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Article

A Natural Approximation to the Complete Elliptic Integral of the First Kind

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Abstract: Let $\mathcal{K}(r)$ be the complete elliptic integral of the first kind. Then, the inequality $2\mathcal{K}(r)/\pi > \tanh^{-1}(r)/\sin^{-1}(r)$ holds for all $r \in (0,1)$. This conclusion does not match those in the existing literature.

Keywords: simple bound; natural approximation; the complete elliptic integral of the first kind

MSC: 33E05; 26E60

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1. Introduction

Legendre's incomplete elliptic integral of the first kind is defined by

$$F(\lambda,k) = \int_0^{\lambda} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

The research on various series expansions and asymptotic approximations for the function $F(\lambda,k)$ is in the ascendant, and interested readers can refer to literature [1–14].

This article focuses on Legendre's complete elliptic integral of the first kind. For $r \in (0,1)$, Legendre's complete elliptic integrals of the first kind (see [15,16]), denoted $\mathcal{K}(r)$, is defined by,

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2\sin^2(t)}},$$

which is the particular case of the Gaussian hypergeometric function and can be a special power series ([17–23]):

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2},\frac{1}{2};1;r^2\right) = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{n!^2} r^{2n}, \tag{1}$$

where $\left(\frac{1}{2}\right)_n = \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\cdots\left(\frac{1}{2}+n-1\right)$.

Many researchers have obtained the upper and lower bounds of this special function ([24–33]). Since 1991, some scholars have considered using the logarithmic function related to r' to approximate $2\mathcal{K}(r)/\pi$. Carlson and Vuorinen [24], Vamanamurhty and Vuorinen [25], Qiu and Vamanamurthy [26] and Alzer [27] proved that the double inequalities

$$A(r) =: \frac{\log r'}{r'-1} < \mathcal{K}(r) < \frac{\pi}{2}\left(\frac{\log r'}{r'-1}\right), \tag{2}$$

$$B(r) =: \left[1+\left(\frac{\pi}{4\log 2}-1\right)r'^2\right]\log\frac{4}{r'} < \mathcal{K}(r) < \left(1+\frac{1}{4}r'^2\right)\log\frac{4}{r'} \tag{3}$$

hold for all $r > 0$, where $r' = \sqrt{1-r^2}$.

Anderson, Vamanamurhty and Vuorinen [28] found and proved that the double inequality

$$C(r) =: \frac{\pi}{2}\left[\frac{\tanh^{-1}(r)}{r}\right]^{1/2} < \mathcal{K}(r) < \frac{\pi}{2}\frac{\tanh^{-1}(r)}{r} \tag{4}$$

holds for all $r \in (0,1)$, where $\tanh^{-1}(r) = \log[(1+r)/(1-r)]/2$ is the inverse hyperbolic tangent function.

The lower bound in (4) was improved by Alzer and Qiu [29], and Yang, Song and Chu [30] independently as follows:

$$\mathcal{K}(r) > \frac{\pi}{2}\left[\frac{\tanh^{-1}(r)}{r}\right]^{3/4} := D(r) \tag{5}$$

holds for all $r \in (0,1)$.

In order to find a new bound function, we first analyze the special function $2\mathcal{K}(r)/\pi$ in the case of two breakpoints:

$$\begin{aligned} \frac{2}{\pi}\mathcal{K}(0^+) &= \frac{2}{\pi}\int_0^{\pi/2} dt = 1, \\ \frac{2}{\pi}\mathcal{K}(1^-) &= \infty. \end{aligned}$$

The last fact is based on the following reasoning: when doing transformation $\sin t = x$, we have

$$\begin{aligned} \frac{2}{\pi}\mathcal{K}(1^-) &= \frac{2}{\pi}\int_0^{\pi/2}\frac{dt}{\sqrt{1-\sin^2(t)}} = \frac{2}{\pi}\int_0^{\pi/2}\frac{\cos t dt}{\cos^2 t} = \frac{2}{\pi}\int_0^{\pi/2}\frac{d\sin t}{1-\sin^2 t} \\ &= \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{2}\left(\frac{1}{1+\sin t}+\frac{1}{1-\sin t}\right)d\sin\ t \\ &= \frac{2}{\pi}\lim_{x\to 1^-}\frac{1}{2}\ln\frac{1+x}{1-x} = \frac{2}{\pi}\lim_{x\to 1^-}\tanh^{-1}(x) = \frac{2}{\pi}\lim_{r\to 1^-}\tanh^{-1}(r). \end{aligned}$$

We then consider

$$\lim_{r\to 1^-}\frac{1}{\sin^{-1}(r)} = \frac{2}{\pi},$$

try to use the function $\tanh^{-1}(r)/\sin^{-1}(r)$ as the bound for the function $2\mathcal{K}(r)/\pi$, and draw the following conclusion.

Theorem 1. *The inequality*

$$\frac{2}{\pi}\mathcal{K}(r) > \frac{\tanh^{-1}(r)}{\sin^{-1}(r)} \tag{6}$$

or

$$\mathcal{K}(r) > \frac{\pi}{2}\frac{\tanh^{-1}(r)}{\sin^{-1}(r)} := G(r)$$

holds for all $r \in (0,1)$.

It must be pointed out that the lower bound for the function $\mathcal{K}(r)$ given in this paper is concise and accurate. In order to express this advantage, we specially list the following two absolute error tables, one is the error table between the function $\mathcal{K}(r)$ and the lower bound $G(r)$ given in this paper (see Table 1), and the other is the error tables between the lower bound $G(r)$ given in this paper and the ones $A(r)$, $B(r)$, $D(r)$ (see Table 2), which given in [24–27,29,30], respectively.

Table 1. The error table between $\mathcal{K}(r)$ and $G(r)$

r	**0.2**	**0.4**	**0.8**	**0.9**
$\mathcal{K}(r) - G(r)$	5.3480×10^{-3}	2.2894×10^{-2}	0.13430	0.21407

Table 2. The error tables between $G(r)$ and $A(r)$, $B(r)$, $D(r)$

r	**0**	**0.4**	**0.9**	**0.99999**
$G(r) - A(r)$	0.57080	0.57288	0.59322	0.68627
$G(r) - B(r)$	2.5244×10^{-29}	-2.1093×10^{-2}	-0.2075	-0.67575
$G(r) - D(r)$	0	-0.02284	-0.20684	2.1124×10^{-2}

The latter table shows the fact that the new lower bound $G(r)$ for $\mathcal{K}(r)$ is better than the one $A(r)$, and is more advantageous than the one $B(r)$ near $r = 0$ while $G(r)$ is more advantageous than the one $D(r)$ near $r = 1$. These facts illustrate the uniqueness of this new lower bound $G(r)$.

2. Lemmas

This article needs the following lemma. Let us give a simple proof of Lemma 1.

Lemma 1 ([34–38])**.** *For* $|x| < 1$,

$$\frac{2x\sin^{-1}(x)}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n\binom{2n}{n}}. \tag{7}$$

Proof. Let

$$h(x) = \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}. \tag{8}$$

Then, $h(0) = 0$, and

$$\sqrt{1-x^2}h(x) = \sin^{-1}(x). \tag{9}$$

Differentiating (9) gives

$$\left(1-x^2\right)h'(x) - xh(x) = 1. \tag{10}$$

Since $h(x)$ is an odd function, we can express it as a power series

$$h(x) = \sum_{n=0}^{\infty} a_n x^{2n+1}. \tag{11}$$

Differentiation of (11) yields

$$h'(x) = \sum_{n=0}^{\infty} (2n+1)a_n x^{2n}.$$

Substituting the series of $h'(x)$ and $h(x)$ into (10) yields

$$a_0 + \sum_{n=0}^{\infty}((2n+3)a_{n+1} - 2(n+1)a_n)x^{2n+2} = 1.$$

Equating coefficients of x^{2n+2}, we can obtain

$$\begin{aligned} a_0 &= 1, \\ a_{n+1} &= 2\frac{n+1}{2n+3}a_n, n \geq 0. \end{aligned}$$

Then,

$$a_n = 2^n \frac{n}{2n+1}\frac{n-1}{2n-1}\cdots\frac{2}{5}\frac{1}{3}1 = 2^n\frac{n!}{(2n+1)!!}, n \geq 0,$$

and

$$\begin{aligned} h(x) &= \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} 2^n \frac{n!}{(2n+1)!!}x^{2n+1} = \sum_{n=1}^{\infty} 2^{n-1}\frac{(n-1)!}{(2n-1)!!}x^{2n-1} \\ &= \sum_{n=1}^{\infty}\frac{n!2^{n-1}}{n(2n-1)!!}x^{2n-1} = \sum_{n=1}^{\infty}\frac{(2x)^{2n-1}}{n\binom{2n}{n}}, \end{aligned} \tag{12}$$

the last equality is founded because of the identity

$$\binom{2n}{n} = \frac{2^n(2n-1)!!}{n!}.$$

□

3. Proof of Main Result

The main conclusion of this paper is equivalent to the following theorem.

Theorem 2. *The inequality*

$$\left[\frac{2}{\pi}\mathcal{K}(r)\right]\sin^{-1}(r) > \tanh^{-1}(r) \tag{13}$$

holds for all $r \in (0,1)$.

Proof. Let

$$\mathcal{F}(r) = \left[\frac{2}{\pi}\mathcal{K}(r)\right]\sin^{-1}(r) - \tanh^{-1}(r),\ 0 < r < 1.$$

Then

$$\mathcal{F}'(r) = \left[\frac{2}{\pi}\mathcal{K}(r)\right]'\sin^{-1}(r) + \left[\frac{2}{\pi}\mathcal{K}(r)\right]\frac{1}{\sqrt{1-r^2}} - \frac{1}{1-r^2},$$

and

$$\frac{\mathcal{F}'(r)}{\sqrt{1-r^2}} = \left[\frac{2}{\pi}\mathcal{K}(r)\right]'\frac{\sin^{-1}(r)}{\sqrt{1-r^2}} + \left[\frac{2}{\pi}\mathcal{K}(r)\right]\frac{1}{(1-r^2)} - \frac{1}{(1-r^2)^{3/2}}.$$

From (1), (7) and Cauchy products we have

$$
\begin{aligned}
\frac{\mathcal{F}'(r)}{\sqrt{1-r^2}} &= \sum_{n=1}^{\infty} \frac{2n\left(\frac{1}{2}\right)_n^2}{n!^2} r^{2n-1} \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n\binom{2n}{n}} r^{2n-1} \\
&\quad + \left[\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{n!^2} r^{2n}\right] \sum_{n=0}^{\infty} r^{2n} - \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n}{n!} r^{2n} \\
&= \frac{1}{r^2} \sum_{n=1}^{\infty} \frac{2n\left(\frac{1}{2}\right)_n^2}{n!^2} r^{2n} \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n\binom{2n}{n}} r^{2n} \\
&\quad + \left[\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{n!^2} r^{2n}\right] \sum_{n=0}^{\infty} r^{2n} - \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n}{n!} r^{2n}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{r^2} \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} \frac{2^{2k-1}}{k\binom{2k}{k}} \frac{2(n-k)\left(\frac{1}{2}\right)_{n-k}^2}{(n-k)!^2}\right] r^{2n} \\
&\quad + \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{\left(\frac{1}{2}\right)_k^2}{k!^2}\right] r^{2n} - \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n}{n!} r^{2n} \\
&= \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} \frac{2^{2k-1}}{k\binom{2k}{k}} \frac{2(n-k)\left(\frac{1}{2}\right)_{n-k}^2}{((n-k)!)^2}\right] r^{2n-2} \\
&\quad + \sum_{n=1}^{\infty} \left[\sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2}\right] r^{2n-2} - \sum_{n=1}^{\infty} \frac{\left(\frac{3}{2}\right)_{n-1}}{(n-1)!} r^{2n-2} \\
&=: \sum_{n=1}^{\infty} f_n r^{2n-2} = \sum_{n=2}^{\infty} f_n r^{2n-2},
\end{aligned}
$$

where

$$
f_n = \sum_{k=1}^{n} \frac{2^{2k-1}}{k\binom{2k}{k}} \frac{2(n-k)\left(\frac{1}{2}\right)_{n-k}^2}{(n-k)!^2} + \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_k^2}{k!^2} - \frac{\left(\frac{3}{2}\right)_{n-1}}{(n-1)!},\ n \geq 2.
$$

Below, we will prove $f_n > 0$ for $n \geq 2$, which is equivalent to

$$
\sum_{k=1}^{n} \frac{2^{2k-1}}{k\binom{2k}{k}} \frac{2(n-k)\left(\frac{1}{2}\right)_{n-k}^2}{(n-k)!^2} > \frac{\left(\frac{3}{2}\right)_{n-1}}{(n-1)!} - \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_k^2}{k!^2},\ n \geq 2. \tag{14}
$$

□

We prove (14) by mathematical induction. It is not difficult to verify that (14) is true for $n = 2$. Assuming that (14) holds for $n \geq 3$, let's prove (14) holds for $n + 1$. Since

$$\begin{aligned}
&\sum_{k=1}^{n+1} \frac{2^{2k-1}}{k\binom{2k}{k}} \frac{2(n+1-k)\left(\frac{1}{2}\right)_{n+1-k}^2}{(n+1-k)!^2} - \sum_{k=1}^{n} \frac{2^{2k-1}}{k\binom{2k}{k}} \frac{2(n-k)\left(\frac{1}{2}\right)_{n-k}^2}{(n-k)!^2} \\
= &\sum_{k=1}^{n} \frac{2^{2k-1}}{k\binom{2k}{k}} \frac{2(n+1-k)\left(\frac{1}{2}\right)_{n+1-k}^2}{(n+1-k)!^2} - \sum_{k=1}^{n} \frac{2^{2k-1}}{k\binom{2k}{k}} \frac{2(n-k)\left(\frac{1}{2}\right)_{n-k}^2}{(n-k)!^2} \\
= &\sum_{k=1}^{n} \frac{2^{2k-1}}{k\binom{2k}{k}} \frac{1}{2(n-k+1)} \frac{\left(\frac{1}{2}\right)_{n-k}^2}{(n-k)!^2} > \frac{2^{2n-1}}{n\binom{2n}{n}} \frac{1}{2} = \frac{2^{2n-2}}{n(2n)!}(n!)^2
\end{aligned}$$

holds for all $n \geq 1$, we have that for $n \geq 3$,

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{2^{2k-1}}{k\binom{2k}{k}} \frac{2(n+1-k)\left(\frac{1}{2}\right)_{n+1-k}^2}{(n+1-k)!^2} &= \sum_{k=1}^{n} \frac{2^{2k-1}}{k\binom{2k}{k}} \frac{2(n+1-k)\left(\frac{1}{2}\right)_{n+1-k}^2}{(n+1-k)!^2} \\
&> \sum_{k=1}^{n} \frac{2^{2k-1}}{k\binom{2k}{k}} \frac{2(n-k)\left(\frac{1}{2}\right)_{n-k}^2}{(n-k)!^2} + \frac{2^{2n-2}(n!)^2}{n(2n)!} \\
&> \frac{\left(\frac{3}{2}\right)_{n-1}}{(n-1)!} - \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_k^2}{k!^2} + \frac{2^{2n-2}(n!)^2}{n(2n)!}.
\end{aligned}$$

The proof of (14) is complete when proving

$$\begin{aligned}
\frac{\left(\frac{3}{2}\right)_{n-1}}{(n-1)!} - \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_k^2}{k!^2} + \frac{2^{2n-2}(n!)^2}{n(2n)!} &> \frac{\left(\frac{3}{2}\right)_n}{n!} - \sum_{k=0}^{n} \frac{\left(\frac{1}{2}\right)_k^2}{k!^2} \\
&\Longleftrightarrow \\
2n\frac{\left(\frac{1}{2}\right)_n}{n!} + \frac{2^{2n-2}(n!)^2}{n(2n)!} &> (2n+1)\frac{\left(\frac{1}{2}\right)_n}{(n)!} - \frac{\left(\frac{1}{2}\right)_n^2}{n!^2} \\
&\Longleftrightarrow
\end{aligned}$$

$$\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2} > \frac{\left(\frac{1}{2}\right)_n}{(n)!} - \frac{2^{2n-2}(n!)^2}{n(2n)!} \tag{15}$$

We prove (15) by mathematical induction too. It is not difficult to verify that (15) is true for $n = 3$. Assuming that (15) holds for $n \geq 4$, let us prove (15) holds for $n + 1$. Since

$$\begin{aligned}
\frac{\left(\frac{1}{2}\right)_{n+1}^2}{(n+1)!^2} &= \frac{(2n+1)^2}{4(n+1)^2} \frac{\left(\frac{1}{2}\right)_n^2}{n!^2} \\
&> \frac{(2n+1)^2}{4(n+1)^2} \frac{\left(\frac{1}{2}\right)_n}{n!} - \frac{(2n+1)^2}{4(n+1)^2} \frac{2^{2n-2}(n!)^2}{n(2n)!},
\end{aligned}$$

we can complete the proof of (15) when proving

$$
\begin{aligned}
\frac{(2n+1)^2}{4(n+1)^2}\frac{\left(\frac{1}{2}\right)_n}{(n)!}-\frac{(2n+1)^2}{4(n+1)^2}\frac{2^{2n-2}(n!)^2}{n(2n)!} &> \frac{\left(\frac{1}{2}\right)_{n+1}}{(n+1)!}-\frac{2^{2n}((n+1)!)^2}{(n+1)(2n+2)!}\\
&\Longleftrightarrow\\
\frac{(2n+1)^2}{4(n+1)^2}\frac{\left(\frac{1}{2}\right)_n}{n!}-\frac{(2n+1)^2}{4(n+1)^2}\frac{2^{2n-2}(n!)^2}{n(2n)!} &> \frac{(2n+1)}{2(n+1)}\frac{\left(\frac{1}{2}\right)_n}{n!}-\frac{2^{2n}((n+1)!)^2}{(n+1)(2n+2)!}\\
&\Longleftrightarrow\\
\frac{1}{4}\frac{2n+4n^2-1}{n(2n+1)(n+1)^2}\frac{(n!)^2 2^{2n-2}}{(2n)!} &> \frac{1}{4}\frac{2n+1}{(n+1)^2}\frac{\left(\frac{1}{2}\right)_n}{n!}\\
&\Longleftrightarrow
\end{aligned}
$$

$$
\frac{\left(\frac{1}{2}\right)_n}{n!} < \frac{2^{2n-2}n!^2(2n+4n^2-1)}{n(2n)!(2n+1)^2}. \tag{16}
$$

By mathematical induction, we can prove (16). It is not difficult to verify that (16) is true for $n=4$. Assuming that (16) holds for $n \geq 5$, let us prove (16) holds for $n+1$. Since

$$
\frac{\left(\frac{1}{2}\right)_{n+1}}{(n+1)!} = \frac{2n+1}{2(n+1)}\frac{\left(\frac{1}{2}\right)_n}{n!} < \frac{2n+1}{2(n+1)}\frac{2^{2n-2}n!^2(2n+4n^2-1)}{n(2n)!(2n+1)^2},
$$

we can complete the proof of (16) when proving

$$
\begin{aligned}
\frac{2n+1}{2(n+1)}\frac{2^{2n-2}n!^2(2n+4n^2-1)}{n(2n)!(2n+1)^2} &< \frac{2^{2n}(n+1)!^2\left[2(n+1)+4(n+1)^2-1\right]}{(n+1)(2n+2)!(2n+3)^2}\\
&= \frac{2^{2n}(n+1)!^2(10n+4n^2+5)}{(2n+2)!(2n+3)^2(n+1)}\\
&\Longleftrightarrow\\
\frac{4n(n+1)(10n+4n^2+5)}{(2n+3)^2(2n+4n^2-1)} &> 1.
\end{aligned}
$$

The last inequality obviously holds due to

$$
4n(n+1)\left(10n+4n^2+5\right)-(2n+3)^2\left(2n+4n^2-1\right)=4n^2+14n+9>0
$$

holds for $n \geq 1$. Based on the above fact $f_n > 0$ for $n \geq 2$, we have $\mathcal{F}'(r) > 0$ for $0 < r < 1$. Since $\mathcal{F}(0^+) = 0$, we obtain that $\mathcal{F}(r) > \mathcal{F}(0^+) = 0$ for $0 < r < 1$. This completes the proof of Theorem 2.

4. Conclusions

Scholars have been exploring the precise and concise bounds of the complete elliptic integral of the first kind $\mathcal{K}(r)$. This paper gives a concise and novel lower bound that conforms to the behavior of the function at two endpoints. I believe that the exploration of this paper provides the possibility to explore more concise and better upper and lower bounds for the complete elliptic integral of the first kind $\mathcal{K}(r)$.

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