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The Generalized Alpha Exponent Power Family of Distributions: Properties and Applications

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Abstract: Here, a new method is recommended to characterize a new continuous distribution class, named the generalized alpha exponent power family of distributions (GAEPFDs). A particular sub-model is presented for exemplifying the objective. The basic statistical properties, such as ordinary moments, the probability weighted moments, mode, quantile, order statistics, entropy measures, and moment generating functions, etc., were explored. To gauge the GAEPPRD parameters, the ML technique was utilized. The estimator behaviour was studied by a Monte Carlo simulation study (MCSS). The effectiveness of GAEPFDs was demonstrated observationally through lifetime data. The applications show that GAEPFDs can offer preferable results over other competitive models.

Keywords: Monte Carlo simulation; moments; G-family; maximum likelihood estimation; power Rayleigh distribution

MSC: 60E05; 62E15

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1. Introduction

The Rayleigh distribution (RD) [1] serves as a model in agriculture and health, analyzing wind speed data, hydrological characteristics, communications theory, engineering, etc. It originates from two parameters. The Weibull distribution (WD), whereas shape parameter is equal to 2; and a random variable (r.v.), Z, follows an RD with scale parameter λ if it has a cumulative distribution function (CDF), such as

$$F(z) = 1 - e^{-\frac{z^2}{2\lambda^2}}; \quad z > 0, \lambda > 0. \tag{1}$$

An r.v. $X = Z^{\frac{1}{\beta}}$ follows a power Rayleigh distribution (PRD) with λ and β as the scale and shape parameters if it has CDF given as

$$F(x) = 1 - e^{-\frac{x^{2\beta}}{2\lambda^2}}; \quad x > 0, \beta, \lambda > 0. \tag{2}$$

In the most last twenty years, many researchers have contributed toward developing new distributions through variable alterations. Moreover, they have used techniques, such as (i) composition, (ii) compounding, and (iii) finite mixtures of distributions. A few generalized families include: Beta-G [2], Weibull-G [3], Kumaraswamy-G [4], McDonald-G [5], exponentiated-G [6], odd-gamma-G [7], Marshall-G [8], moment exponential-G [9], Lomax-G [10]. For more details on well-known G-classes, refer to [11].

The new idea in generalization began when the transformed-transformer (T-X) family of the distribution was proposed by [12]. Let T be a r.v., having a p.d.f $w(t)$, where $T \in [m_1, m_2]$, $-\infty \leq m_1 < m_2 < \infty$, and let X be a r.v., $H[F(x)]$, a function of $F(x)$, fulfill (i)–(iii) conditions as under:

(i) $H[F(x)] \in [m_1, m_2]$;
(ii) $H[F(x)]$ is monotonically increasing and differentiable; and
(iii) $x \to -\infty \Longrightarrow H[F(x)] \to m_1$ and $x \to \infty \Longrightarrow H[F(x)] \to m_2$.

The family of distributions T-X having CDF, $G(x)$, is

$$G(x) = \int_{m_1}^{H[F(x)]} w(t)\,\mathrm{d}t, \quad x \in \mathcal{R}, \tag{3}$$

The p.d.f corresponds to (3) is

$$g(x) = \frac{\partial H[F(x)]}{\partial x} w\{H[f(x)]\}, \quad x \in \mathcal{R}.$$

Recently, a technique, known as the alpha power transformation (APT), presented in [13], added an extra shape parameter to an existing lifetime distribution to bring more flexibility to the given distribution. The APT has CDF, as

$$G(x) = \frac{\alpha^{F(x)} - 1}{\alpha - 1}; \quad \alpha > 0, \alpha \neq 1, x \in \mathcal{R} \tag{4}$$

and the corresponding p.d.f has the form

$$g(x) = \frac{\log \alpha}{\alpha - 1} \alpha^{F(x)} f(x)$$

Here, $F(x)$ and $f(x)$ are the CDF and p.d.f of any continuous baseline distribution with r.v. X. The APT has been used by many researchers to develop new distributions and data analysis purposes. In [13], it is observed that a generated two parameter (shape and scale) alpha power exponential distribution (APED) behaves similar to the two-parameter generalized exponential (GE), gamma, or Weibull distributions, and take similar shapes of p.d.f and HRF. Reference [14] defined a method of introducing two extra parameters to a baseline distribution called the modified APT (MAPT). The CDF of the MAPT is defined as

$$G(x) = \frac{\beta^{F^2(x)} \alpha^{F(x)} - 1}{\alpha\beta - 1}; \alpha, \beta \geq 1, \alpha\beta \neq 1$$

The corresponding p.d.f has the form

$$g(x) = \frac{1}{\alpha\beta - 1} \beta^{F^2(x)} \alpha^{F(x)} [\ln \alpha + 2F(x) \ln \beta] f(x)$$

For three extra parameters in the model, we can use $\beta^{F^\gamma(x)}, \gamma > 0$. In [14], the modified APED (MAPED) with three parameters (α, β as shape and λ as rate) is proposed. The extra parameters in the exponential distribution produced asymmetrical densities to the right and left and change the tail length. The p.d.f of the MAPED is a concave function and has increasing HRF. The MAPED succeeded to fit the real data as compared to APED [13], among others.

The main motivations for developing the new G-family are:

(i) Constructing new G-families as a function of a CDF, $H[F(x)]$.
(ii) The proposed extension of the APT is based on a generator $\left[1 - \theta\left\{\log \bar{F}(x;\xi)\right\}^{-1}\right]^{-1}$ instead of the existing generator $F(x;\xi)$.
(iii) The proposed generator has the power to produce better results of estimates and goodness of fit tests that can make it attractive and distinguishable for applied researchers.

Combining the ideas of (3) and (4), we introduce a new flexible distribution class, called the generalized alpha exponent power-*G* (GAEP-G) family. Let

$$W(t) = \frac{\alpha^t - 1}{\alpha - 1}, \quad t \geq 0 \tag{5}$$

be the CDF of a r.v. T. The p.d.f corresponding to (5) is

$$w(t) = \frac{\log \alpha}{\alpha - 1}\alpha^t, \quad t > 0. \tag{6}$$

Setting $H[F(x)] = \left[1 - \theta\left\{\log \bar{F}(x;\xi)\right\}^{-1}\right]^{-1}$ and $w(t)$ as (6) in (3), we define CDF of the GAEP-*G* family as

$$\begin{aligned} G(x;\alpha,\theta,\xi) &= (\alpha-1)^{-1}\log\alpha \int_0^{\left[1-\theta\left\{\log \bar{F}(x;\xi)\right\}^{-1}\right]^{-1}} \alpha^t \, dt \\ &= (\alpha-1)^{-1}\left\{\alpha^{\left[1-\theta\left\{\log \bar{F}(x;\xi)\right\}^{-1}\right]^{-1}} - 1\right\} \\ &\theta > 0, \alpha > 0, \alpha \neq 1, x \in \mathcal{R} \end{aligned} \tag{7}$$

The p.d.f related to (7) is

$$g(x;\alpha,\theta,\xi) = \frac{(\alpha-1)^{-1}\theta\log\alpha f(x;\xi)\alpha^{\left[1-\theta\left\{\log \bar{F}(x;\xi)\right\}^{-1}\right]^{-1}}}{\bar{F}(x;\xi)\left\{\log \bar{F}(x;\xi)\right\}^2\left[1-\theta\left\{\log \bar{F}(x;\xi)\right\}^{-1}\right]^2} \tag{8}$$

GAEPFDs can serve as leading models to widely applied classes in disasters, agriculture, engineering, health, and hydrological characteristics, such as precipitation, flood, storm events, and wind speed data.

The paper is laid out as follows: in Section 1, GAEPFDs as a new family is presented. In Section 2, a sub-model, GAEPPRD, is characterized and gives plots of p.d.f, CDF, reliability, and HRF. In Section 3, we derive some statistical properties. Estimation of GAEPPRD parameters and ACBs are tended to in Section 4. In Section 5, the MCSS is provided. In Section 6, we describe the importance of GAEPFDs through two applications to lifetime data. Finally, the conclusion is presented in Section 7.

2. The Generalized Alpha Exponent Power Power Rayleigh Distribution (GAEPPRD)

Here, using (2) and (7), we define CDF of the sub-model of the GAEP-*G* family, namely, GAEPPRD as

$$\begin{aligned} G(x;\alpha,\theta,\xi) &= \frac{\alpha^{\left(\frac{x^{2\beta}}{x^{2\beta}+2\theta\lambda^2}\right)} - 1}{\alpha - 1}, \\ x \geq 0, \xi = (\beta,\lambda), &\quad \theta,\beta,\lambda,\alpha > 0, \alpha \neq 1 \end{aligned} \tag{9}$$

The p.d.f of GAEPPRD analogous to (9) is

$$g(x;\alpha,\theta,\xi) = \frac{4\ \log\alpha\ \theta\ \beta\ \lambda^2\ x^{2\beta-1}}{(\alpha-1)(x^{2\beta}+2\theta\lambda^2)^2}\alpha^{\left(\frac{x^{2\beta}}{x^{2\beta}+2\theta\lambda^2}\right)}, \quad x > 0. \tag{10}$$

Using (9), the reliability of GAEPPRD is

$$R(x) = 1 - G(x) = \frac{\alpha - \alpha^{\left(\frac{x^{2\beta}}{x^{2\beta}+2\theta\lambda^2}\right)}}{\alpha - 1} \tag{11}$$

With the help of (10) and (11), the hazard rate function (HRF) of GAEPPRD is

$$h(x) = \frac{g(x)}{R(x)} = \frac{4\ \log\alpha\ \theta\ \beta\ \lambda^2\ x^{2\beta-1}}{(x^{2\beta}+2\theta\lambda^2)^2} \times \frac{1}{\alpha^{\frac{2\theta\lambda^2}{2\theta\lambda^2+x^{2\beta}}}-1} \tag{12}$$

Graphs of p.d.f, CDF, reliability, and HRF of GAEPPRD for some selected values of parameters are shown in Figures 1 and 2. In Figure 1, the shapes of the GAEPPRD p.d.f are reversed-J, right- skewed and unimodal; as expected, the CDF shows the increasing pattern. Figure 2 displays that the reliability graph tends to decrease as the time increases and the HRF curve shows shapes, such as the first increase and then the decrease and reversed-J.

Let a continuous r.v. X with reliability function specified in (11); the mean residual life function (MRLF) of GAEPPRD is the average value of the remaining lifetimes after a time point x. The MRLF is expressed as

$$\begin{aligned}\mu(x) &= \frac{1}{R(x)}\int_x^\infty y\ g(y)\ \mathrm{d}y - x; \quad x \geq 0\\ &= \frac{1}{R(x)}\left\{\mu_1' - \int_0^x y\ g(y)\ \mathrm{d}y\right\} - x\end{aligned} \tag{13}$$

Using the incomplete beta type II distribution, we obtain the first incomplete moment $T_1(x) = \int_0^x y\ g(y)\ \mathrm{d}y$, where

$$\int_0^x y\ g(y)\ \mathrm{d}y = \sum_{i=0}^{\infty} \frac{(\log\alpha)^{i+1} B_t((i+1+\frac{1}{2\beta}),(1-\frac{1}{2\beta}))}{i!\eta^{\frac{1}{2\beta}}(\alpha-1)} \tag{14}$$

$t = \eta\ x^{2\beta}$, $\eta = \frac{1}{2\theta\lambda^2}$ and $B_t(a,b) = \int_0^t \frac{t^{a-1}}{(1+t)^{a+b}}\mathrm{d}t$.

Putting (11), (14) and (19) in (13), we have

$$\mu(x) = \sum_{i=0}^{\infty} \frac{(\log\alpha)^{1+i}}{i!\eta^{\frac{1}{2\beta}}(\alpha-\alpha^{\frac{x^{2\beta}}{x^{2\beta}+2\theta\lambda^2}})}\left[B((1+i+\frac{1}{2\beta}),(1-\frac{1}{2\beta})) - B_t((1+i+\frac{1}{2\beta}),(1-\frac{1}{2\beta}))\right] - x \tag{15}$$

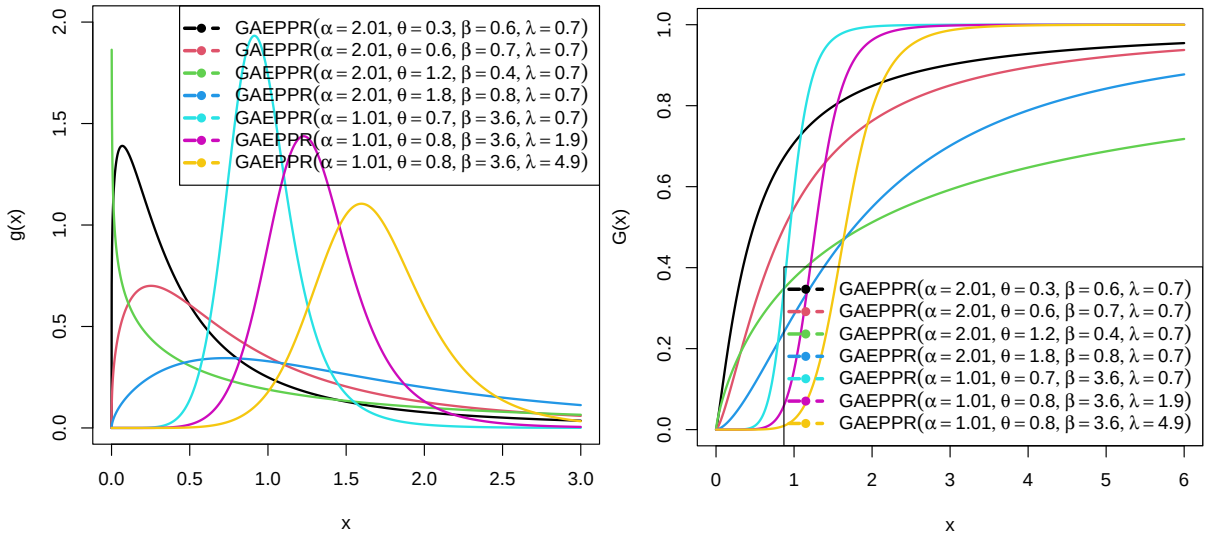


Figure 1. The p.d.f and CDF plots of the GAEPPRD.

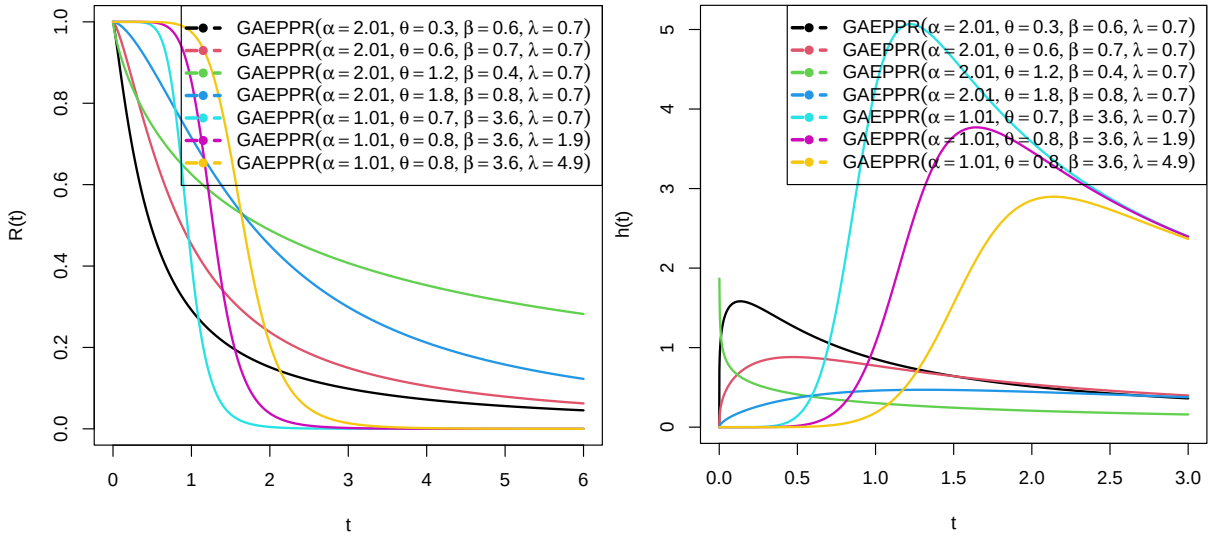


Figure 2. Plots for reliability and HRFs of the GAEPPRD.

3. Statistical Properties

Here, we concentrate on a few statistical properties of the GAEPPRD.

3.1. Moments

About the origin, the *rth* moment of GAEPPRD, having r.v. X and p.d.f (10) is as

$$\begin{aligned}\mu'_r &= E(X^r) = \int_0^\infty x^r g(x)\mathrm{d}x \\ &= \int_0^\infty \frac{4\ \log\alpha\ \theta\ \beta\ \lambda^2\ x^{2\beta+r-1}}{(\alpha-1)(x^{2\beta}+2\theta\lambda^2)^2}\alpha^{\left(\frac{x^{2\beta}}{x^{2\beta}+2\theta\lambda^2}\right)}\mathrm{d}x\end{aligned} \tag{16}$$

Using the series, $\alpha^x = \sum_{i=0}^\infty \frac{x^i(\log\alpha)^i}{i!}$, we have

$$\mu'_r = \sum_{i=0}^\infty \frac{4(\log\alpha)^{i+1}\theta\beta\lambda^2}{(\alpha-1)i!}\int_0^\infty \frac{x^{2\beta(i+1)+r-1}}{(x^{2\beta}+2\theta\lambda^2)^{i+2}}\mathrm{d}x \tag{17}$$

Let $\eta = \frac{1}{2\theta\lambda^2}$, then from (17), we have

$$\mu'_r = \sum_{i=0}^\infty \frac{2(\eta\ \log\alpha)^{i+1}\ \beta}{(\alpha-1)i!}\int_0^\infty \frac{x^{2\beta(i+1)+r-1}}{(1+\eta x^{2\beta})^{i+2}}\mathrm{d}x$$

After simplifying, we have

$$\begin{aligned}\mu'_r &= \sum_{i=0}^\infty \frac{(\log\alpha)^{i+1}}{(\alpha-1)\ i!\ \eta^{\frac{r}{2\beta}}}\int_0^\infty \frac{t^{i+\frac{r}{2\beta}}}{(1+t)^{i+2}}\mathrm{d}t \\ &= \sum_{i=0}^\infty \frac{(\log\alpha)^{i+1}}{\eta^{\frac{r}{2\beta}}\ i!\ (\alpha-1)}\int_0^\infty \frac{t^{(i+\frac{r}{2\beta}+1)-1}}{(1+t)^{(i+\frac{r}{2\beta}+1)+(\frac{2\beta-r}{2\beta})}}\mathrm{d}t\end{aligned}$$

Using the definition of type-II Beta distribution, $B(a,b) = \int_0^\infty \frac{u^{a-1}}{(1+u)^{a+b}}\mathrm{d}u$, we have

$$\mu'_r = \sum_{i=0}^\infty \frac{(\log\alpha)^{i+1}B((i+\frac{r}{2\beta}+1),(\frac{2\beta-r}{2\beta}))}{\eta^{\frac{r}{2\beta}}(\alpha-1)i!} \tag{18}$$

Here, we have the 1st and 2nd raw moment of the GAEPPRD, when $r = 1, 2$ in (18) is given as

$$\mu_1' = \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1} B((i + \frac{1}{2\beta} + 1), (\frac{2\beta-1}{2\beta}))}{\eta^{\frac{1}{2\beta}} (\alpha - 1) i!} \tag{19}$$

$$\mu_2' = \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1} B((i + \frac{2}{2\beta} + 1), (\frac{2\beta-2}{2\beta}))}{\eta^{\frac{2}{2\beta}} (\alpha - 1) i!} \tag{20}$$

Using (19), the mean of GAEPPRD, and (20) in relation (21), we obtain the variance of GAEPPRD.

$$\mu_2 = \mu_2' - (\mu_1')^2 \tag{21}$$

3.2. Moment Generating Function (MGF)

The MGF is given as

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} g(x) \mathrm{d}x \tag{22}$$

Using p.d.f (10), and the series $e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$ and $\alpha^x = \sum_{j=0}^{\infty} \frac{x^j (\log \alpha)^j}{j!}$, $M_X(t)$, the MGF of GAEPPRD, is given as

$$M_X(t) = \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^r (\log \alpha)^{j+1}}{r! j! \eta^{\frac{r}{2\beta}} (\alpha - 1)} B((j + \frac{r}{2\beta} + 1), (\frac{2\beta - r}{2\beta})).$$

3.3. Mode

When GAEPPRD approaches to its maximum point, it is called mode.

Taking the natural logarithm of p.d.f (10), and the differentiating $\log\big(g(x)\big)$ w. r. t. x and equating it to zero yields

$$(1 + 2\beta) x^{4\beta} + 4\theta\lambda^2 (1 - \beta \log \alpha) x^{2\beta} + 4(1 - 2\beta)\theta^2 \lambda^4 = 0 \tag{23}$$

The Equation (23) is a nonlinear equation (NLE) and it does not have an analytic solution w. r. t. x; therefore, it has to be solved numerically, if x_0 is a root of (23), then it must be $f''\big[\log(g(x_0))\big] < 0$.

3.4. Quantile Function (QF)

Let r.v. X follow GAEPPRD having CDF (9). Then $x_p = Q(p) = G^{-1}(p)$, QF of X, is given by inverting (9) as follows

$$x_p = \left\{ \frac{\big[\log(\alpha)\big\{\log\big(p(\alpha - 1) + 1\big)\big\}^{-1} - 1\big]}{2\theta\lambda^2} \right\}^{-\frac{1}{2\beta}} \tag{24}$$

where $p \in (0, 1)$ and GAEPPRD has a closed form expression of QF, which makes it simpler to generate random numbers. In particular, the 1st, 2nd, and 3rd quartiles are obtained by setting $p = 0.25, 0.50$ and 0.75 in (24), respectively.

3.5. Skewness and Kurtosis Using Quantile Approach

Given the QF, as in (24), Moor's kurtosis [15] based on octiles is given as

$$K_M = \frac{Q_{0.875} + Q_{0.375} - (Q_{0.625} + Q_{0.125})}{Q_{0.750} - Q_{0.250}} \tag{25}$$

Moreover, the quartile-based Bowley measure of skewness [16] is given as

$$SK_B = \frac{Q_{0.75} - 2Q_{0.50} + Q_{0.25}}{Q_{0.75} - Q_{0.25}} \tag{26}$$

3.6. The Mean Deviation (MD)

For r.v. X with CDF (9), p.d.f (10), the MD of GAEPPRD about the mean and median can be easily expressed as

$$\delta_1 = \int_x |x - \mu_1'| g(x) \mathrm{d}x$$
$$\delta_1(x) = 2\mu_1' G(\mu_1') - 2T_1(\mu_1') \tag{27}$$

and

$$\delta_2 = \int_x |x - M| g(x) \mathrm{d}x$$
$$\delta_2(x) = \mu_1' - 2T_1(M), \tag{28}$$

respectively, where $\mu_1' = mean$ from (19), $M = x_{0.50}$ is the median from (24), $G(\mu_1')$ from (9) and $T_1(x)$ from (14) is easily obtained.

3.7. Rényi Entropy (RE)

A measure of variation of uncertainty is known as entropy. Renyi [17] defined entropy as

$$I_x(\delta) = (1-\delta)^{-1} \log \int_{-\infty}^{\infty} g(x)^{\delta} \mathrm{d}x, \quad \delta > 0, \delta \neq 1. \tag{29}$$

Using (10), (29) and after simplification, the RE of GAEPPRD is given by

$$I_x(\delta) = \frac{1}{1-\delta} \log \Big[\sum_{i=0}^{\infty} \frac{(\log \alpha)^{\delta+i} (2\beta)^{\delta-1} \delta^i}{(\alpha-1)^{\delta} i! \eta^{\frac{1}{2\beta}(1-\delta)}} B((\delta + i + \frac{1}{2\beta}(1-\delta)), (\delta + \frac{1}{2\beta}(\delta - 1))) \Big] \tag{30}$$

3.8. The Probability Weighted Moments (PWMs)

For r.v. X, the PWMs can be derived by the following expression

$$\tau_{r,s} = E[X^r G(x)^s] = \int_{-\infty}^{\infty} x^r (G(x))^s g(x) \mathrm{d}x. \tag{31}$$

We consider the generalized binomial series, which hold for any real number b and $|Z| < 1$

$$(1-Z)^b = \sum_{k=0}^{\infty} (-1)^k \binom{b}{k} Z^k \tag{32}$$

Using CDF (9) and (32), we obtain

$$(G(x))^s = \frac{1}{(1-\alpha)^s} \sum_{k=0}^{\infty} (-1)^k \binom{s}{k} \alpha^{(\frac{s x^{2\beta}}{x^{2\beta} + 2\theta\lambda^2})} \tag{33}$$

The PWMs of GAEPPRD are obtained by substituting (10) and (33) into (31), as follows ($r \geq 1, s \geq 0$)

$$\tau_{r,s} = \sum_{k,j=0}^{\infty} \frac{(-1)^{k-s} (\log \alpha)^{j+1} (s+1)^j \binom{s}{k}}{j! (\alpha-1)^{s+1} \eta^{\frac{r}{2\beta}}} B((j + 1 + \frac{r}{2\beta}), (1 - \frac{r}{2\beta})) \tag{34}$$

3.9. Distribution of Order Statistic (OS)

In a real-life study and reliability analysis, the OS has been widely used. Let $X^{(1:m)} \leq X^{(2:m)} \leq \cdots \leq X^{(m:m)}$ are the OS of a random sample $X_1 < X_2 < \cdots < X_m$ drawn from GAEPPRD with $G_X(x)$ (9) and $g_X(x)$ (10), then p.d.f of $X^{(k:m)}; k = 1, 2, \cdots, m$ is given by

$$g_{X^{(k:m)}}(x) = \frac{m!}{(k-1)!(m-k)!}[G_X(x)]^{k-1}[1 - G_X(x)]^{m-k} g_X(x) \tag{35}$$

Using (32), we obtain $g_{X^{(k:m)}}(x)$ as

$$g_{X^{(k:m)}}(x) = \sum_{i=0}^{m-k} \frac{(-1)^i m! \binom{m-k}{i}}{(k-1)!(m-k)!}[G_X(x)]^{i+k-1} g_X(x) \tag{36}$$

Here, $[G_X(x)]^{i+k-1}$ is defined as

$$[G_X(x)]^{i+k-1} = \frac{1}{(1-\alpha)^{i+k-1}} \sum_{l=0}^{i+k-1} (-1)^l \binom{i+k-1}{l} \alpha^{\frac{(i+k-1)x^{2\beta}}{x^{2\beta}+2\theta\lambda^2}} \tag{37}$$

The p.d.f of kth OS for GAEPPRD is derived by putting (10) and (37) in (36)

$$g_{X^{(k:m)}}(x) = \sum_{i=0}^{m-k} \sum_{l=0}^{i+k-1} \frac{(-1)^{l-k+1} m! \binom{m-k}{i} \binom{i+k-1}{l}}{(k-1)!(m-k)!} \frac{4 \log \alpha\, \theta\beta\lambda^2 x^{2\beta-1}}{(\alpha-1)^{i+k}(x^{2\beta}+2\theta\lambda^2)^2} \alpha^{\frac{(i+k)x^{2\beta}}{x^{2\beta}+2\theta\lambda^2}} \tag{38}$$

Further, for GAEPPRD, the *rth* moment of kth OS is defined as

$$E(X^r_{(k:m)}) = \int_{-\infty}^{\infty} x^{(k:m)} g(x^{(k:m)}) \mathrm{d}x^{(k:m)} \tag{39}$$

By putting (38) in (39), leads to

$$E(X^r_{(k:m)}) = \sum_{i=0}^{m-k} \sum_{j=0}^{\infty} \sum_{l=0}^{i+k-1} \frac{(-1)^{l-k+1} m!}{j!(k-1)!(m-k)!} \frac{\binom{m-k}{i}\binom{i+k-1}{l}(\log \alpha)^{j+1}(i+k)^j}{\eta^{\frac{r}{2\beta}}(\alpha-1)^{i+k}} B((1+j+\frac{r}{2\beta}), (1-\frac{r}{2\beta})) \tag{40}$$

4. Maximum Likelihood (ML) Estimation

Here, from a complete sample only, we obtain the ML estimators (MLEs) of $\Phi = (\alpha, \theta, \beta, \lambda)^T$ for GAEPPRD. Let the observed values, $x_1, x_2, \ldots, x_n$, be obtained from (10) with parameters Φ. The expression of the log-likelihood function (LLF) for Φ can be given as

$$\begin{aligned} \log L(x; \alpha, \theta, \beta, \lambda) &= n \log(4) - n \log(\alpha - 1) + n \log(\log(\alpha)) + n \log(\theta) + n \log(\beta) + 2n \log(\lambda) + (2\beta - 1) \sum_{i=1}^{n} \log(x_i) \\ &+ \log(\alpha) \sum_{i=1}^{n} \frac{x_i^{2\beta}}{x_i^{2\beta} + 2\theta\lambda^2} - 2 \sum_{i=1}^{n} \log(x_i^{2\beta} + 2\theta\lambda^2) \end{aligned} \tag{41}$$

Partial derivatives of (41) w. r. t. Φ are given by

$$\frac{\partial \log L(x; \alpha, \theta, \beta, \lambda)}{\partial \alpha} = -\frac{n}{\alpha - 1} + \frac{n}{\alpha \log(\alpha)} + \frac{1}{\alpha} \sum_{i=1}^{n} \frac{x_i^{2\beta}}{x_i^{2\beta} + 2\theta\lambda^2} \tag{42}$$

$$\frac{\partial \log L(x; \alpha, \theta, \beta, \lambda)}{\partial \theta} = \frac{n}{\theta} - 2\lambda^2 \log(\alpha) \sum_{i=1}^{n} \frac{x_i^{2\beta}}{(x_i^{2\beta} + 2\theta\lambda^2)^2} - 4\lambda^2 \sum_{i=1}^{n} \frac{1}{x_i^{2\beta} + 2\theta\lambda^2} \tag{43}$$

$$\frac{\partial \log L(x; \alpha, \theta, \beta, \lambda)}{\partial \beta} = \frac{n}{\beta} + 2 \sum_{i=1}^{n} \log(x_i) + \log(\alpha) \sum_{i=1}^{n} \frac{4\theta\lambda^2 x_i^{2\beta} \log(x_i)}{(x_i^{2\beta} + 2\theta\lambda^2)^2} - 4 \sum_{i=1}^{n} \frac{x_i^{2\beta} \log(x_i)}{x_i^{2\beta} + 2\theta\lambda^2} \tag{44}$$

$$\frac{\partial \log L(x;\alpha,\theta,\beta,\lambda)}{\partial \lambda} = \frac{2n}{\lambda} - \sum_{i=1}^{n} \frac{4\theta\lambda \log(\alpha) x_i^{2\beta}}{(x_i^{2\beta} + 2\theta\lambda^2)^2} - 8\theta\lambda \sum_{i=1}^{n} \frac{1}{x_i^{2\beta} + 2\theta\lambda^2} \tag{45}$$

Setting

$$\frac{\partial \log L(x;\ \alpha,\theta,\beta,\lambda)}{\partial \alpha} = 0 \tag{46}$$

$$\frac{\partial \log L(x;\ \alpha,\ \theta,\beta,\lambda)}{\partial \theta} = 0 \tag{47}$$

$$\frac{\partial \log L(x;\ \alpha,\ \theta,\ \beta,\lambda)}{\partial \beta} = 0 \tag{48}$$

$$\frac{\partial \log L(x;\ \alpha,\ \theta,\ \beta,\ \lambda)}{\partial \lambda} = 0 \tag{49}$$

and solving, simultaneously, yields $\hat{\Phi} = (\hat{\alpha}, \hat{\theta}, \hat{\beta}, \hat{\lambda})^T$, the MLEs, of Φ. The analytical solutions of Equations (46)–(49) cannot be obtained. Consequently, to numerically solve these equations, computer software Maple, Mathematica, MATLAB, and R can be utilized.

Asymptotic Confidence Bounds (ACBs) of GAEPPRD

Here, elements of the observed information matrix (OIM), with the order 4×4, are required to estimate the confidence interval's (CIs) of Φ. OIM, $J(\Phi) = \{I_{ij}\}$ (for $i, j = \alpha, \theta, \beta, \lambda$), where I_{ij} can be approximated by

$$I_{ij}(\hat{\Phi}) = E\Big[- \frac{\partial \log L(\Phi)}{\partial \Phi_i \partial \Phi_j} |_{\Phi=\hat{\Phi}} \Big]$$

Using standard regularity conditions, the asymptotic distributions (ADs) of the GAEPPRD parameters are

$$\sqrt{n}(\hat{\Phi}_i - \Phi_i) \simeq N_4(0, J^{-1}(\hat{\Phi}_i)); i = \alpha, \theta, \beta, \lambda$$

where $J^{-1}(\hat{\Phi})$ is the variance covariance matrix (VCM) of Φ. Here, $J(\hat{\Phi})$ evaluated at $\hat{\Phi}$ is the total OIM. Then, the approximate $100(1-\Lambda)\%$ CIs for α, θ, β, and λ, based on the ADs of the GAEPPRD, are determined, respectively, as

$$\hat{\alpha} \pm Z_{\frac{\Lambda}{2}} \times \sqrt{var(\hat{\alpha})},$$

$$\hat{\theta} \pm Z_{\frac{\Lambda}{2}} \times \sqrt{var(\hat{\theta})},$$

$$\hat{\beta} \pm Z_{\frac{\Lambda}{2}} \times \sqrt{var(\hat{\beta})},$$

and

$$\hat{\lambda} \pm Z_{\frac{\Lambda}{2}} \times \sqrt{var(\hat{\lambda})},$$

where, the main diagonal of $J^{-1}(\Phi)$ has $var(.)$'s corresponding to Φ, and $Z_{\frac{\Lambda}{2}}$ is $\frac{\Lambda}{2}$the upper percentile of the standard normal distribution (SND). The derivatives in the OIM $(\alpha, \theta, \beta, \lambda)$ for the unknown parameters are given as

$$\frac{\partial \log L}{\partial \alpha^2} = \frac{n}{(\alpha-1)^2} - \frac{n(1+\log(\alpha))}{(\alpha \log(\alpha))^2} - \frac{1}{\alpha^2} \sum_{i=1}^{n} \frac{x_i^{2\beta}}{x_i^{2\beta} + 2\theta\lambda^2} \tag{50}$$

$$\frac{\partial \log L}{\partial \alpha \partial \beta} = \frac{4\theta\lambda^2}{\alpha} \sum_{i=1}^{n} \frac{\log(x_i) x_i^{2\beta}}{(x_i^{2\beta} + 2\theta\lambda^2)^2} \tag{51}$$

$$\frac{\partial \log L}{\partial \alpha \partial \theta} = -\frac{2\lambda^2}{\alpha} \sum_{i=1}^{n} \frac{x_i^{2\beta}}{(x_i^{2\beta} + 2\theta\lambda^2)^2} \tag{52}$$

$$\frac{\partial \log L}{\partial \alpha \partial \lambda} = -\frac{4\theta\lambda}{\alpha} \sum_{i=1}^{n} \frac{x_i^{2\beta}}{(x_i^{2\beta} + 2\theta\lambda^2)^2} \tag{53}$$

$$\frac{\partial \log L}{\partial \theta^2} = -\frac{n}{\theta^2} + \sum_{i=1}^{n} \frac{8\lambda^4 \log(\alpha) x_i^{2\beta}}{(x_i^{2\beta} + 2\theta\lambda^2)^3} + \sum_{i=1}^{n} \frac{8\lambda^4}{(x_i^{2\beta} + 2\theta\lambda^2)^2} \tag{54}$$

$$\frac{\partial \log L}{\partial \theta \partial \beta} = -2\lambda^2 \log(\alpha) \sum_{i=1}^{n} \frac{x_i^{2\beta}(4\theta\lambda^2 - 2x_i^{2\beta})}{(x_i^{2\beta} + 2\theta\lambda^2)^3} \log(x_i) + \sum_{i=1}^{n} \frac{8\lambda^2 x_i^{2\beta} \log(x_i)}{(x_i^{2\beta} + 2\theta\lambda^2)^2} \tag{55}$$

$$\frac{\partial \log L}{\partial \theta \partial \lambda} = \sum_{i=1}^{n} \frac{16 \log(\alpha)\lambda^3 \theta x_i^{2\beta}}{(x_i^{2\beta} + 2\theta\lambda^2)^3} - \sum_{i=1}^{n} \frac{4\lambda \log(\alpha) x_i^{2\beta}}{(x_i^{2\beta} + 2\theta\lambda^2)^2} + \sum_{i=1}^{n} \frac{16\lambda^3\theta}{(x_i^{2\beta} + 2\theta\lambda^2)^2} - \sum_{i=1}^{n} \frac{8\lambda}{x_i^{2\beta} + 2\theta\lambda^2} \tag{56}$$

$$\frac{\partial \log L}{\partial \beta^2} = -\frac{n}{\beta^2} - 16\theta\lambda^2 \sum_{i=1}^{n} \frac{(\log(x_i))^2 x_i^{2\beta}}{(x_i^{2\beta} + 2\theta\lambda^2)^2} + 16\theta^2\lambda^4 \log(\alpha) \sum_{i=1}^{n} \frac{x_i^{2\beta}(\log(x_i))^2}{(x_i^{2\beta} + 2\theta\lambda^2)^3} \tag{57}$$

$$\frac{\partial \log L}{\partial \beta \partial \lambda} = 16\theta\lambda \sum_{i=1}^{n} \frac{\log(x_i) x_i^{2\beta}}{(x_i^{2\beta} + 2\theta\lambda^2)^2} + \log(\alpha) \sum_{i=1}^{n} \frac{\log(x_i) x_i^{2\beta}}{(x_i^{2\beta} + 2\theta\lambda^2)^2} \left[8\theta\lambda - \frac{32\theta^2\lambda^3}{x_i^{2\beta} + 2\theta\lambda^2}\right] \tag{58}$$

$$\frac{\partial \log L}{\partial \lambda^2} = -\frac{2n}{\lambda^2} + \sum_{i=1}^{n} \frac{x_i^{2\beta}}{(x_i^{2\beta} + 2\theta\lambda^2)^2} \left[\frac{32\theta^2\lambda^2 \log(\alpha)}{x_i^{2\beta} + 2\theta\lambda^2} - 4\theta \log(\alpha)\right] + \sum_{i=1}^{n} \frac{1}{x_i^{2\beta} + 2\theta\lambda^2} \left[\frac{32\theta^2\lambda^2}{x_i^{2\beta} + 2\theta\lambda^2} - 8\theta\right] \tag{59}$$

5. Monte Carlo Simulation Study (MCSS)

Here, we check the behaviour of the MLEs in terms of n (sample size). MCSS is conducted to analyze the MLE behaviour for GAEPPRD. It is utilized for maximizing the function (41). The number of MC replicates are made 1000 times each with $x_1, x_2, \cdots, x_n$ random samples of $n = 25, 50, \cdots, 1000$ sizes. We obtain the average MLEs, mean square errors (MSEs), biases, and absolute biases (ABs) at each sample size using the R package. The mathematical formulae of the average MSE and bias are as under

$$MSE = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\Phi}_i - \Phi)^2$$

$$Bias = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\Phi}_i - \Phi)$$

where $\Phi = (\alpha, \theta, \beta, \lambda)^T$. Empirical results acquired after performing MCSS are reported in Table 1 and graphically arranged in Figures 3–7. We can observe from these tables and figures that:

- MSE and biases decrease as n increases.
- Estimates $\hat{\Phi}$ of GAEPPRD are very steady and are nearer to the true value of parameters as n increases.

Table 1. Results of the simulation (MLE, MSE, and bias) for GAEPPRD are reported.

	Set 1: α = 1.01, θ = 0.35, β = 0.52, λ = 2.10				**Set 2: α = 1.01, θ = 0.35, β = 1.72, λ = 2.10**		
n	**Parameter**	**MLE**	**MSE**	**Bias**	**MLE**	**MSE**	**Bias**
25	$\hat{\alpha}$	1.491538	1.726753	0.481538	1.718767	2.609741	0.708767
	$\hat{\theta}$	0.431429	0.021406	0.081429	0.427190	0.023709	0.077190
	$\hat{\beta}$	0.551041	0.009131	0.031041	1.829149	0.101138	0.109149
	$\hat{\lambda}$	2.140928	0.008750	0.040928	2.143801	0.018360	0.043801
50	$\hat{\alpha}$	1.262727	0.865884	0.252727	1.601448	2.219799	0.591448
	$\hat{\theta}$	0.398809	0.008535	0.048809	0.384631	0.009805	0.034631
	$\hat{\beta}$	0.537213	0.003960	0.017213	1.773581	0.045368	0.053581
	$\hat{\lambda}$	2.131701	0.003116	0.031701	2.125463	0.006916	0.025462
100	$\hat{\alpha}$	1.218750	0.748422	0.208749	1.451656	1.599158	0.441656
	$\hat{\theta}$	0.372961	0.004158	0.022961	0.366762	0.005201	0.016762
	$\hat{\beta}$	0.528802	0.001813	0.008801	1.741680	0.019225	0.021679
	$\hat{\lambda}$	2.118585	0.001805	0.018585	2.119403	0.003339	0.019403
300	$\hat{\alpha}$	1.088609	0.269229	0.078609	1.227820	0.780194	0.217819
	$\hat{\theta}$	0.358947	0.001433	0.008947	0.357817	0.002398	0.007817
	$\hat{\beta}$	0.526016	0.000528	0.006016	1.724552	0.006559	0.004552
	$\hat{\lambda}$	2.107153	0.000462	0.007153	2.112156	0.001238	0.012156
500	$\hat{\alpha}$	1.046439	0.108999	0.036439	1.148274	0.462359	0.138274
	$\hat{\theta}$	0.356214	0.000715	0.006214	0.357542	0.001467	0.007541
	$\hat{\beta}$	0.524586	0.000339	0.004586	1.718011	0.004018	-0.00198
	$\hat{\lambda}$	2.103363	0.000264	0.003363	2.106691	0.002007	0.006690
700	$\hat{\alpha}$	1.020115	0.017619	0.010115	1.121509	0.383672	0.111508
	$\hat{\theta}$	0.354464	0.000397	0.004464	0.356875	0.001113	0.006875
	$\hat{\beta}$	0.523886	0.000243	0.003886	1.723349	0.002748	0.003348
	$\hat{\lambda}$	2.101910	0.000118	0.001910	2.106004	0.000502	0.006003
750	$\hat{\alpha}$	1.019648	0.017153	0.009648	1.109350	0.324241	0.099350
	$\hat{\theta}$	0.354333	0.000361	0.004333	0.356346	0.000887	0.006346
	$\hat{\beta}$	0.524141	0.000234	0.004141	1.723205	0.002346	0.003205
	$\hat{\lambda}$	2.101872	0.000135	0.001872	2.103226	0.001489	0.003226
1000	$\hat{\alpha}$	1.018177	0.016033	0.008176	1.089865	0.256770	0.079865
	$\hat{\theta}$	0.353585	0.000300	0.003584	0.356587	0.000594	0.006587
	$\hat{\beta}$	0.522979	0.000179	0.002978	1.722570	0.001963	0.002570
	$\hat{\lambda}$	2.101511	0.000095	0.001511	2.101308	0.002263	0.001308

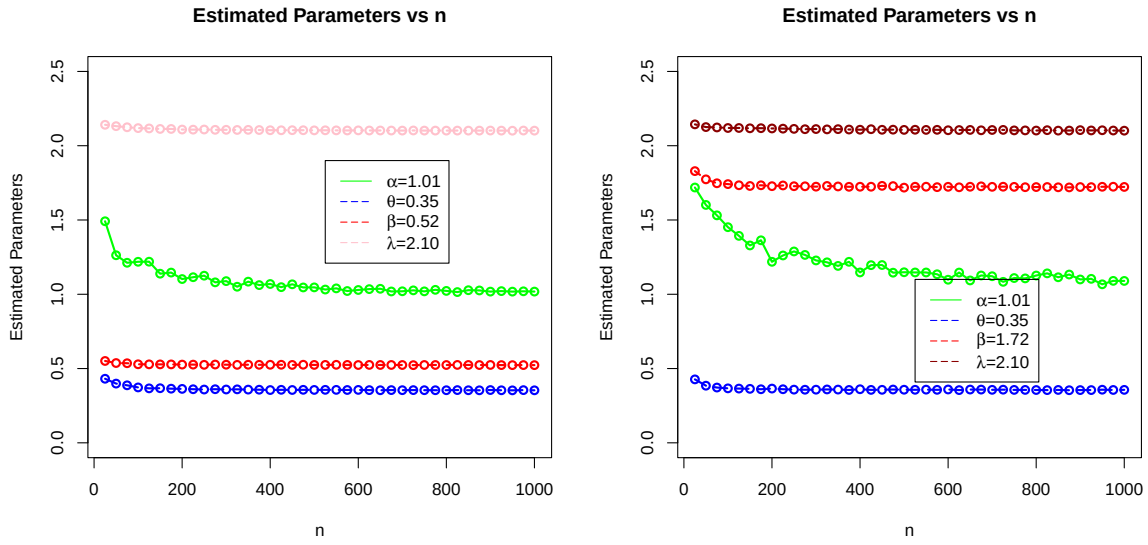


Figure 3. Plots of MLEs of the GAEPPRD for Set 1: $\alpha = 1.01, \theta = 0.35, \beta = 0.52, \lambda = 2.10$ and Set 2: $\alpha = 1.01, \theta = 0.35, \beta = 1.72, \lambda = 2.10$.

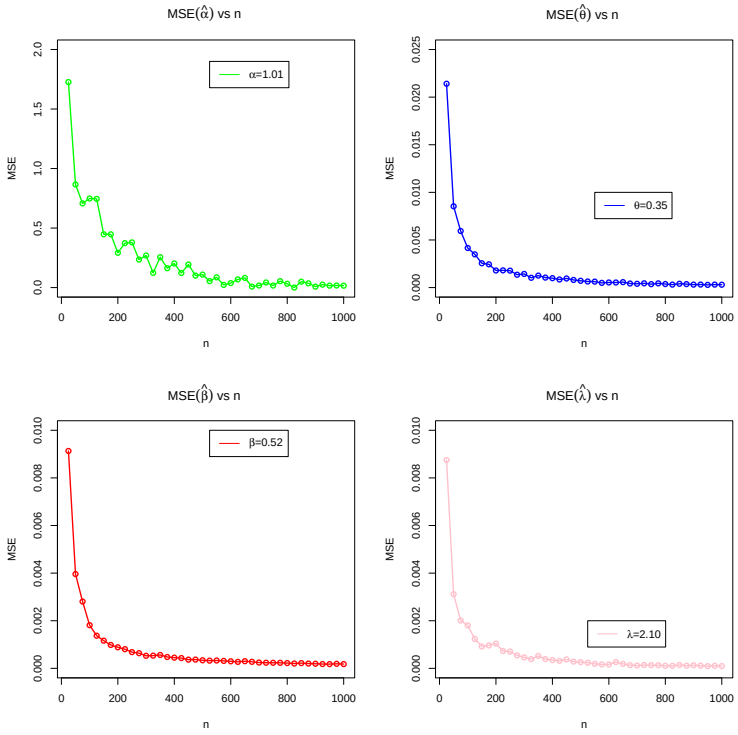


Figure 4. For set 1, MSE plots of GAEPPRD : $\alpha = 1.01, \theta = 0.35, \beta = 0.52, \lambda = 2.10$.

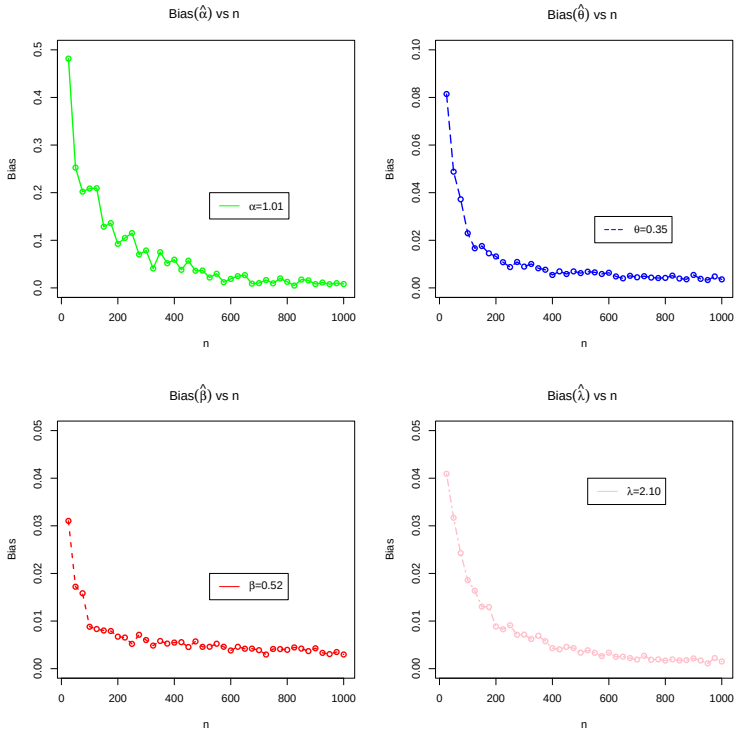


Figure 5. For set 1, bias plots of GAEPPRD: $\alpha = 1.01, \theta = 0.35, \beta = 0.52, \lambda = 2.10$.

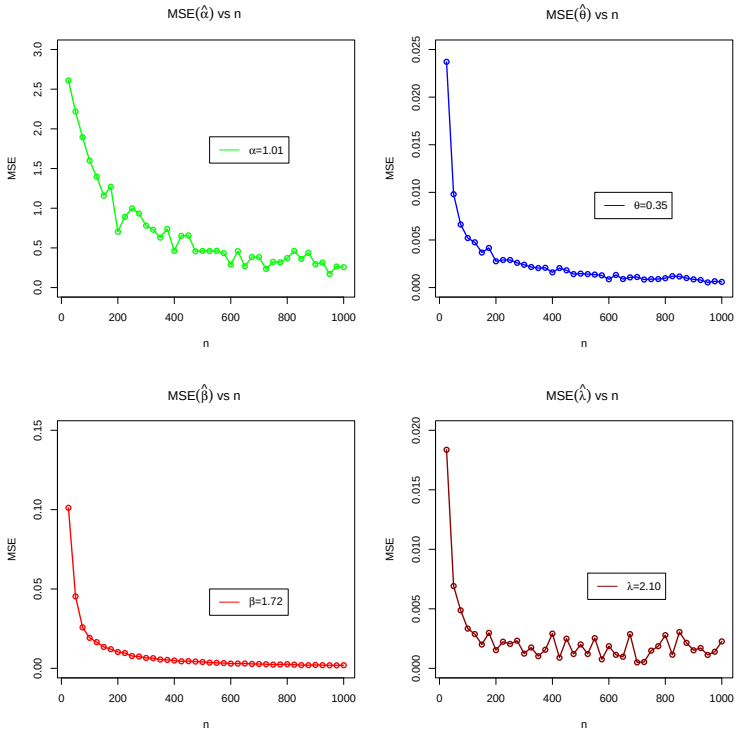


Figure 6. For set 2, $\alpha = 1.01, \theta = 0.35, \beta = 1.72, \lambda = 2.10$, MSE plots of the GAEPPRD.

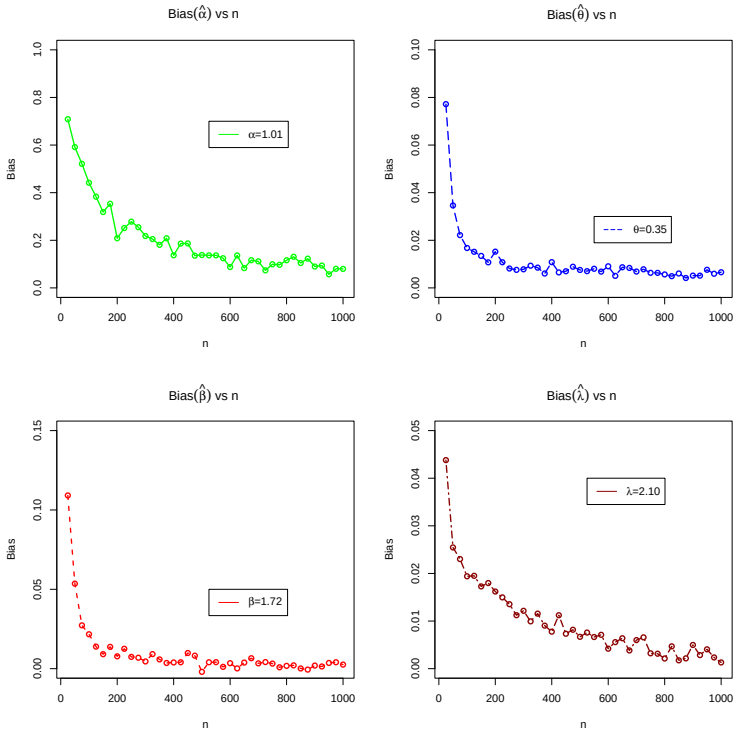


Figure 7. For set 2, $\alpha = 1.01, \theta = 0.35, \beta = 1.72, \lambda = 2.10$, bias plots of the GAEPPRD.

6. Applications

Here, we present applications of derived models using lifetime datasets. We compare fits of the GAEPPRD with other known lifetime models.

To verify which model is suitable for the considered data, we used certain analytical measures (AMs). These AMs included (i) discrimination measures (DMs) and (ii) goodness of fit measures (GFMs). The DMs are given by

- Akaike information criterion (IC)

$$AIC = -2l(\hat{\Phi}) + 2q$$

- Bayesian IC

$$BIC = -2l(\hat{\Phi}) + q\ \log(n)$$

- The Hannan–Quinn IC

$$HQIC = -2l(\hat{\Phi}) + 2q\ \log(\log(n))$$

- Consistent AIC

$$CAIC = -2l(\hat{\Phi}) + \frac{2n\,q}{n-q-1}$$

The GFMs are given by

- Test statistics: Anderson–Darling (AD)

$$AD = -n^{-1}\sum_{i=1}^{n}(-1+2i)\Big[\log\Big\{1-G(x_{n-i+1})\Big\} + \log(G(x_i))\Big] - n$$

- Test statistics: Cramér–von Mises (CM)

$$CM = \sum_{i=1}^{n}\Big[\frac{i}{n} - \frac{1}{2n} - G(x_i)\Big]^2 + \frac{1}{12n}$$

where $l(\hat{\Phi})$ is LLF at MLEs, q (no. of model parameters), n (sample size), and x_i (calculated value at the ith position in the sample point according to ascending order).

6.1. Dataset 1

From [18], the strength of glass fibers, length 1.5 cm, dataset was taken, consisting of 72 observations. The observations are as follows:

12, 95, 15, 96, 22, 98, 99, 109, 24, 24, 110, 32, 32, 33, 34, 38, 38, 43, 44, 121, 48, 127, 52, 129, 53, 54, 54, 131, 55, 56, 143, 146, 146, 57, 58, 58, 59, 60, 60, 175, 175, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 91, 73, 87, 75, 85, 76, 84, 76, 83, 81, 211, 233, 258, 258, 263, 297, 341, 376, 341.

6.2. Dataset 2

As reported by [19], the water runoff (in mm) of 34 storm events; dataset observed from a mall watershed in Korea (west of the city of Suwon) . The observations are as follows:

0.9, 0.6, 16.8, 59.3, 2, 78.2, 30.7, 146.8, 1.8, 3.4, 1.1, 0.8, 2.5, 6.1, 17, 5.1, 216.2, 8.1, 1.6, 2, 2, 0.8, 0.8, 2.9, 7.3, 13.3, 181.7, 20.5, 24.1, 33.5, 89.1, 7.2, 6, 75.9.

We fit datasets 1 and 2 by the GAEPPRD along with very well-known lifetime models, namely; Rayleigh [1], Alpha Power Rayleigh (APR) [18], Exponential, Alpha Power Exponential (APE) [13], Exponentiated Inverse Rayleigh (EIR), Alpha Power Exponentiated Inverse Rayleigh (APEIR) [20], Gumbel and Alpha Power Gumbel (APG) distributions. The competing models have p.d.f s, as folows:

- The RD

$$g(x;\lambda) = \frac{x}{\lambda^2}e^{-\frac{x^2}{2\lambda^2}}, \quad x,\lambda > 0.$$

- The APR distribution

$$g(x;\lambda,\alpha) = (\alpha-1)^{-1}\log(\alpha)\frac{x}{\lambda^2}e^{-\frac{x^2}{2\lambda^2}}\alpha^{1-e^{-\frac{x^2}{2\lambda^2}}},$$

$$x>0,\alpha,\lambda>0,\alpha\neq 1.$$

- The exponential distribution

$$g(x;\lambda) = \lambda e^{-\lambda x}, \quad \lambda>0, x>0.$$

- The APE distribution

$$g(x;\alpha,\lambda) = \frac{\log(\alpha)}{\alpha-1}\lambda e^{-\lambda x}\alpha^{1-e^{-\lambda x}},$$

$$x>0, \quad \alpha,\lambda>0,\alpha\neq 1.$$

- The EIR distribution

$$g(x;\lambda,\beta) = \frac{\beta}{\lambda^2 x^3}e^{-\frac{\beta}{2\lambda^2x^2}}, x>0,\ \beta,\lambda>0.$$

- The APEIR distribution

$$g(x;\lambda,\alpha,\beta) = \log(\alpha)\{\alpha-1\}^{-1}\frac{\beta}{\lambda^2x^3}e^{-\frac{\beta}{2\lambda^2x^2}}\alpha^{-\frac{\beta}{2\lambda^2x^2}},$$

$$x>0, \quad \lambda,\alpha,\beta>0,\alpha\neq 1.$$

- The Gumbel distribution

$$g(x;\sigma,\mu) = \frac{1}{\sigma}e^{-e^{-\frac{x-\mu}{\sigma}}-\frac{x-\mu}{\sigma}},$$

$$-\infty<x,\mu<\infty, \quad \sigma>0.$$

- The APG distribution

$$g(x;\alpha,\mu,\sigma) = \frac{\log(\alpha)}{\sigma(\alpha-1)}e^{-e^{-\frac{x-\mu}{\sigma}}-\frac{x-\mu}{\sigma}}\alpha^{e^{-e^{-\frac{x-\mu}{\sigma}}}}$$

$$-\infty<x,\mu<\infty, \quad \alpha,\sigma>0, \quad \alpha\neq 1.$$

6.3. Illustration

Here, we illustrate GAEPPRD by analyzing datasets 1 and 2. For the analyzed data, the MLEs with the standard error (SE) of GAEPPRD and other considered competing models are recorded in Tables 2 and 3. The AMs of the fitted models are recorded in Tables 4 and 5. In light of the above measures, GAEPPRD has minimum DMs, GFMs, a Kolmogorov–Smirnov (KS) value, and the highest *p*-value. Thus, we can infer that GAEPPRD represents the best fit among the compared models for datasets 1 and 2. In favor of the results recorded in Tables 4 and 5, the estimated CDF and p.d.f of the acquired distributions are plotted in Figures 8 and 9. In Figure 10, the Kaplan–Meier survival (KMS) plots are sketched. From Figures 8 and 9, we can see that the sample histogram is close to the fitted density of GAEPPRD and empirical CDF is nearer to the fitted CDF.

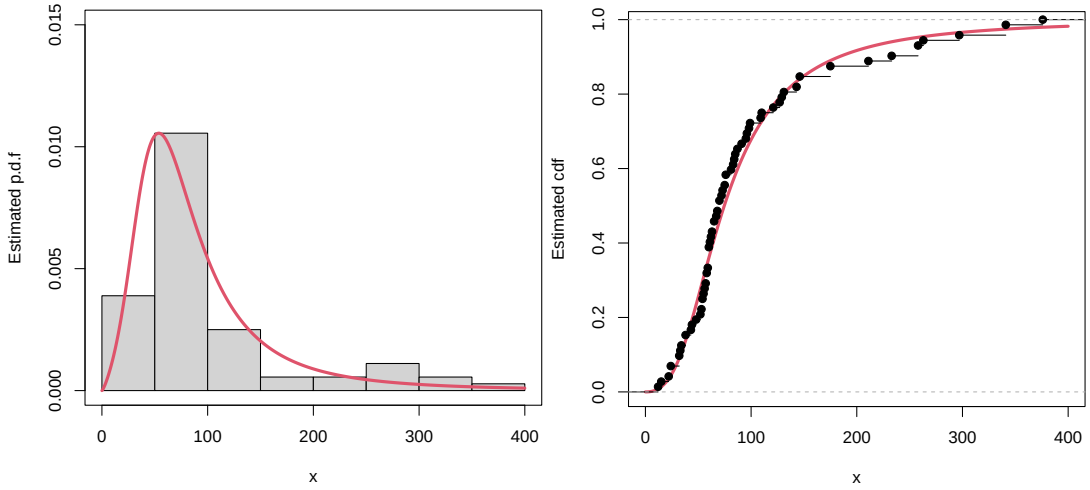


Figure 8. Estimated p.d.f (**left**) and CDF (**right**) of the GAEPPR model corresponding to strength glass fiber data.

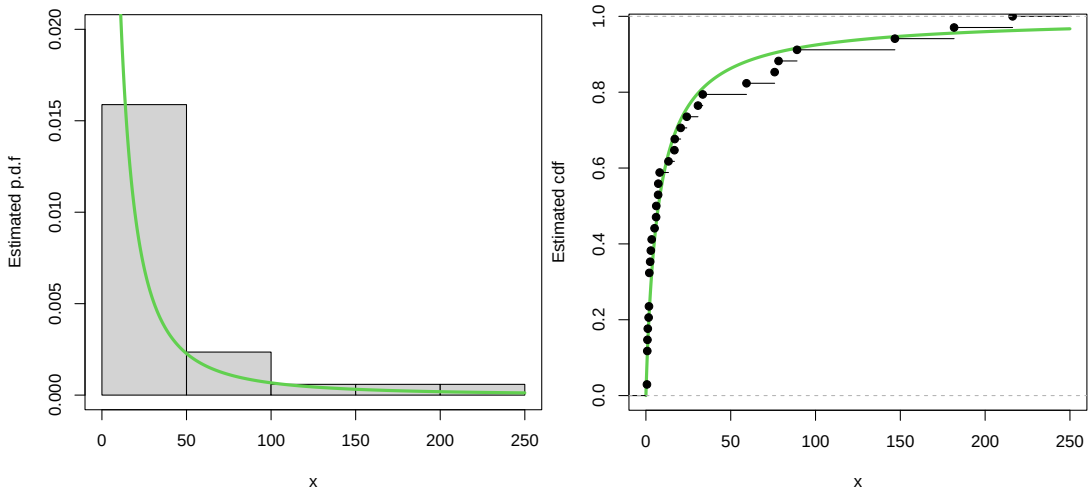


Figure 9. Estimated p.d.f (**left**) and CDF (**right**) of the GAEPPR model corresponding to runoff data.

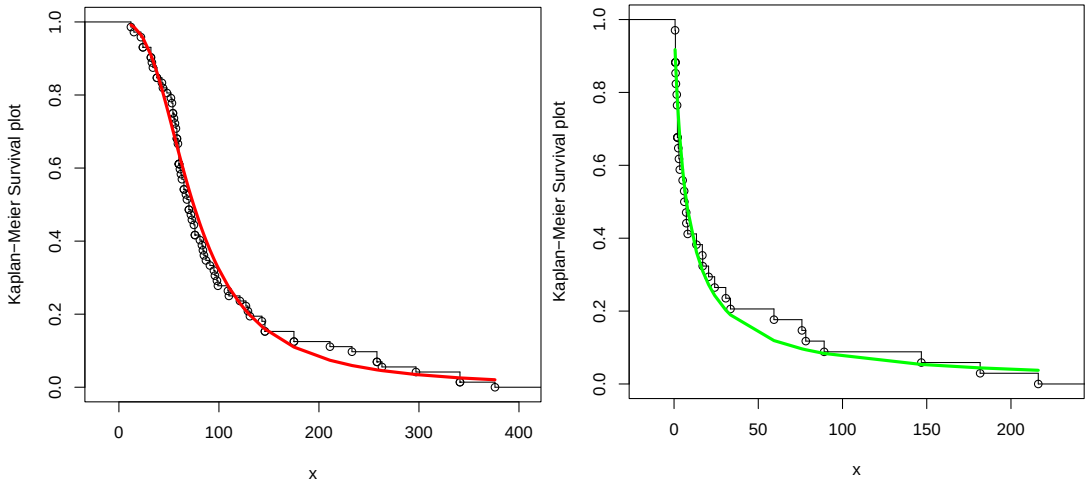


Figure 10. KMS plots of the GAEPPR model for (a) the strength glass fiber data (**left**) (b) the runoff data (**right**).

Table 2. MLEs with SE (in parentheses) of competing models for strength glass fiber data.

Dist.	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\sigma}$
GAEPPR	25.50294	17.64514	1.14882	12.00927		
	(48.30090)	(94.55587)	(0.15376)	(32.76318)		
Rayleigh				90.69159		
				(5.34339)		
APR	0.00203			189.42548		
	(0.00064)			(17.43809)		
Exponential				0.01004		
				(0.00117)		
APE	24.52761			0.01894		
	(20.07510)			(0.00247)		
EIR			11.29459	0.05081		
			(28.88565)	(0.06488)		
APEIR	88.31763		68.05749	0.20623		
	(93.91820)		(971.56445)	(1.47179)		
Gumbel					67.92667	46.97277
					(5.74111)	(4.76069)
APG	0.16026				90.72187	54.92586
	(0.13986)				(12.54540)	(7.55966)

Table 3. MLEs with SE (in parentheses) of competing models for the runoff data.

Dist.	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\sigma}$
GAEPPR	0.99980	0.27685	0.48003	3.50733		
	(2.55687)	(14.65321)	(0.06680)	(92.85352)		
APE	0.04099			0.01802		
	(0.04519)			(0.00578)		
Exponential				0.03190		
				(0.00547)		
APR	0.00458			53.49016		
	(0.00461)			(6.05064)		
Rayleigh				43.51663		
				(3.73153)		
APEIR	37.21066		0.62533	0.40881		
	(36.78043)		(22.80775)	(7.45487)		
EIR			2.55497	0.65753		
			(128.46384)	(16.53021)		
Gumbel					12.19433	25.18078
					(4.42417)	(4.01849)
APG	0.08776				26.40804	29.37897
	(0.08896)				(7.44646)	(5.62504)

Table 4. DMs and GFMs of the competing models for strength glass fiber data.

Dist.	AIC	BIC	CAIC	HQIC	KS	*p*-Value	CM	AD
GAEPPR	786.972	796.078	787.569	790.597	0.081	0.726	0.08915	0.51688
Rayleigh	818.592	820.869	818.649	819.498	0.287	1.385×10^{-5}	0.58751	3.24003
APR	799.508	804.061	799.682	801.321	0.202	0.005752	0.37931	2.10592
Exponential	808.885	811.161	808.942	809.791	0.212	0.003047	0.33759	1.84769
APE	797.160	801.714	797.334	798.973	0.138	0.1302	0.40927	2.27891
EIR	817.472	822.026	817.646	819.285	0.251	0.0002307	0.34022	2.01255
APEIR	790.212	797.042	790.565	792.931	0.115	0.2956	0.12764	0.77645
Gumbel	800.218	804.771	800.392	802.030	0.152	0.07017	0.41974	2.36386
APG	796.915	803.745	797.268	799.634	0.118	0.2672	0.31517	1.77364

Table 5. DMs and GFMs of the competing models for the runoff data.

Dist.	AIC	BIC	CAIC	HQIC	KS	*p*-Value	CM	AD
GAEPPR	285.802	291.908	287.181	287.884	0.101	0.876	0.08707	0.61138
APE	292.423	295.476	292.810	293.464	0.231	0.052	0.15166	0.97717
Exponential	304.287	305.814	304.412	304.808	0.361	0.000290	0.21225	1.31849
APR	400.414	403.466	400.801	401.455	0.528	1.135×10^{-8}	0.26002	1.56803
Rayleigh	441.859	443.385	441.984	442.379	0.603	3.654×10^{-11}	0.30521	1.83314
APEIR	355.477	360.056	356.277	357.039	0.387	7.614×10^{-5}	0.08923	0.61259
EIR	375.059	378.113	375.447	376.101	0.481	2.97×10^{-7}	0.09802	0.68447
Gumbel	343.123	346.175	343.509	344.164	0.279	0.00971	0.69361	3.76934
APG	339.477	344.056	340.277	341.039	0.244	0.03492	0.61374	3.38509

7. Conclusions

In this article, we proposed the GAEP-G family of distributions, which is a new family of continuous distributions. In this regard, a four-parameter special model, GAEPPRD, was thoroughly studied. A few properties of GAEPPRD were derived analytically. A new performance method was checked by MCSS. To demonstrate the capability of the GAEPPRD, we considered two lifetime datasets and matched the DMs and GFMs with other distributions. By considering certain AMs, the GAEPPRD was 'outclass' . We hope that the new contribution will be helpful and will draw in more extensive applications in the field of distribution theory.

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