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Pursuit Differential Game with Slow Pursuers on the 1-Skeleton Graph of the Icosahedron

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Abstract: A differential game of $m, 3 \leq m \leq 6$, pursuers and one evader is studied on an icosahedron in $\mathbb{R}^3$. All the players move only along the 1-skeleton graph of the icosahedron when the maximal speeds of the pursuers are less than the speed of the evader. Pursuit is said to be completed if the state of a pursuer coincides with the state of evader at some time. We give a sufficient condition of the completion of pursuit in the game.

Keywords: graph of polyhedron; differential game; pursuer; evader; strategy

MSC: 05C57; 91A43

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1. Introduction

The main constraints on the controls for differential games are integral constraint and geometric constraint. In the case of geometric constraints on the controls of players, players choose the values of control functions from given convex subsets of $\mathbb{R}^d$ for some $d \geq 1$. Differential games of many players is one of the important chapters of differential games. For the multi-player differential games with geometric constraints, interesting results were obtained by [1–11]. A detailed survey is given in the paper [12] for such differential games.

In [10], a game of m pursuers and one evader is studied. All the players have the same dynamic capability. The game is completed if the states of k pursuers simultaneously coincide with the state of the evader. If fewer than k pursuers reach the evader, then all of these pursuers are destroyed by the pursuer. In the case of a discrete time game, the necessary and sufficient condition of game completion is obtained in terms of k-hull.

Additionally, the paper [2] studies a differential game of a group of rigidly coordinated evaders and many pursuers with equal capabilities. Simultaneous multiple capture occurs in the game if a certain number of pursuers catches the evaders at the same time. The main result of the paper is the sufficient and necessary conditions for simultaneous multiple capture of evaders, where pursuers use piecewise-program counter strategies.

The differential games of many players, where the control resources of players are bounded, are an interesting and difficult area of differential games. In such games, players need to optimize the expenditure of the resource as well (see, for example, refs. [13–15]). In [14], the simple motion differential game of many pursuers and many evaders was studied, and it was proved that if the total energy of the evaders is greater than or equal to that of the pursuers, then evasion is possible from any initial position of the players.

There are many papers devoted to differential games with state constraints (see, for example, refs. [16,17]). An interesting class of differential games with state constraint is differential games on graphs (see, for example, refs. [18–22]). If the graph is connected and finite, then only one fast pursuer can catch the evader. Therefore, the case of many slow pursuers is of importance, and the problems for such games are minimizing the number of pursuers to complete the game and for the construction of pursuit and evasion strategies.

Azamov et al. [22] studied differential games of many pursuers and one evader on the edge graphs of regular polyhedrons in $\mathbb{R}^3$. The speeds of all players do not exceed 1. It was established that the minimum number of pursuers to complete the game for the tetrahedron, octahedron and cube is two, and that for the dodecahedron and icosahedron is three.

Later on Azamov et al. [21] showed that on the edge graphs of the d-dimensional regular simplex, cocube, and cube, and it was shown that the minimum numbers of pursuers to complete the game for these polyhedrons are 2, 2, and $[d/2] + 1$, respectively. In the work [19], it was established that the minimum number of pursuers to complete the game for the regular 24-gone and 120-gone in $\mathbb{R}^4$ is equal to 3.

The paper [23] is devoted to the differential game of many slow pursuers and one evader on the edge graphs of the n-dimensional cocube. The problem of finding the optimal number of pursuers to complete the game was solved in that paper. It should be noted that the Π-strategy (see, for example, ref. [24]) played a key role in constructing the strategies of players.

It should be noted that there are other types of dynamic games on abstract graphs, such as multimove games, where players move from one vertex of the graph to an adjacent one by jumping (see, for example, refs. [25–30] and search games (see, for example, ref. [31]).

The present paper is devoted to studying a pursuit game of m, $3 \leq m \leq 6$, slow pursuers and one evader on the 1-skeleton graph of the regular icosahedron K in the Euclidean space $\mathbb{R}^3$ (Figure 1).

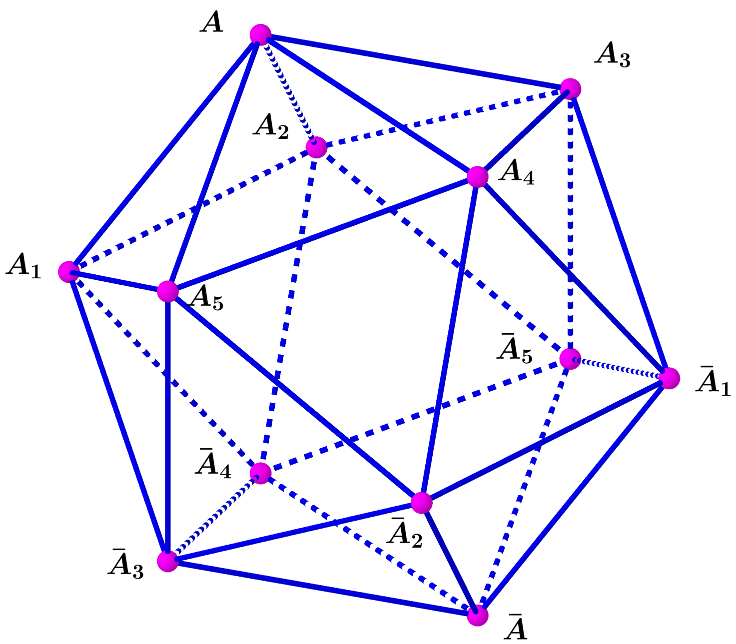

Figure 1. Icosahedron.

We recall that a regular icosahedron is one of the five Platonic solids, and it is a convex polyhedron with 30 edges, 12 vertices, and 20 faces. We construct strategies for the slow pursuers to complete the game.

2. Statement of Problem

In the present paper, a differential game of m pursuers $x_1, x_2, \dots, x_m$ and one evader y is considered. Players move according to the following equations:

$$\begin{aligned} &\dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \quad i = 1, 2 \dots m, \\ &\dot{y} = v, \quad\ \ y(0) = y_0, \end{aligned} \tag{1}$$

where $x_i, y, x_{i0}, y_0 \in \mathbb{R}^3$, $x_{i0} \neq y_0$, $i = 1, 2, \dots, m$; u_1, u_2, …, u_m and v are the control parameters of pursuers $x_1, x_2, \dots, x_m$ and evader, respectively. All the pursuers and the evader move along the edge graphs of icosahedron K. The maximal speed of evader y is σ, and the maximal speeds of the pursuers $x_1, x_2, \dots, x_m$ are $\rho_1, \rho_2, \dots, \rho_m$, respectively, i.e., $|u_i| \leq \rho_i$, $i = 1, 2, \dots, m$, and $|v| \leq \sigma$.

Let $B(r)$ denote the ball of radius r and centered at the origin of $\mathbb{R}^3$.

Definition 1. *We call the function $u_i(\cdot)$, $u_i : [0, \infty) \to B(\rho_i)$ an admissible control of the pursuer x_i, $i \in \{1, 2, \dots, m\}$, if the solution $x_i(\cdot)$ of the initial value problem*

$$\dot{x}_i = u_i, \ x_i(0) = x_{i0},$$

satisfies the condition $x_i(t) \in K$, $t \geq 0$.

Definition 2. *We call the function $v(\cdot)$, $v : [0, \infty) \to B(\sigma)$ an admissible control of the evader y, if for the solution $y(\cdot)$ of the initial value problem*

$$\dot{y} = v, \ y(0) = y_0,$$

we have $y(t) \in K$, $t \geq 0$.

We consider a pursuit differential game where the pursuers apply strategies and the evader uses an arbitrary admissible control. We give a definition for the strategies of pursuers.

Definition 3. *We call the functions $(t, x_1, x_2, \dots, x_m, y, v) \to U_i(t, x_1, x_2, \dots, x_m, y, v)$, $i = 1, 2, \dots, m$, strategies of pursuers x_i, $i = 1, 2, \dots, m$, if, for $u_i = U_i(t, x_1, x_2, \dots, x_m, y, v)$, $i = 1, 2, \dots, m$, and for any admissible control $v = v(t)$ of the evader, the initial value problem (1) has a unique solution $x_1(t), x_2(t), \dots, x_m(t)$, $y(t) \in K$, $t \geq 0$.*

This definition shows that to construct the strategy of the pursuers, the information about the states of players $x_1(t)$, $x_2(t)$, $x_3(t)$, $y(t)$, and velocity of evader $v(t)$ at current time t is needed.

Definition 4. *If, for some number $T > 0$ and any initial states of players $x_{10}, x_{20}, \dots, x_{m0}, y_0 \in K$, there exist strategies of pursuers such that $x_i(\tau) = y(\tau)$ at some $0 < \tau \leq T$ and $i \in \{1, 2, \dots, m\}$, then we say that pursuit is completed in the game in K for the time T.*

The pursuers try to complete the game as early as possible. The evader tries to maintain the inequality $x_i(t) \neq y(t)$ as long as possible.

Problem 1. *The problem is to find a sufficient condition on $\rho_1, \rho_2, \dots, \rho_m$ and σ for the completion of pursuit in the game in K.*

Note that in the case where $\rho_1 = 1$, $\rho_2 > 0$, $\rho_3 > 0$, and $\sigma = 1$ the problem was studied in [20] and it was shown that pursuit can be completed in the game on icosahedron K.

3. Main Result

Let us consider a regular icosahedron $AA_1A_2A_3A_4A_5\bar{A}_1\bar{A}_2\bar{A}_3\bar{A}_4\bar{A}_5\bar{A}$ with the edges of length 1, where $\bar{A}$, $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$, $\bar{A}_4$, and $\bar{A}_5$ are antipodal of vertices A, A_1, A_2, A_3, A_4, and A_5, respectively (Figure 1). We assume that the length of the edges of the icosahedron is equal to 1.

3.1. The Case of Three Pursuers P_1, P_2, P_3

In this subsection, we consider a differential game of three pursuers P_1, P_2, P_3 and one evader on the icosahedron. We prove the following statement.

Theorem 1. *If the maximal speeds of two pursuers belong to $[1/2, 1)$ and the speed of the third pursuer is in the interval $[2/3, 1)$, then pursuit can be completed in the game on icosahedron K.*

Without loss of generality, we assume that $1/2 \leq \rho_1 < 1$, $1/2 \leq \rho_2 < 1$, and $2/3 \leq \rho_3 < 1$. It is sufficient to show that pursuit can be completed when $\rho_1 = \rho_2 = 1/2$, $\rho_3 = 2/3$.

Proof. We use the graph of icosahedron on the plane illustrated in Figure 2. We denote $A_{ij}(k) = A_i + k\overrightarrow{A_iA_j}$. That is, $A_{ij}(k)$ is the point of the edge A_iA_j whose distance from the point A_i is k, $0 \leq k \leq 1$.

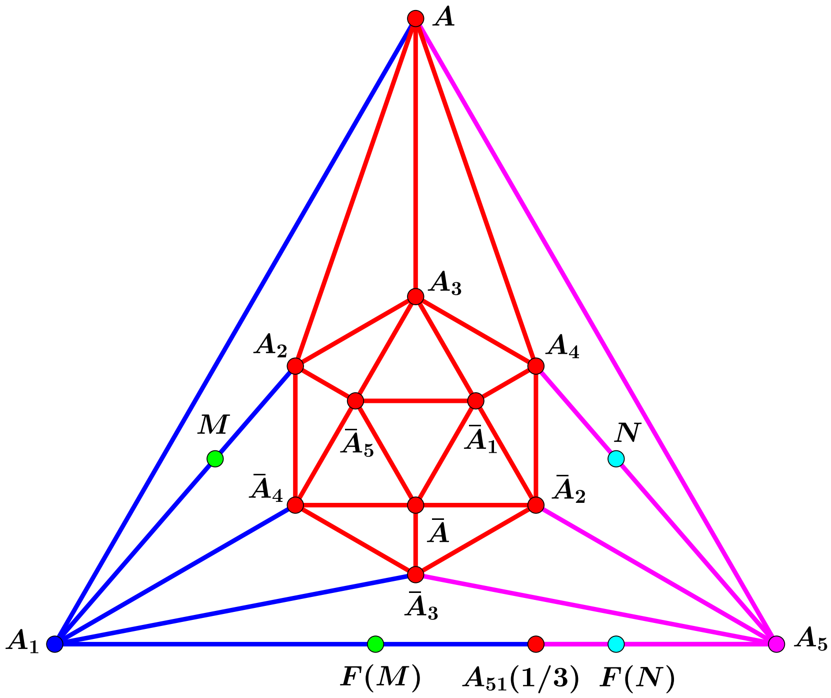

Figure 2. Points $F(M)$ and $F(N)$: $A_1F(M) = \frac{2}{3}A_1M$, $A_5F(N) = \frac{1}{3}A_5N$.

We want to place first the pursuer P_3 of speed $2/3$ on the edge A_1A_5 to guard this edge from the evader. To this end, we define the point $F(M)$ associated with any point M of icosahedron K for which $A_1M \leq 1$ and $M \notin A_1A_5$, that is, with the point on the edges A_1A, A_1A_2, $A_1\bar{A}_3$, $A_1\bar{A}_4$ (highlighted in blue) by the equation $A_1F(M) = \frac{2}{3}A_1M$ (Figure 2). Clearly, if the evader is at the point M and the pursuer P_3 is at the point $F(M)$, then they can reach the point A_1 at the same time. Note that if pursuer P_3 of speed $2/3$ is at the point $F(M)$ and evader E of speed 1 is at the point M, then pursuer P_3 can reach the vertex A_1 not later than evader E.

Similarly, we define the point $F(N)$ associated with any point N, for which $A_5N \leq 1$ and $N \notin A_1A_5$, that is, with any point on the edges A_5A, A_5A_4, $A_5\bar{A}_2$, $A_5\bar{A}_3$ (highlighted in pink) by the equation $A_5F(N) = \frac{1}{3}A_5N$. Clearly, if evader is at the point N and pursuer P_3 is at the point $F(N)$, then they can reach the point A_5 at the same time. Note that if pursuer P_3 of speed $2/3$ is at the point $F(N)$ and evader E of speed 1 is at the point N, then pursuer P_3 can reach the vertex A_5 not later than evader E.

If $A_5M \geq 1$ and $A_1M \geq 1$, that is, if M is on an edge highlighted in red, then $F(M)$ is defined by the equation $F(M) = A_{51}(1/3)$.

Next, we construct strategies for the pursuers P_1, P_2, and P_3 and show that pursuit can be completed in a finite time in the game. We divide this process into two stages. In Stage 1, we bring the pursuers to some points associated with the position of the evader. Then, in Stage 2, we prove that pursuit is completed.

Stage 1. Pursuer P_3 comes to the point A_1 and then moves along A_1A_5 until $P_3(t_1) = F(E(t_1))$ at some time t_1. Starting from the time t_1 to some unspecified time t_2 to be defined later, pursuer P_3 maintains the equation $P_3(t) = F(E(t))$.

Next, to describe a strategy for the pursuer P_2, we define the point $G(Q) \in \bar{A}\bar{A}_3$ associated with each point Q for which $\bar{A}Q \leq 1$ and $Q \notin \bar{A}\bar{A}_3$, that is, with each point on the edges $\bar{A}\bar{A}_1$, $\bar{A}\bar{A}_2$, $\bar{A}\bar{A}_4$, $\bar{A}\bar{A}_5$ (highlighted in cyan) by the equation $\bar{A}G(Q) = \frac{1}{2}\bar{A}Q$. Clearly, the evader of speed 1 at the point Q and pursuer P_2 of speed $1/2$ at the point $G(Q)$ can reach the point $\bar{A}$ at the same time.

We also define the point $G(Q) \in \bar{A}\bar{A}_3$ associated with each point Q for which $\bar{A}_3Q \leq 1$ and $Q \notin \bar{A}\bar{A}_3$, that is, with each point on the edges $\bar{A}_3A_1$, $\bar{A}_3A_5$, $\bar{A}_3\bar{A}_2$, $\bar{A}_3\bar{A}_4$ (highlighted in violet) by the equation $\bar{A}_3G(Q) = \frac{1}{2}\bar{A}_3Q$ (Figure 3). Clearly, the evader of speed 1 is at the point Q and pursuer P_2 of speed $1/2$ at the point $G(Q)$ can reach the point $\bar{A}_3$ at the same time. If $\bar{A}Q > 1$ and $\bar{A}_3Q > 1$, then we define $G(Q)$ as the mid point of the edge $\bar{A}\bar{A}_3$. To indicate that point $G(Q)$ is the point of the edge $\bar{A}\bar{A}_3$, we use the symbol $G_{\bar{A}\bar{A}_3}(Q)$.

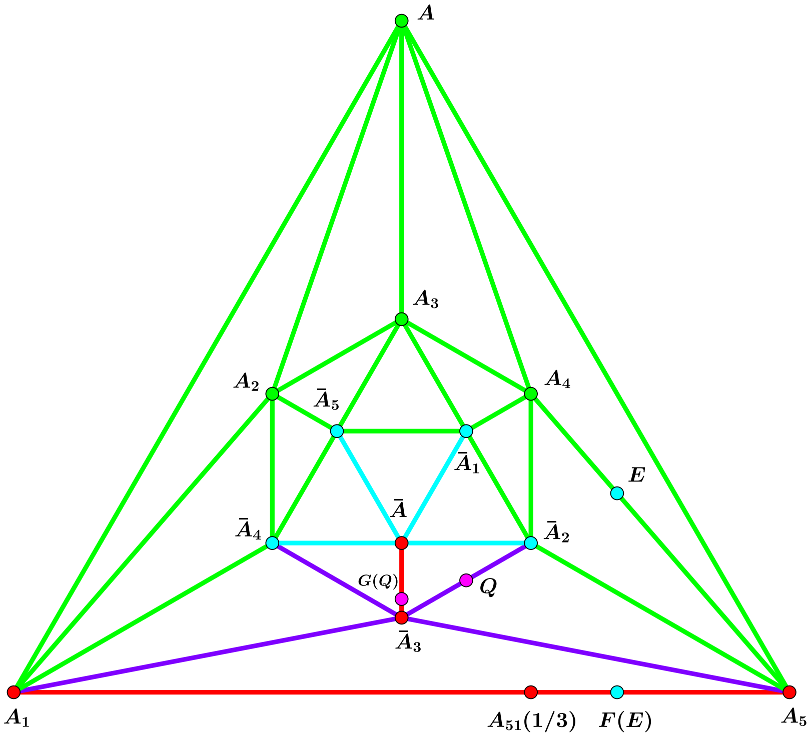

Figure 3. Point $G(Q)$: $\bar{A}G(Q) = \frac{1}{2}\bar{A}Q$ or $\bar{A}_3G(Q) = \frac{1}{2}\bar{A}_3Q$.

We let the pursuer P_2 first come to the point $\bar{A}$ and move toward $\bar{A}_3$ until $P_2(t_1') = G(E(t_1'))$ at some time t_1'. Pursuer P_2 further moves on the point $G(E(t))$ keeping it until the time t_2, which will be defined later.

Next, to describe a strategy for the pursuer P_1, we define point $H(L)$ on the triangle $\bar{A}_1\bar{A}\bar{A}_5$ associated with each point L on the icosahedron K, except for the edge A_1A_5 (Figure 4). First, we define $H(L) \in \bar{A}_1\bar{A}_5$ for $L \notin \bar{A}_1\bar{A}_5$. We let

(1) $H(A_2) = \bar{A}_5$, $H(A_4) = \bar{A}_1$,
(2) $H(L) = \bar{A}_1$ if $L \in \bar{A}_1A_4$, and $H(L) = \bar{A}_5$ if $L \in \bar{A}_5A_2$,
(3) $H(L) = A_{\bar{1}\bar{5}}(1/2)$ if $L \in AA_1 \cup AA_3 \cup AA_5$ (recall $A_{\bar{1}\bar{5}}(1/2)$ is midpoint of $\bar{A}_1\bar{A}_5$). This set is highlighted in yellow in Figure 4.
(4) If $L \in AA_4$ (Figure 4), then $H(L) \in \bar{A}_1\bar{A}_5$ is defined by the equation $\bar{A}_1H(L) = \frac{1}{2}A_4L$. Similarly, if L is on one of the edges A_4A, A_2A, A_2A_3, $\bar{A}_5A_3$, $\bar{A}_1A_3$, A_4A_3 whose vertex is $X \in \{A_2, \bar{A}_5, \bar{A}_1, A_4\}$, then $H(L) \in \bar{A}_1\bar{A}_5$ is defined by the equation $H(X)H(L) = \frac{1}{2}XL$.

In a similar fashion, we define $H(L) \in \bar{A}\bar{A}_1$ and $H(L) \in \bar{A}\bar{A}_5$ if point L is on the edges highlighted in cyan and blue.

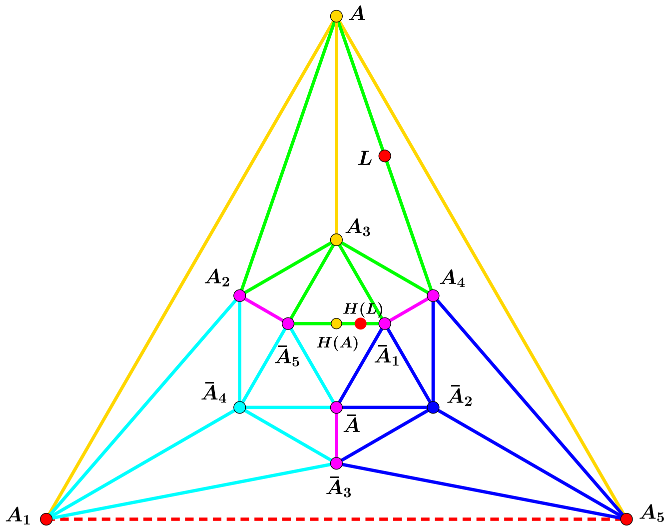

Figure 4. Point $H(L)$: $\bar{A}_1H(L) = \frac{1}{2}A_4L$.

Note that if the pursuer P_1 of speed $1/2$ catches the point $H(E)$ for some position of evader E at some time, then P_1 can further move on the point $H(E)$. Therefore, pursuer P_1 first tries to catch the point $H(E)$ (Figure 5).

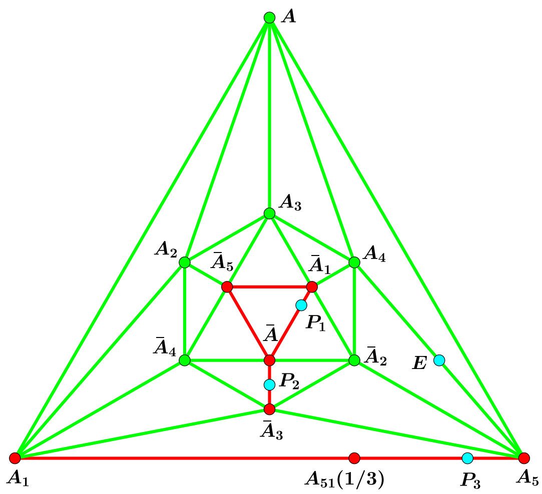

Figure 5. Points $P_1 = H(E)$, $P_2 = G(E)$, $P_3 = F(E)$.

To this end, pursuer P_1 first comes to the vertex $\bar{A}$ and starting from the time $\max\{t_1, t_1'\}$ moves along the sides of the triangle $\bar{A}\bar{A}_5\bar{A}_1$ in the direction $\bar{A} \to \bar{A}_5 \to \bar{A}_1 \to \bar{A}$. Since the speed of the point $H(E)$ does not exceed $1/2$, and the evader cannot pass through the points $A_1, A_5, \bar{A}, \bar{A}_3$, therefore we have $P_1(t_2) = H(E(t_2))$ at some t_2.

Stage 2. Starting from the time t_2, pursuers chase the evader as follows. Pursuer P_1 maintains the equation $P_1(t) = H(E(t))$, $t \geq t_2$. Clearly, if the evader comes to a point of the triangle $\bar{A}\bar{A}_5\bar{A}_1$, then the evader is captured by the pursuer P_1. In other words, pursuer P_1 can guard the triangle $\bar{A}\bar{A}_5\bar{A}_1$.

Pursuer P_3 continues to apply their strategy to guard the edge A_1A_5, that is $P_3(t) = F(E(t))$, $t \geq t_2$.

Pursuer P_2, starting from time t_2, moves to the point $\bar{A}_3$ and comes to the point $\bar{A}_3$ at some time t_2'. If at this time the evader is on one of the edges $\bar{A}_3A_1$ and $\bar{A}_3A_5$, then E is trapped by pursuers P_2 and P_3 and the evader will be captured. If E is not on the edges $\bar{A}_3A_1$ and $\bar{A}_3A_5$, then E is on an edge whose one endpoint is either A or A_2 or A_3 or A_4 or $\bar{A}_2$ or $\bar{A}_4$.

If E is on an edge with an endpoint $\bar{A}_2$ or A_4 at the time t_2', then pursuer P_2 moves along the path $\bar{A}_3 \to \bar{A}_2 \to A_4$ and then moves by the edge A_4A or A_4A_3 where E is, forcing the evader to come to one of the points, A and A_3. Evader E may also be trapped by P_2 and one of the pursuers P_1 and P_3 if E stays on an edge with endpoint at $\bar{A}_2$ or A_4 highlighted in blue (Figure 6) when P_2 reaches the green endpoint of the edge. In the latter case, the evader will be captured. Therefore, we assume that E comes to one of the points A and A_3.

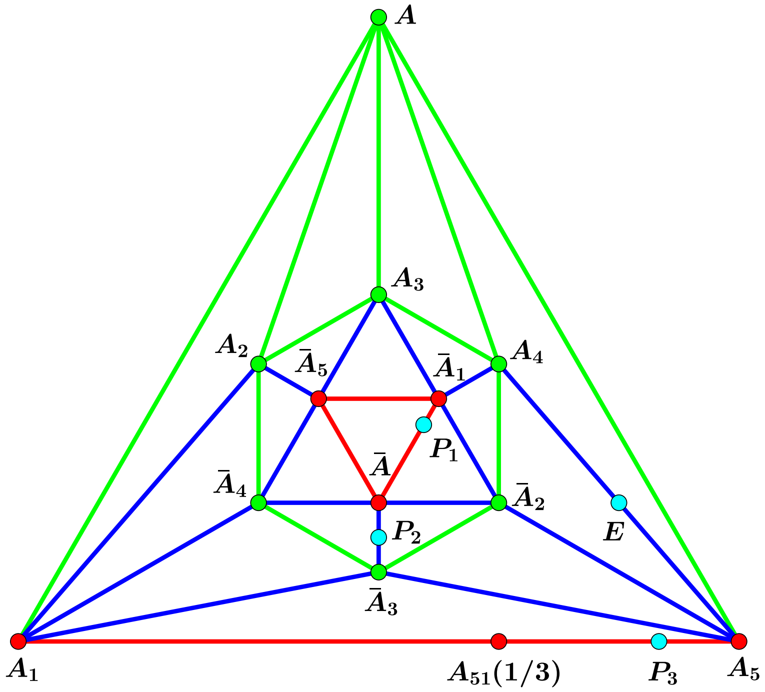

Figure 6. Blue edges joined to hexagon $A_2A_3A_4\bar{A}_2\bar{A}_3\bar{A}_4$.

As E reaches one of the points A and A_3 at some time t_3, pursuer P_2 returns back and moves along the sides of hexagon $A_2A_3A_4\bar{A}_2\bar{A}_3\bar{A}_4$ clockwise and comes to the point A_2.

Recall that when E reaches one of the points A and A_3 at time t_3, by the strategy of pursuer P_3, they will be at the point $A_{51}(1/3)$. Starting from time t_3, pursuer P_3 applies the following strategy. If evader E moves along the path $A \to A_4 \to \bar{A}_2$ or $A_3 \to A_4 \to \bar{A}_2$ toward the vertex $\bar{A}_2$ after time t_3, then pursuer P_3 moves along the path $A_{51}(1/3) \to A_5 \to \bar{A}_2$ maintaining the equation $\frac{2}{3}d(A, E) = d(A_{51}(1/3), P_3)$ or $\frac{2}{3}d(A_3, E) = d(A_{51}(1/3), P_3)$, respectively, where $d(A_3, E)$ denotes the distance of evader E from A_3 along the path

$A_3 \to A_4 \to \bar{A}_2$. If $F(E(t)) \in A_1A_{51}(1/3)$, then $P_3(t) = F(E(t))$. This strategy of pursuer P_3 allows them to guard the edges A_1A_5 and $A_5\bar{A}_2$ from the evader for $t \geq t_3$ (Figure 7).

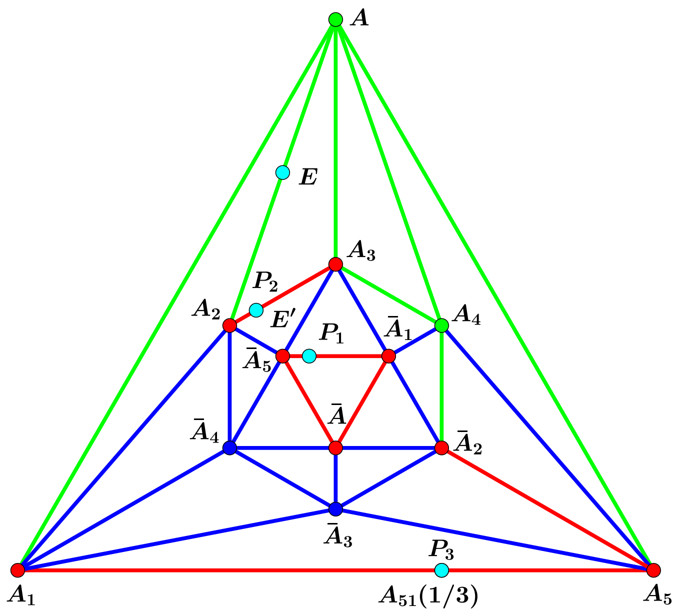

Figure 7. The projection E' of evader $E \in S = AA_1 \cup AA_2 \cup AA_3 \cup AA_4 \cup AA_5$.

If E is on an edge with an endpoint $\bar{A}_4$ or A_2 or A but not on the edge with the endpoint A_4 at time t_2', then pursuer P_2 moves along the path $\bar{A}_3 \to \bar{A}_4 \to A_2$. We assume again that E is not trapped by two pursuers before P_2 reaches A_2.

What is the behavior of pursuer P_2 after reaching point A_2? To describe further the strategy of pursuer P_2, let us define the projection E' of $E \in S = AA_1 \cup AA_2 \cup AA_3 \cup AA_4 \cup AA_5$ on the edge A_2A_3 by the following equations: (a) $A_2E' = \frac{1}{2}A_2E$ if $E \in AA_2$, (b) $A_3E' = \frac{1}{2}A_3E$ if $E \in AA_3$, (c) $E' = A_{23}(1/2)$, that is, E' is the midpoint of A_2A_3, if $E \in AA_1 \cup AA_5$, and (d) $A_3E' = \frac{1}{2}A_4E$ if $E \in AA_4$.

As the pursuer reaches the point A_2, they move along the edge A_2A_3 until pursuer P_2 coincides with the projection E' of $E \in S$ on the edge A_2A_3 at some time t_4. When pursuer P_2 coincides with E', they further move, maintaining the equation $P_2(t) = E'(t)$ provided $E \in S$. This is possible since the speed of E' does not exceed $1/2$.

This strategy of the pursuer P_2 does not allow the evader to pass through the points A_2 and A_3 for $t \geq t_4$, that is, P_2 guards the edge A_2A_3 from the evader. If the evader reaches one of these points, it will be captured by P_2.

The evader can leave the set S only from the point A_4 at some time since the points A_1, A_5 are guarded by P_3, and A_2, A_3 are guarded by P_2. When the evader comes to the point A_4, according to the strategy above, pursuer P_2 will be at point A_3 at that time. We let the pursuer P_2 further stay at A_3 until E enters S through point A_4. As E enters S through the point A_4 we let again $P_2(t) = E'(t)$.

Starting from the time $T = \max\{t_3, t_4\}$ pursuer P_1 moves toward point $\bar{A}_1$. As P_1 reaches the point $\bar{A}_1$, P_1 moves toward the point A_4 along the edge $\bar{A}_1A_4$. When P_1 reaches point A_4, the evader E is either on the edge $A_4\bar{A}_2$, on A_4A_5 or on A_4A_3, or on an edge with the vertex A.

If $E \in A_4\bar{A}_2$ or $E \in A_4A_5$, then E is trapped by pursuers P_1 and P_3 and soon will be captured. If $E \in A_4A_3$, then E is trapped by pursuers P_1 and P_2 and will be captured. If the evader E is on an edge with the vertex A, pursuer P_1 moves along the edge A_4A then toward the evader, and the evader will be captured. The proof of the theorem is complete. $\square$

3.2. The Case of More than Three Pursuers

In this subsection, we consider a differential game of m, $m \geq 4$ pursuers and one evader. For the case of four pursuers P_1, P_2, P_3, P_4 ($m = 4$) and one evader on the icosahedron K, we prove the following statement.

Theorem 2. *If the maximal speeds of two pursuers belong to* $[1/2, 1)$ *and the maximal speeds of the other two pursuers are positive, then the pursuit can be completed in the game on icosahedron K.*

Proof. Without loss of generality, we assume that $1/2 \leq \rho_1 < 1$, $1/2 \leq \rho_2 < 1$, and $\rho_3, \rho_4 > 0$. It is sufficient to show that the pursuit can be completed when $\rho_1 = \rho_2 = 1/2$ and $\rho_3, \rho_4 > 0$.

We construct strategies for the pursuers P_1, P_2, P_3, and P_4 and show that pursuit can be completed in a finite time in the game. We divide this process into two stages. In Stage 1, we bring the pursuers to some specific points. Then, in the Stage 2, we prove that pursuit is completed.

Stage 1. We use the definition of the point $G(Q)$ defined in the proof of Theorem 1. We let the pursuer P_1 come to the vertex A_2, then move along A_2A_3 toward A_3 until the state of $P_1(t)$ coincides with the point $G_{A_2A_3}(E(t))$, and further move on this point keeping it. Then, clearly, if the evader reaches one of the points A_2 or A_3 at some time, it is captured at that time (Figure 8). Note that when the pursuer P_1 moves along the edge A_2A_3, they can collide with the evader E as well. Then, clearly, the evader is captured. Therefore, we exclude such a collision.

We let the pursuer P_2 come to the vertex A_1, then move along $A_1\bar{A}_3$ toward the point $\bar{A}_3$ until the state of $P_2(t)$ coincides with the point $G_{A_1\bar{A}_3}(E(t))$, and further move on this point, keeping it. Let the pursuers P_3 and P_4 come to the points $\bar{A}_2$ and A_4, respectively, and stay there until the first time $t = t_1$ when all the equations $P_1(t) = G_{A_2A_3}(E(t))$, $P_2(t) = G_{A_1\bar{A}_3}(E(t))$, $P_3(t) = \bar{A}_2$, and $P_4(t) = A_4$ are satisfied.

Stage 2. We show that pursuit can be completed. Indeed, if the evader is on one of the edges A_1A_2, A_3A_4, $A_4\bar{A}_2$, or $\bar{A}_2\bar{A}$ (highlighted in pink) at time t_1, then the evader is trapped by two pursuers, and so the evader will be captured. For example, if $E(t_1) \in A_1A_2$, then starting from time t_1, the pursuer P_2 of speed ρ_2 comes to the point A_1 and then moves toward the point A_2 along A_1A_2 to catch the evader.

If at time t_1 the evader is on an edge with an endpoint at A or A_5 (highlighted in blue), then we let the pursuer P_3 of speed ρ_3 come to the point A_5 at some time t_2, (that is, $P_3(t_2) = A_5$) and pursuers P_1 and P_2 further control the edges A_2A_3 and $A_1\bar{A}_3$, respectively, and $P_4(t) = A_4$. If the evader is on one of the edges A_5A_1, $A_5\bar{A}_3$, $A_5\bar{A}_2$, A_5A_4 at the time t_2, then the evader is trapped by two pursuers, and so the evader will be captured.

If the evader is not on any of these edges, then P_3 of speed ρ_3 starting from time t_2 moves toward point A, and we let $P_3(t_3) = A$ at some time t_3. Then at time t_3, the evader is trapped by two pursuers, and so the evader will be captured.

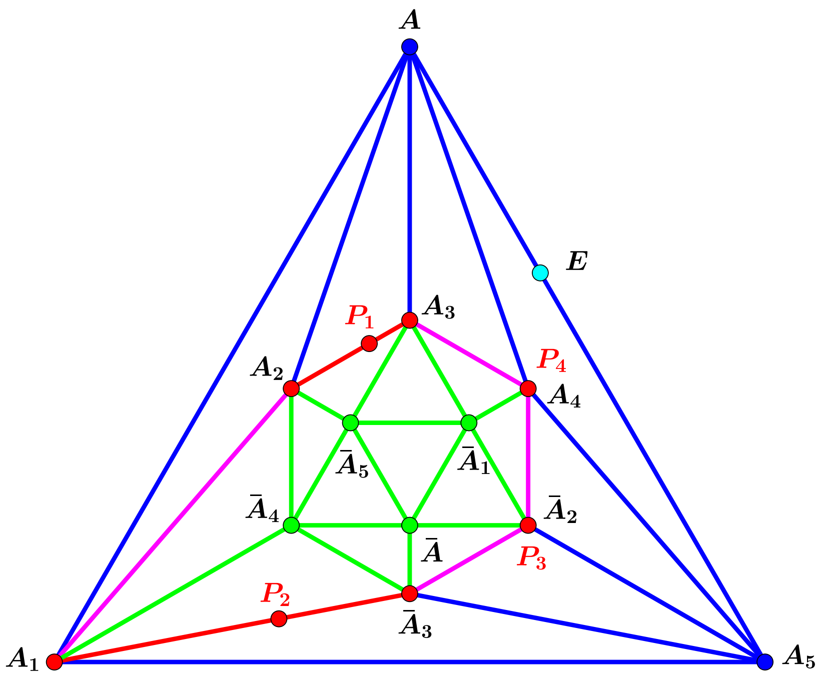

Figure 8. Location of the pursuers.

If the evader is on an edge with the vertex $\bar{A}$ or $\bar{A}_1$ or $\bar{A}_4$ or $\bar{A}_5$ (highlighted in green) at the time t_1, then we let the pursuer P_4 of speed ρ_4 come to the vertex $\bar{A}_1$ at some time t_2'. The evader is trapped by two pursuers at time t_2' if it is on the edge $\bar{A}_1\bar{A}_2$ or $\bar{A}_1 A_3$. If the evader is not on these edges, we let the pursuer P_3 of speed ρ_3 come to the vertex $\bar{A}$ at some time t_3'. If the evader is not trapped by two pursuers at the time t_3', then pursuer P_4 moves along the path $\bar{A}_1 \to \bar{A}_5 \to \bar{A}_4$, and P_3 stays at the vertex $\bar{A}$ for $t \geq t_3'$. Then, clearly, the evader is trapped by two pursuers when pursuer P_4 reaches $\bar{A}_5$ or $\bar{A}_4$, and so it will be captured. The proof of the theorem is complete. □

Next, for the case of five pursuers P_1, P_2, P_3, P_4, P_5 ($m = 5$) and one evader on the icosahedron K, the following statement is true.

Theorem 3. *If the maximal speed of one pursuer belong to* $[1/2, 1)$ *and the maximal speeds of the other three pursuers are positive, then the pursuit can be completed in the game on icosahedron* K.

The proof of this theorem is very similar to the proof of the previous theorem. Therefore, we just give the main idea of the proof. Let $\rho_1 = 1/2$ and $\rho_2, \rho_3, \rho_4, \rho_5 > 0$. First, the pursuer P_5 catches the point $G_{A_2 A_3}(E(t))$ and further moves on this point, and pursuers P_1, P_2, P_3, P_4 come to the points A_1, $\bar{A}_3$, $\bar{A}_2$, A_4, respectively. Then, we let pursuer P_3 move along the path $A_2 \to A_5 \to A$ until the evader is trapped by two pursuers if the evader is on an edge with an endpoint at A or A_5. If the evader is on an edge with an endpoint at $\bar{A}$ or $\bar{A}_1$ or $\bar{A}_4$ or $\bar{A}_5$, then P_4 moves from A_4 to $\bar{A}_1$, then P_3 moves from $\bar{A}_2$ to $\bar{A}$, and then P_4 moves along the path $\bar{A}_1 \to \bar{A}_5 \to \bar{A}_4$ until the evader is trapped by two pursuers. For the case of six pursuers $P_1, P_2, P_3, P_4, P_5, P_6$ ($m = 6$) and one evader on the icosahedron K, the following statement is true.

Theorem 4. *If the maximal speeds of pursuers are positive, then pursuit can be completed in the game on icosahedron* K.

We let the pursuers $P_1, P_2, P_3, P_4, P_5, P_6$ move to the points A_1, $\bar{A}_3$, $\bar{A}_2$, A_4, A_3, and A_2, respectively. The rest of the reasoning is as above.

4. Conclusions and Discussion

We studied a differential game of three slow pursuers and one evader of speed 1 on 1-skeleton of an icosahedron. Previous research [20] shows that three pursuers of speeds $\rho_1 = 1$, $\rho_2 > 0$, $\rho_3 > 0$ can catch one evader of speed 1, but two pursuers of speed 1 cannot catch one evader of speed 1 on icosahedron K. Therefore, in the present paper, we studied the case of slow pursuers. We obtained a sufficient condition for speeds of pursuers to complete the game when the number of pursuers is $m = 3, 4, 5, 6$. We also constructed strategies for the pursuers to complete the game. According to Theorem 4, six pursuers with positive speeds can catch one evader of speed 1.

If the speeds of the pursuers are not in the intervals considered in Theorems 1–4, then differential game has not been studied yet. For the further investigation, we give some open problems for the differential game on the graph edge of an icosahedron. Can three pursuers catch one evader of speed 1 if the speeds of the pursuers are (1) $\rho_1 < 1/2$, $\rho_2 = 1/2$, $\rho_3 = 2/3$, and (2) $\rho_1 = 1/2$, $\rho_2 = 1/2$, $\rho_3 < 2/3$?

Additionally, the case where there are obstacles along the edges is very interesting. In particular, such an obstacle can be considered a pursuer with the speed equal to 0. Such a pursuer cannot move, but if the state of the evader coincides with that of this pursuer, the pursuit is completed.

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References

1. Bakolas, E.; Tsiotras, P. Relay Pursuit of a Maneuvering Target Using Dynamic Voronoi Diagrams. *Automatica* **2012**, *48*, 2213–2220. [CrossRef]
2. Blagodatskikh, A.I.; Petrov, N.N. Simultaneous Multiple Capture of Rigidly Coordinated Evaders. *Dyn. Games Appl.* **2019**, *9*, 594–613. [CrossRef]
3. Blagodatskikh, A.I. Multiple capture of rigidly coordinated evaders. *Bull. Udmurt. Univ. Math. Mech. Comput. Sci.* **2016**, *26*, 46–57.
4. Borowko, P.; Rzymowski, W.; Stachura, A. Evasion from many pursuers in the simple motion case. *J. Math. Anal. Appl.* **1988**, *135*, 75–80. [CrossRef]
5. Chernous'ko, F.L.; Zak, V.L. On differential games of evasion from many pursuers. *J. Optim. Theory Appl.* **1985**, *46*, 461–470. [CrossRef]
6. Chikrii, A.A.; Prokopovich, P.V. Simple pursuit of one evader by a group. *Cybern. Syst. Anal.* **1992**, *28*, 438–444.. [CrossRef]
7. Chodun W. Differential games of evasion with many pursuers. *J. Math. Anal. Appl.* **1989**, *142*, 370–389. [CrossRef]
8. Ramana, M.V.; Mangal, K. Pursuit strategy to capture high-speed evaders using multiple pursuers. *J. Guid. Control Dyn.* **2017**, *40*, 139–149. [CrossRef]
9. Scott, W.L.; Leonard, N.E. Optimal evasive strategies for multiple interacting agents with motion constraints. *Autom. J. IFAC* **2018**, *94*, 26–34. [CrossRef]
10. Shaunak, B.; Subhash, S. k-capture in multiagent pursuit evasion, or the lion and the hyenas. *Theor. Comput. Sci.* **2014**, *522*, 13–23
11. Sun, W.; Tsiotras, P.; Yezzi, A.J. Multiplayer Pursuit-Evasion Games in Three-Dimensional Flow Fields. *Dyn. Games Appl.* **2019**, *9*, 1188–1207. [CrossRef]
12. Kumkov, S.S.; Stéphane, L.M.; Patsko, V.P. Zero-sum pursuit-evasion differential games with many objects: Survey of publications. *Dyn. Games Appl.* **2017**, *7*, 609–633. [CrossRef]

13. Alias, I.A.; Ibragimov, G.I.; Rakhmanov, A.T. Evasion Differential Game of Infinitely Many Evaders from Infinitely Many Pursuers in Hilbert Space. *Dyn. Games Appl.* **2016**, *6*, 1–13. [CrossRef]
14. Ibragimov, G.I.; Ferrara, M.; Kuchkarov, A.S.; Bruno, A.P. Simple motion evasion differential game of many pursuers and evaders with integral constraints. *Dyn. Games Appl.* **2018**, *8*, 352–378. [CrossRef]
15. Ibragimov, G.I.; Salleh, Y. Simple motion evasion differential game of many pursuers and one evader with integral constraints on control functions of players. *J. Appl. Math.* **2012**, *8*, 748096. [CrossRef]
16. Alexander, S.; Bishop, R.; Christ, R. Capture pursuit games on unbounded domain. *LÈnseignement MathÈmatique* **2009**, *55*, 103–125. [CrossRef]
17. Kuchkarov, A.S.; Risman, M.H.; Malik, A.H. Differential games with many pursuers when evader moves on the surface of a cylinder. *ANZIAM J.* **2012**, *53*, E1–E20. [CrossRef]
18. Andreae, T.; Hartenstein, F.; Wolter, A. A two-person game on graphs where each player tries to encircle his opponent's men. *Theoret. Comput. Sci.* **1999**, *215*, 305–323. [CrossRef]
19. Azamov, A.A.; Kuchkarov, A.S.; Holboyev, A.G. The pursuit-evasion game on the 1-skeleton graph of the regular polyhedron. III. *Mat. Teor. Igr Pril.* **2019**, *11*, 5–23
20. Azamov, A.A.; Kuchkarov, A.S.; Holboyev, A.G. Pursuit-evasion game on edge graphs of regular polyhedrons in presence of slow pursuers. *Uzb. Math. J.* **2017**, *1*, 140–145.
21. Azamov, A.A.; Kuchkarov, A.S.; Holboyev, A.G. The pursuit-evasion game on the 326 1-skeleton graph of the regular polyhedron. II. *Mat. Teor. Igr Pril.* **2016**, *8*, 3–13 327. Translated in *Automation and Remote Control* **2019**, *80*, 164–170. (In Russian)
22. Azamov, A.A.; Kuchkarov, ASh, Holboyev, A.G. The pursuit-evasion game on the 1-skeleton graph of the regular polyhedron. I. *Mat. Teor. Igr Pril.* **2015**, *7*, 3–15. Translated in *Automation and Remote Control* **2017**, *78*, 754–761. (In Russian) [CrossRef]
23. Azamov, A.A.; Ibaydullaev, T.; Ibragimov, G.I.; Alias, I.A. Optimal number of pursuers in the differential games on the 1-skeleton of orthoplex. *Symmetry* **2021**, *13*, 2170. [CrossRef]
24. Azamov, A.A.; Samatov, B.T. The Π-strategy: Analogies and applications. In *Contribution to Game Theory and Management*; St-Petersburg University: Saint Petersburg, Russia, 2011; Volume IV, pp. 33–46.
25. Bulgakova, M.A.; Petrosyan, L.A. Multistage games with pairwise interactions on full 345 graph. *Mat. Teor. Igr Pril.* **2019**, *11*, 3–20. Translated in *Automation and Remote Control* **2020**, *81*, 1519–1530. (In Russian)
26. Bonato, A.; Golovach, P.; Hahn, G.; Kratochvil, J. The capture time of a graph. *Discret. Math.* **2009**, *309*, 5588–5595. [CrossRef]
27. Bonato, A.; Nowakowski, R.J. The game of cops and robbers on graphs. In *Student Mathematical Library*; American Mathematical Society: Providence, RI, USA, 2011; Volume 61, p. 276.
28. Gavenčiak, T. Cop-win graphs with maximum capture-time. *Discret. Math.* **2010**, *310*, 1557–1563. [CrossRef]
29. Nowakowski, R.J. Unsolved problems in combinatorial games. In *Games of No Chance*; Cambridge University Press: Cambridge, MA, USA, 2019; Volume 5, pp. 125–168.
30. Petrosyan, L.A.; Sedakov, A.A. Multi-step network game with full information. *Math. Theory Games Its Appl.* **2009**, *1*, 66–81. Translated in *Automation and Remote Control* **2014**, *75*, 1532–1540. (In Russian)
31. Azamov A.A. Lower bound for the advantage coefficient in the graph search problem. *Differ. Equations* **2008**, *44*, 1764–1767. [CrossRef]