78} + G_{9,10} + G_{11,12} + G_{13,14} - G_{15,16};\\
e_3e_4 &\mapsto w_2 = -G_{12} + G_{34} - G_{56} + G_{78} + G_{9,10} + G_{11,12} - G_{13,14} + G_{15,16};\\
e_5e_6 &\mapsto w_3 = -G_{12} - G_{34} + G_{56} + G_{78} + G_{9,10} - G_{11,12} + G_{13,14} + G_{15,16};\\
e_7e_8 &\mapsto w_4 = G_{12} + G_{34} + G_{56} + G_{78} - G_{9,10} + G_{11,12} + G_{13,14} + G_{15,16}.
\end{aligned}$$

On the other side, the negative of the Killing form of $\mathfrak{so}(17)$ is $B(X,Y) = -15\mathrm{tr}(XY)$, where $X, Y \in \mathfrak{so}(17)$. Then, we have the Lie algebra decomposition, with respect to $B(\cdot,\cdot)$,

$$\mathfrak{so}(17) = \mathfrak{so}(16) \oplus \mathfrak{m}_2 = \mathfrak{spin}(9) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where $\mathfrak{m}_1$ is the orthogonal complement of $\mathfrak{spin}(9)$ in $\mathfrak{so}(16)$ and $\mathfrak{m}_2 = \mathrm{span}_{\mathbb{R}}\{G_{a,17} | 1 \leq a \leq 16\}$.

Proof of Theorem 2 for the $(\mathrm{SO}(17), \mathrm{SO}(16), \mathrm{Spin}(9))$ Case. Take $u_1 = u_2 = \frac{\sqrt{30}}{30}G_{1,17} \in \mathfrak{m}_2$, $v_1 = v_2 = \frac{\sqrt{30}}{30}G_{2,17} \in \mathfrak{m}_2$ and $w_5 = G_{12} + G_{34} + G_{56} - G_{78} + 2G_{9,10} \in \mathfrak{m}_1$. It is not difficult to check $B([u_1,v_1],w_5) \neq 0$ and $B([u_2,v_2],w_1) \neq 0$. Hence, by Proposition 3, the left-invariant metric $\langle\cdot,\cdot\rangle_{t_1,t_2}$ on SO(17) is a $\mathcal{B}$-metric if—and only if—it is an Einstein metric. Since SO(17) admits no non-naturally reductive left-invariant Einstein-like metric, such as Equation (1) [4], there is no non-naturally reductive left-invariant $\mathcal{B}$-metric, such as Equation (1) on SO(17). $\square$

3.8. The $(\mathrm{SO}(43), \mathrm{SO}(42), \mathrm{Sp}(4))$ *Case*

In this case, the Lie algebra $\mathfrak{sp}(4)$ is embedded into $\mathfrak{so}(42)$ by its irreducible representation with the highest weight $\lambda = L_1 + L_2 + L_3 + L_4$ [3,4]. Consider the standard representation ρ of $\mathfrak{sp}(4)$ and the highest weight of $\bigwedge^4 \rho$ is $L_1 + L_2 + L_3 + L_4$ [19,21]. However, $\bigwedge^4 \rho$ is not an irreducible representation. In fact, define a contraction map $\varphi_3 : \bigwedge^4 \mathbb{C}^6 \to \bigwedge^2 \mathbb{C}^6$ as follows

$$\begin{aligned}\varphi_4(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = {} & b_0(v_1, v_2)v_3 \wedge v_4 - b_0(v_1, v_3)v_2 \wedge v_4 + b_0(v_1, v_4)v_2 \wedge v_3 \\ & + b_0(v_2, v_3)v_1 \wedge v_4 - b_0(v_2, v_4)v_1 \wedge v_3 + b_0(v_3, v_4)v_1 \wedge v_2,\end{aligned}$$

where $b_0(\cdot, \cdot)$ is a bilinear form on $\mathbb{C}^8$ that corresponds to $J = \begin{pmatrix} 0 & -\mathrm{Id}_4 \\ \mathrm{Id}_4 & 0 \end{pmatrix}$. It follows that the kernel W of the map φ_4 is exactly the irreducible complex representation of $\mathfrak{sp}(4)$ with the highest weight $L_1 + L_2 + L_3 + L_4$ [19,21]. Let $\{e_1, \cdots, e_8\}$ be a standard basis of $\mathbb{C}^8$, and set

$$\begin{aligned}
& N_1 = e_1e_2e_3e_4, N_2 = e_2e_3e_4e_5, N_3 = e_1e_3e_4e_6, N_4 = e_1e_2e_4e_7, \\
& N_5 = e_1e_2e_3e_8, N_6 = e_3e_4e_5e_6, N_7 = e_2e_4e_5e_7, N_8 = e_2e_3e_5e_8, \\
& N_9 = e_1e_2e_3e_7 - e_1e_2e_4e_8, N_{10} = e_2e_3e_5e_7 - e_2e_4e_5e_8, N_{11} = -e_1e_2e_4e_6 + e_1e_3e_4e_7, \\
& N_{12} = -e_2e_4e_5e_6 + e_3e_4e_5e_7, N_{13} = -e_1e_2e_3e_6 - e_1e_3e_4e_8, N_{14} = -e_2e_3e_5e_6 - e_3e_4e_5e_8, \\
& N_{15} = e_1e_2e_3e_5 - e_2e_3e_4e_8, N_{16} = e_1e_3e_5e_6 - e_3e_4e_6e_8, N_{17} = e_1e_2e_4e_5 + e_2e_3e_4e_7, \\
& N_{18} = e_1e_4e_5e_6 + e_3e_4e_6e_7, N_{19} = e_1e_3e_4e_5 - e_2e_3e_4e_6, N_{20} = e_1e_4e_5e_7 - e_2e_4e_6e_7, \\
& N_{21} = e_1e_2e_5e_6 - e_1e_4e_5e_8 - e_2e_3e_6e_7 + e_3e_4e_7e_8, \\
& N_{22} = e_1e_2e_5e_6 - e_1e_3e_5e_7 - e_2e_4e_6e_8 + e_3e_4e_7e_8, \\
& N_{23} = e_5e_6e_7e_8, N_{24} = e_1e_6e_7e_8, N_{25} = e_2e_5e_7e_8, N_{26} = e_3e_5e_6e_8, \\
& N_{27} = e_4e_5e_6e_7, N_{28} = e_1e_2e_7e_8, N_{29} = e_1e_3e_6e_8, N_{30} = e_1e_4e_6e_7, \\
& N_{31} = -e_4e_5e_6e_8 + e_3e_5e_6e_7, N_{32} = -e_1e_4e_6e_8 + e_1e_3e_6e_7, N_{33} = e_3e_5e_7e_8 - e_2e_5e_6e_8, \\
& N_{34} = e_1e_3e_7e_8 - e_1e_2e_6e_8, N_{35} = -e_4e_5e_7e_8 - e_2e_5e_6e_7, N_{36} = -e_1e_4e_7e_8 - e_1e_2e_6e_7, \\
& N_{37} = -e_4e_6e_7e_8 + e_1e_5e_6e_7, N_{38} = -e_2e_4e_7e_8 + e_1e_2e_5e_7, N_{39} = e_3e_6e_7e_8 + e_1e_5e_6e_8, \\
& N_{40} = e_2e_3e_7e_8 + e_1e_2e_5e_8, N_{41} = -e_2e_6e_7e_8 + e_1e_5e_7e_8, N_{42} = -e_2e_3e_6e_8 + e_1e_3e_5e_8,
\end{aligned}$$

where we omit the symbol $\wedge$ for convenience. Then, the kernel of the map φ_4 is given by $W = \mathrm{span}_{\mathbb{C}}\{N_a | 1 \le a \le 42\}$.

On the other hand, the complex representation W is a real type [19,21]. To see this, define a complex linear map $\psi : W \to W$ by $\psi(N_l) = N_{l+22}$, $\psi(N_m) = N_m$ and $\psi(N_n) = N_{n-22}$, where $1 \le l \le 20, 21 \le m \le 22$ and $23 \le n \le 42$. Moreover, the conjugate map γ on W is defined by $\gamma(\sum_{a=1}^{42} \lambda_a N_a) = \sum_{a=1}^{42} \bar{\lambda}_a N_a$, where $\lambda_a \in \mathbb{C}$ and $\bar{\lambda}_a$ is the conjugate of λ_a. Then, W is a real type with the structure map $\mathcal{F} = \psi \circ \gamma$ and the real form W_0 of the representation W is fixed by $\mathcal{F}$. Induce a Hermitian inner product $\langle \cdot, \cdot \rangle_W$ on W by the standard Hermitian inner product on $\bigotimes^4 \mathbb{C}^8$ and set the following:

- $\tilde{N}_i = 2\sqrt{6}(N_i + N_{22+i})$, $N' = 2\sqrt{6}(\sqrt{-1}N_i - \sqrt{-1}N_{22+i})$ for $1 \le i \le 8$;
- $\tilde{N}_j = 2\sqrt{3}(N_j + N_{22+j})$, $N' = 2\sqrt{3}(\sqrt{-1}N_j - \sqrt{-1}N_{22+j})$ for $9 \le j \le 20$.

Then there is an orthonormal basis of the real form W_0, namely $\{\tilde{N}_\alpha, \sqrt{6}N_{21}, \sqrt{6}N_{22}, N'_\alpha |1 \le \alpha \le 20\}$. By the representation W_0, the Lie algebra $\mathfrak{sp}(4)$ is identified with a Lie subalgebra, which is also denoted by $\mathfrak{sp}(4)$, in $\mathfrak{so}(42)$. Particularly, vector $G_{15} \in \mathfrak{sp}(4)$ corresponds to

$$w_1 = G'_{12} - G'_{36} - G'_{47} - G'_{58} - G'_{9,10} - G'_{11,12} - G'_{13,14} \in \mathfrak{so}(42).$$

Recall that the negative of the Killing form of $\mathfrak{so}(43)$ is $B(X, Y) = -41\mathrm{tr}(XY)$, where $X, Y \in \mathfrak{so}(43)$. Then, there is a Lie algebra decomposition with respect to $B(\cdot, \cdot)$,

$$\mathfrak{so}(43) = \mathfrak{so}(42) \oplus \mathfrak{m}_2 = \mathfrak{sp}(4) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where $\mathfrak{m}_1$ is the orthogonal complement of $\mathfrak{sp}(4)$ in $\mathfrak{so}(42)$ and $\mathfrak{m}_2 = \mathrm{span}_{\mathbb{R}}\{G_{a,43} | 1 \leq a \leq 42\}$.

Proof of Theorem 2 for the $(\mathbf{SO(43), SO(42), Sp(4)})$ Case. Take $u_1 = u_2 = G_{1,43} \in \mathfrak{m}_2$, $v_1 = v_2 = G_{2,43} \in \mathfrak{m}_2$ and $w_2 = G_{12} + G_{36} \in \mathfrak{m}_1$. It is not difficult to check $B([u_1, v_1], w_2) \neq 0$ and $B([u_2, v_2], w_1) \neq 0$. Hence, by Proposition 3, the left-invariant metric $\langle \cdot, \cdot \rangle_{t_1,t_2}$ on $\mathrm{SO}(43)$ is a $\mathcal{B}$-metric if—and only if—it is an Einstein metric. Since $\mathrm{SO}(43)$ admits no non-naturally reductive left-invariant Einstein-like metric such as Equation (1) [14], there is no non-naturally reductive left-invariant $\mathcal{B}$-metric, such as Equation (1) on $\mathrm{SO}(43)$. □

3.9. The $(\mathrm{SO}(129), \mathrm{SO}(128), \mathrm{Spin}(16))$ *Case*

In the this case, the Lie algebra $\mathfrak{spin}(16)$ is embedded into $\mathfrak{so}(128)$ by its half-spin representation Δ^+. Now, we recall the construction of Δ^+ first [23]. Let $\{e_1, \cdots, e_{16}\}$ be the standard basis of $\mathbb{R}^{16}$ with respect to an inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^{16}}$ and $Cl(\mathbb{R}^{16})$ be the corresponding Clifford algebra over $\mathbb{R}^{16}$. It is well known that products in $Cl(\mathbb{R}^{16})$ are determined by $e_r^2 = -1$ and $e_r e_s = -e_s e_r$ for $1 \leq r \neq s \leq 16$, and the Lie algebra $\mathfrak{spin}(16) = \mathrm{span}_{\mathbb{R}}\{e_r e_s | 1 \leq r < s \leq 16\}$. Moreover, there is a basis $\{\prod_{s=1}^{16} e_s^{r_s} | r_s = 0 \text{ or } 1\}$ of $Cl(\mathbb{R}^{16})$, and considering the subalgebra $Cl(\mathbb{R}^{16})_0$, spanned by elements in which $\sum_s r_s$ is even, of $Cl(\mathbb{R}^{16})$. Directly, take a subgroup,

$$E_0 = \{\pm \prod_{s=1}^{16} e_s^{r_s} | r_s = 0 \text{ or } 1, \ \sum_{s=1}^{16} r_s \text{ is even}\},$$

of $Cl(\mathbb{R}^{16})_0$, and let $F \subset E_0$ be the subgroup generated by $\{e_{2p-1}e_{2p} | 1 \leq p \leq 8\}$. Then, we will begin with a one-dimensional complex representation W of F to construct the half-spin representation Δ^+ of $\mathfrak{spin}(16)$.

Consider the complexification of $Cl(\mathbb{R}^{16})$, i.e., $Cl(\mathbb{C}^{16}) = Cl(\mathbb{R}^{16}) \otimes_{\mathbb{R}} \mathbb{C}$. One can check that $Cl(\mathbb{C}^{16})$ is a Clifford algebra over $\mathbb{C}^{16}$ with the dot product [20]. Moreover, products in $Cl(\mathbb{C}^{16})$ are induced by those in $Cl(\mathbb{R}^{16})$. Define a one-dimensional complex vector space $W := \{\lambda w | \lambda \in \mathbb{C}\}$, where

$$w = \sum_{\substack{0 \leq q \leq 8 \\ 1 \leq p_1 < \cdots < p_q \leq 8}} (-\sqrt{-1})^q (e_{2p_1-1}e_{2p_1}) \cdots (e_{2p_q-1}e_{2p_q}) \in Cl(\mathbb{C}^{16}).$$

Then there is a one-dimensional complex representation ρ of F on W induced by products in $Cl(\mathbb{C}^{16})$, and it is easy to check $\rho(e_{2p-1}e_{2p})w = \sqrt{-1}w$ for $1 \leq p \leq 8$.

For any $\sigma \in E_0/F$, we use σW to denote the one-dimensional complex vector space generated by gw, where $g \in \sigma$ and gw is the product of g, w in $Cl(\mathbb{C}^{16})$. One can check that σW does not depend on the choice of g. Hence, we can construct a 128-dimensional complex vector space directly,

$$\Delta^+ = \bigoplus_{\sigma \in E_0/F} \sigma W,$$

and any element of E_0 permutes $\{\sigma W | \sigma \in E_0/F\}$ by products of $Cl(\mathbb{C}^{16})$. As a consequence, we obtain a 128-dimensional complex representation ρ' of E_0 on Δ^+.

Recall that the group ring of E_0 over $\mathbb{R}$ is defined by $\mathbb{R}(E_0) := \{\sum_t \lambda_t X_t | \lambda_t \in \mathbb{R}, X_t \in E_0\}$. We obtain a complex representation $\rho'_{\mathbb{R}(E_0)}$ of $\mathbb{R}(E_0)$ on Δ^+ by extending the representation ρ' from E_0 to $\mathbb{R}(E_0)$. To avoid confusion, we write μ for $-1 \in Cl(\mathbb{R}^{16})_0$ and J for the ideal of $\mathbb{R}(E_0)$ generated by $\mu + 1$. By the definition $Cl(\mathbb{R}^{16})_0 = \mathbb{R}(E_0)/J$, there is a representation ρ'' of $Cl(\mathbb{R}^{16})_0$ on Δ^+ induced by $\rho'_{\mathbb{R}(E_0)}$. Finally, we obtain a 128-dimensional complex representation of $\mathrm{Spin}(16)$ on Δ^+ by the restriction $\rho''|_{\mathrm{Spin}(16)}$ which induces a 128-dimensional complex representation $\tilde{\rho}$ of $\mathfrak{spin}(16)$ on Δ^+ called the half-spin representation of $\mathfrak{spin}(16)$.

In detail, for any $X \in \mathfrak{spin}(16)$ and $u \in \Delta^+$, $\rho(X)u = Xu$, where Xu is the product of X and u in $Cl(\mathbb{C}^{16})$.

Then, we illustrate the real form Δ_0^+, which is fixed by the structure map $\mathcal{F}$, of the representation space Δ^+. Denote $e_1e_3\cdots e_{15}$ by e acting on Δ^+ by products in $Cl(\mathbb{C}^{16})$ and set

$$V = \mathrm{span}_{\mathbb{C}}\{w, e_{2p-1}e_{2q-1}w, e_1e_{2i-1}e_{2j-1}e_{2k-1}w | 1 \leq p < q \leq 8, 2 \leq i < j < k \leq 8\}.$$

By the discussions above, the representation space Δ^+ is isomorphic to $V \oplus eV$. More precisely, organize basis vectors of V in the lexicographical order and let N_α be the αth basis vector. Then, define $N_{64+\alpha} = eN_\alpha$ for $1 \leq \alpha \leq 64$, and we obtain $\Delta^+ = \mathrm{span}_{\mathbb{C}}\{N_a | 1 \leq a \leq 128\}$.

On the other hand, the representation Δ^+ is a real representation with the structure map $\mathcal{F} = e \circ \gamma$, where γ is the conjugate map defined by $\gamma(\sum_{a=1}^{128} \lambda_a N_a) = \sum_{a=1}^{128} \bar{\lambda}_a N_a$, with $\lambda_a \in \mathbb{C}$, and $\bar{\lambda}_a$ is the conjugate of λ_a [21]. Set $\tilde{N}_\alpha = \frac{N_\alpha + eN_\alpha}{\sqrt{2}}$ and $N'_\alpha = \frac{\sqrt{-1}N_\alpha - \sqrt{-1}eN_\alpha}{\sqrt{2}}$, where $1 \leq \alpha \leq 64$. It is not difficult to check that $\tilde{N}_\alpha$ and N'_α are fixed by $\mathcal{F}$. Hence, the real form Δ_0^+ is spanned by $\{\tilde{N}_\alpha, N'_\alpha | 1 \leq \alpha \leq 64\}$ with an inner product $\langle \cdot, \cdot \rangle_{\Delta_0^+}$ on Δ_0^+ defined by assuming $\{\tilde{N}_\alpha, N'_\alpha | 1 \leq \alpha \leq 64\}$ is orthonormal.

In fact, by the half-spin representation of $\mathfrak{spin}(16)$ on Δ_0^+, we identify $\mathfrak{spin}(16)$ with a Lie subalgebra, which is also denoted by $\mathfrak{spin}(16)$, in $\mathfrak{so}(128)$. Particularly, the vector e_1e_3 corresponds to $w_1 = G_{12} + G_{39} + (*) \in \mathfrak{so}(128)$, where $(*)$ denotes other terms which do not include G_{12} or G_{39}.

Recall the negative of the Killing form of $\mathfrak{so}(129)$ is $B(X, Y) = -127\mathrm{tr}(XY)$, where $X, Y \in \mathfrak{so}(129)$. Then, there is a Lie algebra decomposition,

$$\mathfrak{so}(129) = \mathfrak{so}(128) \oplus \mathfrak{m}_2 = \mathfrak{spin}(16) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where $\mathfrak{m}_1$ is the orthogonal complement of $\mathfrak{spin}(16)$ in $\mathfrak{so}(128)$ and $\mathfrak{m}_2 = \mathrm{span}_{\mathbb{R}}\{G_{a,129} | 1 \leq a \leq 128\}$.

Proof of Theorem 2 for the $(\mathbf{SO(129), SO(128), Spin(16)})$ Case. Take $u_1 = u_2 = G_{1,129} \in \mathfrak{m}_2, v_1 = v_2 = G_{2,129} \in \mathfrak{m}_2$ and $w_2 = G_{12} - G_{39} \in \mathfrak{m}_1$. It is not difficult to check $B([u_1, v_1], w_2) \neq 0$ and $B([u_2, v_2], w_1) \neq 0$. Hence, by Proposition 3, the left-invariant metric $\langle \cdot, \cdot \rangle_{t_1,t_2}$ on SO(129) is a $\mathcal{B}$-metric if—and only if—it is an Einstein metric. Since SO(129) admits no non-naturally reductive left-invariant Einstein-like metric, such as Equation (1) [14], there is no non-naturally reductive left-invariant $\mathcal{B}$-metric, such as Equation (1) on SO(129). $\square$

3.10. The (Spin(9), Spin(8), Spin(7)) *Case*

In this case, we refer [22] for the embedding of $\mathfrak{spin}(7)$ into $\mathfrak{spin}(8)$. In fact, any $X \in \mathfrak{spin}(7)$ is a linear combination of the following elements: $\alpha_1G_{43} + \beta_1G_{65} + \gamma_1G_{78} + \delta_1G_{21}$, $\alpha_2G_{24} + \beta_2G_{75} + \gamma_2G_{86} + \delta_2G_{31}$, $\alpha_3G_{32} + \beta_3G_{85} + \gamma_3G_{67} + \delta_3G_{41}$, $\alpha_4G_{26} + \beta_4G_{37} + \gamma_4G_{48} + \delta_4G_{51}$, $\alpha_5G_{52} + \beta_5G_{38} + \gamma_5G_{74} + \delta_5G_{61}$, $\alpha_6G_{82} + \beta_6G_{53} + \gamma_6G_{46} + \delta_6G_{71}$, $\alpha_7G_{27} + \beta_7G_{63} + \gamma_7G_{54} + \delta_7G_{81}$, where $\alpha_i + \beta_i + \gamma_i + \delta_i = 0$ for $(i = 1, \cdots, 7)$.

On the other hand, the negative of the Killing form of $\mathfrak{spin}(9)$ is $B(X, Y) = -7\mathrm{tr}(XY)$, where $X, Y \in \mathfrak{spin}(9)$. Moreover, there is an orthonormal basis of $\mathfrak{spin}(9)$ with respect to $B(\cdot, \cdot)$ as follows,

$$h_1 = \frac{2G_{32}-G_{54}-G_{67}}{2\sqrt{21}},\ h_2 = \frac{G_{54}-G_{67}}{2\sqrt{7}},\ h_3 = \frac{-G_{32}-G_{54}-G_{67}+3G_{10}}{2\sqrt{42}},$$
$$h_4 = \frac{2G_{13}-G_{64}-G_{75}}{2\sqrt{21}},\ h_5 = \frac{G_{64}-G_{75}}{2\sqrt{7}},\ h_6 = \frac{-G_{13}-G_{64}-G_{75}+3G_{20}}{2\sqrt{42}},$$
$$h_7 = \frac{2G_{21}-G_{74}-G_{56}}{2\sqrt{21}},\ h_8 = \frac{G_{74}-G_{56}}{2\sqrt{7}},\ h_9 = \frac{-G_{21}-G_{74}-G_{56}+3G_{30}}{2\sqrt{42}},$$
$$h_{10} = \frac{2G_{15}-G_{26}-G_{37}}{2\sqrt{21}},\ h_{11} = \frac{G_{26}-G_{37}}{2\sqrt{7}},\ h_{12} = \frac{-G_{15}-G_{26}-G_{37}+3G_{40}}{2\sqrt{42}},$$
$$h_{13} = \frac{2G_{41}-G_{27}-G_{63}}{2\sqrt{21}},\ h_{14} = \frac{G_{27}-G_{63}}{2\sqrt{7}},\ h_{15} = \frac{-G_{41}-G_{27}-G_{63}+3G_{50}}{2\sqrt{42}},$$
$$h_{16} = \frac{2G_{71}-G_{42}-G_{35}}{2\sqrt{21}},\ h_{17} = \frac{G_{42}-G_{35}}{2\sqrt{7}},\ h_{18} = \frac{-G_{71}-G_{42}-G_{35}+3G_{60}}{2\sqrt{42}},$$
$$h_{19} = \frac{2G_{16}-G_{52}-G_{43}}{2\sqrt{21}},\ h_{20} = \frac{G_{52}-G_{43}}{2\sqrt{7}},\ h_{21} = \frac{-G_{16}-G_{52}-G_{43}+3G_{70}}{2\sqrt{42}},$$
$$u_1 = \frac{G_{32}+G_{54}+G_{67}+G_{10}}{2\sqrt{14}},\ u_2 = \frac{G_{13}+G_{64}+G_{75}+G_{20}}{2\sqrt{14}},\ u_3 = \frac{G_{21}+G_{74}+G_{56}+G_{30}}{2\sqrt{14}},$$
$$u_4 = \frac{G_{15}+G_{26}+G_{37}+G_{40}}{2\sqrt{14}},\ u_5 = \frac{G_{41}+G_{27}+G_{63}+G_{50}}{2\sqrt{14}},\ u_6 = \frac{G_{71}+G_{42}+G_{35}+G_{60}}{2\sqrt{14}},$$
$$u_7 = \frac{G_{16}+G_{52}+G_{43}+G_{70}}{2\sqrt{14}},\ p_\alpha = \frac{G_{\alpha 9}}{\sqrt{14}},\ \text{for } 0 \leq \alpha \leq 7.$$

In addition, we have a Lie algebra decomposition with respect to $B(\cdot,\cdot)$,

$$\mathfrak{spin}(9) = \mathfrak{spin}(8) \oplus \mathfrak{m}_2 = \mathfrak{spin}(7) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where $\mathfrak{spin}(7) = \mathrm{span}\{h_i | 1 \leq i \leq 21\}$, $\mathfrak{m}_1 = \mathrm{span}\{u_r | 1 \leq r \leq 7\}$ and $\mathfrak{m}_2 = \mathrm{span}\{p_\alpha | 0 \leq \alpha \leq 7\}$.

Proof of Theorem 2 for the $(\mathbf{Spin(9), Spin(8), Spin(7)})$ Case. Calculated directly, we find $[p_1, p_2] = -\frac{\sqrt{42}}{28}h_3 + \frac{\sqrt{14}}{28}u_1$. Then, taking $u_1 = u_2 = p_1$ and $v_1 = v_2 = p_2$, we obtain $[u_1, v_1]_{\mathfrak{m}_1} \neq 0$ and $[u_2, v_2]_{\mathfrak{h}} \neq 0$. Hence, by Proposition 3, the left-invariant metric $\langle\cdot,\cdot\rangle_{t_1,t_2}$ on Spin(9) is a $\mathcal{B}$-metric if—and only if—it is an Einstein metric. Since Spin(9) admits no non-naturally reductive left-invariant Einstein-like metric, such as Equation (1) [14], there is no non-naturally reductive left-invariant $\mathcal{B}$-metric, such as Equation (1) on Spin(9). $\square$

4. Conclusions

In the current paper, we study left-invariant Einstein-like metrics on compact Lie groups. Now, we summarize our main results and remaining questions, as follows:

- We prove in Theorem 1 that any left-invariant metric $\langle\cdot,\cdot\rangle_{t_1,t_2}$ defined by Equation (1) on the compact Lie group G must be an $\mathcal{A}$-metric. Recall that a Riemannian metric g is simultaneously $\mathcal{A}$-metric and $\mathcal{B}$-metric if—and only if—it is Ricci-parallel. Then, the left-invariant metric $\langle\cdot,\cdot\rangle_{t_1,t_2}$ on G is a $\mathcal{B}$-metric if—and only if—it is Ricci-parallel. In other words, we prove that Conjecture 1.1 is true for the compact Lie group G endowed with the left-invariant metric $\langle\cdot,\cdot\rangle_{t_1,t_2}$.
- We prove in Theorem 2 that any compact Lie group G in Table 1 admits no non-naturally reductive left-invariant $\mathcal{B}$-metric, such as Equation (1). Moreover, we find Proposition 3, which can help us to avoid cumbersome calculations, and every case in Table 1 is discussed.
- In the future, we will focus on the compact Lie group G, in which G/K is not an irreducible symmetric space any more. Hence, Lie bracket relations are more complex, so it is difficult to prove Conjecture 1.1. On the other hand, it will also be interesting if we can find some counter-examples of Conjecture 1.1.

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