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Fixed Point Results for a New Rational Contraction in Double Controlled Metric-like Spaces

Jia Deng [1], Xiaolan Liu [1,2,3,*], Yan Sun [1] and Mi Zhou [4,5,6,7,*]

[1] College of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong 643000, China; 31907010402@stu.suse.edu.cn (J.D.); 320070108107@stu.suse.edu.cn (Y.S.)
[2] Key Laboratory of Higher Education of Sichuan Province for Enterprise Informationalization and Internet of Things, Zigong 643000, China
[3] South Sichuan Center for Applied Mathematics, Zigong 643000, China
[4] School of Science and Techology, University of Sanya, Sanya 572022, China
[5] Center for Mathematical Reaserch, University of Sanya, Sanya 572022, China
[6] Academician Guoliang Chen Team Innovation Center, University of Sanya, Sanya 572022, China
[7] Academician Chunming Rong Workstation, University of Sanya, Sanya 572022, China
* Correspondence: xiaolanliu@suse.edu.cn (X.L.); mizhou@sanyau.edu.cn (M.Z.)

Abstract: In this paper, we present a new type of rational contraction in double controlled metric-like spaces and improve recent results of such spaces. Moreover, there is an example to verify the correctness of our results. Finally, we also obtain some new fixed point results, which can be derive directly from our main theorem.

Keywords: fixed point; rational contraction; controlled metric spaces; double controlled metric-like spaces

MSC: 47H10; 54H25

Citation: Deng, J.; Liu, X.; Sun, Y.; Zhou, M. Fixed Point Results for a New Rational Contraction in Double Controlled Metric-like Spaces. *Mathematics* **2022**, *10*, 1439. https://doi.org/10.3390/math10091439

Academic Editors: Salvador Romaguera, Dimitrios Georgiou and Manuel Sanchis

Received: 15 March 2022
Accepted: 21 April 2022
Published: 24 April 2022

1. Introduction

In recent decades, many researchers in the field of fixed point theory have made a number of generalizations of the usual metric space. Different types of generalized metric spaces lead to different theoretical achievements in general. With the emergence of these abstract generalized metric spaces, many authors can continue to study and enrich the fixed point theory according to different types of contraction conditions. As early as 1993, Czerwik [1] formally introduced the notion of b-metric space by putting a constant on the right-hand side of the triangle inequality, which is an interesting generalized metric space. The recent results for b-metric spaces can be seen in [2–4] and references therein. In 2017, Kamran et al. [5] replaced the coefficient of b-metric with a binary function and introduced a type of generalized b-metric space, namely extended b-metric spaces. Such spaces have inspired researchers of fixed point theory. For some recent fixed point results in several generalized metric spaces, refer to [6–11]. In 2018, Mlaiki et al. [12] defined a new class of extension of b-metric space, called it as controlled metric space, which is different from extended b-metric space. They proved the Banach fixed point result in controlled metric spaces. Later, some investigators studied such spaces. In 2019, Lateef [13] obtained the existence of a fixed point for Kannan-type contractions in controlled metric spaces. Shortly after, Ahmad et al. [14] introduced Reich type contractions and (α, F)-contractions, and established related fixed point theorems in controlled metric spaces. In a summary, Mlaiki et al. [15] investigated and improved some recent fixed point results in controlled metric spaces, and presented some new fixed point results. On the basis of controlled metric spaces, Abdeljawad et al. [16] presented double controlled metric spaces. Very recently, Mlaiki [17] has introduced double controlled metric-like spaces, which are produced by the fact that self-distance doesn't need to be zero.

Motivated by the existing results, we aim to further investigate the fixed point results in double controlled metric-like spaces. In this paper, we define a new type of rational contraction, and provide sufficient conditions for the existence and uniqueness of a fixed point of this kind of contraction. There is an example to verify the correctness of our important results. On the one hand, we improve some fixed point results in double controlled metric-like spaces; on the other hand, we attain a lot of corollaries related to the new rational contraction. As an application, we apply our result to the problem of Fredholm integral equation.

2. Preliminaries

In the beginning, let us retrace the definitions of several type generalzied metric spaces, which are useful to the next work.

As early as in 1993, Czerwik [1] formally defined and studied b-metric spaces by adding a constant in the right of triangle inequality of metric.

Definition 1 ([1])**.** *Let X be a nonempty set, $b \geqslant 1$ be a constant and $d : X \times X \to [0, \infty)$ be a function. If the following conditions hold:*

(d1) $d(u, v) = 0$ *if and only if* $u = v$*;*
(d2) $d(u, v) = d(v, u)$*, for all* $u, v \in X$*;*
(d3) $d(u, v) \leq b[d(u, w) + d(w, v)]$*, for all* $u, w, v \in X$*.*

Then d is said to be a b-metric, (X, d) is called to be a b-metric space, where b is the coefficient of b-metric.

The coefficient of b-metric binary function θ was proposed by Kamran et al. [5] in 2017, and the notion of extended b-metric space was presented.

Definition 2 ([5])**.** *Let X be a nonempty set, $\theta : X \times X \to [1, \infty)$ and $d_\theta : X \times X \to [0, \infty)$ be two functions. If the following conditions hold:*

(d_θ1) $d_\theta(u, v) = 0$ *if and only if* $u = v$*;*
(d_θ2) $d_\theta(u, v) = d_\theta(v, u)$*, for all* $u, v \in X$*;*
(d_θ3) $d_\theta(u, v) \leq \theta(u, v)[d_\theta(u, w) + d_\theta(w, v)]$*, for all* $u, v, w \in X$*.*

Then d_θ is said to be an extended b-metric, (X, d_θ) is an extended b-metric space with θ.

In 2018, Mlaiki et al. [12] proposed a new kind of metric space called controlled metric space. As a new extension of b-metric space, it is different from extended b-metric space.

Definition 3 ([12])**.** *Let X be a nonempty set, $\chi : X \times X \to [1, \infty)$ and $d_\chi : X \times X \to [0, \infty)$ be two functions. If the following conditions hold:*

(d_χ1) $d_\chi(u, v) = 0$ *if and only if* $u = v$*;*
(d_χ2) $d_\chi(u, v) = d_\chi(v, u)$*;*
(d_χ3) $d_\chi(u, v) \leq \chi(u, w) d_\chi(u, w) + \chi(w, v) d_\chi(w, v)$*.*

Then d_χ is a controlled metric, (X, d_χ) is called to be a controlled metric space.

In the same year, Abdeljawad et al. [16] presented double controlled metric spaces on the basics of controlled metric spaces.

Definition 4 ([16])**.** *Let X be a nonempty set, $\chi_1, \chi_2 : X \times X \to [1, \infty)$ and $d_\chi : X \times X \to [0, \infty)$ be three functions. If the following conditions hold:*

(d_χ1) $d_\chi(u, v) = 0$ *if and only if* $u = v$*;*
(d_χ2) $d_\chi(u, v) = d_\chi(v, u)$*;*
(d_χ3) $d_\chi(u, v) \leq \chi_1(u, w) d_\chi(u, w) + \chi_2(w, v) d_\chi(w, v)$*.*

Then d_χ is a double controlled metric, (X, d_χ) is said to be a double controlled metric space.

Very recently, Mlaiki [17] proposed the notion of double controlled metric-like spaces by weakening the condition $(d_\chi 1)$ such that self-distance does not need to be zero.

Definition 5 ([17]). *Let X be a nonempty set, $\chi_1, \chi_2 : X \times X \to [1, \infty)$ and $d_\chi : X \times X \to [0, \infty)$ are three functions. If it satisfies the following conditions:*

$(d_\chi 1)$ $d_\chi(u, v) = 0 \Rightarrow u = v$;
$(d_\chi 2)$ $d_\chi(u, v) = d_\chi(v, u)$;
$(d_\chi 3)$ $d_\chi(u, v) \leq \chi_1(u, w)d_\chi(u, w) + \chi_2(w, v)d_\chi(w, v)$.

Then d_χ is said to be a double controlled metric-like, and (X, d_χ) is said to be a double controlled metric-like space.

Remark 1. *Note that double controlled metric-like space includes double controlled metric space as its particular case, but the converse in general is not true.*

Now, we recall two examples of double controlled metric-like space.

Example 1 ([17]). *Let $X = [0, \infty)$. Defined $d_\chi : X \times X \to [0, \infty)$ by:*

$$d_\chi(u, v) = \begin{cases} 0, & \text{if } u = v \neq 0, \\ \frac{1}{2}, & \text{if } u = v = 0, \\ \frac{1}{u}, & \text{if } u \geq 1 \text{ and } v \in [0, 1), \\ \frac{1}{v}, & \text{if } v \geq 1 \text{ and } u \in [0, 1), \\ 1, & \text{otherwise}. \end{cases}$$

Let $\chi_1, \chi_2 : X \times X \to [1, \infty)$ be

$$\chi_1(u, v) = \begin{cases} u, & \text{if } u, v \geq 1, \\ v, & \text{otherwise}. \end{cases}$$

and

$$\chi_2(u, v) = \begin{cases} u, & \text{if } u, v < 1, \\ \max\{u, v\}, & \text{otherwise}. \end{cases}$$

Then it's easy to verify that (X, d_χ) is a double controlled metric-like space. However, (X, d_χ) is not a double controlled metric space. For more details, refer to [17].

Example 2 ([17]). *Let $X = \{a, b, c\}$ and define $d_\chi : X \times X \to [0, \infty)$ by*

$$\begin{aligned} &d_\chi(a, a) = d_\chi(b, b) = 0, d_\chi(c, c) = \frac{1}{10}, \\ &d_\chi(a, b) = d_\chi(b, a) = 1, \\ &d_\chi(a, c) = d_\chi(c, a) = \frac{1}{2}, \\ &d_\chi(b, c) = d_\chi(c, b) = \frac{2}{5}, \end{aligned}$$

Take $\chi_1, \chi_2 : X \times X \to [1, \infty)$ are two symmetric functions and defined by
$\chi_1(a, a) = \chi_1(b, b) = \chi_1(c, c) = \chi_1(a, c) = 1$, $\chi_1(a, b) = \frac{11}{10}$, $\chi_1(b, c) = \frac{8}{5}$;
$\chi_2(a, a) = \chi_2(b, b) = \chi_2(c, c) = 1$, $\chi_2(a, b) = \frac{11}{10}$, $\chi_2(a, c) = \frac{3}{2}$, $\chi_2(b, c) = \frac{5}{4}$.

Then (X, d_χ) is a double controlled metric-like space. However, $d_\chi(c, c) = \frac{1}{10}$, so (X, d_χ) is not a double controlled metric space. For more details, refer to [17].

In the following, we recall some properties of double controlled metric-like spaces.

Definition 6 ([17])**.** *Let (X, d_χ) be a double controlled metric-like space. For each sequence $\{u_n\} \in X$,*

(1) if $\lim_{n,m\to\infty} d_\chi(u_n, u_m)$ exists and is finite, then we say that $\{u_n\}$ a Cauchy sequence;

(2) if $\lim_{n\to\infty} d_\chi(u_n, u) = d_\chi(u, u)$, then we say that $\{u_n\}$ converges to u;

(3) if every Cauchy sequence in X is convergent to some point in X, then we say that (X, d_χ) is complete.

Very recently, Tas [18] has introduced Reich contraction in the double controlled metric-like spaces.

Theorem 1 ([18])**.** *Let (X, d_χ) be a complete double controlled metric-like space and $\chi_1, \chi_2 : X \times X \to [1, \infty)$. Let $f : X \to X$ be a self mapping such that*

$$d_\chi(fu, fv) \le \alpha d_\chi(u, v) + \beta d_\chi(u, fu) + \gamma d_\chi(v, fv),$$

for all $u, v \in X$, where $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$.

For $u_0 \in X$, take $u_n = f^n u_0$. Let $\lambda = \frac{\alpha+\beta}{1-\gamma}$. Suppose that

$$\sup_{m\ge1} \lim_{i\to\infty} \frac{\chi_1(u_{i+1}, u_{i+2})}{\chi_1(u_i, u_{i+1})} \chi_2(u_{i+1}, u_m) < \frac{1}{\lambda}.$$

Moreover, for each $u \in X$, $\lim_{n\to\infty} \chi_1(u, u_n)$ exists and is finite; $\lim_{n\to\infty} \chi_2(u_n, u) < \frac{1}{\gamma}$, then there exists a unique $u^* \in X$ such that $fu^* = u^*$.

3. Main Results

In this section, we firstly prove our main results for a rational contraction in double controlled metric-like spaces.

Theorem 2. *Let (X, d_χ) be a complete double controlled metric-like space and $\chi_1, \chi_2 : X \times X \to [1, \infty)$. Let $f : X \to X$ be a self mapping such that*

$$d_\chi(fu, fv) \le \alpha d_\chi(u, v) + \beta d_\chi(u, fu) + \gamma d_\chi(v, fv) + \eta \frac{d_\chi(u, fu) d_\chi(v, fv)}{1 + d_\chi(u, v)}, \tag{1}$$

for all $u, v \in X$, where $\alpha, \beta, \gamma, \eta \in [0, 1)$ with $\alpha + \beta + \gamma + \eta < 1$.
For $u_0 \in X$, take $u_n = f^n u_0$. Let $\lambda = \frac{\alpha+\beta}{1-\gamma-\eta}$. Suppose that

$$\sup_{m\ge1} \lim_{i\to\infty} \frac{\chi_1(u_{i+1}, u_{i+2})}{\chi_1(u_i, u_{i+1})} \chi_2(u_{i+1}, u_m) < \frac{1}{\lambda}. \tag{2}$$

Moreover, for each $u \in X$, $\lim_{n\to\infty} \chi_1(u, u_n)$ exists and is finite; $\lim_{n\to\infty} \chi_2(u_n, u) < \frac{1}{\gamma}$ for $\gamma > 0$. In particular, when $\gamma = 0$, the latter condition becomes $\lim_{n\to\infty} \chi_2(u_n, u) < \infty$, then there exists a unique $u^* \in X$ such that $fu^* = u^*$.

Proof. Let $u_0 \in M$. We can construct a sequence $\{u_n\}$ by $u_{n+1} = fu_n$ for all $n \in \mathbb{N}$. If $u_{n_0} = u_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then $fu_{n_0} = u_{n_0}$. That implies u_{n_0} is a fixed point of f. Hence, suppose that for all $n \in \mathbb{N}$, $u_n \ne u_{n+1}$, i.e., $d_\chi(u_n, u_{n+1}) > 0$. Apply $u = u_n$ and $v = u_{n+1}$ to (1), we have

$$\begin{aligned}
d_\chi(u_{n+1},u_{n+2}) &= d_\chi(fu_n, fu_{n+1}) \\
&\leq \alpha d_\chi(u_n,u_{n+1}) + \beta d_\chi(u_n, fu_n) + \gamma d_\chi(u_{n+1}, fu_{n+1}) \\
&\quad + \eta \frac{d_\chi(u_n, fu_n) d_\chi(u_{n+1}, fu_{n+1})}{1 + d_\chi(u_n,u_{n+1})} \\
&= \alpha d_\chi(u_n,u_{n+1}) + \beta d_\chi(u_n,u_{n+1}) + \gamma d_\chi(u_{n+1},u_{n+2}) \\
&\quad + \eta \frac{d_\chi(u_n,u_{n+1}) d_\chi(u_{n+1},u_{n+2})}{1 + d_\chi(u_n,u_{n+1})} \\
&\leq (\alpha+\beta) d_\chi(u_n,u_{n+1}) + (\gamma+\eta) d_\chi(u_{n+1},u_{n+2}).
\end{aligned}$$

which implies that

$$d_\chi(u_{n+1},u_{n+2}) \leq \frac{\alpha+\beta}{1-\gamma-\eta} d_\chi(u_n,u_{n+1}) = \lambda d_\chi(u_n,u_{n+1}),$$

and further we have

$$d_\chi(u_{n+1},u_{n+2}) \leq \lambda d_\chi(u_n,u_{n+1}) \leq \lambda^2 d_\chi(u_{n-1},u_n) \leq \cdots \leq \lambda^{n+1} d_\chi(u_0,u_1), \text{ for all } n \in \mathbb{N}. \tag{3}$$

Now we shall prove the $\{u_n\}$ is a Cauchy sequence. For all $n, m \in \mathbb{N}$ with $n < m$,

$$\begin{aligned}
d_\chi(u_n,u_m) &\leq \chi_1(u_n,u_{n+1}) d_\chi(u_n,u_{n+1}) + \chi_2(u_{n+1},u_m) d_\chi(u_{n+1},u_m) \\
&\leq \chi_1(u_n,u_{n+1}) d_\chi(u_n,u_{n+1}) + \chi_2(u_{n+1},u_m)\chi_1(u_{n+1},u_{n+2}) d_\chi(u_{n+1},u_{n+2}) \\
&\quad + \chi_2(u_{n+1},u_m)\chi_2(u_{n+2},u_m) d_\chi(u_{n+2},u_m) \\
&\leq \chi_1(u_n,u_{n+1}) d_\chi(u_n,u_{n+1}) + \chi_2(u_{n+1},u_m)\chi_1(u_{n+1},u_{n+2}) d_\chi(u_{n+1},u_{n+2}) \\
&\quad + \chi_2(u_{n+1},u_m)\chi_2(u_{n+2},u_m)\chi_1(u_{n+2},u_{n+3}) d_\chi(u_{n+2},u_{n+3}) \\
&\quad + \chi_2(u_{n+1},u_m)\chi_2(u_{n+2},u_m)\chi_2(u_{n+3},u_m) d_\chi(u_{n+3},u_m) \\
&\leq \cdots \\
&\leq \chi_1(u_n,u_{n+1}) d_\chi(u_n,u_{n+1}) + \sum_{i=n+1}^{m-2} \Big(\prod_{j=n+1}^{i} \chi_2(u_j,u_m)\Big) \chi_1(u_i,u_{i+1}) d_\chi(u_i,u_{i+1}) \\
&\quad + \prod_{i=n+1}^{m-1} \chi_2(u_i,u_m) d_\chi(u_{m-1},u_m).
\end{aligned}$$

We make use of $\chi_i(u,v) \geqslant 1, i = 1,2$, by (3), then it follows that

$$\begin{aligned}
d_\chi(u_n,u_m) &\leq \chi_1(u_n,u_{n+1}) d_\chi(u_n,u_{n+1}) + \sum_{i=n+1}^{m-2} \Big(\prod_{j=n+1}^{i} \chi_2(u_j,u_m)\Big) \chi_1(u_i,u_{i+1}) d_\chi(u_i,u_{i+1}) \\
&\quad + \prod_{i=n+1}^{m-1} \chi_2(u_i,u_m))\chi_2(u_{m-1},u_m)) d_\chi(u_{m-1},u_m) \\
&\leq \chi_1(u_n,u_{n+1}) \lambda^n d_\chi(u_0,u_1) + \sum_{i=n+1}^{m-2} \Big(\prod_{j=n+1}^{i} \chi_2(u_j,u_m)\Big) \chi_1(u_i,u_{i+1}) \lambda^i d_\chi(u_0,u_1) \\
&\quad + \prod_{i=n+1}^{m-1} \chi_2(u_i,u_m)\chi_1(u_{m-1},u_m) \lambda^{m-1} d_\chi(u_0,u_1) \\
&= \chi_1(u_n,u_{n+1}) \lambda^n d_\chi(u_0,u_1) + \sum_{i=n+1}^{m-1} \Big(\prod_{j=n+1}^{i} \chi_2(u_j,u_m)\Big) \chi_1(u_i,u_{i+1}) \lambda^i d_\chi(u_0,u_1) \\
&\leq \chi_1(u_n,u_{n+1}) \lambda^n d_\chi(u_0,u_1) + \sum_{i=n+1}^{m-1} \Big(\prod_{j=0}^{i} \chi_2(u_j,u_m)\Big) \chi_1(u_i,u_{i+1}) \lambda^i d_\chi(u_0,u_1).
\end{aligned} \tag{4}$$

Now if we define, for all $q \in \mathbb{N}$,

$$S_q = \sum_{i=0}^{q} (\prod_{j=0}^{i} \chi_2(u_j, u_m)) \chi_1(u_i, u_{i+1}) \lambda^i d_\chi(u_0, u_1).$$

From (4), we can deduce that

$$d_\chi(u_n, u_m) \leq d_\chi(u_0, u_1)[\lambda^n \chi_1(u_n, u_{n+1}) + (S_{m-1} - S_n)]. \tag{5}$$

Consider the series $\sum_{i=0}^{\infty} (\prod_{j=0}^{i} \chi_2(u_j, u_m)) \chi_1(u_i, u_{i+1}) \lambda^i d_\chi(u_0, u_1)$.

Let $\rho_n = \lambda^n d_\chi(u_0, u_1) \prod_{j=0}^{n} \chi_2(u_j, u_m) \chi_1(u_n, u_{n+1})$, then we have

$$\frac{\rho_{n+1}}{\rho_n} = \frac{\lambda^{n+1} d_\chi(u_0, u_1) \prod_{j=0}^{n+1} \chi_2(u_j, u_m) \chi_1(u_{n+1}, u_{n+2})}{\lambda^n d_\chi(u_0, u_1) \prod_{j=0}^{n} \chi_2(u_j, u_m) \chi_1(u_n, u_{n+1})} = \lambda \frac{\chi_1(u_{n+1}, u_{n+2})}{\chi_1(u_n, u_{n+1})} \chi_2(u_{n+1}, u_m).$$

In view of (2), we conclude that the series $\sum_{i=0}^{\infty} (\prod_{j=0}^{i} \chi_2(u_j, u_m)) \chi_1(u_i, u_{i+1}) \lambda^i d_\chi(u_0, u_1)$ is convergent (by the ratio test of positive series). Consequently, if we take the limit as $n, m \to \infty$ in the inequality (5), then we get that

$$\lim_{n,m \to \infty} d_\chi(u_n, u_m) = 0. \tag{6}$$

Hence, $\{u_n\}$ is a Cauchy sequence in (X, d_χ). In view of the completeness of (X, d_χ), so there must exists a point $u^* \in X$ such that

$$\lim_{n \to \infty} d_\chi(u_n, u^*) = d_\chi(u^*, u^*) = \lim_{n,m \to \infty} d_\chi(u_n, u_m) = 0. \tag{7}$$

Next we prove that $fu^* = u^*$. Consider the condition $(d_\chi 3)$ and (1), then

$$\begin{aligned}
d_\chi(u^*, fu^*) &\leq \chi_1(u^*, u_{n+1}) d_\chi(u^*, u_{n+1}) + \chi_2(u_{n+1}, fu^*) d_\chi(u_{n+1}, fu^*) \\
&= \chi_1(u^*, u_{n+1}) d_\chi(u^*, u_{n+1}) + \chi_2(u_{n+1}, fu^*) d_\chi(fu_n, fu^*) \\
&\leq \chi_1(u^*, u_{n+1}) d_\chi(u^*, u_{n+1}) + \chi_2(u_{n+1}, fu^*)[\alpha d_\chi(u_n, u^*) + \beta d_\chi(u_n, fu_n) \\
&\quad + \gamma d_\chi(u^*, fu^*) + \eta \frac{d_\chi(u_n, fu_n) d_\chi(u^*, fu^*)}{1 + d_\chi(u_n, u^*)}] \\
&= \chi_1(u^*, u_{n+1}) d_\chi(u^*, u_{n+1}) + \chi_2(u_{n+1}, fu^*)[\alpha d_\chi(u_n, u^*) + \beta d_\chi(u_n, u_{n+1}) \\
&\quad + \gamma d_\chi(u^*, fu^*) + \eta \frac{d_\chi(u_n, u_{n+1}) d_\chi(u^*, fu^*)}{1 + d_\chi(u_n, u^*)}].
\end{aligned}$$

Taking account into (6) and $\lim_{n \to \infty} \chi_1(u, u_n)$ exists and is finite, take the limits as $n \to \infty$ on the two sides of the above inequality, we obtain

$$d_\chi(u^*, fu^*) \leq \lim_{n \to \infty} \chi_2(u_{n+1}, fu^*)[\gamma d_\chi(u^*, fu^*)]. \tag{8}$$

Based on the assumptions of Theorem 2, we discuss the following two cases:

Case 1. $\lim_{n \to \infty} \chi_2(u_{n+1}, fu^*) < \infty$ for $\gamma = 0$.
By (8), we obtain $d_\chi(u^*, fu^*) = 0$;

Case 2. $\lim_{n \to \infty} \chi_2(u_{n+1}, fu^*) < \frac{1}{\gamma}$ for $\gamma > 0$.

In this case, if $d_\chi(u^*, fu^*) > 0$, then (8) is equal to

$$d_\chi(u^*, fu^*) \leq \lim_{n\to\infty} \chi_2(u_{n+1}, fu^*)[\gamma d_\chi(u^*, fu^*)] < \frac{1}{\gamma}[\gamma d_\chi(u^*, fu^*)] = d_\chi(u^*, fu^*).$$

Therefore, we get a contradiction, then $d_\chi(u^*, fu^*) = 0$.

In the both cases, $d_\chi(u^*, fu^*) = 0$, so $fu^* = u^*$. Now, we prove that if p is the fixed point of f, then $d_\chi(p, p) = 0$. Indeed, assume that p is the fixed point of f. Apply $u = p$ and $v = p$ to (1), and we have

$$\begin{aligned} d_\chi(p, p) &= d_\chi(fp, fp) \\ &\leq \alpha d_\chi(p, p) + \beta d_\chi(p, fp) + \gamma d_\chi(p, fp) + \eta \frac{d_\chi(p, fp) d_\chi(p, fp)}{1 + d_\chi(p, p)} \\ &\leq \alpha d_\chi(p, p) + \beta d_\chi(p, p) + \gamma d_\chi(p, p) + \eta d_\chi(p, p) \\ &= (\alpha + \beta + \gamma + \eta) d_\chi(p, p). \end{aligned}$$

Since $\alpha + \beta + \gamma + \eta \in [0, 1)$, so we get a contrary to $d_\chi(p, p) > 0$, thus $d_\chi(p, p) = 0$.

Eventually, suppose that f has two different fixed points, say u^* and v^*. Apply $u = u^*$ and $v = v^*$ to (1), then it easily follows that

$$\begin{aligned} d_\chi(u^*, v^*) &= d_\chi(fu^*, fv^*) \\ &\leq \alpha d_\chi(u^*, v^*) + \beta d_\chi(u^*, fu^*) + \gamma d_\chi(v^*, fv^*) + \eta \frac{d_\chi(u^*, fu^*) d_\chi(v^*, fv^*)}{1 + d_\chi(u^*, v^*)} \\ &= \alpha d_\chi(u^*, v^*) + \beta d_\chi(u^*, u^*) + \gamma d_\chi(v^*, v^*) + \eta \frac{d_\chi(u^*, u^*) d_\chi(v^*, v^*)}{1 + d_\chi(u^*, v^*)} \\ &= \alpha d_\chi(u^*, v^*) < d_\chi(u^*, v^*). \end{aligned}$$

Which is a contradiction. Hence, f has a unique fixed point u^*. □

Remark 2.

(1). *Our results is an improvement of the results of Lateef [19]. On the one hand, if $\beta = \gamma = 0$, notice that our contraction becomes a Fisher contraction in [19], the conclusion still holds, i.e., f has a fixed point; on the other hand, we get the result in double controlled metric-like spaces, instead of controlled metric spaces. In other words, we extend the result to double controlled metric-like spaces.*

(2). *Special cases:*

case 1. if $\eta = 0$ in Theorem 3.1, then we obtain the results of Tas [18];

case 2. if $\beta = \gamma = \eta = 0$, then we obtain Banach contraction, which is the results of Mlaiki [17];

case 3. if $\beta = \gamma$ and $\alpha = \eta = 0$, then we obtain Kannan contraction, which is the results of Mlaiki [17].

(3). *By Remark 1, we know that every double controlled metric space is a double controlled metric-like space, and self-distance in the latter does not need to be zero. Thus, our results still hold in double controlled metric spaces.*

We give a simple example to verify the correctness of Theorem 2.

Example 3. *Let $X = \{0,1,2\}$ and define $d_\chi : X \times X \to [0,\infty)$ by*

$$\begin{aligned} &d_\chi(0,0) = d_\chi(1,1) = 0, d_\chi(2,2) = 1, \\ &d_\chi(0,1) = d_\chi(1,0) = 10, \\ &d_\chi(0,2) = d_\chi(2,0) = 6, \\ &d_\chi(1,2) = d_\chi(2,1) = 4, \end{aligned}$$

Consider $\chi_1, \chi_2 : X \times X \to [1,\infty)$ are two symmetric function and defined as follows:
$\chi_1(0,0) = \chi_1(1,1) = \chi_1(2,2) = 1$, $\chi_1(0,2) = 1$, $\chi_1(0,1) = \frac{11}{10}$, $\chi_1(1,2) = \frac{6}{5}$;
$\chi_2(0,0) = \chi_2(1,1) = \chi_2(2,2) = 1$, $\chi_2(0,1) = \frac{11}{10}$, $\chi_2(0,2) = \frac{7}{6}$, $\chi_2(1,2) = 1$.

Define a mapping $f : X \to X$ by

$$fu = \begin{cases} 2, & \text{if } u = 0, \\ 1, & \text{if } u \in \{1,2\}. \end{cases}$$

Firstly, we show that (X, d_χ) is a double controlled metric-like space. By the symmetry of d_χ, χ_1 and χ_2, then $(d_\chi 1)$ and $(d_\chi 2)$ in the sense of Definition 5 are obviously satisfied, so we focus on the following three cases, and the others are clearly true.
Case 1. $u = 0, v = 1$, $d_\chi(0,1) = 10 \leq 10 = \chi_1(0,2)d_\chi(0,2) + \chi_2(2,1)d_\chi(2,1)$;
Case 2. $u = 0, v = 2$, $d_\chi(0,2) = 6 \leq 15 = \chi_1(0,1)d_\chi(0,1) + \chi_2(1,2)d_\chi(1,2)$;
Case 3. $u = 1, v = 2$, $d_\chi(1,2) = 4 \leq 18 = \chi_1(1,0)d_\chi(1,0) + \chi_2(0,2)d_\chi(0,2)$,
then it must be $d_\chi(u,v) \leq \chi_1(u,w)d_\chi(u,w) + \chi_2(w,v)d_\chi(w,v)$, for all $u, w, v \in X$, i.e., $(d_\chi 3)$ holds. So (X, d_χ) is a double controlled metric-like space.

Next, we take $\alpha = \frac{1}{5}$, $\beta = \frac{1}{3}$, $\gamma = \frac{1}{4}$, $\eta = \frac{1}{10}$. Now, let us talk about the following three cases:
Case 1. $u = 0, v = 1$, $d_\chi(f0, f1) = d_\chi(2,1) = 4 \leq 4 = \frac{1}{5}d_\chi(0,1) + \frac{1}{3}d_\chi(0,2) + \frac{1}{4}d_\chi(1,1) + \frac{1}{10}\frac{d_\chi(0,2)d_\chi(1,1)}{1+d_\chi(0,1)}$;
Case 2. $u = 0, v = 2$, $d_\chi(f0, f2) = d_\chi(2,1) = 4 < \frac{159}{35} = \frac{1}{5}d_\chi(0,2) + \frac{1}{3}d_\chi(0,2) + \frac{1}{4}d_\chi(2,1) + \frac{1}{10}\frac{d_\chi(0,2)d_\chi(2,1)}{1+d_\chi(0,2)}$;
Case 3. $u = 1, v = 2$, $d_\chi(f1, f2) = d_\chi(1,1) = 0 < \frac{32}{15} = \frac{1}{5}d_\chi(1,2) + \frac{1}{3}d_\chi(1,1) + \frac{1}{4}d_\chi(2,1) + \frac{1}{10}\frac{d_\chi(1,1)d_\chi(2,1)}{1+d_\chi(1,1)}$.

So (1) is satisfied. For any $u_0 \in X$, (2) is also satisfied. Indeed, we have

$$\sup_{m\geq 1}\lim_{i\to\infty} \frac{\chi_1(u_{i+1}, u_{i+2})}{\chi_1(u_i, u_{i+1})}\chi_2(u_{i+1}, u_m) = 1 < \frac{39}{32} = \frac{1}{\lambda} = \frac{1-\gamma-\eta}{\alpha+\beta}.$$

It is easy to verify the other assumptions in Theorem 2 also hold. So it meets requirements of Theorem 2, in this case, we can find that the unique fixed point of f is $u = 1$.

By our main results, we can obtain some fixed point results in a few metric-type spaces.

Corollary 1. *Let (X, d_χ) be a complete controlled metric space and $\chi : X \times X \to [1,\infty)$. If a self-mapping $f : X \to X$ such that*

$$d_\chi(fu, fv) \leq \alpha d_\chi(u,v) + \beta d_\chi(u, fu) + \gamma d_\chi(v, fv) + \eta\frac{d_\chi(u, fu)d_\chi(v, fv)}{1 + d_\chi(u,v)}, \tag{9}$$

for all $u, v \in X$, where $\alpha, \beta, \gamma, \eta \in [0,1)$ with $\alpha + \beta + \gamma + \eta < 1$.

For $u_0 \in X$, take $u_n = f^n u_0$. Let $\lambda = \frac{\alpha+\beta}{1-\gamma-\eta}$. Suppose that

$$\sup_{m\geq 1}\lim_{i\to\infty} \frac{\chi(u_{i+1}, u_{i+2})}{\chi(u_i, u_{i+1})}\chi(u_{i+1}, u_m) < \frac{1}{\lambda}. \tag{10}$$

Moreover, for each $u \in X$, $\lim_{n\to\infty} \chi(u, u_n)$ exist and finite and $\lim_{n\to\infty} \chi(u_n, u) < \frac{1}{\gamma}$, then there exists a unique $u^* \in X$ such that $fu^* = u^*$.

Proof. Take $\chi_1 = \chi_2 = \chi$ in Theorem 2. □

Corollary 2. *Let (X, d) be a complete b-metric space. If a self-mapping $f : X \to X$ such that*

$$d(fu, fv) \leq \alpha d(u, v) + \beta d(u, fu) + \gamma d(v, fv) + \eta \frac{d(u, fu)d(v, fv)}{1 + d(u, v)}, \tag{11}$$

for all $u, v \in X$, where $\alpha, \beta, \gamma, \eta \in (0, 1)$ with $\alpha + \beta + \gamma + \eta < 1$.

Suppose that

$$b < \min\{\frac{1 - \gamma - \eta}{\alpha + \beta}, \frac{1}{\gamma}\}.$$

Then there exists a unique $u^* \in M$ such that $fu^* = u^*$.

Proof. Let $\chi_1 = \chi_2 = b$ in Theorem 2, where b is the cofficient of b-metric space (X, d). □

Corollary 3. *Let (X, d) be a complete metric space. If a self-mapping $f : X \to X$ such that*

$$d(fu, fv) \leq \alpha d(u, v) + \beta d(u, fu) + \gamma d(v, fv) + \eta \frac{d(u, fu)d(v, fv)}{1 + d(u, v)}, \tag{12}$$

for all $u, v \in X$, where $\alpha, \beta, \gamma, \eta \in (0, 1)$ with $\alpha + \beta + \gamma + \eta < 1$.

Then there exists a unique $u^* \in M$ such that $fu^* = u^*$.

Proof. In Corollary 2, a b-metric space reduces to a metric space when $b = 1$, so the conclusion holds clearly. □

4. Application

In this section, we aim to apply Theorem 2 to solve the existence and uniqueness problems of solutions for the Fredholm integral equations, which defined as follows:

$$F(x) = \int_a^b f(x, y, F(y))dy + M(x), \text{ for all } x, y \in [a, b], \tag{13}$$

where $M : [a, b] \to \mathbb{R}$ and $f : [a, b] \times [a, b] \times \mathbb{R}$ are two continuous functions. Let $X = C([a, b])$ be the set of all continuous real value functions defined on $[a, b]$. Set

$$d_\chi(F_1(t), F_2(t)) = \sup_{t\in[a,b]} |F_1(t) - F_2(t)|^2.$$

Take

$$\chi_1(F_1, F_2) = \begin{cases} 1 + \sup_{t\in[a,b]} |F_1(t) - F_2(t)|, & \text{if } F_1 \neq F_2, \\ 1, & \text{if } F_1 = F_2 \end{cases}$$

and

$$\chi_2(F_1, F_2) = 1, \text{for all } F_1, F_2 \in X.$$

Obviously, it can easily follows that (X, d_χ) is a complete double controlled metric-like space.

Theorem 3. *Define $T : X \to X$ by*

$$T(F(x)) = \int_a^b f(x, y, F(y))dy + M(x), \textit{ for all } x, y \in [a, b]. \tag{14}$$

Suppose that for all $F_1, F_2 \in X$, $x, y \in [a, b]$,

$$|f(x, y, F_1(y)) - f(x, y, F_2(y))| \leq \frac{1}{b-a}\sqrt{M},$$

where

$$M = \alpha d_\chi(F_1(t), F_2(t)) + \beta d_\chi(F_1(t), T(F_1(t))) + \gamma d(F_2(t), T(F_2(t))) \\ + \eta \frac{d_\chi(F_1(t), T(F_1(t)))d_\chi(F_2(t), T(F_2(t)))}{1 + d_\chi(F_1(t), F_2(t))}$$

with $\alpha + \beta + \gamma + \eta < 1$. *Then, Equation* (13) *has a unique solution.*

Proof. Now, we prove T satisfies the all conditions of Theorem 2. For all $F_1, F_2 \in X$, we get

$$\begin{aligned} |T(F_1(x)) - T(F_2(x))|^2 &\leq (\int_a^b |[f(x, y, F_1 y) - f(x, y, F_1 y)]| dy)^2 \\ &\leq (\int_a^b \frac{1}{(b-a)}\sqrt{M} dy)^2 \\ &= \frac{1}{(b-a)^2}(\int_a^b dy)^2 M \\ &= \alpha d_\chi(F_1(t), F_2(t)) + \beta d_\chi(F_1(t), T(F_1(t))) + \gamma d(F_2(t), T(F_2(t))) \\ &+ \eta \frac{d_\chi(F_1(t), T(F_1(t)))d_\chi(F_2(t), T(F_2(t)))}{1 + d_\chi(F_1(t), F_2(t))}. \end{aligned}$$

Hence,

$$\begin{aligned} d_\chi(F_1(x), F_2(x)) &= \sup_{t \in [a,b]} |T(F_1(x)) - T(F_2(x))|^2 \\ &\leq \alpha d_\chi(F_1(t), F_2(t)) + \beta d_\chi(F_1(t), T(F_1(t))) + \gamma d(F_2(t), T(F_2(t))) \\ &+ \eta \frac{d_\chi(F_1(t), T(F_1(t)))d_\chi(F_2(t), T(F_2(t)))}{1 + d_\chi(F_1(t), F_2(t))}. \end{aligned}$$

Therefore, all conditions of Theorem 2 are satisfied, so Equation (13) has a unique solution. □

5. Conclusions

In this paper, we define a new type of rational contraction in double controlled metric-like spaces. On the one hand, we obtain the fixed point theorems of this kind of contraction. On the other hand, we improve some recent results in double controlled metric-like spaces. Next, we use a simple example to show the validity of our main results. Finally, we get a lot of corollaries related to the new rational contraction. There is no doubt about the importance of fixed point theory. Considering the study of contraction condition and generalized metric space are two important research directions of fixed point theory, based on this paper, we give some ideas for the future.

There are some possible works in the future:

- (i) Consider replacing the rational expression in this article with another rational expression;
- (ii) Extend our results to another metric space, like double controlled quasi metric-like space [20], fuzzy double controlled metric space [21], triple controlled metric space [22], and so on;
- (iii) The four constants on the right hand side of the rational contraction inequality may be changed to special functions.

Author Contributions: Resources, J.D., X.L. and Y.S.; Software, X.L. and M.Z.; Writing—original draft, J.D., X.L. and Y.S.; Writing—review & editing, J.D., X.L., Y.S. and M.Z. All authors contributed equally in writing this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This work is partially supported by National Natural Science Foundation of China (Grant No.11872043), Central Government Funds of Guiding Local Scientific and Technological Development for Sichuan Province (Grant No.2021ZYD0017), Zigong Science and Technology Program (Grant No.2020YGJC03), the Opening Project of Key Laboratory of Higher Education of Sichuan Province for Enterprise Informationalization and Internet of Things (Grant No.2020WYJ01), 2020 Graduate Innovation Project of Sichuan University of Science and Engineering (Grant No.y2020078), 2021 Innovation and Entrepreneurship Training Program for College Students of Sichuan University of Science and Engineering(Grant No.cx2021150).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: All the authors are grateful to the editor and anonymous referees for their excellent suggestions, which greatly improved this paper.

Conflicts of Interest: The authors declare no conflict of interest.

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