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On the Existence of Self-Similar Solutions in the Thermostatted Kinetic Theory with Unbounded Activity Domain

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Abstract: This paper is devoted to the mathematical analysis of a spatially homogeneous thermostatted kinetic theory framework with an unbounded activity domain. The framework consists of a partial integro-differential equation with quadratic nonlinearity where the domain of the activity variable is the whole real line. Specifically the mathematical analysis refers firstly to the existence and uniqueness of the solution for the related initial boundary value problem; Secondly the investigations are addressed to the existence of a class of self-similar solutions by employing the Fourier transform method. In particular the main result is obtained for a nonconstant interaction rate and a nonconstant force field. Conclusions and perspectives are discussed in the last section of the paper.

Keywords: kinetic theory; complex system; nonlinearity; Fourier transform; IBV problem

MSC: 35A01; 45M05; 37N25

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1. Introduction

Nonlinear evolution equations have recently been proposed for the modeling of complex phenomena in biology, vehicular traffic, crowds and swarms dynamics, microgrid composed of sources of energy, see the books [1,2]. The existence and uniqueness of solutions and their numerical simulations have been the target of multidisciplinary research activity [3]. In particular, in mathematical physics, important research activity has been focused on the existence and analysis of self-similar solutions, e.g., a solution may be obtained from another solution by the result of a uniform scaling (enlarging or shrinking).

A self-similar solution in fluid dynamics is a form of solution which is similar to itself if the independent and dependent variables are appropriately scaled [4–6]. This type of solution is also called a self-similar solution of the first kind [7]; indeed, the self-similar solution of the second kind exists which cannot be derived from dimensional analysis, see [8–10]. The most important investigations of self-similar solutions in mathematical physics date back to the classical Boltzmann equation [11], see, among others [12–14] and for the self-similar asymptotics the papers [15–18].

This paper aims at proving the existence of a class of *self-similar* solutions for the thermostatted kinetic theory for active particles, which has been proposed in the last decade for the modeling of complex systems subjected to an external force field. According to this theory, the complex system is divided into different subsystems, called functional subsystems, composed of particles expressing the same strategy, called activity. The evolution equations of each functional subsystem is obtained by balancing, into the elementary volume of the microscopic states, the inlet and outlet flows of active particles. The interested reader is referred to papers [19–22] for the theoretical results and to papers [23,24] for the recent applications.

This paper is devoted to the existence of self-similar solutions of a mathematical framework of the thermostatted kinetic theory. Specifically the framework is assumed to be homogeneous with respect to the space and velocity variables; the interactions are assumed to be conservative (conservation of the density and the activation energy), the activity domain is assumed to be unbounded and more precisely the whole real line. In particular, the initial boundary value problem is defined and the existence and uniqueness of the solution is first investigated. The self-similar solutions of the initial boundary value problem are investigated by employing the Fourier transform method. The main result consists of the proof of the theorem of the existence of self-similar solutions and asymptotic self-similar solutions are also investigated.

It is worth stressing that the self-similar solutions are investigated for both constant and nonconstant interaction rates and external force fields. This is a generalization with respect to the previous published papers where the roles of the interaction rates and the force field have always been assumed to be constants, see [22].

The contents of the present paper are organized into five more sections which follow this introduction. Specifically, Section 2 deals with the thermostatted kinetic theory and the related stationary problem; Section 3 is devoted to the initial boundary value problem and the existence and uniqueness of the solution. The existence of self-similar solutions are investigated in Sections 4 and 5, where the analysis is performed in the case of constant and nonconstant interaction rates and force fields, respectively. Conclusions and perspectives are postponed to Section 6.

2. The Thermostatted Kinetic Theory for an Unbounded Activity Domain

This section briefly reviews the main elements of the thermostatted kinetic theory for a complex system described by a distribution function $f = f(t,u) : [0,+\infty[\times\mathbb{R} \to \mathbb{R}^+$, where u denotes the activity variable which models the strategy of the particles composing the system and it is assumed to take values in an unbounded domain, and specifically $\mathbb{R}$. Let $\mathcal{A}(u_*,u^*,u) : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$, $\eta(u_*,u^*) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$, and $F(u) : \mathbb{R} \to \mathbb{R}^+$, f is the solution of the following partial integro-differential equation with quadratic nonlinearity:

$$\partial_t f(t,u) + T_F[f](t,u) = J[f](t,u), \tag{1}$$

where

- $J[f](t,u) = G[f](t,u) - L[f](t,u)$ models the *net flux* of particles (gain G and loss L):

$$G[f](t,u) = \int_{\mathbb{R}\times\mathbb{R}} \eta(u_*,u^*)\,\mathcal{A}(u_*,u^*,u) f(t,u_*) f(t,u^*)\,du_*\,du^*,$$

$$L[f](t,u) = f(t,u)\int_{\mathbb{R}} \eta(u_*,u^*)\,f(t,u^*)\,du^*,$$

- $T_F[f](t,u)$ defines the following thermostat term (according to [25,26]):

$$T_F[f](t,u) = \partial_u\left(\left(F(u) - u\int_{\mathbb{R}} u\,F(u)\,f(t,u)\,du\right) f(t,u)\right),$$

which allows us to keep constant the *activation energy* [19]:

$$\mathbb{E}_2[f] = \int_{\mathbb{R}} u^2\,f(t,u)\,du.$$

Assume that

$$\begin{aligned} &\eta(u_*,u^*) = \eta > 0, \qquad \forall u_*,u^* \in \mathbb{R},\\ &F(u) = F > 0, \qquad \forall u \in \mathbb{R}, \end{aligned}$$

then the mathematical framework (1) can be rewritten as follows:

$$\begin{aligned}\partial_t f(t,u) + F\partial_u((1-u\,\mathbb{E}_1[f](t))f(t,u)) &= \\ \int_{\mathbb{R}\times\mathbb{R}} \eta\,\mathcal{A}(u_*,u^*,u)f(t,u_*)f(t,u^*)\,du_*\,du^* &- \eta\,f(t,u)\,\mathbb{E}_0[f](t),\end{aligned} \tag{2}$$

where

$$\mathbb{E}_0[f](t) = \int_{\mathbb{R}} f(t,u)\,du, \quad \mathbb{E}_1[f](t) = \int_{\mathbb{R}} u\,f(t,u)\,du$$

The mathematical framework is conservative, indeed $\mathbb{E}_0[f](t)$ is constant. In what follows it is assumed that $\mathbb{E}_0[f](t) = 1$, for $t > 0$.

A *nonequilibrium stationary solution* of (2) is a function $g(u) : \mathbb{R} \to \mathbb{R}^+$, which satisfies the related *stationary problem*:

$$J[g](u) - T_F[g](u) = 0. \tag{3}$$

The existence of a positive stationary solution $g(u) \in C(\mathbb{R} \setminus \{\alpha\})$, with $\alpha = \frac{1}{\mathbb{E}_1[f]}$, of the stationary problem (3) has been proved in [27]. In particular, [21]:

$$\mathbb{E}_1[f](t) = \int_{\mathbb{R}} uf(t,u)\,du \xrightarrow{t\to+\infty} \mathbb{E}_1^+ = \frac{-\eta + \sqrt{\eta^2+4F^2}}{2F}.$$

3. The Initial Boundary Value Problem: Existence and Uniqueness

Let f_0 be a suitable function (*initial data*). The *Initial Boundary Value Problem* (*IBVP*) related to Equation (2) writes:

$$\begin{cases} \partial_t f(t,u) + F\,\partial_u\left(\left(1 - u\displaystyle\int_{\mathbb{R}} u\,f(t,u)\,du\right)f(t,u)\right) = J(t,u) & (t,u) \in [0,+\infty[\times\mathbb{R} \\ f(0,u) = f_0(u) & u \in \mathbb{R} \\ \displaystyle\lim_{|u|\to+\infty} f(t,u) = 0. \end{cases} \tag{4}$$

Let $\mathcal{K}(D_u) = \{f = f(t,u) : [0,+\infty[\times D_u \to \mathbb{R}^+, f(t,u) \in C([0,+\infty[;L^1(D_u)) : \mathbb{E}_0[f](t) = \mathbb{E}_2[f](t) = 1\}$. The following theorem holds true:

Theorem 1. *Assume that:*

(i) $\mathcal{A}(u_*,u^*,u)f(t,u_*)f(t,u^*)$ *is an integrable function with respect to the elementary measure* du_*du^**;*
(ii) $\displaystyle\int_{\mathbb{R}} \mathcal{A}(u_*,u^*,u)\,du = 1$*, for all* $u_*,u^* \in \mathbb{R}$*;*
(iii) $\displaystyle\int_{\mathbb{R}} u^2\mathcal{A}(u_*,u^*,u)\,du = u_*^2$*, for all* $u_*,u^* \in \mathbb{R}$*;*
(iv) $\displaystyle\int_{\mathbb{R}} u\mathcal{A}(u_*,u^*,u)\,du = 0$*, for all* $u_*,u^* \in \mathbb{R}$*;*
(v) $f_0(u) \in L^1(D_u)$*;*
(vi) $\displaystyle\int_{\mathbb{R}} f_0(u)\,du = \int_{\mathbb{R}} u^2 f_0(u)\,du = 1.$

Then there exists a unique function $f \in \mathcal{K}(\mathbb{R})$*, which is a solution of the IBVP (4). Moreover the solution depends continuously on the initial data* f_0*.*

Proof. Firstly, assume that $D_u = [-a, a]$, $a > 0$. The existence and uniqueness of the solution $f(t,u) \in \mathcal{K}(D_u)$ of the IBVP:

$$\begin{cases} \partial_t f(t,u) + F\partial_u\left(\left(1 - u\int_{D_u} u\, f(t,u)\, du\right) f(t,u)\right) = J[f](t,u) & (t,u) \in [0,+\infty[\times D_u \\ f(0,u) = f_0(u) & u \in D_u \\ f(t,u) = 0 & u \in \partial D_u, \end{cases} \tag{5}$$

has been proved in [28] under the assumptions $(i) - (iii)$, and the continuous dependence with respect to the initial data $f_0(u)$ has been gained in [29]. Accordingly, for every $R > 0$, there exists a unique solution of the IBVP (5) for $D_u =]-R, R[$. Let $\varphi_R(u)$ be the following cut-off function:

$$\varphi_R(u) := \begin{cases} 1 & u \in]-R, R[\\ 0 & \text{otherwise}. \end{cases} \tag{6}$$

Multiplying the two sides of the $(5)_1$ by $\varphi_R(u)$, the IBVP (4) is restricted to $[-R, R]$, $R \in \mathbb{N}$. That is, $f_R(t,u) = 0$ for $|u| = R$ (i.e., on the boundary of $[-R, R]$). This is reasonable since it is required that $f \in L^1(D_u)$. Then there exists a unique function $f_R(t,u) \in \mathcal{K}(]-R, R[)$ solution of the restricted problem. Let $R_2 > R_1 > 0$, and $f_{R_1}(t,u)$ the solution related to $]-R_1, R_1[$ and $f_{R_2}(t,u)$ the solution related to $]-R_2, R_2[$. By using the uniqueness, it follows, for $t > 0$:

$$f_{R_1}(t,u) = f_{R_2}(t,u) \qquad |u| < R_1.$$

Indeed in the set $|u| < R_1$ one has:

$$\varphi_{R_1}(u) = \varphi_{R_2}(u),$$

and then

$$f_{R_1}(t,u) = \varphi_{R_1}(u) f(t,u) = \varphi_{R_2}(u) f(t,u) = f_{R_2}(t,u), \qquad t > 0.$$

For every $R > 0$,

$$\mathbb{E}_0[f_R](t) = \int_{\mathbb{R}} f_R(t,u)\, du = \int_{[-R,R]} f_R(t,u)\, du$$

is a monotone sequence of R since

$$\int_{\mathbb{R}} f_{R_2}\, du \geq \int_{\mathbb{R}} f_{R_1}\, du$$

for $R_2 \geq R_1$. Then it admits a limit $l \in \mathbb{R}$, i.e.,

$$\mathbb{E}_0[f_R](t) = \int_{[-R,R]} f_R(t,u)\, du \xrightarrow{R\to\infty} l.$$

It is necessary to prove that there exists a function $f(t,u) \in L^1(\mathbb{R})$ such that:

$$l = \int_{\mathbb{R}} f(t,u)\, du.$$

In order to prove it, observe that there exists $f(t,u) \in L^1(\mathbb{R})$:

$$\mathbb{E}_0[f_R](t) = \int_{[-R,R]} f_R(t,u)\, du = \int_{[-R,R]} \varphi_R(u)\, f(t,u). \tag{7}$$

Indeed, if $R_2 > R_1 > 0$, since $f_{R_1}(t,u) = f_{R_2}(t,u)$, for $|u| \leq R_1$, we have that:

$$\varphi_{R_1}(u)\, f_1(t,u) = f_{R_1}(t,u) = f_{R_2}(t,u) = \varphi_{R_2}(u)\, f_2(t,u),$$

and then

$$f_1(t,u) = f_2(t,u), \qquad |u| \le R_1.$$

By passing to the limit, as R goes to infinity, into the (7), one has:

$$\mathbb{E}_0[f_R](t) = \int_{[-R,R]} f_R(t,u)\, du \xrightarrow{R\to\infty} \int_{\mathbb{R}} f(t,u)\, du.$$

The existence of a solution $f(t,u) \in \mathcal{K}(\mathbb{R})$ of the IBVP (4) is thus gained.
The uniqueness has to be proved. Let $f_1(t,u)$ and $f_2(t,u)$ be two of the solutions of $(5)_1$ in the space $\mathcal{K}(\mathbb{R})$, the following quantity has to be estimated:

$$\int_{\mathbb{R}} |f_1(t,u) - f_2(t,u)|\, du.$$

By using the cut-off function $\varphi_R(u)$, for every $R > 0$, the following functions can be considered:

$$f_{1R}(t,u) := \varphi_R(u)\, f_1(t,u)$$
$$f_{2R}(t,u) := \varphi_R(u)\, f_2(t,u).$$

By uniqueness in the bounded domain $D_u = [-R,R]$, it follows that:

$$\int_{[-R,R]} |f_{1R}(t,u) - f_{2R}(t,u)|\, du = 0, \quad \forall R > 0. \tag{8}$$

By using the continuity of the norm and passing to the limit, as R goes to infinity, in the (8), one has:

$$0 = \int_{[-R,R]} |f_{1R}(t,u) - f_{2R}(t,u)|\, du \xrightarrow{R\to\infty} \int_{\mathbb{R}} |f_1(t,u) - f_2(t,u)|\, du. \tag{9}$$

Then,

$$\int_{\mathbb{R}} |f_1(t,u) - f_2(t,u)|\, du = 0,$$

and the uniqueness of the solution of the IBVP (4) is thus gained.

Finally, by improving the technique of [29] for the bounded case $D_u = [-R, R]$, the continuous dependence on the initial data for the framework (4) can be gained.

As a matter of fact, assume that:

$$\left\| f_1^0(u) - f_2^0(u) \right\|_{L^1(D_u)} \le \delta,$$

for $\delta > 0$. There exist $f_1(t,u)$, $f_2(t,u) \in \mathcal{K}(\mathbb{R})$ solutions of the problem (5) with initial data, respectively, f_1^0 and f_2^0. Then,

$$\begin{aligned} f_1(t,u) - f_2(t,u) = {} & \left(f_1^0(u) - f_2^0(u) \right) \\ & + \int_0^t (J[f_1,f_1](\tau,u) - J[f_2,f_2](\tau,u))\, d\tau \\ & + F \int_0^t \partial_u((1 - u\mathbb{E}_1[f_2](\tau)) f_2(\tau,u))\, d\tau \\ & - F \int_0^t \partial_u((1 - u\mathbb{E}_1[f_1](\tau)) f_1(\tau,u)), d\tau. \end{aligned} \tag{10}$$

By estimating the terms of the right-hand side of the (10) one has:

$$
\begin{aligned}
&\int_{D_u} |f_1(t,u) - f_2(t,u)|\, du \le \delta \\
&+ 2\,\eta \int_0^t \left(\int_{D_u} |f_2(\tau,u) - f_1(\tau,u)|\, du \right) d\tau \\
&+ \eta \int_0^t \left(\int_{D_u} |f_2(\tau,u) - f_1(\tau,u)|\, du \right) d\tau \\
&+ 2F\big(C(\eta,F) + \mathbb{E}_1^+\big) \int_0^t \left(\int_{D_u} |f_2(\tau,u) - f_1(\tau,u)|\, du \right) d\tau.
\end{aligned}
\tag{11}
$$

By reordering the the terms of this last relation and using the constant

$$
\bar{C} := \big(3\,\eta + 2\,F\big(C(\eta,F) + \mathbb{E}_1^+\big)\big),
$$

we get

$$
\begin{aligned}
&\int_{D_u} |f_1(t,u) - f_2(t,u)|\, du \\
&\le \delta + \bar{C} \int_0^t \left(\int_{D_u} |f_2(\tau,u) - f_1(\tau,u)|\, du \right) d\tau.
\end{aligned}
\tag{12}
$$

Finally, the Gronwall inequality [30] and inequality (12) yield

$$
\int_{D_u} |f_1(t,u) - f_2(t,u)|\, du \le \delta\, e^{\bar{C}\,t},
\tag{13}
$$

for all $t > 0$, so that, for $T > 0$,

$$
\max_{t\in[0,T]} \left(\int_{D_u} |f_1(t,u) - f_2(t,u)|\, du \right) \le \delta\, e^{\bar{C}\,t},
$$

and our claim is proved. □

4. Existence of a Class of Self-Similar Solutions

Let $\lambda > 0$ and $f_\lambda(t,u)$ be the scaled solution of the framework (1):

$$
f_\lambda(t,u) := f\left(t, \frac{u}{\lambda}\right).
\tag{14}
$$

Let $f(t,u)$ be the solution of Equation (1) and $\hat{f}(t,k)$ its *Fourier transform* defined as follows:

$$
\hat{f}(t,k) = \int_{\mathbb{R}} f(t,u) e^{-2\pi i k u}\, du, \quad k \in \mathbb{R},
\tag{15}
$$

and $\hat{g}(k)$ the Fourier transform of $g(u)$ (solution of the related nonequilibrium stationary problem)

$$
\hat{g}(k) = \int_{\mathbb{R}} g(u) e^{-2\pi i k u}\, du, \quad k \in \mathbb{R}.
\tag{16}
$$

In [21] the following result has been proved:

$$
\left|\hat{f}(t,k) - \hat{g}(k)\right| \le \left(|k|^{2+p} + |k|\right) e^{-\frac{\sqrt{\eta^2+4F^2}}{F}t}.
$$

Specifically, $\hat{f}(t,k)$ converges exponentially fast, as $t \to +\infty$, to $\hat{g}(k)$.
The Fourier transform of the Equation (1) writes as follows:

$$
\partial_t \hat{f}(t,k) + 2\pi i k F \hat{f}(t,k) + F k \mathbb{E}_1[f](t) \partial_k \hat{f}(t,k) + \eta \hat{f}(t,k) = \hat{G}[\hat{f}],
\tag{17}
$$

where

$$\hat{G}[\hat{f}] = \int_{\mathbb{R}} \hat{f}(t, k\alpha(u)) * \hat{\mathcal{A}}(u_*, u^*, u) * \hat{f}(t, k\beta(u))\, du,$$

and $*$ denotes the convolution product. Specifically, the state u is assumed to be a linear combination of u_* and u^* and then

$$u = \alpha(u)u_* + \beta(u)u^*.$$

The following result is the main aim of the present paper.

Theorem 2. *Let the assumptions* (i), (ii), (iii) *hold true. Then there exists a solution of the problem* (4)*, such that:*

$$\hat{f}_\lambda(t,k) = \lambda\, \hat{f}(t, \lambda\, k) \qquad k \in \mathbb{R}, \quad \lambda > 0. \tag{18}$$

Moreover, if the initial data $f_0(u)$ *is self-similar, i.e.,* $f_0(u) = f_0(|u|)$*, and there exists* $\alpha > 1$ *such that*

$$\hat{f}_\lambda(t, \lambda k) = o\left(\frac{1}{\lambda^\alpha}\right), \qquad \lambda \to +\infty,$$

then there exists an asymptotic self-similar solution $\gamma(|k|)$ *such that:*

$$\hat{f}_\lambda(t,k) \xrightarrow{t \to +\infty} \gamma(|k|). \tag{19}$$

Proof. Let $f(t,u) \in \mathcal{K}(\mathbb{R})$ be the solution of the IBVP (4), $\psi(t,u) \in \mathcal{K}(\mathbb{R})$ and $\lambda > 0$. Let $S[\psi](t,k)$ and $Z[\psi,\lambda](t,k)$ be the following operators:

$$S[\psi](t,k) := \int_{\mathbb{R}} \psi(u)\, e^{-2\pi i k\, u}\, du, \tag{20}$$

$$Z[\psi,\lambda](t,k) := \int_{\mathbb{R}} \psi\left(\frac{u}{\lambda}\right) e^{-2\pi i k\, u}\, du. \tag{21}$$

By using the (20), one has:

$$S[f](t,k) = \hat{f}(t,k),$$

which solves Equation (17).

Let $f(t,u)$ be the solution of (4). Bearing the expression of $f_\lambda(t,u)$ in mind, the (21) rewrites:

$$Z[f,\lambda](t,k) = \int_{\mathbb{R}} f\left(t, \frac{u}{\lambda}\right) e^{-2\pi i k\, u}\, du = \int_{\mathbb{R}} f_\lambda(t,u) e^{-2\pi i k\, u}\, du = \hat{f}_\lambda(t,k).$$

One observes that $\hat{f}_\lambda(t,k)$ is the solution of the Fourier transform of the rescaled Equation (1). Specifically:

$$\begin{aligned} \int_{\mathbb{R}} \partial_t f_\lambda(t,u) e^{-2\pi i k\, u}\, du + \int_{\mathbb{R}} \partial_u (F(1 - u\mathbb{E}_1[f](t)) f_\lambda(t,u)) e^{-2\pi i k\, u}\, du \\ = \int_{\mathbb{R}} J[f_\lambda](t,u) e^{-2\pi i k\, u}\, du. \end{aligned} \tag{22}$$

The first term on the right hand side of the (22) writes:

$$\int_{\mathbb{R}} \partial_t f_\lambda(t,u) e^{-2\pi i k\, u}\, du = \partial_t \hat{f}_\lambda(t,k). \tag{23}$$

By straightforward calculations, the second term of the right hand side of the Equation (22) rewrites:

$$\begin{aligned}
\int_{\mathbb{R}} \partial_u \Big(F\Big(1 - u\mathbb{E}_{[}f](t)\Big) f\Big(t, \frac{u}{\lambda}\Big)\Big) e^{-2\pi i k\, u}\, du \\
&= F \int_{\mathbb{R}} \partial_u \Big((1 - u\mathbb{E}_1[f](t)) f\Big(t, \frac{u}{\lambda}\Big)\Big) e^{-2\pi i k\, u}\, du \\
&= -F \int_{\mathbb{R}} (1 - u\, \mathbb{E}_1[f](t)) f\Big(t, \frac{u}{\lambda}\Big) \Big(-2\pi i \frac{k}{\lambda}\Big) e^{-2\pi i k\, u}\, du \\
&= (2\pi i k)\, F \int_{\mathbb{R}} (1 - u\, \mathbb{E}_1[f](t)) f\Big(t, \frac{u}{\lambda}\Big) e^{-2\pi i k\, u}\, du \\
&= 2\pi i k\, F\, \hat{f}_\lambda(t,k) - 2\pi i k F \mathbb{E}_1[f](t) \int_{\mathbb{R}} u\, f\Big(t, \frac{u}{\lambda}\Big) e^{-2\pi i k u}\, du.
\end{aligned} \tag{24}$$

The last term of the right hand side of the (24) rewrites:

$$\begin{aligned}
-2\pi i k F \mathbb{E}_1[f](t) \int_{\mathbb{R}} u\, f\Big(t, \frac{u}{\lambda}\Big) e^{-2\pi i k u}\, du &= -Fk\mathbb{E}_1[f](t) \int_{\mathbb{R}} (2\pi i u)\, f\Big(t, \frac{u}{\lambda}\Big) e^{-2\pi i k u}\, du \\
&= Fk\mathbb{E}_1[f](t) \int_{\mathbb{R}} f\Big(t, \frac{u}{\lambda}\Big)\, \partial_k \Big(e^{-2\pi i k u}\Big)\, du \\
&= Fk\mathbb{E}_1[f](t) \partial_k \hat{f}_\lambda(t,k).
\end{aligned} \tag{25}$$

By using the (25), the (24) rewrites:

$$\int_{\mathbb{R}} \partial_u \Big(F(1 - u\mathbb{E}_1[f](t)) f\Big(t, \frac{u}{\lambda}\Big)\Big) e^{-2\pi i k\, u}\, du = 2\pi i k\, F\, \hat{f}_\lambda(t,k) + Fk\mathbb{E}_1[f](t)\partial_k \hat{f}_\lambda(t,k). \tag{26}$$

The right hand side of the (22) rewrites:

$$\int_{\mathbb{R}} J[f_\lambda](t,u) e^{-2\pi i k\, u}\, du = \hat{G}[\hat{f}_\lambda](t,k) - \eta \hat{f}_\lambda(t,k), \tag{27}$$

where

$$\hat{G}[\hat{f}_\lambda](t,k) := \int_{\mathbb{R}} \hat{f}_\lambda(t, k\alpha(u)) * \hat{\mathcal{A}}(k\alpha(u), k\beta(u), u) * \hat{f}_\lambda(t, k\beta(u))\, du.$$

By using the (23), (26) and (27), the (22) rewrites:

$$\partial_t \hat{f}_\lambda(t,k) + 2\pi i k\, F\, \hat{f}_\lambda(t,k) + Fk\mathbb{E}_1[f](t)\partial_k \hat{f}_\lambda(t,k) + \eta \hat{f}_\lambda(t,k) = \hat{G}[\hat{f}_\lambda](t,k). \tag{28}$$

Accordingly ([21]), $\hat{f}(t,k)$ is a solution of (4), and $\hat{f}_\lambda(t,k)$ is a solution (28).
By using the (20) and the (21) and the change of variables $s = \frac{u}{\lambda}$, one has:

$$\begin{aligned}
Z[f,\lambda](t,k) &= \hat{f}_\lambda(t,k) \\
&= \int_{\mathbb{R}} f\Big(t, \frac{u}{\lambda}\Big)\, e^{-2\pi i k\, u}\, du \\
&= \lambda \int_{\mathbb{R}} f(t,s)\, e^{-2\pi i \lambda\, k\, s}\, ds \\
&= \lambda\, \hat{f}(t, \lambda k),
\end{aligned} \tag{29}$$

then the first part of the proof is concluded.
Assume that the initial data $f_0(u)$ is self-similar, i.e., $f_0(u) = f_0(|u|)$. Bearing the expression of $\hat{f}$ in mind and (29), one has:

$$\hat{f}_\lambda(t,k) = \frac{\hat{f}(t,\lambda k)}{\frac{1}{\lambda}}.$$

Since $\hat{f}(t, \lambda k) = o\left(\frac{1}{\lambda^\alpha}\right)$, for $\alpha > 1$, as $\lambda \to +\infty$, then it follows:

$$\hat{f}_\lambda(t,k) \xrightarrow[\lambda\to+\infty]{t\to\pm\infty} 0. \tag{30}$$

Then for $\lambda \to +\infty$, there exists a function $\chi(t,k)$ such that:

$$\hat{f}_\lambda(t,k) = \frac{1}{\lambda^\alpha}\,\chi(t,k). \tag{31}$$

In particular (for $t = 0$) by using the assumption on the self-similarity of $f_0(u)$, (31) rewrites:

$$\hat{f}_\lambda(0,k) = \frac{1}{\lambda^\alpha}\,\chi(0,|k|). \tag{32}$$

By using the (32), there exists a function $\gamma(k)$, such that:

$$\hat{f}_\lambda(0,k) = \frac{c}{\lambda^\alpha}\,\gamma(|k|), \tag{33}$$

where $c = \theta(0)$ is a positive constant and $\theta(t)$ is a positive time-dependent function. By using the asymptotic behaviour (30) and the (33), it follows that, for $t \to \pm\infty$ and $\lambda \to +\infty$:

$$\hat{f}_\lambda(t,|k|) = \frac{\theta(t)}{\lambda^\alpha}\,\gamma(|k|). \tag{34}$$

The (33) is an asymptotic self-similar solution for Equation (4) and the proof is thus concluded. □

Remark 1. *By passing to the limit, as time goes to infinity, in the* (28)*, one has:*

$$\hat{g_\lambda}(k) = \lambda\,\hat{g}(\lambda k), \quad k \in \mathbb{R}, \tag{35}$$

regardless of the choice of λ*.*

However, the function $\gamma(|k|)$ *of Equation* (19) *of Theorem* 2 *can be seen as an asymptotic self-similar solution of the nonequilibrium stationary problem related to the framework* (3)*.*

5. The Nonconstant Interaction Rate-Force Field Case

This section deals with the asymptotic self-similar solution analysis for the thermostatted framework (1) for a nonconstant external force field and nonconstant interaction rate. Specifically, let $F(u) : \mathbb{R} \to \mathbb{R}^+$ be the external force field, and $\eta(u_*, u^*) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$ the interaction rate between the particle with state u_* and the particle with state u^*. The related continuous thermostatted framework writes reads:

$$\begin{cases} \partial_t f(t,u) + \partial_u\left(\left(F(u) - u\displaystyle\int_{\mathbb{R}} uF(u)f(t,u)\,du\right)f(t,u)\right) \\ = \displaystyle\int_{\mathbb{R}\times\mathbb{R}} \eta(u_*,u^*)\,\mathcal{A}(u_*,u^*,u)f(t,u_*)f(t,u^*)\,du_*\,du^* - f(t,u)\int_{\mathbb{R}} \eta(u,u^*)\,f(t,u^*)\,du^*. \\ \qquad\qquad (t,u) \in [0,+\infty[\,\times\mathbb{R} \\ f(0,u) = f_0(u) \qquad\qquad u \in \mathbb{R} \\ \displaystyle\lim_{|u|\to\pm\infty} f(t,u) = 0. \end{cases} \tag{36}$$

The following *weighted moment* is defined:

$$\mathbb{E}_1[f,F](t) := \int_{\mathbb{R}} uF(u)f(t,u)\,du.$$

The main result of this section follows.

Theorem 3. *Let $f(t,u)$ be a solution of the IBVP (36). Assume that:*

(i) $f(t,u) \in C^1\big([0,+\infty[; L^1(\mathbb{R})\big)$;
(ii) $\mathbb{E}_1[f,F](t) < +\infty$, *for all* $t > 0$;
(iii) *There exists* $F > 0$ *such that* $F(u) \leq F$, *for all* $u \in \mathbb{R}$;
(iv) *There exists* $\eta > 0$ *such that* $\eta(u_*, u^*) \leq \eta$, *for all* u_*, $u^* \in \mathbb{R}$.

Then there exists an asymptotic self-similar solution of the Fourier Transform of (36).

Proof. Let
$$f_\lambda(t,u) := f\Big(t, \frac{u}{\lambda}\Big). \tag{37}$$
By applying the Fourier transform to Equation (36), the first term on the left-hand side reads:
$$\partial_t \hat{f(t,u)} = \partial_t \hat{f}(t,k). \tag{38}$$
By straightforward calculations, the thermostat term of the (36) is written as follows:
$$\begin{aligned}
&\int_{\mathbb{R}} \partial_u \bigg(\Big(F(u) - u \int_{\mathbb{R}} uF(u) f(t,u)\, du \Big) f(t,u) \bigg) e^{-2\pi iku}\, du \\
&= 2\pi ik \bigg[\int_{\mathbb{R}} (F(u) - u\mathbb{E}_1[f,F](t)) f(t,u) e^{-2\pi iku}\, du \bigg] \\
&= 2\pi ik \bigg[\int_{\mathbb{R}} (F(u) f(t,u)) e^{-2\pi iku}\, du - \int_{\mathbb{R}} \mathbb{E}_1[f,F](t) u\, f(t,u) e^{-2\pi iku}\, du \bigg] \\
&= 2\pi ik \bigg[\hat{f}(t,k) \star \hat{F}(k) - \mathbb{E}_1[f,F](t) \int_{\mathbb{R}} u\, f(t,u) e^{-2\pi iku}\, du \bigg].
\end{aligned} \tag{39}$$
Bearing the right hand side of (36) in mind, one has:
$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}\times\mathbb{R}} \eta(u_*, u^*)\, \mathcal{A}(u_*, u^*, u) f(t,u_*) f(t,u^*) e^{-2\pi iku}\, du_*\, du^*\, du \\
&= \int_{\mathbb{R}\times\mathbb{R}} \eta(u_*, u^*) f(t,u_*) f(t,u^*) \Big(\int_{\mathbb{R}} \mathcal{A}(u_*, u^*, u) e^{-2\pi iku}\, du \Big) du_*\, du^* \\
&= \int_{\mathbb{R}\times\mathbb{R}} \eta(u_*, u^*) \hat{\mathcal{A}}(u_*, u^*, u) f(t,u_*) f(t,u^*)\, du_*\, du^*,
\end{aligned} \tag{40}$$
and
$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} f(t,u)\, \eta(u,u^*)\, f(t,u^*) e^{-2\pi iku}\, du\, du^* \\
&= \int_{\mathbb{R}} f(t,u^*) \Big(\int_{\mathbb{R}} \eta(u,u^*) f(t,u) e^{-2\pi iku}\, du \Big)\, du^*.
\end{aligned} \tag{41}$$
By using the (38)–(41), Equation (36) rewrites:
$$\begin{aligned}
&\partial_t \hat{f}(t,k) + 2\pi ik \bigg[\hat{f}(t,k) \star \hat{F}(k) - \mathbb{E}_1[f,F](t) \int_{\mathbb{R}} u\, f(t,u) e^{-2\pi iku}\, du \bigg] \\
&= \int_{\mathbb{R}\times\mathbb{R}} \eta(u_*, u^*) \hat{\mathcal{A}}(u_*, u^*, u) f(t,u_*) f(t,u^*)\, du_*\, du^* \\
&- \int_{\mathbb{R}} f(t,u^*) \Big(\int_{\mathbb{R}} \eta(u,u^*) f(t,u) e^{-2\pi iku}\, du \Big)\, du^*.
\end{aligned} \tag{42}$$
By observing that:
$$\hat{f}_\lambda(t,k) = \lambda \hat{f}(t,\lambda k),$$
one has:
$$\partial_t \hat{f}_\lambda(t,k) = \lambda \partial_t \hat{f}(t,\lambda k). \tag{43}$$

Straightforward calculations show that:

$$\begin{aligned} \hat{f}_\lambda(t,k) \star \hat{F}_\lambda(k) &= \lambda \hat{f}(t,\lambda k) \star \lambda \hat{F}(k) \\ &= \lambda^2 \hat{f}(t,\lambda k) \star \hat{F}(k), \end{aligned} \tag{44}$$

and, by using the change of variables $\frac{u}{\lambda} = y$, one has:

$$\begin{aligned} \mathbb{E}_1[f_\lambda, F_\lambda] \int_{\mathbb{R}} u f_\lambda(t,u) e^{-2\pi i k u}\, du &= \mathbb{E}_1[f_\lambda, F_\lambda] \lambda \int_{\mathbb{R}} y f(t,y) e^{-2\pi i \lambda k y}\, dy \\ &= \mathbb{E}_1[f_\lambda, F_\lambda] \lambda \widehat{u f(t,u)}(t,k). \end{aligned} \tag{45}$$

Let us consider the scaled version of the right-hand side of the (42).One has:

$$\int_{\mathbb{R}\times\mathbb{R}} \eta\left(\frac{u_*}{\lambda}, \frac{u^*}{\lambda}\right) \hat{\mathcal{A}}_\lambda\left(\frac{u_*}{\lambda}, \frac{u^*}{\lambda}, k\right) f\left(t, \frac{u_*}{\lambda}\right) f\left(t, \frac{u^*}{\lambda}\right) du_*\, du^*. \tag{46}$$

By using the following change of coordinates:

$$\begin{aligned} \frac{u_*}{\lambda} &= y \\ \frac{u^*}{\lambda} &= z, \end{aligned}$$

the (46) is rewritten as follows:

$$\begin{aligned} &\lambda^2 \int_{\mathbb{R}\times\mathbb{R}} \eta(\lambda u_*, \lambda u^*) \hat{\mathcal{A}}_\lambda(\lambda u_*, \lambda u^*, k) f(t,\lambda u_*) f(t,\lambda u^*)\, du_*\, du^* \\ &= \lambda^2 \int_{\mathbb{R}\times\mathbb{R}} \eta(\lambda u_*, \lambda u^*) \lambda \hat{\mathcal{A}}(\lambda u_*, \lambda u^*, \lambda k) f(t,\lambda u_*) f(t,\lambda u^*)\, du_*\, du^* \\ &= \lambda^3 \int_{\mathbb{R}\times\mathbb{R}} \eta(\lambda u_*, \lambda u^*) \hat{\mathcal{A}}(\lambda u_*, \lambda u^*, \lambda k) f(t,\lambda u_*) f(t,\lambda u^*)\, du_*\, du^*. \end{aligned} \tag{47}$$

By using the same argument, one has:

$$\begin{aligned} &\int_{\mathbb{R}} f\left(t, \frac{u^*}{\lambda}\right) \left(\int_{\mathbb{R}} \eta\left(\frac{u}{\lambda}, \frac{u^*}{\lambda}\right) f\left(t, \frac{u}{\lambda}\right) e^{-2\pi i k u}\, du \right) du^* \\ &= \lambda \int_{\mathbb{R}} f(t,\lambda u^*) \left(\int_{\mathbb{R}} \eta\left(\frac{u}{\lambda}, \lambda u^*\right) f\left(t, \frac{u}{\lambda}\right) e^{-2\pi i k u}\, du \right) du^* \\ &= \lambda^2 \int_{\mathbb{R}} f(t,\lambda u^*) \left(\int_{\mathbb{R}} \eta(\lambda u, \lambda u^*) f(t,\lambda u) e^{-2\pi i k u}\, du \right) du^*. \end{aligned} \tag{48}$$

By using the (43)–(45), (47) and the (48), it is possible to conclude that $\hat{f}_\lambda$ solves the following scaled equation:

$$\begin{aligned} &\lambda \hat{f}(t,\lambda k) + 2\pi i k \left[\lambda^2 \hat{f}(t,\lambda k) \star \hat{F}(\lambda k) - \mathbb{E}_1[f_\lambda, F_\lambda] \lambda \widehat{u f(t,u)}(t,\lambda k) \right] \\ &= \lambda^3 \int_{\mathbb{R}\times\mathbb{R}} \eta(\lambda u_*, \lambda u^*) \hat{\mathcal{A}}(\lambda u_*, \lambda u^*, \lambda k) f(t,\lambda u_*) f(t,\lambda u^*)\, du_*\, du^* \\ &- \lambda^2 \int_{\mathbb{R}} f(t,\lambda u^*) \left(\int_{\mathbb{R}} \eta(\lambda u, \lambda u^*) f(t,\lambda u) e^{-2\pi i k u}\, du \right) du^*. \end{aligned} \tag{49}$$

The Equation (50) is rewritten as follows:

$$\begin{aligned}
&\lambda \hat{f}(t,\lambda k)\\
&= -2\pi i k\Big[\lambda^2 \hat{f}(t,\lambda k)\star \hat{F}(\lambda k) - \mathbb{E}_1[f_\lambda, F_\lambda]\lambda \widehat{u f(t,u)}(t,\lambda k)\Big]\\
&+ \lambda^3 \int_{\mathbb{R}\times\mathbb{R}} \eta(\lambda u_*, \lambda u^*)\hat{\mathcal{A}}(\lambda u_*, \lambda u^*, \lambda k) f(t,\lambda u_*) f(t,\lambda u^*)\, du_*\, du^*\\
&- \lambda^2 \int_{\mathbb{R}} f(t,\lambda u^*)\left(\int_{\mathbb{R}} \eta(\lambda u, \lambda u^*) f(t,\lambda u) e^{-2\pi i k u}\, du\right) du^*.
\end{aligned} \tag{50}$$

The *asymptotic estimate* of the terms on the right hand side of the (50) is the aim. By using the previous regularity assumptions, as time t goes to infinity, there exist four constants $\alpha, \beta, \gamma, \delta > 0$ and four functions $R_1(t,k)$, $R_2(t,k)$, $R_3(t,k)$, $R_4(t,k)$ such that:

$$\begin{aligned}
&2\pi i k \lambda^2 \hat{f}(t,\lambda k)\star \hat{F}(\lambda k) = \lambda^\alpha R_1(t,k),\\
&2\pi i k \mathbb{E}_1[f_\lambda, F_\lambda]\lambda \widehat{u f(t,u)}(t,\lambda k) = \lambda^\beta R_2(t,k),\\
&\lambda^3 \int_{\mathbb{R}\times\mathbb{R}} \eta(\lambda u_*, \lambda u^*)\hat{\mathcal{A}}(\lambda u_*, \lambda u^*, \lambda k) f(t,\lambda u_*) f(t,\lambda u^*)\, du_*\, du^* = \lambda^\gamma R_3(t,k),\\
&\lambda^2 \int_{\mathbb{R}} f(t,\lambda u^*)\left(\int_{\mathbb{R}} \eta(\lambda u, \lambda u^*) f(t,\lambda u) e^{-2\pi i k u}\, du\right) du^* = \lambda^\delta R_4(t,k).
\end{aligned}$$

Then, as times t goes to infinity, (50) is rewritten as follows:

$$\lambda \hat{f}(t,\lambda k) = \lambda^\alpha R(t,k), \tag{51}$$

where the positive constant $\alpha > 0$ and the function $R(t,k)$ have been used with abuse of notation. By using the change of variables,

$$\lambda k = \lambda e^{\theta t},$$

for $\theta > 0$, (51) is rewritten as follows:

$$\lambda f(t,\lambda e^{\theta t}) = \lambda^\alpha \Phi(t). \tag{52}$$

Bearing the scaling argument (37) in mind, one has:

$$\hat{f}_\lambda(t, e^{\theta t}) = \Phi(t), \tag{53}$$

then the function $\Phi(t)$ is an asymptotic self-similar solution of the thermostatted framework (36). □

6. Conclusions and Research Perspectives

The present paper can be considered a further generalization of the thermostatted kinetic theory proposed in [19]; indeed the activity domain is assumed to be the whole real line and the related initial-boundary-value problem is investigated. However, the main contribution of this paper is the definition and analysis of self-similar solutions whose existence is obtained by employing the Fourier transform method.

From the modeling point of view the variable $u \in \mathbb{R}$, called *activity*, describes the microscopic state of the particles of the system (*active particles*) and represents the strategy or function expressed by the particles: η is the *interaction rate*; $\mathcal{A}(u_*, u^*, u)$ is the *transition probability function*, i.e., the probability that the active particle with state u_* falls into the state u after interacting with the active particle with state u^*, and F denotes an *external force field* which acts on the system. The framework is thus proposed for the modeling of complex systems where the role of the interactions is at the basis of the emerging phenomena (see [31,32]).

From the perspective point of view, a further investigation would be the generalization of the results to a complex system composed by n functional subsystems. Accordingly, the framework consists of n partial-integro differential equations with quadratic nonlinearity and the thermostat term would also be modified. The introduction of space and velocity variables can also be pursued and consequently the analysis of self-similar solutions could be concerned with the other microscopic state variables.

The analysis presented in this paper can also be generalized to the case where the activity variable can attain discrete values and then for a discrete thermostatted kinetic theory [20].

The existence of all moments for the self-similar solution remains open and the results will be presented in due course.

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