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On $\mathbb{BID}$-Cone b-Metric Spaces over Banach Algebras: New Topological Properties and Fixed Point Theorems

Huaping Huang [1,†], Wei-Shih Du [2,*,†] and Jen-Yuan Chen [2,†]

[1] School of Mathematics and Statistics, Chongqing Three Gorges University, Wanzhou, Chongqing 404100, China; huaping@sanxiau.edu.cn
[2] Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan; jchen@nknu.edu.tw
* Correspondence: wsdu@mail.nknu.edu.tw
† These authors contributed equally to this work.

Abstract: In this paper, we introduce the concepts of an inferior idempotent cone and a $\mathbb{BID}$-cone b-metric space over Banach algebra. We establish some new existence theorems and fixed point theorems in the setting of complete $\mathbb{BID}$-cone b-metric spaces over Banach algebra. Some fundamental questions and examples are also given.

Keywords: $\mathbb{BID}$-cone b-metric space over Banach algebra; inferior idempotent cone; c-sequence; θ-sequence; $\mathbb{BID}$-cone b-Cauchy sequence; fixed point theorem

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1. Introduction and Preliminaries

In recent decades, the problem of nonlinear analysis with its relation to fixed point theory has emerged as a rapidly growing area of research based on its applications in game theory, optimization problem, control theory, integral and differential equations and inclusions, dynamic systems theory, signal and image processing, and so on. Fixed point theory is a beautiful mixture of analysis, topology and geometry. It has been revealed as a very powerful and important tool in the study of nonlinear phenomena. Since Polish mathematician Banach invented the Banach contractive mapping principle (for short, BCMP) in 1922 (see [1]), many scholars have attempted to generalize BCMP to different spaces. One of the prominent generalizations is to establish fixed-point results from metric spaces to generalized metric spaces such as b-metric space [2,3], modular metric space [4], cone metric space [5], fuzzy metric space [6], partial metric space [7], etc. It is worth mentioning that one of interesting properties of generalized metric spaces with regard to fixed point problems is topological properties, which have become an indispensable object of study. For distinct spaces, there have been all kinds of topological properties. In 2007, Huang and Zhang [5] defined cone metric space by substituting a normed space instead of the real line, but went further, defining convergent and Cauchy sequences in the terms of interior points of the underlying cone. They also investigated topological properties on cones. Afterwards, some topological properties of cone metric spaces developed into one of the centers of strong research activities (see [8–18]). In 2014, Xu and Radenović [19] considered topological properties on cones and algebraic cones in the setting of cone metric spaces over Banach algebras introduced by Liu and Xu [20]. In 2019, Huang et al. [21] presented some new topological properties of cone metric spaces over Banach algebras, such as the property of c-sequence, the P-property, the T-stability of Picard's iteration, the well-posedness of fixed point problems, etc. In the same year, Huang [22] gave the topological properties of E-metric spaces with applications to fixed point theory. On the other hand, in 2015, Huang and Radenović [23] introduced the notion of cone b-metric spaces over Banach algebra, which greatly generalizes b-metric spaces and cone metric spaces over Banach algebra. However, concerning the systemic discussion about the crucial topological properties of cone b-metric spaces over Banach algebras, as far as we know, it has been vacant so far.

The paper is divided into five sections. In Section 2, we introduce the concept of inferior idempotent cone and give some fundamental questions and examples. In Section 3, the concept of $\mathbb{BID}$-cone b-metric space over Banach algebra with inferior idempotent cones, $\mathbb{BID}$-cone b-convergent sequence and $\mathbb{BID}$-cone b-Cauchy sequence are introduced. Finally, in Section 4, we establish some new auxiliary theorems and fixed point theorems in the setting of complete $\mathbb{BID}$-cone b-metric spaces over Banach algebra. Our new results are original and quite different from the well-known generalizations on the topic in the literature.

2. Inferior Idempotent Cones and Fundamental Questions

A topological vector space (t.v.s. for short) is a vector space with a topology such that the vector space operations (addition and scalar multiplication) are continuous. A topological vector space is locally convex if its origin has a basis of neighborhoods that are convex. Let V be a t.v.s. with its zero vector θ_V. A nonempty subset C of V is said to be (i) *proper* if $C \neq \emptyset$; (ii) a *cone* if $\lambda C \subseteq C$ for $\lambda \geq 0$; (iii) a *convex cone* if $C + C \subseteq C$ and $\lambda C \subseteq C$ for $\lambda \geq 0$. A cone C in V is said to be *pointed* if $C \cap (-C) = \{\theta_V\}$. For a given proper, pointed and convex cone C in V, we can define a partial ordering $\precsim$ (or $\precsim_C$) with respect to C by

$$x \precsim y \Longleftrightarrow y - x \in C.$$

$x \prec y$ will stand for $x \precsim y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intC$, where $intC$ denotes the interior of C. As usual, we allow the use of these symbols: (i) $x \precsim y \Longleftrightarrow y \succsim x$, (ii) $x \prec y \Longleftrightarrow y \succ x$, and (iii) $x \ll y \Longleftrightarrow y \gg x$.

Remark 1. *If C is a convex cone in V, then $\theta_V \in C$.*

Lemma 1. *Let V be a t.v.s. with its zero vector θ_V and C be a nonempty subset of V with $intC \neq \emptyset$. Then the following statements hold.*

(a) If C is a cone, then $\lambda intC = intC$ for all $\lambda > 0$;
(b) If C is a convex cone, then $intC + C = intC + intC = intC$.

Proof. First, we verify conclusion (a). Assume that C is a cone. Then

$$\lambda intC = int(\lambda intC) \subseteq int(\lambda C) \subseteq intC \quad \text{for any } \lambda > 0. \tag{1}$$

Let $\lambda > 0$ be given. Therefore, $\lambda^{-1} > 0$. By using (1), we obtain

$$intC = \lambda\left(\lambda^{-1} intC\right) \subseteq \lambda intC. \tag{2}$$

Combining (1) with (2), we show that $\lambda intC = intC$. To see (b), we first claim

$$intC + C = intC. \tag{3}$$

Since C is a convex cone, we get

$$intC + C = int(intC + C) \subseteq int(C + C) \subseteq intC.$$

Due to $\theta_V \in C$, we obtain

$$intC = \{\theta_V\} + intC \subseteq C + intC.$$

Hence our claim (3) is proved. Next, we prove that $intC + intC = intC$. By (3), we obtain

$$intC + intC \subseteq intC + C \subseteq intC. \tag{4}$$

On the other hand, by (a), we get

$$intC \subseteq \frac{1}{2}intC + \frac{1}{2}intC = intC + intC. \tag{5}$$

Taking into account (4) and (5), we show that $intC + intC = intC$. The proof is completed. □

Lemma 2. *Let V be a t.v.s. with its zero vector θ_V, C be a pointed convex cone with $intC \neq \emptyset$ in V and $x, y, z, w \in V$. Then the following statements hold.*

(i) If $x \precsim y \precsim z$ (that is, $x \precsim y$ and $y \precsim z$), then $x \precsim z$;
(ii) If $x \precsim y \ll z$ (that is, $x \precsim y$ and $y \ll z$), then $x \ll z$;
(iii) If $x \ll y \precsim z$ (that is, $x \ll y$ and $y \precsim z$), then $x \ll z$;
(iv) If $x \ll y \ll z$ (that is, $x \ll y$ and $y \ll z$), then $x \ll z$;
(v) If $x \precsim w$ and $y \precsim w$, then $x + y \precsim 2w$;
(vi) If $x \precsim w$ and $y \ll w$, then $x + y \ll 2w$;
(vii) If $x \ll w$ and $y \precsim w$, then $x + y \ll 2w$;
(viii) If $x \ll w$ and $y \ll w$, then $x + y \ll 2w$.

Proof. It is obvious that conclusion (i) holds. We only verify conclusion (ii) and a similar argument could be made for conclusions (iii)–(viii). Since $x \precsim y \iff y - x \in C$ and $y \ll z \iff z - y \in intC$, by (b) of Lemma 1, we get

$$z - x = (z - y) + (y - x) \in intC + C \subseteq intC,$$

which means that $x \ll z$. □

Lemma 3. *Let V be a t.v.s. with its zero vector θ_V and C be a pointed and closed cone with $intC \neq \emptyset$ in V. If $\theta_V \precsim u \ll \epsilon$ for each $\epsilon \gg \theta_V$, then $u = \theta_V$.*

Proof. For each $\epsilon \gg \theta_V$, that is $\epsilon \in intC$, applying Lemma 1(a), we obtain $\frac{\epsilon}{n} \in intC$ for any $n \in \mathbb{N}$. By our hypothesis, we have $\theta_V \precsim u \ll \frac{\epsilon}{n}$ for any $n \in \mathbb{N}$. Thus $\frac{\epsilon}{n} - u \in intC$, which leads to $\frac{\epsilon}{n} - u \in C$ for any $n \in \mathbb{N}$. Taking limit as $n \to \infty$ from this expression and noting that C is a closed subset of V, we get $-u \in C$. Therefore,

$$u \in C \cap (-C) = \{\theta_V\},$$

and hence $u = \theta_V$. □

Recall that $\mathbb{A}$ is called a *Banach algebra* if $\mathbb{A}$ is a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and there is an associative and distributive multiplication $* : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ satisfying

$$\|x * y\| \leq \|x\|\|y\| \quad \text{for all } x, y \in \mathbb{A}$$

and

$$c(x * y) = (cx) * y = x * (cy)$$

for all $x, y \in \mathbb{A}$ and $c \in \mathbb{K}$. A point e is called the (*multiplicative*) *unit element* of $\mathbb{A}$ if $e * x = x * e = x$ for all $x \in \mathbb{A}$. Without loss of generality, we may assume that $\|e\| = 1$ (see, e.g., [24]).

In this paper, we introduce the concept of an inferior idempotent cone.

Definition 1. *Let $\mathbb{A}$ be a real Banach algebra with its zero element θ and e be the unit element of $\mathbb{A}$. A nonempty subset K of $\mathbb{A}$ is called an inferior idempotent cone if it is a convex cone satisfying*

(i) $e \in K$;
(ii) $K^2 (:= KK) \subseteq K$.

Example 1. *Let $\mathbb{A} = \mathbb{R}^2$ with a norm defined by*

$$\|\mathbf{x}\| = \max\{|x_1|, |x_2|\} \quad \text{for } \mathbf{x} = (x_1, x_2) \in \mathbb{A}.$$

Define an associative and distributive multiplication $* : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ by

$$\mathbf{x} * \mathbf{y} = (x_1 \cdot y_1, x_2 \cdot y_2) \quad \text{for } \mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{A}.$$

Then $\mathbb{A}$ is a real Banach algebra and $e := (1, 1)$ is its unit element. Let

$$K = \Big\{ (x, y) \in \mathbb{R}^2 : x, y \geq 0 \Big\}$$

and

$$H = \Big\{ (x, y) \in \mathbb{R}^2 : y \geq 0 \Big\}.$$

Then K and H are inferior idempotent cones. Note that K is pointed, but H is not pointed.

Example 2. *Let*

$$\mathbb{A} = \big\{ a = (a_{ij})_{3\times 3} : a_{ij} \in \mathbb{R},\ 1 \leq i, j \leq 3 \big\}.$$

Define

$$\|a\| = \frac{1}{3} \sum_{1 \leq i,j \leq 3} |a_{ij}| \quad \text{for } a \in \mathbb{A}.$$

Then $\mathbb{A}$ is a Banach algebra with its zero element $\theta := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *is the unit element of $\mathbb{A}$. Let*

$$K = \big\{ a \in \mathbb{A} : a_{ij} \geq 0,\ 1 \leq i, j \leq 3 \big\}.$$

Hence, K is an inferior idempotent cone in $\mathbb{A}$.

Remark 2. *In fact, the concept of an inferior idempotent cone is different from the usual convex cone. For example, let $\mathbb{A} = C([0, 2])$ be the set of all continuous functions on $[0, 2]$ with the usual multiplication and norm $\|\cdot\|$. Clearly, $\mathbb{A}$ is a Banach algebra with its unit element 1. If we set*

$$K = \{ x \in \mathbb{A} : x(t) \geq 0, t \in [0, 2] \},$$

then K is either a convex cone or an inferior idempotent cone. If we put

$$C = \left\{ x \in K : \int_0^1 x(t)\,\mathrm{d}t \geq \frac{1}{16}\|x\| \right\},$$

then C is a convex cone, but not an inferior idempotent cone. As a matter of fact, if we choose $x(t) = t$, $y(t) = t^2$, then $x(t), y(t) \in C$, but $x(t)y(t) \notin C$, then (ii) of our Definition 1 is not satisfied. Hence, the concept of an inferior idempotent cone is quite an interesting subject for consideration.

Here, we give some fundamental questions for Banach algebras equipped with distinct inferior idempotent cones.

Question A. Let $\mathbb{A}$ be a Banach algebra with its zero element θ and K be an inferior idempotent cone in $\mathbb{A}$. Suppose that $a, u \in K$ and $a \neq \theta$. Can either $au = \theta$ or $ua = \theta$ so that $u = \theta$?

Solution: The answer is no. In fact, $u = \theta$ does not necessarily hold if $au = \theta$ or $ua = \theta$. For example, let

$$\mathbb{A} = \{a = (a_{ij})_{2\times 2} : a_{ij} \in \mathbb{R},\ 1 \le i, j \le 2\}.$$

Define

$$\|a\| = \frac{1}{2} \sum_{1 \le i,j \le 2} |a_{ij}| \ \text{ for } a \in \mathbb{A}.$$

Then $\mathbb{A}$ is a Banach algebra with its zero element $\theta := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Let

$$K = \{a \in \mathbb{A} : a_{ij} \ge 0,\ 1 \le i, j \le 2\}.$$

Thus, K is an inferior idempotent cone in $\mathbb{A}$. Put

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } u = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that $a, u \in K$, $a \neq \theta$ and $au = \theta$, but $u \neq \theta$.

Question B. Let $\mathbb{A}$ be a Banach algebra with its zero element θ and K be an inferior idempotent cone in $\mathbb{A}$.

(a) Is there an invertible element for any point in $intK$?
(b) Is $K(intK) \subseteq intK$ true?

Solution:

(a) The answer is no. For example, let $\mathbb{A}$, $\|\cdot\|$ and K be defined as in Example 2. Take

$$a := \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Then $a \in intK := \{a \in \mathbb{A} : a_{ij} > 0,\ 1 \le i, j \le 3\}$. However, a is not invertible because of $\det(a) = 0$;

(b) The answer is no. For example, let $\mathbb{A}$, $\|\cdot\|$ and K be defined as in Example 2. Put

$$u = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{4} & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 1 \\ \frac{1}{4} & \frac{1}{2} & 2 \\ 1 & 1 & \frac{1}{2} \end{pmatrix}.$$

Then $u \in K$ and $v \in intK$. However, we have

$$uv = \begin{pmatrix} 0 & 0 & 0 \\ \frac{5}{4} & \frac{21}{16} & \frac{7}{4} \\ \frac{21}{16} & \frac{5}{4} & \frac{3}{2} \end{pmatrix} \notin intK$$

which implies $K(intK) \nsubseteq intK$.

3. $\mathbb{BID}$-Cone *b*-Metric Spaces over Banach Algebra and Basic Topological Properties

Let $(E, \|\cdot\|)$ be a real Banach space with its zero vector θ and $\{u_n\}$ be a sequence in E. Recall that $\{u_n\}$ is called

(i) A *θ-sequence* if $\lim\limits_{n\to\infty} u_n = \theta$;

(ii) A *c-sequence* if for any $c \gg \theta$, there exists a natural number N such that for all $n \geq N$, one has $u_n \ll c$.

The following result is important for our proofs. Although its proof is similar to the proof of ([21], Proposition 2.2), we give it here for the sake of completeness and the reader's convenience.

Lemma 4. *Let $(E, \|\cdot\|)$ be a real Banach space and C be a convex cone in E. If $\{u_n\}$ is a θ-sequence in E, then $\{u_n\}$ is a c-sequence.*

Proof. Let $c \gg \theta$ be given. Since $c \in intC$, there exists $\delta_1 > 0$ such that

$$U(c, \delta_1) := \{x \in E : \|x - c\| < \delta_1\} \subseteq C.$$

Since $u_n \to \theta$ as $n \to \infty$, for any $\delta_2 \in (0, \delta_1)$, there exists $N \in \mathbb{N}$ such that $\|u_n\| < \delta_2$ for all $n \geq N$. Hence, we get

$$\|(c - u_n) - c\| = \|u_n\| < \delta_2 < \delta_1,$$

which deduces $c - u_n \in U(c, \delta_1) \subseteq C$ and hence

$$c - u_n \in intC \iff u_n \ll c.$$

Therefore, $\{u_n\}$ is a c-sequence in E. □

In 2013, Du and Karapinar [10] introduced the concept of TVS-cone b-metric space as follows.

Definition 2 ([10])**.** *Let Y be a locally convex Hausdorff t.v.s. with its zero vector θ_Y, C be a proper, closed, convex and pointed cone in Y with $intC \neq \emptyset$ and $\precsim$ be a partial ordering with respect to C. Let X be a nonempty set and $s \geq 1$ be a given real number. A vector-valued function $p : X \times X \to Y$ is said to be TVS-cone b-metric if the following conditions are satisfied:*

(i) $\theta_Y \precsim p(x, y)$ *for all* $x, y, \in X$ *and* $p(x, y) = \theta_Y$ *if and only if* $x = y$;
(ii) $p(x, y) = p(y, x)$ *for all* $x, y \in X$;
(iii) $p(x, z) \precsim s[p(x, y) + p(y, z)]$ *for all* $x, y, z \in X$.

The pair (X, p) is called a TVS-cone b-metric space.

If we take $Y = \mathbb{R}$ and $C = [0, \infty)$ in Definition 2, then we get the notion of b-metric spaces in the sense of Bakhtin [3] and Czerwik [2].

In 2015, Huang and Radenović [23] introduced the concept of cone b-metric space over Banach algebra as follows.

Definition 3 ([23])**.** *Let $\mathbb{A}$ be a real Banach algebra with its zero vector θ, C be a geometric cone in $\mathbb{A}$ with $intC \neq \emptyset$ and $\precsim$ be a partial ordering with respect to C. Let X be a nonempty set and $s \geq 1$ be a given real number. Let $p : X \times X \to \mathbb{A}$ be a vector-valued function. The pair (X, p) is called a cone b-metric space over Banach algebra if the following conditions are satisfied:*

(i) $\theta \precsim p(x, y)$ *for all* $x, y, \in X$ *and* $p(x, y) = \theta$ *if and only if* $x = y$;
(ii) $p(x, y) = p(y, x)$;
(iii) $p(x, z) \precsim s[p(x, y) + p(y, z)]$ *for all* $x, y, z \in X$.

In this paper, we introduce the concept of $\mathbb{BID}$-cone b-metric space over Banach algebra with inferior idempotent cones.

Definition 4. *Let $\mathbb{A}$ be a real Banach algebra with its zero vector θ, K be an inferior idempotent cone in $\mathbb{A}$ with $intK \neq \emptyset$ and $\precsim$ be a partial ordering with respect to K. Let X be a nonempty set and $s \geq 1$ be a given real number. A vector-valued function $\beta : X \times X \to \mathbb{A}$ is said to be $\mathbb{BID}$-cone b-metric if the following conditions are satisfied:*

(i) $\theta \precsim \beta(x, y)$ *for all* $x, y, \in X$ *and* $\beta(x, y) = \theta$ *if and only if* $x = y$;
(ii) $\beta(x, y) = \beta(y, x)$;
(iii) $\beta(x, z) \precsim s[\beta(x, y) + \beta(y, z)]$ *for all* $x, y, z \in X$.

The pair (X, β) is called a $\mathbb{BID}$-cone b-metric space over Banach algebra.

Some basic definitions and notations (convergence, Cauchy sequence, completeness, etc.) in a $\mathbb{BID}$-cone b-metric space over Banach algebra can be defined by the mimic of [25] as follows:

Definition 5. *Let (X, β) be a $\mathbb{BID}$-cone b-metric space over Banach algebra and $\{x_n\}$ be a sequence in X.*

(i) $\{x_n\}$ *is called a $\mathbb{BID}$-cone b-convergent sequence if it is a c-sequence;*
(ii) $\{x_n\}$ *is called a $\mathbb{BID}$-cone b-Cauchy sequence if $\{\beta(x_n, x_m)\}$ is a c-sequence for n, m;*
(iii) (X, β) *is called complete if every $\mathbb{BID}$-cone b-Cauchy sequence in X is $\mathbb{BID}$-cone b-convergent.*

The following lemmas will be needed in this article.

Lemma 5 ([24])**.** *Let $\mathbb{A}$ be a real Banach algebra and $u \in \mathbb{A}$. Denote by $\rho(u)$ the spectral radius of u. Then*

$$\rho(u) = \lim_{n\to\infty} \|u^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|u^n\|^{\frac{1}{n}}.$$

If $\rho(u) < 1$, then $e - u$ is invertible and

$$(e-u)^{-1} = \sum_{n=0}^{\infty} u^n.$$

Lemma 6 ([21])**.** *Let $\mathbb{A}$ be a real Banach algebra and $k \in \mathbb{A}$. Then $\rho(k) < 1$ if and only if $\{k^n\}$ is a θ-sequence.*

Lemma 7 ([23])**.** *Let $\mathbb{A}$ be a real Banach algebra and $k \in \mathbb{A}$. If $\rho(k) < 1$, then $\{k^n\}$ is a c-sequence.*

The proof of the following lemma is similar to the proof of ([26], Lemma 1.8), so we omit it.

Lemma 8 ([26])**.** *Let $\mathbb{A}$ be a real Banach algebra and K be a convex cone in $\mathbb{A}$. If $\alpha, \beta \in K$, $\{u_n\}$ and $\{v_n\}$ are c-sequences in $\mathbb{A}$, then $\{\alpha u_n + \beta v_n\}$ is also a c-sequence in $\mathbb{A}$.*

Question C. Let $\mathbb{A}$ be a Banach algebra with its zero element θ and K be an inferior idempotent cone in $\mathbb{A}$.

(a) Let $u \in K$ with $\rho(u) < 1$. Does $(e-u)^{-1} \in K$ imply $e - u \in K$?
(b) Let $\theta \precsim k \prec e$. Is $\{k^n\}$ a θ-sequence?

Solution:

(a) The answer is no. For example, define the same $\mathbb{A}$, $\|\cdot\|$ and K as in the solution of Question B. Put

$$u = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Then $u \in K$, $\rho(u) \leq \|u\| = \frac{2}{3} < 1$, and

$$e - u = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \notin K.$$

It is easy to get that

$$(e-u)^{-1} = \begin{pmatrix} \frac{8}{3} & \frac{4}{3} & 0 \\ \frac{4}{3} & \frac{8}{3} & 0 \\ 0 & 0 & 2 \end{pmatrix} \in K.$$

(b) The answer is no. For example, let $\mathbb{A}$, e, $\|\cdot\|$, K and an associative and distributive multiplication be defined as in Example 1. Take $\kappa = (1, \frac{1}{2})$. Hence, $\kappa \prec e$. However, we obtain

$$\kappa^n = \left(1, \frac{1}{2^n}\right) \nrightarrow \theta = (0,0) \text{ as } n \to \infty.$$

which means that $\{\kappa^n\}$ is not a θ-sequence.

4. Some New Judgement Theorems and Fixed Point Theorems

Let U be a nonempty set and $T : U \to U$ be a selfmapping. A point z in U is a *fixed point* of T if $Tz = z$. The set of fixed points of T is denoted by $\mathcal{F}(T)$.

In this section, we first establish the following new existence theorem.

Theorem 1. *Let $(E, \|\cdot\|)$ be a real Banach space, C be a cone in E and $\{a_n\}$ be a c-sequence in intK. Then $\{a_n\}$ exists a nonincreasing subsequence with respect to $\gg$.*

Proof. Due to $a_1 \in intK$, by (a) of Lemma 1, we have $\frac{a_1}{2} \in intK$. Since $\{a_n\}$ is a c-sequence in $intK$, there is a natural number n_1, such that

$$\theta \ll a_{n_1} \ll \frac{a_1}{2}.$$

Since $a_{n_1} \in intK$, $\frac{a_{n_1}}{2} \in intK$ and hence there exists a natural number $n_2 > n_1$ such that

$$\theta \ll a_{n_2} \ll \frac{a_{n_1}}{2}. \tag{6}$$

Similarly, there exists a natural number $n_3 > n_2 > n_1$ such that

$$\theta \ll a_{n_3} \ll \frac{a_{n_2}}{2}. \tag{7}$$

Taking into account (6) and (7), we obtain

$$\theta \ll a_{n_3} \ll \frac{a_{n_2}}{2} \ll \frac{a_{n_1}}{2^2}.$$

By the mathematical induction, there exists a natural number n_k with $n_k > n_{k-1} > n_{k-2} > \cdots > n_1$ such that

$$\theta \ll a_{n_k} \ll \frac{a_{n_{k-1}}}{2} \ll \frac{a_{n_{k-2}}}{2^2} \ll \cdots \ll \frac{a_{n_1}}{2^{k-1}},$$

which implies

$$a_{n_1} \gg 2a_{n_2} \gg 2^2 a_{n_3} \gg \cdots \gg 2^{k-1} a_{n_k}.$$

Continuing this process, we can obtain a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ satisfying

$$a_{n_1} \gg 2a_{n_2} \gg 2^2 a_{n_3} \gg \cdots \gg 2^{k-1} a_{n_k} \gg \cdots,$$

which deduces
$$a_{n_1} \gg a_{n_2} \gg a_{n_3} \gg \cdots \gg a_{n_k} \gg \cdots .$$

Therefore, $\{a_{n_k}\}$ is a nonincreasing subsequence of $\{a_n\}$ with respect to $\gg$. $\square$

The following result is an immediate consequence of Theorem 1.

Corollary 1. *Let $\{a_n\}$ be a positive sequence satisfying $\lim_{n\to\infty} a_n = 0$, Then $\{a_n\}$ exists a strictly decreasing subsequence.*

Theorem 2. *Let (X, β) be a $\mathbb{BID}$-cone b-metric space over Banach algebra with coefficient $s \geq 1$ and K be an inferior idempotent cone in $\mathbb{A}$. Suppose that $\{x_n\}$ is a sequence in X such that $\{\beta(x_n, x_{n+1})\}$ is a c-sequence, $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and $\{x_{n_k}\}$ is a $\mathbb{BID}$-cone b-Cauchy sequence in X. If $\sup_{k\in\mathbb{N}}(n_{k+1} - n_k) < \infty$, then $\{x_n\}$ is a $\mathbb{BID}$-cone b-Cauchy sequence in X.*

Proof. Let $c \gg \theta$ be given. Since $\{\beta(x_n, x_{n+1})\}$ is a c-sequence, there exists $K_1 \in \mathbb{N}$ such that for all $k_1 > K_1$, we have
$$\beta(x_{n_{k_1}}, x_{n_{k_1}+1}) \ll c. \tag{8}$$

By the assumption that $\{x_{n_k}\}$ is a $\mathbb{BID}$-cone b-Cauchy sequence in X, then for the aforementioned $c \gg \theta$, there exists $K_2 \in \mathbb{N}$ such that if $k_2 \geq k_1 > K_2$. Therefore
$$\beta(x_{n_{k_1}}, x_{n_{k_2}}) \ll c. \tag{9}$$

Put $K = \max\{K_1, K_2\}$. We claim that (8) and (9) are satisfied when $k_2 \geq k_1 > K$. Now for any $n \geq n_{K+1}$ and any $p \in \mathbb{N}$, there exist $k_1, k_2 \in \mathbb{N}$ with $k_1 \leq k_2$ such that $n_{k_1} \leq n < n_{k_1+1}$ and $n_{k_2} \leq n + p < n_{k_2+1}$. Clearly, $k_1 > K$. Indeed, suppose the contrary, $k_1 \leq K$, then $n_{K+1} \leq n < n_{k_1+1} \leq n_{K+1}$. This is a contradiction. By our hypothesis, we may write $\sup_{k\in\mathbb{N}}(n_{k+1} - n_k) = M$. Hence, we obtain

$$\begin{aligned}
\beta(x_n, x_{n+p}) &\precsim s[\beta(x_{n_{k_1}}, x_n) + \beta(x_{n_{k_1}}, x_{n+p})] \\
&\precsim s\beta(x_{n_{k_1}}, x_n) + s^2\beta(x_{n_{k_1}}, x_{n_{k_2}}) + s^2\beta(x_{n_{k_2}}, x_{n+p}) \\
&\precsim s[s\beta(x_{n_{k_1}}, x_{n_{k_1}+1}) + s\beta(x_{n_{k_1}+1}, x_n)] + s^2\beta(x_{n_{k_1}}, x_{n_{k_2}}) \\
&\quad + s^2[s\beta(x_{n_{k_2}}, x_{n_{k_2}+1}) + s\beta(x_{n_{k_2}+1}, x_{n+p})] \\
&\precsim s[s\beta(x_{n_{k_1}}, x_{n_{k_1}+1}) + s^2\beta(x_{n_{k_1}+1}, x_{n_{k_1}+2}) + s^2\beta(x_{n_{k_1}+2}, x_n)] + s^2\beta(x_{n_{k_1}}, x_{n_{k_2}}) \\
&\quad + s^2[s\beta(x_{n_{k_2}}, x_{n_{k_2}+1}) + s^2\beta(x_{n_{k_2}+1}, x_{n_{k_2}+2}) + s^2\beta(x_{n_{k_2}+2}, x_{n+p})] \\
&\precsim s[s\beta(x_{n_{k_1}}, x_{n_{k_1}+1}) + s^2\beta(x_{n_{k_1}+1}, x_{n_{k_1}+2}) + s^3\beta(x_{n_{k_1}+2}, x_{n_{k_1}+3}) + \cdots \\
&\quad + s^{n-n_{k_1}-1}\beta(x_{n-2}, x_{n-1}) + s^{n-n_{k_1}-1}\beta(x_{n-1}, x_n)] + s^2\beta(x_{n_{k_1}}, x_{n_{k_2}}) \\
&\quad + s^2[s\beta(x_{n_{k_2}}, x_{n_{k_2}+1}) + s^2\beta(x_{n_{k_2}+1}, x_{n_{k_2}+2}) + s^3\beta(x_{n_{k_2}+2}, x_{n_{k_2}+3}) \\
&\quad + \cdots + s^{n+p-n_{k_2}-1}\beta(x_{n+p-2}, x_{n+p-1}) + s^{n+p-n_{k_2}-1}\beta(x_{n+p-1}, x_{n+p})] \\
&\precsim s^2[\beta(x_{n_{k_1}}, x_{n_{k_1}+1}) + s\beta(x_{n_{k_1}+1}, x_{n_{k_1}+2}) + s^2\beta(x_{n_{k_1}+2}, x_{n_{k_1}+3}) + \cdots \\
&\quad + s^{n-n_{k_1}-2}\beta(x_{n-2}, x_{n-1}) + s^{n-n_{k_1}-1}\beta(x_{n-1}, x_n)] + s^2\beta(x_{n_{k_1}}, x_{n_{k_2}}) \\
&\quad + s^3[\beta(x_{n_{k_2}}, x_{n_{k_2}+1}) + s\beta(x_{n_{k_2}+1}, x_{n_{k_2}+2}) + s^2\beta(x_{n_{k_2}+2}, x_{n_{k_2}+3}) \\
&\quad + \cdots + s^{n+p-n_{k_2}-2}\beta(x_{n+p-2}, x_{n+p-1}) + s^{n+p-n_{k_2}-1}\beta(x_{n+p-1}, x_{n+p})] \\
&\ll s^2(1 + s + s^2 + \cdots + s^{n-n_{k_1}-2} + s^{n-n_{k_1}-1})c + s^2c \\
&\quad + s^3(1 + s + s^2 + \cdots + s^{n+p-n_{k_2}-2} + s^{n+p-n_{k_2}-1})c
\end{aligned}$$

$$
\begin{aligned}
&= \frac{s^2}{s-1}(s^{n-n_{k_1}}-1)c + s^2c + \frac{s^3}{s-1}(s^{n+p-n_{k_2}}-1)c \\
&\precsim \frac{s^2}{s-1}(s^M-1)c + s^2c + \frac{s^3}{s-1}(s^M-1)c \\
&= s^2\left[\frac{s+1}{s-1}(s^M-1)+1\right]c,
\end{aligned}
$$

which means that $\{x_n\}$ is a $\mathbb{BID}$-cone b-Cauchy sequence in X. The proof is completed. □

The following conclusions are immediate from Lemma 4 and Theorem 2.

Corollary 2. *Let (X, β) be a $\mathbb{BID}$-cone b-metric space over Banach algebra with coefficient $s \geq 1$ and K be an inferior idempotent cone in $\mathbb{A}$. Suppose that $\{x_n\}$ is a sequence in X such that $\{\beta(x_n, x_{n+1})\}$ is a θ-sequence, $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and $\{x_{n_k}\}$ is a $\mathbb{BID}$-cone b-Cauchy sequence in X. If $\sup_{k\in\mathbb{N}}(n_{k+1} - n_k) < \infty$, then $\{x_n\}$ is a $\mathbb{BID}$-cone b-Cauchy sequence in X.*

Corollary 3. *Let (X, β) be a $\mathbb{BID}$-cone b-metric space over Banach algebra with coefficient $s \geq 1$ and K be an inferior idempotent cone in $\mathbb{A}$. Suppose that $\{x_n\}$ is a sequence in X such that $\{\beta(x_n, x_{n+1})\}$ is a c-sequence. If $\{x_{2n}\}$ or $\{x_{2n-1}\}$ is a $\mathbb{BID}$-cone b-Cauchy sequence in X, then $\{x_n\}$ is also a $\mathbb{BID}$-cone b-Cauchy sequence in X.*

Theorem 3. *Let (X, β) be a $\mathbb{BID}$-cone b-metric space over Banach algebra with coefficient $s \geq 1$ and K be an inferior idempotent cone in $\mathbb{A}$. Suppose that $\{x_n\}$ is a sequence in X satisfying*

$$\beta(x_{n+1}, x_{n+2}) \precsim k\beta(x_n, x_{n+1}) \quad \textit{for any } n \in \mathbb{N}, \tag{10}$$

where $k \in K$ and $\{k^n\}$ is a θ-sequence, then $\{x_n\}$ is a $\mathbb{BID}$-cone b-Cauchy sequence in X.

Proof. By (10), it follows that

$$\beta(x_{n+1}, x_{n+2}) \precsim k\beta(x_n, x_{n+1}) \precsim k^2\beta(x_{n-1}, x_n) \precsim \cdots \precsim k^n\beta(x_1, x_2). \tag{11}$$

Since $\{k^n\}$ is a θ-sequence, by Lemma 6, $\rho(k) \in [0, 1)$. Hence, there is an $\alpha \in \mathbb{N}$ such that $s[\rho(k)]^\alpha < 1$. For any $n, m \in \mathbb{N}$ with $n < m$, we consider the following two possible cases:

(i) Assume that $m - n \leq \alpha$. In view of (11), it is valid that

$$
\begin{aligned}
\beta(x_n, x_m) &\precsim s\beta(x_n, x_{n+1}) + s\beta(x_{n+1}, x_m) \\
&\precsim s\beta(x_n, x_{n+1}) + s^2\beta(x_{n+1}, x_{n+2}) + s^2\beta(x_{n+2}, x_m) \\
&\precsim s\beta(x_n, x_{n+1}) + s^2\beta(x_{n+1}, x_{n+2}) + s^3\beta(x_{n+2}, x_{n+3}) \\
&\quad + \cdots + s^{m-n-1}\beta(x_{m-2}, x_{m-1}) + s^{m-n-1}\beta(x_{m-1}, x_m) \\
&\precsim sk^{n-1}\beta(x_1, x_2) + s^2k^n\beta(x_1, x_2) + s^3k^{n+1}\beta(x_1, x_2) \\
&\quad + \cdots + s^{m-n-1}k^{m-3}\beta(x_1, x_2) + s^{m-n-1}k^{m-2}\beta(x_1, x_2) \\
&\precsim sk^{n-1}[e + sk + (sk)^2 + \cdots + (sk)^{m-n-1}]\beta(x_1, x_2) \\
&\precsim sk^{n-1}[e + sk + (sk)^2 + \cdots + (sk)^{\alpha-1}]\beta(x_1, x_2) \\
&= k^{n-1}\varpi,
\end{aligned}
$$

where

$$\varpi = s[e + sk + (sk)^2 + \cdots + (sk)^{\alpha-1}]\beta(x_1, x_2).$$

(ii) Assume that $m-n>\alpha$. Put $\gamma=-\lfloor -\frac{m-n}{\alpha}\rfloor-1$, where $\lfloor\cdot\rfloor$ indicates the floor function or the greatest integer function. It may be verified that $\gamma<\frac{m-n}{\alpha}\leq\gamma+1$ and $0<m-(n+\gamma\alpha)\leq\alpha$. Then

$$\begin{aligned}\beta(x_n,x_m)&\precsim s\beta(x_n,x_{n+\alpha})+s\beta(x_{n+\alpha},x_m)\\&\precsim s\beta(x_n,x_{n+\alpha})+s^2\beta(x_{n+\alpha},x_{n+2\alpha})+s^2\beta(x_{n+2\alpha},x_m)\\&\precsim s\beta(x_n,x_{n+\alpha})+s^2\beta(x_{n+\alpha},x_{n+2\alpha})+s^3\beta(x_{n+2\alpha},x_{n+3\alpha})\\&\quad+\cdots+s^{\gamma}\beta(x_{n+(\gamma-1)\alpha},x_{n+\gamma\alpha})+s^{\gamma+1}\beta(x_{n+\gamma\alpha},x_m).\end{aligned}\tag{12}$$

As a consequence of $\rho(sk^\alpha)=s\rho(k^\alpha)=s[\rho(k)]^\alpha<1$, by Lemma 5, we obtain

$$(e-sk^\alpha)^{-1}=e+sk^\alpha+\cdots+(sk^\alpha)^{\gamma-1}+(sk^\alpha)^{\gamma}+\cdots.\tag{13}$$

Since $0<m-(n+\gamma\alpha)\leq\alpha$, by taking into account (12) and (13), we get

$$\begin{aligned}\beta(x_n,x_m)&\precsim sk^{n-1}\varpi+s^2k^{n+\alpha-1}\varpi+\cdots+s^{\gamma}k^{n-1+(\gamma-1)\alpha}\varpi+s^{\gamma+1}k^{n-1+\gamma\alpha}\varpi\\&=sk^{n-1}\varpi[e+sk^\alpha+\cdots+(sk^\alpha)^{\gamma-1}+(sk^\alpha)^{\gamma}]\\&\precsim sk^{n-1}\varpi[e+sk^\alpha+\cdots+(sk^\alpha)^{\gamma-1}+(sk^\alpha)^{\gamma}+\cdots]\\&=sk^{n-1}\varpi(e-sk^\alpha)^{-1}.\end{aligned}$$

Owing to $\rho(k)<1$, by Lemmas 7 and 8, $\{k^{n-1}\varpi\}$ and $\{sk^{n-1}\varpi(e-sk^\alpha)^{-1}\}$ are c-sequences. Finally, by applying Lemma 2 to the cases (i) and (ii), we prove that $\{x_n\}$ is a $\mathbb{BID}$-cone b-Cauchy sequence. The proof is completed. □

Corollary 4. *Let (X,β) be a $\mathbb{BID}$-cone b-metric space over Banach algebra with coefficient $s\geq1$ and K be an inferior idempotent cone in $\mathbb{A}$. Suppose that $\{x_n\}$ is a sequence in X such that (10) is satisfied, where $k\in K$ and $\rho(k)<1$. Then $\{x_n\}$ is a $\mathbb{BID}$-cone b-Cauchy sequence in X.*

Proof. Applying Lemma 6 and Theorem 3, we can show the desired result. □

Remark 3. *Theorem 3 generalizes ([21], Theorem 2.18) and Corollary 4 extends ([21], Corollary 2.19). Moreover, Corollary 4 improves and generalizes ([22], Lemma 2.2), ([27], Lemma 2.2), ([28], Lemma 2.3) and ([29], Lemma 6). In addition, our proof method is sharply simpler than that of the previous results.*

Theorem 4. *Let (X,β) be a complete $\mathbb{BID}$-cone b-metric space over Banach algebra with coefficient $s\geq1$ and K be an inferior idempotent cone in $\mathbb{A}$. Suppose that $T:X\to X$ is a mapping satisfying*

$$\beta(Tx,Ty)\precsim k\beta(x,y)\quad\text{for any } x,y\in X,\tag{14}$$

where $k\in K$ with $\rho(k)<1$. Then $\mathcal{F}(T)\neq\emptyset$.

Moreover, if K is pointed and closed, then $\mathcal{F}(T)$ is a singleton set and for any $x\in X$, the iterative sequence $\{T^nx\}_{n\in\mathbb{N}}$ $\mathbb{BID}$-cone b-converges to the unique fixed point of T.

Proof. Choose $x_0\in X$ and put $x_{n+1}=Tx_n$ for $n\in\mathbb{N}$. By (14), it is not hard to get

$$\beta(x_{n+1},x_{n+2})\precsim k\beta(x_n,x_{n+1})\quad\text{for any } n\in\mathbb{N}.$$

Hence, by Corollary 4, ones deduce that $\{x_n\}$ is a $\mathbb{BID}$-cone b-Cauchy sequence in X. By the completeness of (X,β), there exists $x^*\in X$ such that $\lim\limits_{n\to\infty}x_n=x^*$. Next, we claim that x^* is a fixed point of T. Indeed, it follows from (14) that

$$\beta(x_{n+1},Tx^*)\precsim k\beta(x_n,x^*).\tag{15}$$

Since $\{\beta(x_n, x^*)\}$ is a c-sequence, by applying Lemmas 2 and 8 to (15), we show that $\{x_n\}$ b-converges to Tx^*. Since the limit of a $\mathbb{BID}$-cone b-convergent sequence is unique, we obtain $Tx^* = x^*$. In other words, $x^* \in \mathcal{F}(T)$.

Moreover, we assume that K is pointed and closed. We want to verify that x^* is the unique fixed point of T. In fact, we suppose for absurd that there exists another fixed point y^*, that is, $Ty^* = y^*$. Then via (14), we get

$$\beta(x^*, y^*) = \beta(Tx^*, Ty^*) \precsim k\beta(x^*, y^*) \precsim k^2\beta(x^*, y^*) \precsim \cdots \precsim k^n\beta(x^*, y^*).$$

Making full use of Lemmas 3, 7 and 8, we obtain $\beta(x^*, y^*) = \theta$ or $y^* = x^*$. The proof is completed. □

Remark 4. *Theorem 4 improves and generalizes ([25], Theorem 2.1).*

5. Conclusions

The main contributions in this paper are as follows.

(a) We introduce the concept of inferior idempotent cones (see Definition 1) and give some fundamental questions and examples;

(b) The concept of $\mathbb{BID}$-cone b-metric space over Banach algebra with inferior idempotent cones, $\mathbb{BID}$-cone b-convergent sequence and $\mathbb{BID}$-cone b-Cauchy sequence are introduced (for more details, see Section 3);

(c) In Section 4, we establish some new auxiliary theorems and fixed point theorems in the setting of complete $\mathbb{BID}$-cone b-metric spaces over Banach algebra.

In summary, our new results are original and quite different from the well-known generalizations on the topic in the literature.

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