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Transposition Regular AG-Groupoids and Their Decomposition Theorems

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Abstract: In this paper, we introduce transposition regularity into AG-groupoids, and a variety of transposition regular AG-groupoids (L1/R1/LR, L2/R2/L3/R3-groupoids) are obtained. Their properties and structures are discussed by their decomposition theorems: (1) L1/R1-transposition regular AG-groupoids are equivalent to each other, and they can be decomposed into the union of disjoint Abelian subgroups; (2) L1/R1-transposition regular AG-groupoids are LR-transposition regular AG-groupoids, and an example is given to illustrate that not every LR-transposition regular AG-groupoid is an L1/R1-transposition regular AG-groupoid; (3) an AG-groupoid is an L1/R1-transposition regular AG-groupoid if it is an LR-transposition regular AG-groupoid satisfying a certain condition; (4) strong L2/R3-transposition regular AG-groupoids are equivalent to each other, and they are union of disjoint Abelian subgroups; (5) strong L3/R2-transposition regular AG-groupoids are equivalent to each other and they can be decomposed into union of disjoint AG subgroups. Their relations are discussed. Finally, we introduce various transposition regular AG-groupoid semigroups and discuss the relationships among them and the commutative Clifford semigroup as well as the Abelian group.

Keywords: semigroup; transposition regular AG-groupoid; decomposition theorem; Abelian group

MSC: 17A30; 17A60

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1. Introduction

The AG-groupoid (also known as the LA-semigroup [1]), first proposed by Kazim and Naseeruddin in 1972 (see [2]), is a groupoid that is a set together with the left invertive law. For example, let $G = \{a, b, c, d, e, f, g\}$ and define the operation on G as shown in Table 1:

Table 1. This is an AG-groupoid.

$*$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	a	c	c	a	c	c
c	a	b	a	b	a	b	b
d	a	b	c	d	e	f	g
e	a	a	a	e	e	a	a
f	a	b	c	f	a	f	f
g	a	b	c	g	a	f	g

In Table 1, for any elements $x, y, z \in G$, there is $(xy)z = (zy)x$; that is, G satisfies the left invertive law. So, G is an AG-groupoid.

The concept of a regular AG-groupoid is proposed in [3], the ideals of the AG-groupoid and regular AG-groupoid are studied, the definitions of completely regular AG-groupoid and fully regular AG-groupoid are given in [4], and the necessary and sufficient conditions

for an AG-groupoid to be a fully regular AG-groupoid are explored. AG-groups and right AG-groups are proposed in [5], and AG-groups are studied more deeply (see [6–8]). At present, through the continuous research of many scholars, the nature and structure of the AG-groupoid have been deeply revealed, and many research results have been accumulated (see [9–16]).

The AG-groupoid is a kind of non-commutative and non-associative algebraic system midway a groupoid and commutative semigroup, but it retains the good properties in the structure of commutative and associative algebra. If an AG-groupoid contains a right identity, then it becomes a commutative monoid (see [17]). An AG-groupoid is the generalization of a semigroup theory and has vast applications in collaboration with semigroups (see [18]). Based on the close connections between semigroups and AG-groupoids, we introduce the transposition regularity proposed by Xiaohong Zhang and Yudan Du in [19] into AG-groupoids.

In this paper, we mainly introduce various transposition regular AG-groupoids, and their structures are characterized. The arrangement of this paper is as follows. In Section 2, we introduce the L1-transposition regular AG-groupoid and R1-transposition regular AG-groupoid, and their structures are described by decomposition theorems. In Section 3, we introduce the LR-transposition regular AG-groupoid and strong LR-transposition regular AG-groupoid, and their properties and structures are charaterzied. In Section 4, the concepts of the L2-transposition regular AG-groupoid and R2-transposition regular AG-groupoid are given, and their decompositon theorems are proved. In Section 5, we introduce the L3-transposition regular AG-groupoid and R3-transposition regular AG-groupoid and discuss the relationships among various transposition regular AG-groupoids. In Section 6, we introduce various transposition regular AG-groupoid semigroups, and their relationships among the commutative Clifford semigroup and Abelian group are given.

2. L1-Transposition Regular AG-Groupoid and R1-Transposition Regular AG-Groupoid

In this section, we give the definitions of transposition regular AG-groupoid and R1-transposition regular AG-groupoid.

Definition 1. *Let G be a groupoid. $a \in G$.*

(1) If there exists $e \in G$ such that $ea = a (ae = a)$, e is said to be the local left (right) identity element of a. e is said to be the local identity element if e is both the local left identity element and local right identity element.

(2) Let e be a local left identity element/right identity element/identity elemnt of a. If there exists b such that $ba = e (ab = e)$, b is said to be a local left(right) inverse element of a relative to e. b is the local inverse element of a relative to e if b is both the local left inverse element of a relative to e and the local right inverse element of a relative to e.

Definition 2 ([2]). *A groupoid S satisfying the left invertive law is called as an AG-groupoid.*

$$(ab)c = (cb)a, \forall a, b, c \in S.$$

In an AG-groupoid S, it satisfies the medial law:

$$(ab)(cd) = (ac)(bd), \forall a, b, c, d \in S.$$

Definition 3. *Let G be an AG-groupoid, $a \in G$. An element a is L1-transposition regular in G if $\exists p \in G$ s.t. $(pa)a = a = a(pa)$. The AG-groupoid G is said to be L1-transposition regular if all its elements are L1-transposition regular.*

Example 1. *Let $G = \{a, b, c, d\}$. Define the operation on G as shown in Table 2. Clearly, $(G, *)$ is an L1-transposition regular AG-groupoid. That is,*

$$(a * a) * a = a = a * (a * a), (b * b) * b = b = b * (b * b),$$

$$(c*c)*c = c = c*(c*c), (d*d)*d = d = d*(d*d).$$

Table 2. L1-transposition regular AG-groupoid.

$*$	**a**	**b**	**c**	**d**
a	a	d	b	c
b	c	b	d	a
c	d	a	c	b
d	b	c	a	d

Clearly, G is not commutative.

Proposition 1. *Let G be an L1-transposition regular AG-groupoid. For any $a \in G$, let $pa = e, f = ee, m = ff$. Then, $\exists q \in G$, such that $ma = a = am, qm = q = mq, aq = m = qa$.*

Proof. For any a in G, let $pa = e$. Then, $ea = a = ae$. So, e is a local identity element of a, and p is a local inverse element of a relative to e. According to the left invertive law,

$$(ee)a = (ae)e = ae = a, a(ee) = (ea)(ee) = (ee)(ae) = (ee)a = ae = a.$$

Furthermore,

$$ap = (ea)p = (pa)e = ee.$$

$$(pe)a = (ae)p = ap = ee.$$

So

$$a(pe) = (ea)(pe) = (ep)(ae) = (ep)a = (ap)e = (ee)e.$$

Furthermore,

$$a(pe) = (ae)(pe) = (ap)(ee) = (ee)(ee).$$

Then, $(ee)e = (ee)(ee)$.
Let $ee = f$. Then, $fa = a = af, ap = f = (pe)a, fe = ff$.
According to left invertive law and medial law,

$$(ff)a = (af)f = af = a, a(ff) = (fa)(ff) = (ff)(af) = (ff)a = a.$$

Then $(ff)a = a = a(ff)$.
However,

$$a(ep) = (ea)(ep) = (ee)(ap) = (ee)(ee) = ff.$$

$$(ep)a = (ep)(ea) = (ee)(pa) = (ee)e = fe = ff.$$

That is, $a(ep) = ff = (ep)a$. Furthermore, ff is the local identity element of a, and ep is local inverse element of a relative to ff. Let $m = ff$. Then, $ma = a = am$, $(ep)a = m = a(ep)$.
So, $mm = ((ep)a)m = (ma)(ep) = a(ep) = m$, and m is idempotent.
So,

$$m(m(ep)) = ((ep)a)(m(ep)) = ((ep)m)(a(ep)) = ((ep)m)m = (mm)(ep) = m(ep).$$

$$\begin{aligned}(m(ep))m &= (m(ep)(mm) = (mm)((ep)m) = m((ep)m)\\ &= ((ep)a)((ep)m) = ((ep)(ep))(am)\\ &= ((ep)(ep))a = (a(ep))(ep) = m(ep).\end{aligned}$$

$$a(m(ep)) = (ma)(m(ep)) = (mm)(a(ep)) = mm = m.$$

$$(m(ep))a = (a(ep))m = mm = m.$$

Then, m is an identity element of $m(ep)$, and a is the local inverse element of $m(ep)$ relative to m. □

Theorem 1. *Let G be an L1-transposition regular AG-groupoid. Define the binary relation $\approx$ on G as follows:*

$$a \approx b \Leftrightarrow e_a = e_b, \forall a, b \in G$$

where e_a is a local identity element of a. Then
(1) The binary relation $\approx$ on G is equivalence relation, and we denote the equivalence class containing x by $[x]_\approx$;
(2) $\forall x \in G$, $[x]_\approx$ is an Abelian subgroup;
(3) $G = \bigcup_{x \in G} [x]_\approx$, that is, every L1-transposition regular AG-groupoid is the disjoint union of Abelian subgroups.

Proof. (1) Clearly, $\forall x \in G$, $e_x = e_x$. That is, $x \approx x$.

Assume that $x \approx y$, then $e_x = e_y$, and $e_y = e_x$. So $y \approx x$.

If $x \approx y$ and $y \approx z$, then $e_x = e_y$, and $e_x = e_y$. Clearly, $e_x = e_z$. That is, $x \approx z$. So, $\approx$ is a equivalence relation on G.

(2) $\forall a, b \in [x]_\approx$, assume that $e_a = e_b = e$. $ab = (ae)b = (be)a = ba$. That is, $[x]_\approx$ satisfies the commutative law. Clearly, $[x]_\approx$ satisfies the left invertive law. Then, for any $a, b, c \in G$, $(ab)c = (cb)a = a(cb) = a(bc)$. So $[x]_\approx$ satisfies the associative law.

According to Proposition 1, there exists $p \in G$ s.t. $pa = e = ap, ea = a = ae, ep = p = pe$. $\exists q \in G$ s.t. $qb = e = bq, eb = b = be, eq = q = qe$. Then,

$$(ab)e = (eb)a = ba = ab, \text{and } e(ab) = (ab)e = ab.$$

That is, $ab \in [x]_\approx$.

For any $a \in [x]_\approx$, $ee = e(ap) = (ea)p = ap = e$, then $e \in [x]_\approx$.

Because $ee = (pa)e = (ea)p = ap = e$, there is $e \in [x]_\approx$.

Because $ep = p = pe$, then $p \in [x]_\approx$. According to the definition of group, $[x]_\approx$is an Abelian subgroup of G.

(3) For any $a \in [x]_\approx$, the local identity element of a is unique, then $G = \bigcup_{x \in G} [x]_\approx$ and $\bigcap_{x \in G} [x]_\approx = \varnothing$. That is, every L1-transposition regular AG-groupoid is the disjoint union of subgroups. □

Definition 4. *Let G be an AG-groupoid. An element a is R1-transposition regular in G if $\exists p \in G$ s.t. $(ap)a = a = a(ap)$. The AG-groupoid G is said to be R1-transposition regular if all its elements are R1-transposition regular.*

Example 2. *Let $G = \{a, b, c, d\}$. Define the operation on G as Table 2; then, we can get that $(G, *)$ is a R1-transposition regular AG-groupoid. There is,*

$$(a * a) * a = a * a = a = a * (a * a), (b * b) * b = a * b = b = b * a = b * (b * b),$$

$$(c * d) * c = a * c = c = c * a = c * (c * d), (d * c) * d = a * d = d = d * a = d * (d * c).$$

Theorem 2. *G is an L1-transposition regular AG-groupoid if and only if it is an R1-transposition regular AG-groupoid.*

Proof. ($\Rightarrow$) Let G be an L1-transposition regular AG-groupoid. For any $a \in G$, there exists $p \in G$ such that $(pa)a = a = a(pa)$. Let $pa = e$. There is, $ea = a = ae$. That is, e is the local identity element of a, and p is the local left inverse element of a relative to e. According to the left invertive law,

$$(ee)a = (ae)e = ae = a, a(ee) = (ea)(ee) = (ee)(ae) = (ee)a = ae = a.$$

Furthermore,

$$ap = (ea)p = (pa)e = ee.$$

$$(pe)a = (ae)p = ap = ee.$$

At this time,

$$(ee)e = (ap)e = (ep)a = (ep)(ae) = (ea)(pe) = a(pe) = (ae)(pe) = (ap)(ee) = (ee)(ee).$$

Let $ee = f$. There is $fa = a = af$, and $ap = f = (pe)a$. Then, $(ap)a = a = a(ap)$. So, a is R1-transposition regular, and G is an R1-transposition regular AG-groupoid.

($\Leftarrow$) Let G be an R1-transposition regular AG-groupoid. For any $a \in G$, there exists $p \in G$ such that $a(ap) = a = (ap)a$. Let $ap = e$, there is $ea = a = ae$. Then, e is a local identity element of a, p is the local right inverse element of a relative to e. According to the left invertive law, there is, $(pe)a = (ae)p = ap = e$.

Furthermore, $(ee)a = (ae)e = ae = a, a(ee) = (ea)(ee) = (ee)(ae) = (ee)a = a$. $(ep)a = (ap)e = ee$.

Then,

$$((ep)a)a = a = a((ep)a).$$

That is, a is L1-transposition regular, and G is an L1-transposition regular AG-groupoid. □

3. LR-Transposition Regular AG-Groupoid and Strong LR-Transposition Regular AG-Groupoid

In [19], for a semigroup S, for any element $a \in S$, there exists $p, q \in S$ such that $(pa)a = a = a(aq)$. However, in an AG-groupoid, we can get the whole equation through the right equation. So, we give the following definition:

Definition 5. *Let G be an AG-groupoid, $a \in G$. An element a is LR-transposition regular in G $\exists q \in G$ s.t. $a = a(aq)$. G is an LR-transposition regular AG-groupoid if all elements of G are LR-transposition regular.*

Proposition 2. *Let G be an LR-transposition regular AG-groupoid. $\forall a \in G$, there exists $q \in G$ s.t. $a = a(aq)$. Then, there exists $p \in G$ s.t. $(pa)a = a$.*

Proof. Assume that $p = q^2a$. Then,

$$(pa)a = ((q^2a)a)a = ((aa)q^2)a = ((aa)(qq))a = ((aq)(aq))a = (a(aq))(aq) = a(aq) = a.$$

At this time, $(pa)a = a$. □

Example 3. *Let $G = \{a, b, c, d, e\}$. Define the operation $*$ on G in Table 3. Clearly, G is an LR-transposition regular AG-groupoid. That is,*

$$(a * a) * a = a * a = a = a * a = a * (a * a),$$

$$(b * b) * b = a * b = b = b * a = b * (b * b),$$

$$(d * c) * c = a * c = c = c * b = c * (c * c),$$

$$(c * d) * d = a * d = d = d * b = d * (d * d),$$

$$(e * e) * e = e * e = e = e * e = e * (e * e).$$

Table 3. This is an LR-transposition regular AG-groupoid.

$*$	**a**	**b**	**c**	**d**	**e**
a	a	b	c	d	e
b	b	a	d	c	b
c	d	c	b	a	d
d	c	d	a	b	c
e	a	b	c	d	e

Theorem 3. *Let G be an LR-transposition regular AG-groupoid. $\forall a \in G$, there exists $q \in G$ s.t. $a = a(aq)$. $(q^2a)a = aq$ if and only if it is an L1-transposition regular AG-groupoid.*

Proof. ($\Rightarrow$) For any element a in G, let $p = q^2a$, and $pa = e = aq$. That is, $(pa)a = a = a(pa)$. Then, G is an L1-transposition regular AG-groupoid.

($\Leftarrow$) Let G be an L1-transposition regular AG-groupoid. For any $a \in G$, there exists $p \in G$ s.t. $(pa)a = a = a(pa)$. According to Proposition 1, let $e = pa, f = ee, ffff = ff$, and $a(a(ep)) = a, a(ep) = ff$.

Then, let $q = ep$, there is $((q^2a)a)a = a, aq = ff$. So,

$$(q^2a)a = ((qq)a)a = (aa)(qq) = (aq)(aq) = (ff)(ff) = ff = aq.$$

That is, $(q^2a)a = aq$. □

Example 4 shows that not every LR-transposition regular AG-groupoid is an L1-transposition regular AG-groupoid.

Example 4. *Let $G = \{a, b, c, d, e\}$. Define the operation on G as Table 3. Then, $(G, *)$ is an LR-transposition regular AG-groupoid. Where $(a * c) * c = b \neq c$, $(b * c) * c = a \neq c$, $(c * c) * c = d \neq c$, $(e * c) * c = b \neq c * (e * c)$, and $(d * c) * c = c \neq d = c * (d * c)$, that is, there don't exist $p \in G$ s.t. $(p * c) * c = c = c * (p * c)$. So, G is not an L1-transposition regular AG-groupoid.*

Definition 6. *Let G be an LR-transposition regular AG-groupoid, $a \in G$. An element a is strong LR-transposition regular in G if $q^2a = q$. G is an LR-transposition regular AG-groupoid if every element of G is strong LR-transposition regular.*

Example 5 shows that not every LR-transposition regular AG-groupoid is a strong LR-transposition regular AG-groupoid.

Example 5. *Let $G = \{a, b, c, d, e\}$. Define the operation on G in Table 3. Then, $(G, *)$ is an LR-transposition regular AG-groupoid. However, $c * (c * c) = c * b = c$, and $c^2 * c = b * c = d \neq c$. That is, $(G, *)$ is not a strong LR-transposition regular AG-groupoid.*

Example 6. *Let $G = \{a, b, c, d, e, f, g, h\}$. Define the operation on G in Table 4. Then, $(G, *)$ is a strong LR-transposition regular AG-groupoid. Let*

$$p_a = \{p \in G | a = a(ap) \text{ and } p^2a = p, a \in G\}.$$

Then

$$p_a = b, p_b = a, p_c = d, p_d = c, p_e = f, p_f = e, p_g = h, p_h = d.$$

Table 4. This is a strong LR-transposition regular AG-groupoid.

∗	a	b	c	d	e	f	g	h
a	b	d	c	a	g	e	f	h
b	c	a	b	d	f	h	g	e
c	a	c	d	b	h	f	e	g
d	d	b	a	c	e	g	h	f
e	h	f	e	g	f	h	g	e
f	e	g	h	f	g	e	f	h
g	g	e	f	h	e	g	h	f
h	f	h	g	e	h	f	e	g

Theorem 4. *Let G be a strong LR-transposition regular AG-groupoid. $\forall a \in G$, there exists $q \in G$ s.t. $a = a(aq)$ and $q^2a = q$. $qa = aq$ if and only if it is an L1-transposition regular AG-groupoid.*

Proof. ($\Rightarrow$) For any a in G, there exists $q \in G$ s.t. $a = a(aq)$ and $q^2a = q$. According to Proposition 2, $(qa)a = a = a(aq)$. Because $qa = aq$, then $(qa)a = a = a(qa)$. So, G is an L1-transposition regular AG-groupoid.

($\Leftarrow$) Let G be an L1-transposition regular AG-groupoid. For any $a \in G$, there exists $p \in G$ s.t. $(pa)a = a = a(pa)$. According to Proposition 1, let $e = pa, f = ee, m = ff$, and $(m(ep))a = m = a((m(ep)), m(m(ep)) = m(ep)$. Let $q = m(ep)$. Then, $qa = m = aq, mq = q$. Then,

$$a(aq) = am = a, q^2a = (qq)a = (aq)q = mq = q.$$

Then, G is a strong LR-transposition regular AG-groupoid, and $qa = aq$. □

Example 7 shows that not every strong LR-transposition regular AG-groupoid is an L1-transposition regular AG-groupoid.

Example 7. *Let $G = \{a, b, c, d, e, f, g, h\}$. Define the operation on G in Table 4. Then, $(G, *)$ is a strong LR-transposition regular AG-groupoid. However, $a * (a * p_a) = a * (a * b) = a * d = a, p_a^2 * a = (b * b) * a = a * a = b = p_a, p_a * a = b * a = c \neq a = a * d = a * p_a$. So, $(G, *)$ is not an L1-transposition regular AG-groupoid.*

4. L2-Transposition Regular AG-Groupoid and R2-Transposition Regular AG-Groupoid

In this section, we define the L2-transposition regular AG-groupoid and R2-transposition regular AG-groupoid.

Definition 7. *Let G be an AG-groupoid, $a \in G$. An element a is L2-transposition regular in G if there exists unique $x \in G$ s.t. $a(xa) = a$ and $x(xa)a = xa$. G is an L2-transposition regular AG-groupoid if all elements of G are L2-transposition regular.*

Proposition 3. *Let G be L2-transposition regular AG-groupoid. $\forall a \in G$, there exists unique $x \in G$ s.t. $a(xa) = a$ and $x(xa)a = xa$. Let $xa = e$. Then, $xa = e = ax$, and $xe = x$.*

Proof. For any a in G, let $xa = e$. Then, $ae = a(xa) = a$. According to left invertive law, there is, $x(xa)a = (xe)a = (ae)x = ax = xa = e$. Then, $a(ax) = a$. There is, $(ee)a = (ae)e = ae = a$.

Then,

$$a((xe)a) = ae = a,$$

$$((xe)((xe)a))a = ((xe)e)a = (ae)(xe) = (ax)(ee) = (xa)(ee) = (xe)(ae) = (xe)a = e.$$

Because x is unique, then $xe = x$. □

Example 8. *Let $G = \{a, b, c, d\}$. Define the operation on G in Table 5. Then, $(G, *)$ is an L2-transposition regular AG-groupoid. Let*

x_a={$x \in G$|*there exists unique* $x \in G$ *s.t.* $a(xa) = a$ *and* $x(xa)a = xa$, $a \in G$}

There is,

$$x_a = a, x_b = b, x_c = c, x_d = d.$$

Table 5. This is an L2-transposition regular AG-groupoid.

∗	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	d	c	b	a
d	c	d	a	b

Definition 8. *Let G be an L2-transposition regular AG-groupoid. G is a strong L2-transposition regular AG-groupoid if* $(xa)^2 = xa$.

Example 9 shows that not every L2-transposition regular AG-groupoid is a strong L2-transposition regular AG-groupoid.

Example 9. *Let* $G = \{a, b, c, d\}$. *Define the operation on G in Table 5. That is,* $x_c * c = c * c = b$, *but* $b^2 \neq b$. *So,* $(G, *)$ *is not a strong L2-transposition regular AG-groupoid.*

Theorem 5. *Let G be a strong L2-transposition regular AG-groupoid. Then, for any* $a \in G$,
(1) The local identity element of a is idempotent;
(2) The local identity element of a is unique;
(3) The local inverse element of a is unique;
(4) The local inverse element of local inverse element of a is a.

Proof. (1) For any $a \in G$, let $xa = e$. Then, $ae = a$, $(xe)a = (ae)x = ax = xa = e$. Because $(xa)^2 = xa$, and $e^2 = e$. According to left invertive law, there is, $ea = (ee)a = (ae)e = ae = a$. Then, $ea = a = ae, ax = e = xa$. That is, e is the local identity element of a, x is the local inverse element of a relative to e. So, e is idempotent. That is to say, for any $a \in G$, the local identity element of a is idempotent.

(2) Assume that the local identity of a is not unique. Then, there exists $e, x, f, y \in G$ such that $ea = a = ae$, $xa = e = ax$. Furthermore, $fa = a = af, ya = f = ay$. According to the left invertive law,

$$ef = (xa)f = (fa)x = ax = e,$$

$$fe = (ya)e = (ea)y = ay = f,$$

$$e = ef = (ee)f = (fe)e = fe = f.$$

Then, for any $a \in G$, the local identity element of a is unique.

(3) According to (1), x is the local inverse element of a relative to e. By Definition 7, x is unique. That is, for $a \in G$, the local inverse element of a is unique.

(4) By (1), for any $a \in G$, let $xa = e$. Then, $ae = a = ea$, and $ax = xa = e$. By Proposition 3, $x(ax) = xe = x, ex = (ee)x = (xe)e = xe = x$. So, the local inverse element of a is a. □

Theorem 6. *Let G be a strong L2-transposition regular AG-groupoid. Define the binary relation* $\approx$ *on G as follows:*

$$a \approx b \Leftrightarrow e_a = e_b, \forall a, b \in G$$

where e_a *is the local identity element of a. Then,*
(1) The binary relation $\approx$ *on G is an equivalence relation, and we denote the equivalence class containing x by* $[x]_\approx$;
(2) $\forall x \in G$, $[x]_\approx$ *is a subgroup;*

(3) $G = \bigcup_{x \in G} [x]_{\approx}$*, that is, every strong L2-transposition regular AG-groupoid is the disjoint union of subgroups.*

Proof. (1) Clearly, $\forall x \in G, e_x = e_x$. That is, $x \approx x$.

Assume that $x \approx y$, then $e_x = e_y$, and $e_y = e_x$. So, $y \approx x$.

If $x \approx y$ and $y \approx z$, then $e_x = e_y$, and $e_x = e_y$. Clearly, $e_x = e_z$. That is, $x \approx z$. So, $\approx$ is an equivalence relation on G.

(2) $\forall a, b \in [x]_{\approx}$, assume that $e_a = e_b = e$. $ab = (ae)b = (be)a = ba$. That is, $[x]_{\approx}$ satisfies the commutative law. Clearly, $[x]_{\approx}$ satisfies the left invertive law. Then, for any $a, b, c \in G$, $(ab)c = (cb)a = a(cb) = a(bc)$. So, $[x]_{\approx}$ satisfies the associative law.

According to Proposition 1, there exists $p \in G$ s.t. $pa = e = ap, ea = a = ae, ep = p = pe$. $\exists q \in G$ s.t. $qb = e = bq, eb = b = be, eq = q = qe$. Then,

$$(ab)e = (eb)a = ba = ab, \text{ and } e(ab) = (ab)e = ab.$$

That is, $ab \in [x]_{\approx}$.

For any $a \in [x]_{\approx}$, $ee = e(ap) = (ea)p = ap = e$, then $e \in [x]_{\approx}$.

Because $ee = (pa)e = (ea)p = ap = e$, there is $e \in [x]_{\approx}$.

Because $ep = p = pe$, then $p \in [x]_{\approx}$. According to the definition of the group, $[x]_{\approx}$is an Abelian subgroup of G.

(3) For any $a \in [x]_{\approx}$, the local identity element of a is unique; then, $G = \bigcup_{x \in G} [x]_{\approx}$ and $\bigcap_{x \in G} [x]_{\approx} = \varnothing$. That is, every strong L2-transposition regular AG-groupoid is the disjoint union of subgroups. □

Example 10. *Let* $G = \{a, b, c, d\}$*. Define the operation on G as Table 6. Then,* $(G, *)$ *is a strong L2-transpostion regular AG-groupoid. Let*

$$x_a = \{x \in G | . \text{ There exists } x \in G \text{ s.t. } a(xa) = a, x(xa)a = xa \text{ and } (xa)^2 = xa, a \in G\}.$$

And

$$x_a = a, x_b = b, x_c = c, x_d = d.$$

Let A, B, C, D *be a set composed of elements whose local identity is* a, b, c, d *respectively. Then,* $A = \{a\}, B = \{b\}, C = \{c\}, D = \{d\}$.

Table 6. This is a strong L2-transpostion regular AG-groupoid.

*	a	b	c	d
a	*a*	*c*	*d*	*b*
b	*b*	*b*	*a*	*c*
c	*c*	*d*	*c*	*a*
d	*d*	*a*	*b*	*d*

Clearly, $G = A \cup B \cup C \cup D$*, where* A, B, C, D *are subgroups of G.*

According to Theorem 5 (2), every strong L2-transpostion regular AG-groupoid is an L1/R1-transpostion regular AG-groupoid. Example 11 shows that not every L1/R1-transpostion regular AG-groupoid is a strong L2-transpostion regular AG-groupoid.

Example 11. *Let* $G = \{a, b, c, d, e\}$*. Define the operation on G as Table 7. Then,* $(G, *)$ *is an L1/R1-transpostion regular AG-groupoid.*

Table 7. This is an L1/R1-transposition regular AG-groupoid.

∗	a	b	c	d	e
a	*a*	*a*	*a*	*d*	*e*
b	*a*	*b*	*c*	*d*	*e*
c	*a*	*c*	*b*	*d*	*e*
d	*d*	*d*	*d*	*e*	*a*
e	*e*	*e*	*e*	*a*	*d*

Because $a*(a*a)=a,(a*(a*a))*a=a*a,a*(b*a)=a*a=a,(b*(b*a))*a=(b*a)*a=a*a=b*a$, *and* $a\neq b$. *So,* $(G,*)$ *is not a strong L2-transpostion regular AG-groupoid.*

Definition 9. *Let G be an AG-groupoid,* $a\in G$. *An element a is an R2-transposition regular in G if there exists a unique* $x\in G$ *s.t.* $(ax)a=a$ *and* $a((ax)x)=ax$. *G is R2-transposition regular AG-groupoid if all its elements are R2-transposition regular.*

Proposition 4. *Let G be an R2-transposition regular AG-groupoid.* $\forall a\in G$, *there exists unique* $x\in G$ *s.t.* $(ax)a=a$ *and* $a((ax)x)=ax$, *let* $ax=e$. *Then,* $x=ex$.

Proof. Let $ax=e$. Then, $ea=a,a(ex)=ax=e.ee=(ax)e=(ex)a$. There is,

$$(a(ex))a=ea=a,$$

$$a((a(ex)(ex))=a(e(ex))=(ea)(e(ex))=(ee)(a(ex))=(ee)e=((ex)a)e=(ea)(ex)=a(ex).$$

Because x is unique, then $ex=x$. □

Example 12. *Let* $X=\{(a,b)|a\in R,$ *and* $a\neq 0,b\in\{1,-1\}\}$, *define* $(a,b)*(c,d)=(ac,-b/d)$. *Then*

$$[(a,b)*(c,d)]*(e,f)=(ac,-b/d)*(e,f)=(ace,b/df)$$

$$[(e,f)*(c,d)]*(a,b)=(ec,-f/d)*(a,b)=(ace,f/bd)$$

Because $b,f\in\{-1,1\}$, $b^2=f^2$, *and* $b/f=f/b$. *We can verify* $b/df=f/bd$. *Thus*

$$[(a,b)*(c,d)]*(e,f)=[(e,f)*(c,d)]*(a,b)$$

satisfies the left invertive law. So, $(X,*)$ *is an AG-groupoid.*

For any $a\in R$,

$$(a,1)*(\frac{1}{a},1)*(a,1)=(1,-1)*(a,1)=(a,1),$$

$$(a,1)*((1,-1)*(\frac{1}{a},1))=(a,1)*(\frac{1}{a},1);$$

$$(a,-1)*(\frac{1}{a},-1)*(a,-1)=(1,-1)*(a,-1)=(a,-1),$$

$$(a,-1)*((1,-1)*(\frac{1}{a},-1))=(a,-1)*(\frac{1}{a},-1).$$

We can get that $(X,*)$ *is an R2-transposition regular AG-groupoid.*

Definition 10. *Let G be an R2-transposition regular AG-groupoid. G is a strong R2-transposition regular AG-groupoid if* $(ax)^2=ax$.

Example 13 shows that not every R2-transposition regular AG-groupoid is a strong R2-transposition regular AG-groupoid.

Example 13. *In Example 12, for any $a \in R$, $(1,-1)$ is the local left identity element of $(a,1)$, and $(1,-1)$ is the local left identity element of $(a,-1)$. However, $(1,1)*(1,1) = (1,-1)$; that is, $(1,1)$ is not idempotent. Then, G is not a strong R2-transposition regular AG-groupoid.*

Theorem 7. *Let G be a strong R2-transposition regular AG-groupoid. Then, for any $a \in G$,*
(1) The local left identity element of a is idempotent;
(2) The local left identity element of a is unique;
(3) The local inverse element of a is unique;
(4) The local inverse element of the local inverse element of a is a.

Proof. (1) For any element a in G, let $ax = e$. Then, $ea = a$. According to the definition, $e^2 = e$.

(2) For any element a in G, assume that the local left identity element of a is not unique. Then, there exist $e, x, f, y \in G$ such that $ea = a, ax = e. fa = a, ay = f$, and $ex = x, fy = y$. So,

$$xa = (ex)a = (ax)e = ee = e, ya = (fy)a = (ay)f = ff = f.$$

Then,

$$ef = (xa)f = (fa)x = ax = e,$$
$$fe = (ya)e = (ea)y = ay = f.$$
$$e = ef = (ee)f = (fe)e = fe = f.$$

So, for any $a \in G$, the local left identity element of a is unique.

(3) According to Definition 9 and (1), $xa = e = ax$, and the local inverse element of a exists, because the local right inverse element of a is unique. So, the local inverse element of a is unique.

(4) According to Proposition 4, let $ax = e$, then $ex = x$. So, the local inverse element of a is a. □

Theorem 8. *Let G be a strong R2-transposition regular AG-groupoid. Define the binary relation $\approx$ on G as follows:*

$$a \approx b \Leftrightarrow e_a = e_b, \forall a, b \in G$$

where e_a is the local left identity element of a. Then
(1) The binary relation $\approx$ on G is the equivalence relation, and we denote the equivalence class containing x by $[x]_\approx$;
(2) $\forall x \in G$, $[x]_\approx$ is an AG-subgroup;
(3) $G = \bigcup_{x \in G} [x]_\approx$, that is, every strong R2-transposition regular AG-groupoid is the disjoint union of AG subgroups.

Proof. (1) Clearly, $\forall x \in G, e_x = e_x$. That is, $x \approx x$.

Assume that $x \approx y$, then $e_x = e_y$, and $e_y = e_x$. So $y \approx x$.

If $x \approx y$ and $y \approx z$, then $e_x = e_y$, and $e_x = e_y$. Clearly, $e_x = e_z$. That is, $x \approx z$. So, $\approx$ is a equivalence relation on G.

(2) For any $a \in [x]_\approx$, let $e_a = e$. Then, $ea = a, am = e$.

$\forall a, b \in [x]_\approx$, assume that $e_a = e_b = e$. $ea = a, am = e. eb = b, by = e$. Then,

$$e(ab) = (ee)(ab) = (ea)(eb) = ab.$$

That is, $ab \in [x]_\approx$.

Clearly, $e \in [x]_{\approx}$.

According to Proposition 4, there is $em = m$. Furthermore, $ma = (em)a = (am)e = ee = e$. Then, $m \in [x]_{\approx}$. So $[x]_{\approx}$ is an AG-subgroup of G.

(3) For any $a \in [x]_{\approx}$, the local left identity element of a is unique; then, $G = \bigcup_{x \in G} [x]_{\approx}$ and $\bigcap_{x \in G} [x]_{\approx} = \varnothing$. That is, every strong R2-transposition regular AG-groupoid is the disjoint union of AG subgroups. □

Theorem 9. *Let G be a strong R2-transposition regular AG-groupoid. Let $G = \bigcup_{n \in I} G_n$, and $i, j \in I$, $G_i \bigcap_{i \neq j} G_j = \varnothing$, where G_n is an AG-subgroup of G, and I is the index set. Then, $|n| \neq 2$ and $|n| \neq 3$.*

Proof. Assume that $|n| = 2$; then, $G = G_1 \bigcup G_2$. Let a_1 be the left identity element of G_1, and $a_1 a_1 = a_1$. Let b_1 be the identity element of G_2, and $b_1 b_1 = b_1$. Because the subset composed of idempotents in the AG-groupoid is the AG-subgroupoid, $a_1 b_1 = a_1$ or $a_1 b_1 = b_1$, it is divided into the following two conditions:

Case 1: Assume that $a_1 b_1 = a_1$; at that time, $(a_1 b_1)a_1 = a_1 a_1 = a_1, a_1((a_1 b_1)b_1) = a_1(a_1 b_1) = a_1 a_1 = a_1 b_1$. Because $(a_1 a_1)a_1 = a_1 a_1 = a_1, a_1((a_1 a_1)a_1) = a_1(a_1 a_1) = a_1 a_1$. Then, $a_1 = b_1$.

Case 2: Assume that $a_1 b_1 = b_1$; in a similar way, $b_1 = a_1$.

Above all, $|n| \neq 2$. Then, $|n| = 1$.

Then, we consider $|n| = 3$, $G = G_1 \bigcup G_2 \bigcup G_3$. Let a_1 be the left identity element of G_1, and $a_1 a_1 = a_1$. Let b_1 be the identity element of G_2, and $b_1 b_1 = b_1$. Let c_1 be the identity element of G_3, and $c_1 c_1 = c_1$. According to Case 1 and Case 2, if in any case of $a_1 b_1 = a_1$, $a_1 b_1 = b_1$, $b_1 a_1 = a_1$, $b_1 a_1 = b_1$, $|n| = 2$, then following Case 1 and Case 2. So, we should consider $a_1 b_1 = c_1$ and $b_1 a_1 = c_1$:

We can get that $(a_1 b_1)b_1 = (b_1 b_1)a_1 = b_1 a_1 = c_1$, because $a_1 b_1 = c_1$, $c_1 b_1 = c_1$. Then, $|n| = 2$, according to Case 1 and Case 2, $|n| = 1$. □

Similarly, if G is a strong L2-transposition regular AG-groupoid, let $G = \bigcup_{n \in I} G_n$ and $\forall i, j \in I$, $G_i \bigcap_{i \neq j} G_j = \varnothing$, where G_n is the subgroup of G, and I is the index set. Then, $|n| \neq 2$ and $|n| \neq 3$.

Example 10 shows the example of the strong R2-transposition regular AG-groupoid where $|n| = 4$.

Example 14. *Let $G = \{a, b, c, d\}$. Define the operation on G as shown in Table 6. Then, $(G, *)$ is a strong R2-transposition regular AG-groupoid. Let*

$$x_a = \{x \in G | \text{ there exists unique } x \in G \text{ s.t. } (ax)a = a, a((ax)x) = ax \text{ and } (ax)^2 = ax, a \in G\}.$$

There is,

$$x_a = a, x_b = b, x_c = c, x_d = d.$$

Let A, B, C, D be a set composed of elements whose local left identity element is a, b, c, d respectively. Then, $A = \{a\}, B = \{b\}, C = \{c\}, D = \{d\}$.

Then, $G = A \cup B \cup C \cup D$, where A, B, C, D is the AG-subgroup of G.

Theorem 10. *Let G be a strong L2-transposition regular AG-groupoid. Then, it is a strong R2-transposition regular AG-groupoid.*

Proof. Let G be a strong L2-transposition regular AG-groupoid. Then, for any $a \in G$, let $xa = e$, according to Proposition 3 and Theorem 6, $ae = a = ea, xa = e = ax$. So, $(ax)a = ea = a, a((ax)x) = a(ex) = (ea)(ex) = (ee)(ax) = e$. Uniqueness holds; G is a strong R2-transposition regular AG-groupoid. □

However, not every strong R2-regular AG-groupoid is a strong L2-transposition regular AG-groupoid in Example 15.

Example 15. *Let $G = \{0, 1, 2, 3\}$. Define the operation on G as shown in Table 8. Clearly, $(G, *)$ is a strong R2-transposition regular AG-groupoid.*

Table 8. This is a strong R2-transposition regular AG-groupoid.

$*$	**0**	**1**	**2**	**3**
0	0	1	2	3
1	3	0	1	2
2	2	3	0	1
3	1	2	3	0

However, G is not a strong L2-transposition regular AG-groupoid, since $1 * (3 * 1) = 1$, $(3 * (3 * 1)) * 1 = 3 * 1$, but $3 * 1 = 2$ and $2 * 2 \neq 2$.

Table 8 shows a distance matrix that contains information about distances (Figure 1).

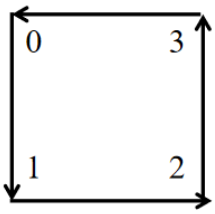


Figure 1. A connected directed unweighted graph.

5. L3-Transposition Regular AG-Groupoid and R3-Transposition Regular AG-Groupoid

In this section, we propose the notion of an L3-transposition regular AG-groupoid and R3-transposition regular AG-groupoid.

Definition 11. *Let G be an AG-groupoid, $a \in G$. An element a is L3-transposition regular in G if there exists unique $x \in G$ s.t.$(xa)a = a$. G is an L3-transposition regular AG-groupoid if all its elements are L3-transposition regular.*

Example 16. *Let $G = \{a, b, c, d\}$; define the operation on G as shown in Table 9. Then, $(G, *)$ is an L3-transposition regular AG-groupoid. Let*

$$x_a = \{x \in G | \text{ there exists unique } x \in G \text{ s.t. } (xa)a = a, a \in G\}.$$

Furthermore,

$$x_a = b, x_b = a, x_c = d, x_d = c.$$

Table 9. This is an L3-transposition regular AG-groupoid.

$*$	**a**	**b**	**c**	**d**
a	*b*	*d*	*c*	*a*
b	*c*	*a*	*b*	*d*
c	*a*	*c*	*d*	*b*
d	*d*	*b*	*a*	*c*

Definition 12. *Let G be an L3-transposition regular AG-groupoid. G is a strong L3-transposition regular AG-groupoid if $(xa)^2 = xa$.*

Example 17 shows that not every L3-transposition regular AG-grooupoid is a strong L3-transposition regular AG-groupoid.

Example 17. *Let $G = \{a, b, c, d\}$, define the operation on G as shown in Table 9. $x_a * a = b * a = c$, and c is not idempotent. Then, G is not a strong L3-transposition regular AG-groupoid.*

Theorem 11. *Let G be an AG-groupoid. G is a strong L3-transposition regular AG-groupoid if and only if it is a strong R2-transposition regular AG-groupoid.*

Proof. ($\Rightarrow$) Let G be a strong L3-transposition regular AG-groupoid. For any element a in G, let $xa = e$, then $ea = a$, and $ee = e$. Then,

$$ax = (ea)x = (xa)e = ee = e.$$

Then, e is the local left identity element of a, and x is the local inverse element of a relative to e, because $((ex)a)a = ((ax)e)a = (ee)a = ea = a$. According to the definition of a strong R3-transposition regular AG-groupoid, $ex = x$.

According to the definition, x, that is, the local left inverse element of a is unique, x is the local inverse element of a; thus, the local inverse element of a is unique. Assume that the local right inverse element of a is not unique; that is, there exists $y \in G$ and $x \neq y$ such that $(ay)a = a$. Let $ay = f$, there is, $fa = a$, because $ef = (xa)f = (fa)x = ax = e$, and $(ee)e = ee = e$, but $(fe)e = (ee)f = ef = e$. According to the definition of a strong L3-transposition regular AG-groupoid, $e = f$. Then, $ay = e$. That is, $((ey)a)a = (aa)(ey) = (ae)(ay) = (ae)e = (ee)a = ea = a$. According to the definition of a strong L3-transposition regular AG-groupoid, $ey = ex = x$.

At this time, $(ax)x = ex = x$, and $(((ya)a)x)x = ((xa)(ya))x = (e(ya))x = ((ee)(ya))x = ((ey)(ea))x = (xa)x = ex = x$. According to definition of a strong L3-transposition regular AG-groupoid, $(ya)a = a$. Then, $x = y$; that is, the local right inverse element of a is unique. So

$$(ax)a = ea = a, a((ax)x) = a(ex) = (ea)(ex) = (ee)(ax) = ee = e = ax \text{ and } ee = e.$$

Furthermore, x is unique, so G is a strong R2-transposition regular AG-groupoid.

($\Leftarrow$) Let G be a strong R2-transposition regular AG-groupoid. Then, for any $a \in G$, there exists a unique $x \in G$ such that $(ax)a = a, a((ax)x) = ax$, and $(ax)^2 = ax$. Let $ax = e$; that is, $ea = a, a(ex) = ax = e, ee = e$. According to Proposition 4, $ex = x$. That is, $xa = (ex)a = (ax)e = ee = e$. According to definition of a strong R2-transposition regular AG-groupoid, the local right inverse element of a is unique.

Assume that the local left inverse element of a relative to the local left identity element is not unique; that is, there exists $y \in G$ and $x \neq y$, such that $(ya)a = a$. Let $ya = f$, there is $fa = a$. However, $ef = (xa)f = (fa)x = ax = e$, and $(ee)e = ee = e, e((ee)e) = e(ee) = ee$, $(ef)e = ee = e, e((ef)f) = e(ef) = ee = e$. According to the definition of a strong R2-transposition regular AG-groupoid, $e = f$, that is, $ya = e$. At this time, $ay = (ea)y = (ya)e = ee = e$, and $(ay)a = ea = a, a((ay)y) = a(ey) = (ea)(ey) = (ee)(ay) = ee = e = ay$. According to the definition of a strong R2-transposition regular AG-groupoid, $x = y$. That is, the local left inverse element of a is relative to the local left identity element.

Because

$$(xa)a = ea = a, \text{ and } ee = e.$$

Furthermore, x is unique, and G is strong L3-transposition regular AG-groupoid. □

Definition 13. *Let G be an AG-groupoid, $a \in G$. An element a is R3-transposition regular in G if there exists a unique $x \in G$ s.t. $a(ax) = a$. G is an R3-transposition regular AG-groupoid if all its elements are R3-transposition regular.*

Example 18. *Let $G = \{a, b, c, d\}$; define the operation on G as shown in Table 9. Then, $(G, *)$ is an R3-transposition regular AG-groupoid. Let*

$$x_a = \{x \in G | \text{ there exists a unique } x \in G \text{ s.t. } a(ax) = a, a \in G\}.$$

Furthermore,

$$x_a = b, x_b = a, x_c = d, x_d = c.$$

Definition 14. *Let G be an R3-transposition regular AG-groupoid. G is a strong R3-transposition regular AG-groupoid if $(ax)^2 = ax$.*

Example 19. *Let $G = \{a, b, c, d\}$; define the operation on G as shown in Table 9. Here, a, b, c, d are not idempotent. Then, $(G, *)$ is not a strong R3-transposition regular AG-groupoid.*

Theorem 12. *Let G be an AG-groupoid. G is a strong L2-transposition regular AG-groupoid if and only if it is a strong R3-transposition regular AG-groupoid.*

Proof. ($\Rightarrow$) Let G be a strong L2-transposition regular AG-groupoid. For any element a of G, let $xa = e$, then $ae = a$, $(x(xa))a = (xe)a = (ae)x = ax = xa$. Then, $a(ax) = a(xa) = a$. If x is unique, G is a strong R3-transposition regular AG-groupoid.

($\Leftarrow$) Let G be a strong R3-transposition regular AG-groupoid. Then, for any $a \in G$, there exists a unique $x \in G$ s.t. $a(ax) = a$, and $(ax)^2 = ax$. Let $ax = e$, then $ae = a$, and $ee = e$. That is, $ea = (ee)a = (ae)e = ae = a$. Because $a(ex) = (ea)(ex) = (ee)(ax) = (ee)e = ee = e$, $a(xe) = (ae)(xe) = (ax)(ee) = e(ee) = ee = e$. According to the definition of a strong R3-transposition regular AG-groupoid, $ex = x = xe$. Then, $xa = (ex)a = (ax)e = ee = e$. That is, $ea = a = ae$, $xa = e = ax$, $xe = x = ex$. According to the definition of a strong R3-transposition regular AG-groupoid, the local right inverse element of a and the relative local identity element are unique.

Assume that the local left inverse element of a relative to the local identity element is not unique; that is, there exists $y \in G$ and $x \neq y$ such that $ya = f$, and $fa = a = af$. Then, $ef = (xa)f = (fa)x = ax = e$. Then, $e(ee) = ee = e$, $e(ef) = ee = e$. So, $e = f$. That is, $ya = e$. Then, $ay = (ea)y = (ya)e = ee = e$. According to the definition of a strong R3-transposition regular AG-groupoid, $x = y$. That is, the local left inverse element of a relative to the local identity element is unique.

At this moment, $a(xa) = ae$, $(x(xa))a = (xe)a = xa = e$, and $(xa)^2 = xa$, and the local left inverse element is unique; then, G is a strong L2-transposition regular AG-groupoid. □

Theorem 13. *Let G be an AG-groupoid. If G is an L2-transposition regular AG-groupoid, G is an R3-transposition regular AG-groupoid.*

Proof. Let G be an L2-transposition regular AG-groupoid. Then, for any element a in G, there exists a unique x such that $a(xa) = a, x(xa)a = xa$. Let $xa = e$; then, $ae = a, (xe)a = (ae)x = ax = xa$. Then, $a(ax) = a$, and x is unique. Then, G is an R3-transposition regular AG-groupoid. □

Example 20. *Let $G = \{a, b, c, d\}$ define the operation on G as shown in Table 9. Clearly, G is an R3-transposition regular AG-groupoid. However, G is not an L2-transposition regular AG-groupoid, since $a * (d * a) = a$, $(d * (d * a)) * a = (d * d) * a = c * a = a \neq d = d * a$.*

Figure 2 shows the relation among various transposition regular AG-groupoids, where A represents a strong L2-transposition regular AG-groupoid and a strong R3-transposition regular AG-groupoid;

B represents an L1/R1-transposition regular AG-groupoid that is not a strong L2-transposition regular AG-groupoid; see Example 11;

C represents an L2-transposition regular AG-groupoid that is not a strong L2-transposition regular AG-groupoid; see Example 9;

D represents an R3-transposition regular AG-groupoid that is not an L2-transposition regular AG-groupoid; see Example 20;

E represents a strong R2/L3-transposition regular AG-groupoid that is not a strong L2-transposition regular AG-groupoid; see Example 15;

F represents an R2-transposition regular AG-groupoid that is not a strong R2-transposition regular AG-groupoid; see Example 13;

G represents an L3-transposition regular AG-groupoid that is not a strong L3-transposition regular AG-groupoid; see Example 17.

Then, A + B is an L1/R1-transposition regular AG-groupoid, A + C is an L2-transposition regular AG-groupoid, A + C + D is an R3-transposition regular AG-groupoid, A + E is a strong R2/L3-transposition regular AG-groupoid, A + E + F is an R2-transposition regular AG-groupoid, and A + E + G is an L3-transposition regular AG-groupoid.

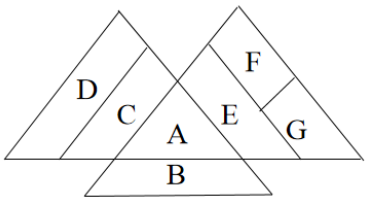


Figure 2. The relation among various transposition regular AG-groupoids.

6. Transposition Regular AG-Groupoid Semigroup

In [20], they introduced the definition of an AG-groupoid semigroup, that is, algebra that is both semigroup and AG-groupoid. In this paper, we reproved that the theorem of a regular AG-groupoid semigroup is a commutative semigroup. We also introduce transposition regularity into the AG-groupoid semigroup, and we discuss the relations among various transpositions of regular AG-groupoid semigroups, commutative Clifford semigroups, and Abelian groups as well as the decomposition theorem of AG-groupoid semigroups.

Definition 15 ([21]). *Let R be a algebra. Subalgebra A is an anti-ideal of R, if for any $x \in R \backslash A$, there is, $xA \cap A \cap Ax = \varnothing$.*

Definition 16 ([22]). *Let $(G, *)$ be a generalized group. Then, for any $x, y \in G$, if $x * y = y * x$, then G is a group.*

Definition 17 ([23]). *S is a Clifford semigroup, if it is a completely regular semigroup, and for any $x, y \in S$,*

$$(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1}).$$

For any element in S, the element c in S is a central element, if for any $s \in S$, there is $cs = sc$. The set of all central elements in S is the center in S.

Theorem 14. *Let S be an AG-groupoid semigroup. If S is regular, then S is a commutative semigroup.*

Proof. Let S be a regular AG-groupoid semigroup. Then, for any elements $a, b \in S$ such that $aba = a$. Then, $\forall c \in S$, according to medial law and left invertive law as well as associative law, there is,

$$ac = (aba)c = (ab)(ac) = (aa)(bc) = (a(ab))c = c(ab)a = c(aba) = ca.$$

Because S is regular, then for any regular elements m in S, for any element n in S, there is $mn = nm$. That is, S is a commutative semigroup. □

The following shows the concepts of various transposition regular AG-groupoid semigroups as well as the connections among them and the commutative Clifford semigroup as well as Abelian group.

In [19], they introduced various transposition regular semigroups and prove that L1/R1/LR-transposition regular semigroups and a completely regular semigroup are equivalent to each other.

Definition 18. *Let G be a semigroup. G is an L1-transposition regular AG-groupoid semigroup if it is an L1-transposition regular AG-groupoid. Similarly, there is a R1/LR, L2/R2/L3/R3-transposition regular AG-groupoid semigroup.*

According to Definition 18, L1/R1/LR-transposition regular AG-groupoid semigroups are equivalent to each other.

Theorem 15. *Let G be an AG-groupoid semigroup. It is an L1-transposition regular AG-groupoid semigroup if and only if it is a commutative Clifford semigroup.*

Proof. ($\Rightarrow$) Let G be an L1-transposition regular AG-groupoid semigroup. Clearly, it is regular, according to Theorem 14, and G is a commutative semigroup. Then, for any $x, y \in G$, there is, $(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1})$. Then, G is a commutative Clifford semigroup.

($\Leftarrow$) Let G be a commutative Clifford semigroup. According to the definition of a Clifford semigroup, G is a completely regular semigroup. Then, G is an L1-transposition regular semigroup. $\forall a, b, c \in G$, there is,

$$(ab)c = c(ab) = c(ba) = (cb)a,$$

Then, G satisfies the left invertive law. Then, G is an L1-transposition regular AG-groupoid semigroup. □

Theorem 16. *Let G be an AG-groupoid semigroup. Then, the folllowing conditions are equivalent:*
(1) G is an L2/R2-transposition regular AG-groupoid semigroup;
(2) G is an L3/R3-transposition regular AG-groupoid semigroup;
(3) G is an Abelian group.

Proof. (1) $\Rightarrow$ (2) Let G be an L2/R2-transposition regular AG-groupoid semigroup. According to the decomposition theorem of them, G is a group. Clearly, G is regular; then, G is a commutative and Abelian group. That is, the L2/R2-transposition regular AG-groupoid semigroup is an Abelian group. $\forall a \in G$, there exists a unique $e, a^{-1} \in G$, such that $ae = a = ea$, $a^{-1}a = e = aa^{-1}$. Then, $(a^{-1}a)a = ea = a$. Then, G is an L3-transposition regular AG-groupoid semigroup. Similarly, $a(aa^{-1}) = ae = a$. Then, G is an R3-transposition regular AG-groupoid semigroup.

(2) $\Rightarrow$ (3) Let G be an L3-transposition regular AG-groupoid semigroup. Then, it is a generalized group. According to Theorem 14, G is commutative. So, it is a group. Then, G is an Abelian group. Similarly, the R3-transposition regular AG-groupoid semigroup is a generalized group, so the L3-transposition regular AG-groupoid semigroup is an Abelian group.

(3) $\Rightarrow$ (1) Let G be an Abelian group. Then, G is an L2/R2-transposition regular AG-groupoid semigroup. $\forall a, b, c \in G$, there is,

$$(ab)c = c(ab) = c(ba) = (cb)a,$$

Then, G satisfies the left invertive law. Then, G is an L2/R2-transposition regular AG-groupoid semigroup. □

Figure 3 shows the relation among them.

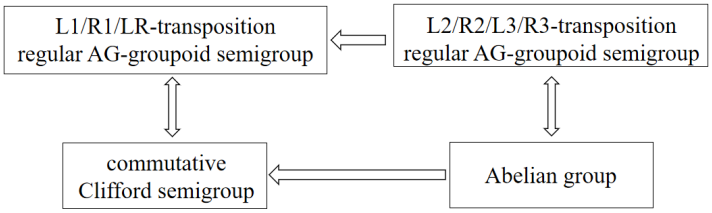


Figure 3. The relation among various transposition regular AG-groupoid semigroups.

In the following, we introduce the properties and decomposition theorem of an AG-groupoid semigroup.

Theorem 17. *Let G be an AG-groupoid semigroup. Let H be the subset of G, and $H = \{a|aba = a, a \in G, b \in G\}$. Then,*
(1) H is the subsemigroup of G;
(2) H is the center;
(3) H is a Clifford semigroup.

Proof. (1) Let G be an AG-groupoid semigroup, $a, x \in H$. Then, a, x is the regular element of G. Then, there exists $m \in G$, s.t. $ama = a$. Then, $a(mam)a = (ama)ma = ama = a$, $(mam)a(mam) = m(ama)mam = mamam = mam$. Then, mam is a regular element. Let $b = mam$, there is $aba = a$, $bab = b$, and there exists $y \in H$ such that $xyx = x$, $yxy = y$. Then,

$$\begin{aligned} ax &= (aba)(xyx) = (ab)(ax)(yx) = (((ax)b)a)(yx) \\ &= (ax)((ba)(yx)) = (ax)((by)(ax)) = (ax)(by)(ax). \end{aligned}$$

That is, for $ax \in G$, there exists $by \in G$ s.t. $(ax)(by)(ax) = ax$. That is, ax is a regular element. Then, $ax \in H$. So H is a subsemigroup of G.

(2) For any regular elements a in H, assume that there exists $b \in G$ s.t. $aba = a$. Then, for any element x that is not regular in G, there is,

$$\begin{aligned} xa &= x(aba) = ((xa)b)a = ((ba)x)a = (ax)(ba) = ((ax)b)a \\ &= ((bx)a)a = (aa)(bx) = (ab)(ax) = (aba)x = ax. \end{aligned}$$

Then, a is a central element; that is, any regular element in G is a central element. Then, H is the center.

(3) By (1), $ax = (ax)(by)(ax)$. Then,

$$(by)(ax)(by) = ((ba)(yx))(by) = ((ba)b)((yx)y)) = by.$$

Furthermore, $(ax)(by) = (by)(ax)$. So, H is a completely regular semigroup. According to Theorem 14, the regular AG-groupoid semigroup is a commutative Clifford semigroup. Then, H is commutative, so for any $x, y \in H$, there is $(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1})$. Then, H is a commutative Clifford semigroup. □

Theorem 18. *Let G be an AG-groupoid semigroup and B be a center of G. Then,*
(1) B is an ideal of G;
(2) $G \backslash B$ is an anti-ideal of G.

Proof. (1) Clearly, B is a center. For any element a, b of B, there is $ab = ba$. $\forall y \in G$, there is, $ay = ya$, $by = yb$. Then,

$$(ab)y = (yb)a = y(ba) = y(ab).$$

Then, ab is a central element. That is, $ab \in B$.

For any element a in B, any element x in G. Because a is a central element, then $ax = xa$. $\forall y \in G$, there is,

$$(ax)y = (ya)x = y(xa) = y(ax).(xa)y = (ya)x = y(ax) = y(xa).$$

Then, ax and xa are central elements, that is, $ax \in B$, $xa \in B$. Because a and x represent any element of B and G, respectively, $BG \subseteq B$, $GB \subseteq B$. Then, B is an ideal of G.

(2) Let A be a set of elements that are not central elements in G. Then, $A \cap B = \varnothing$. So, $A = G \backslash B$. For any element x in B, any element y in A, there is, $xy \in B$, $yx \in B$. Then, $xA \subseteq B$, $Ax \subseteq B$. So, $xA \cap A \cap Ax = \varnothing$. Then, A is anti-ideal. □

The following example shows that not every central element is a regular element.

Example 21. *Let $G = \{a, b, c, d\}$ be an AG-groupoid semigroup, the operation on G as shown in Table 10, where a, b are central elements, but b is not a regular element.*

Table 10. AG-groupoid semigroup that is not commutative.

$*$	**a**	**b**	**c**	**d**
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	a	b

7. Discussion

In Example 15, we show the relationship between Table 8 and a connected directed unweighted graph. So, we can discuss applications in the graph for AG-groupoids. In this paper, we mainly talk about the various transposition regular AG-groupoids. We can discuss the relationships among transposition regular AG-groupoids, hypersemigroups and T2CA-groupoids as well as non-classical logical algebras (see [24–28]).

8. Conclusions

In this paper, we introduce transposition regularity into AG-groupoids, and various transposition regular AG-groupoids are obtained. Their properties and structures are discussed by studying their decomposition theorem: (1) an L1-transposition regular AG-groupoid is equivalent to an R1-transposition regular AG-groupoid and they are a disjoint union of Abelian subgroups; (2) an LR/strong LR-transposition regular AG-groupoid satisfying a certain condition is equivalent to the L1/R1-transposition regular AG-groupoid; (3) a strong L2-transposition regular AG-groupoid and strong R3-transposition regular AG-groupoid are equivalent to each other and can be decomposed into the union of Abelian subgroups; (4) a strong L3-transposition regular AG-groupoid and strong R2-transposition regular AG-groupoid are equivalent to each other and can be decomposed into the union of AG subgroups. Finally, we introduce transposition regularity into the AG-groupoid semigroup and prove that: (1) L1/R1/LR-transposition regular AG-groupoid semigroups are equivalent to commutative Clifford semigroups; (2) L2/R2/L3/R3-transposition regular AG-groupoid semigroups are equivalent to Abelian groups; (3) an AG-groupoid semigroup is a union of ideal and anti-ideal.

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