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On Some Properties of the First Brocard Triangle in the Isotropic Plane

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Abstract: In this paper we introduce the first Brocard triangle of an allowable triangle in the isotropic plane and derive the coordinates of its vertices in the case of a standard triangle. We prove that the first Brocard triangle is homologous to the given triangle and that these two triangles are parallelogic. We consider the relationships between the first Brocard triangle and the Steiner axis, the Steiner point, and the Kiepert parabola of the triangle. We also investigate some other interesting properties of this triangle and consider relationships between the Euclidean and the isotropic case.

Keywords: isotropic plane; first Brocard triangle; Brocard circle; parallelogy

MSC: 51N25

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Citation: Volenec, V.; Kolar-Begović, Z.; Kolar-Šuper, R. On Some Properties of the First Brocard Triangle in the Isotropic Plane. *Mathematics* **2022**, *10*, 1381. https://doi.org/10.3390/math10091381

Academic Editor: Yang-Hui He

Received: 9 March 2022
Accepted: 19 April 2022
Published: 20 April 2022

1. Introduction and Motivation

The isotropic (or Galilean) plane is a projective-metric plane, where the absolute consists of one line—the absolute line ω_A, and one point on that line—the absolute point Ω_A.

When using homogeneous coordinates in the projective plane, $P = (x_0 : x_1 : x_2)$, $x_0^2 + x_1^2 + x_2^2 \neq 0$, then we choose the absolute point $\Omega_A = (0 : 1 : 0)$ and the absolute line ω_A having the equation $x_2 = 0$. Points incident to the absolute line ω_A are called *isotropic points* and lines incident to the absolute point Ω_A are called *isotropic lines*. We will mention a few well known metric quantities in the isotropic plane for which we assume that $x = \frac{x_0}{x_2}$ and $y = \frac{x_1}{x_2}$.

Two lines are called *parallel* if they have the same isotropic point. Points which lie on the same isotropic line are said to be parallel.

For two non-parallel points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ the *isotropic distance* is defined as $d(P_1, P_2) := x_2 - x_1$. Notice that the isotropic distance is directed. For two parallel points $P_1 = (x_1, y_1)$ and $P_2 = (x_1, y_2)$, the *isotropic span* is defined as $s(P_1, P_2) := y_2 - y_1$. The midpoint of the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is defined as $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

The angle formed by non-isotropic lines l_1 and l_2 given by $y = m_1x + b_1$ and $y = m_2x + b_2$ is defined by $\varphi = \angle(l_1, l_2) := m_2 - m_1$, and it is directed. The bisector of the lines l_1 and l_2 is given by the equation $y = \frac{m_1+m_2}{2}x + \frac{b_1+b_2}{2}$. A normal line to a line l at a point P is the isotropic line n passing through P.

All projective transformations that preserve the absolute figure are of the form

$$\begin{aligned} \overline{x} &= a + px, \qquad a, b, c, p, q \in \mathbf{R}, \\ \overline{y} &= b + cx + qy, \quad pq \neq 0, \end{aligned}$$

and form the 5-parametric group G_5 known as the *group of similarities* of the isotropic plane (see [1]).

Distances, spans and angles are kept invariant under the subgroup G_3 of G_5 which consists of transformations of the form

$$\begin{aligned} \overline{x} &= a + x \\ \overline{y} &= b + cx + y, \qquad a, b, c \in \mathbf{R}. \end{aligned}$$

G_3 is called the *motion group* of the isotropic plane.

Metric quantities and all the facts related to the geometry of the isotropic plane can be found in [1,2].

A triangle is called *allowable* if none of its sides are isotropic [1]. As it is explained in [3], according to [1], to any allowable triangle in the isotropic plane there is exactly one circumscribed circle. The equation of this circle is of the form $y = ux^2 + vx + w$, $u \neq 0$. Choosing a suitable coordinate system and applying the group of similarities, we may assume that the equation of this circle is $y = x^2$, and that the vertices of the allowable triangle ABC are $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$, where a, b, and c are mutually different numbers. For convenience, we will frequently use abbreviations $abc = p$ and $ab + bc + ca = q$. Choosing, without loss of generality, that $a + b + c = 0$, the diameter of the circle circumscribed to the triangle ABC, passing through its centroid $G = \left(\frac{a+b+c}{3}, \frac{a^2+b^2+c^2}{3}\right) = \left(0, -\frac{2}{3}q\right)$, lies on the y-axis, while the x-axis is tangent to this circle at the endpoint of that diameter.

For each allowable triangle ABC, one can, in the described way, achieve that its circumscribed circle has the equation $y = x^2$, and its vertices are of the form $A = (a, a^2)$, $B = (b, b^2)$, and $C = (c, c^2)$, with $a + b + c = 0$. We shall say that such a triangle is in the *standard position*, or shorter, that the triangle ABC is the *standard triangle*. To prove geometric facts for allowable triangles, it is sufficient to give a proof for a standard triangle. Its sides BC, CA, and AB have equations $y = -ax - bc$, $y = -bx - ca$, and $y = -cx - ab$. Using the mentioned notations it can be proved that $q = bc - a^2$ and $(c - a)(a - b) = 2q - 3bc$.

The *tangential triangle* of a given triangle ABC is the triangle $A_tB_tC_t$ determined by the three tangents to the circumscribed circle of the triangle ABC at its vertices. It can be proved that the lines AA_t, BB_t, and CC_t are symmetric, with respect to bisectors of the angles A, B, and C, to the medians AG, BG, and CG of the triangle ABC. The lines AA_t, BB_t, and CC_t meet at the point K which is called the *symmedian center* of the triangle ABC.

Let ABC be a standard triangle and let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively $\mathcal{A}'$, $\mathcal{B}'$, and $\mathcal{C}'$, be lines through the points A, B, and C such that $\angle(AB, \mathcal{A}) = \angle(BC, \mathcal{B}) = \angle(CA, \mathcal{C}) =: \varphi$, $\angle(\mathcal{A}', AC) = \angle(\mathcal{B}', BA) = \angle(\mathcal{C}', CB) =: \psi$. In [4] it is proved that the lines $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ pass through a common point, say Ω_1, if and only if $\varphi = \omega$, and the lines $\mathcal{A}'$, $\mathcal{B}'$, and $\mathcal{C}'$ pass through a common point, say Ω_2, if and only if $\psi = \omega$, where ω is given by $\omega = -\frac{1}{3q}(b - c)(c - a)(a - b)$. The points Ω_1 and Ω_2 are called *Crelle–Brocard points*, and ω is called the *Brocard angle*.

The isotropic analogue of Brocard's theorem was first obtained in [2].

The standard triangle ABC has, by [4], the symmedian center K and Crelle–Brocard points Ω_1 and Ω_2 given by

$$K = \left(\frac{3p}{2q}, -\frac{q}{3}\right), \; \Omega_1 = \left(\frac{p - p_1}{q}, \frac{27{p_1}^2 - 2q^3}{9q^2}\right), \; \Omega_2 = \left(\frac{p - p_2}{q}, \frac{27{p_2}^2 - 2q^3}{9q^2}\right) \tag{1}$$

where $p_1 = \frac{1}{3}(bc^2 + ca^2 + ab^2)$, $p_2 = \frac{1}{3}(b^2c + c^2a + a^2b)$. One can prove that $p_1 + p_2 + p = 0$, $p_1^2 + p_1p_2 + p_2^2 = -\frac{q^3}{9}$, $p^2 + pp_1 + p_1^2 = -\frac{q^3}{9}$, and $p^2 + pp_2 + p_2^2 = -\frac{q^3}{9}$.

These three points lie, according to [5], on the *Brocard circle* of the triangle ABC (see Figure 1), given by

$$y = 2x^2 - \frac{3p}{q}x - \frac{q}{3}. \tag{2}$$

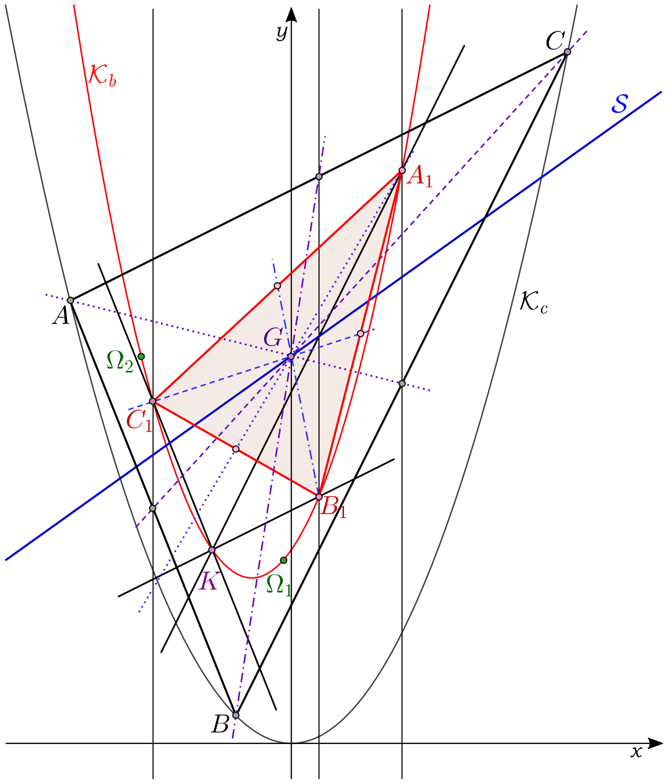

Figure 1. Brocard circle $\mathcal{K}_b$, the Steiner axis $\mathcal{S}$, and the first Brocard triangle $A_1B_1C_1$ of the triangle ABC. Visualization of statements of Theorems 1, 3 and 4.

If A_m, B_m, and C_m are midpoints of the sides BC, CA, and AB of the allowable triangle ABC, and A', B' and C' are points on the perpendicular bisectors of these sides such that spans $s(A_m, A')$, $s(B_m, B')$, and $s(C_m, C')$ are proportional to the lengths of sides BC, CA, and AB, then the points $BC \cap B'C'$, $CA \cap C'A'$, and $AB \cap A'B'$ lie on a line, call it $\mathcal{T}$. Triangles $A'B'C'$ are the so-called *Kiepert triangles* of the triangle ABC, and the line $\mathcal{T}$ is the axis of homology of the triangle ABC and the corresponding Kiepert triangle $A'B'C'$. Axes of homology of an allowable triangle ABC and its Kiepert triangles envelope a parabola which is called the *Kiepert parabola* [6].

The inscribed and the circumscribed Steiner's ellipses of an allowable triangle have the same nonisotropic axis, which passes through the centroid G of that triangle and which in the case of a standard triangle has equation $y = -\frac{3p}{2q}x - \frac{2}{3}q$. This axis is called the *Steiner's axis* of the considered triangle. In [7], the *Steiner point* of the allowable triangle ABC is defined as the fourth (the first three being A, B, and C) common point S of the circumscribed circle and the circumscribed Steiner ellipse of that triangle. If ABC is a standard triangle then $S = \left(-\frac{3p}{q}, \frac{9p^2}{q^2}\right)$.

2. The First Brocard Triangle of a Triangle in the Isotropic Plane

In this section we will define the first Brocard triangle of a triangle in the isotropic plane.

Theorem 1. *Given a standard triangle ABC, the lines through its symmedian center K and parallel to its sides BC, CA, and AB meet the Brocard circle, besides the point K, at points*

$$\begin{aligned} A_1 &= \Big(-\frac{a}{2}, \frac{1}{6q}(9ap + 3a^2q - 2q^2)\Big), \\ B_1 &= \Big(-\frac{b}{2}, \frac{1}{6q}(9bp + 3b^2q - 2q^2)\Big), \\ C_1 &= \Big(-\frac{c}{2}, \frac{1}{6q}(9cp + 3c^2q - 2q^2)\Big), \end{aligned} \tag{3}$$

which lie on bisectors of the sides BC, CA, and AB, respectively (see Figure 1).

Proof. The point A_1 obviously lies on the bisector of BC, and with $x = -\frac{a}{2}$, from (2) and

$$y = -ax + \tfrac{3ap}{2q} - \tfrac{q}{3}, \tag{4}$$

we get the ordinate of the point A_1

$$y = \tfrac{a^2}{2} + \tfrac{3ap}{2q} - \tfrac{q}{3} = \tfrac{1}{6q}(9ap + 3a^2q - 2q^2).$$

Therefore, A_1 is the intersection of that line and the Brocard circle (2). This line is parallel to BC and passes through K, see (1). □

The points A_1, B_1, and C_1 from Theorem 1 determine the *first Brocard triangle* of the triangle ABC (see Figure 1).

Theorem 2. *The sides of the first Brocard triangle $A_1B_1C_1$ of the standard triangle ABC are given by*

$$\begin{aligned} B_1C_1 \quad \ldots \quad y &= \Big(a - \frac{3p}{q}\Big)x - \frac{1}{6}(2q + 3bc), \\ C_1A_1 \quad \ldots \quad y &= \Big(b - \frac{3p}{q}\Big)x - \frac{1}{6}(2q + 3ca), \\ A_1B_1 \quad \ldots \quad y &= \Big(c - \frac{3p}{q}\Big)x - \frac{1}{6}(2q + 3ab). \end{aligned} \tag{5}$$

Proof. The point B_1 satisfies the first equation in (5) because

$$\Big(a - \frac{3p}{q}\Big)\Big(-\frac{b}{2}\Big) - \frac{1}{6}(2q + 3bc) = \frac{1}{6q}(9bp + 3b^2q - 2q^2),$$

and so does the point C_1. □

Theorem 3. *A triangle and its first Brocard triangle have the same centroid (see Figure 1).*

Proof. According to [3], the triangle ABC has the centroid $G = \big(0, -\frac{2}{3}q\big)$. The triangle $A_1B_1C_1$ with vertices (3), has the same centroid because $a + b + c = 0$ and

$$\frac{1}{3} \cdot \frac{1}{6q}\big(9(a+b+c)p + 3(a^2+b^2+c^2)q - 6q^2\big) = \frac{1}{18q}\big(3q(-2q) - 6q^2\big) = -\frac{2}{3}q. \quad \square$$

3. The First Brocard Triangle and Some Other Significant Elements

In this section we consider the relationships between the first Brocard triangle and some other objects related to a triangle in the isotropic plane.

Theorem 4. *Let G be the centroid and $A_1B_1C_1$ the first Brocard triangle of an allowable triangle ABC. Then the pairs of lines GA, GA_1; GB, GB_1; and GC, GC_1 have the same bisector. This bisector is the Steiner axis of the triangle ABC (Figure 1).*

For the Euclidean case see [8,9].

Proof. The lines GA and GA_1 have slopes

$$\frac{a^2 + \frac{2}{3}q}{a} = a + \frac{2q}{3a},$$
$$\frac{\frac{1}{6q}(9ap + 3a^2q - 2q^2) + \frac{2}{3}q}{-\frac{a}{2}} = -\frac{1}{3aq}(9ap + 3a^2q + 2q^2) = -\frac{3p}{q} - a - \frac{2q}{3a}.$$

The sum of slopes is equal to $-\frac{3p}{q}$, therefore the bisector of these lines has the slope $-\frac{3p}{2q}$. The Steiner axis of the triangle ABC is given by $y = -\frac{3p}{2q}x - \frac{2}{3}q$ [7], which passes through the centroid $G = \left(0, -\frac{2}{3}q\right)$, and coincides with this bisector. $\square$

Theorem 5. *The first Brocard triangle $A_1B_1C_1$ of an allowable triangle ABC is homologous with this triangle, and the center of homology is K'—the point reciprocal to the symmedian center of the triangle ABC (see Figure 2).*

For the Euclidean case see [9].

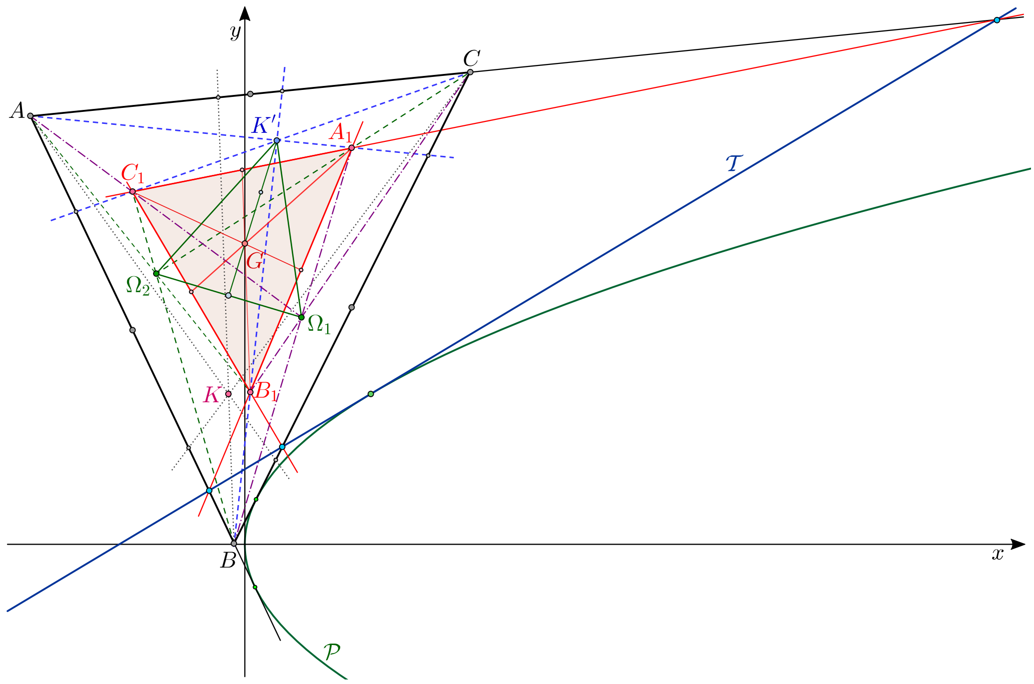

Figure 2. The axis of homology $\mathcal{T}$ of the triangle ABC and its first Brocard triangle $A_1B_1C_1$, and the Kiepert parabola $\mathcal{P}$ of the triangle ABC. Visualization of statements of Theorems 5 and 7.

Proof. The line
$$9aqy = (2q^2 + 3a^2q - 9ap)x + 9a^2p + 6a^3q - 2aq^2$$
passes through the point $A = (a, a^2)$ and also through the point A_1, hence it is the line AA_1. In addition, this line passes through the point

$$K' = \left(-\frac{3p}{q}, \frac{27p^2 - 8q^3}{9q^2}\right), \tag{6}$$

which is, by [5], reciprocal to the symmedian center K and anticomplementary to the midpoint of Crelle–Brocard points Ω_1 and Ω_2. The same also holds for analogous lines BB_1 and CC_1. $\square$

Theorem 6. *In case of the standard triangle ABC, the axis of homology of triangles ABC and $A_1B_1C_1$ from Theorem 5, has the equation*

$$y = -\tfrac{6p}{q}x - \tfrac{q}{6}. \tag{7}$$

Proof. It is enough to prove e.g., that the point $\left(\frac{q}{6a}, -\frac{q}{6} - bc\right)$ lies on lines BC and B_1C_1, and on the line defined by (7).

Indeed, we have

$$\begin{aligned}
-a \cdot \tfrac{q}{6a} - bc &= -\tfrac{q}{6} - bc, \\
\left(a - \tfrac{3p}{q}\right) \cdot \tfrac{q}{6a} - \tfrac{1}{6}(2q + 3bc) &= \tfrac{q}{6} - \tfrac{bc}{2} - \tfrac{q}{3} - \tfrac{bc}{2} = -\tfrac{q}{6} - bc, \\
-\tfrac{6p}{q} \cdot \tfrac{q}{6a} - \tfrac{q}{6} &= -\tfrac{q}{6} - bc.
\end{aligned}$$

$\square$

In the discussion following Theorem 2 in [6] it is shown that the Kiepert triangle $A'B'C'$ of a triangle ABC, with $t = -\omega$, coincides with the first Brocard triangle $A_1B_1C_1$ of the triangle ABC and the axis of homology of triangles ABC and $A_1B_1C_1$ touches the Kiepert parabola of the triangle ABC (see Figure 2).

Theorem 7. *The triangle ABC and its first Brocard triangle $A_1B_1C_1$ are three-homologous, and the centers of homologies are the point K' and Crelle–Brocard points Ω_1 and Ω_2 of that triangle. Triangles ABC and $K'\Omega_1\Omega_2$ have the same centroid G [9] (see Figure 2).*

Proof. According to [4], the lines $A\Omega_1$ and $A\Omega_2$ have equations

$$\begin{aligned}
A\Omega_1 \quad &\ldots \quad y = (\omega - c)x - a\omega - ab, \\
A\Omega_2 \quad &\ldots \quad y = -(\omega + b)x + a\omega - ca.
\end{aligned}$$

The point C_1 lies on the first line, and the point B_1 lies on the second one because

$$\begin{aligned}
(\omega - c) \cdot (-\tfrac{c}{2}) - a\omega - ab &= \tfrac{1}{2}\big(c^2 - 2ab - \omega(2a + c)\big) \\
&= \tfrac{1}{2}\big(c^2 - 2(q + c^2) + \tfrac{1}{3q}(b - c)(c - a)(a - b) \cdot (a - b)\big) \\
&= \tfrac{1}{6q}\big(-3q(2q + c^2) + (2q - 3bc)(2q - 3ca)\big) \\
&= \tfrac{1}{6q}\big(-2q^2 - 3c^2q - 6q(bc + ca) + 9abc^2\big) \\
&= \tfrac{1}{6q}(9cp + 3c^2q - 2q^2),
\end{aligned}$$

$$\begin{aligned}
-(\omega + b) \cdot (-\tfrac{b}{2}) + a\omega - ca &= \tfrac{1}{2}\big(b^2 - 2ca + \omega(2a + b)\big) \\
&= \tfrac{1}{2}\big(b^2 - 2(q + b^2) + \tfrac{1}{3q}(b - c)(c - a)(a - b) \cdot (c - a)\big) \\
&= \tfrac{1}{6q}\big(-3q(2q + b^2) + (2q - 3ab)(2q - 3bc)\big) \\
&= \tfrac{1}{6q}(9bp + 3b^2q - 2q^2).
\end{aligned}$$

Therefore, points Ω_1 and Ω_2 lie on lines AC_1 and AB_1, respectively, the point Ω_1 lies on lines BA_1 and CB_1, and the point Ω_2 lies on lines BC_1 and CA_1. Points Ω_1, Ω_2 and K' have the centroid $G = \left(0, -\frac{2}{3}q\right)$ because

$$\frac{1}{3}\left(\frac{p-p_1}{q} + \frac{p-p_2}{q} - \frac{3p}{q}\right) = -\frac{p+p_1+p_2}{3q} = 0$$

and

$$\begin{aligned}\frac{1}{3}\left(\frac{27p_1^2-2q^3}{9q^2} + \frac{27p_2^2-2q^3}{9q^2} + \frac{27p^2-8q^3}{9q^2}\right) &= \frac{1}{9q^2}\left(9(p_1+p_2)^2 + 9p_1^2 + 9p_2^2 - 4q^3\right) \\ &= \frac{1}{9q^2}\left(-2q^3 - 4q^3\right) = -\frac{2}{3}q. \end{aligned}$$ □

The line AS has the slope $a - \frac{3p}{q}$, where S is the Steiner point of the triangle ABC [7]. $AS \parallel B_1C_1$ because the line B_1C_1 has the same slope as AS. Similarly, we get $BS \parallel C_1A_1$ and $CS \parallel A_1B_1$. Because of Theorem 1, we have $A_1K \parallel BC$, $B_1K \parallel CA$, and $C_1K \parallel AB$. Therefore, triangles ABC and $A_1B_1C_1$ have the property that lines through the vertices of the first triangle parallel to the corresponding sides of the second triangle pass through a common point, and lines through the vertices of the second triangle parallel to the corresponding sides of the first triangle pass through another common point. These two triangles are called *parallelogic*, and the two mentioned points are the *centers of parallelogy* of these triangles. So we have the following theorem.

Theorem 8. *A triangle and its first Brocard triangle are parallelogic with the centers of parallelogy at the Steiner point and at the symmedian center of this triangle (see Figure 3).*

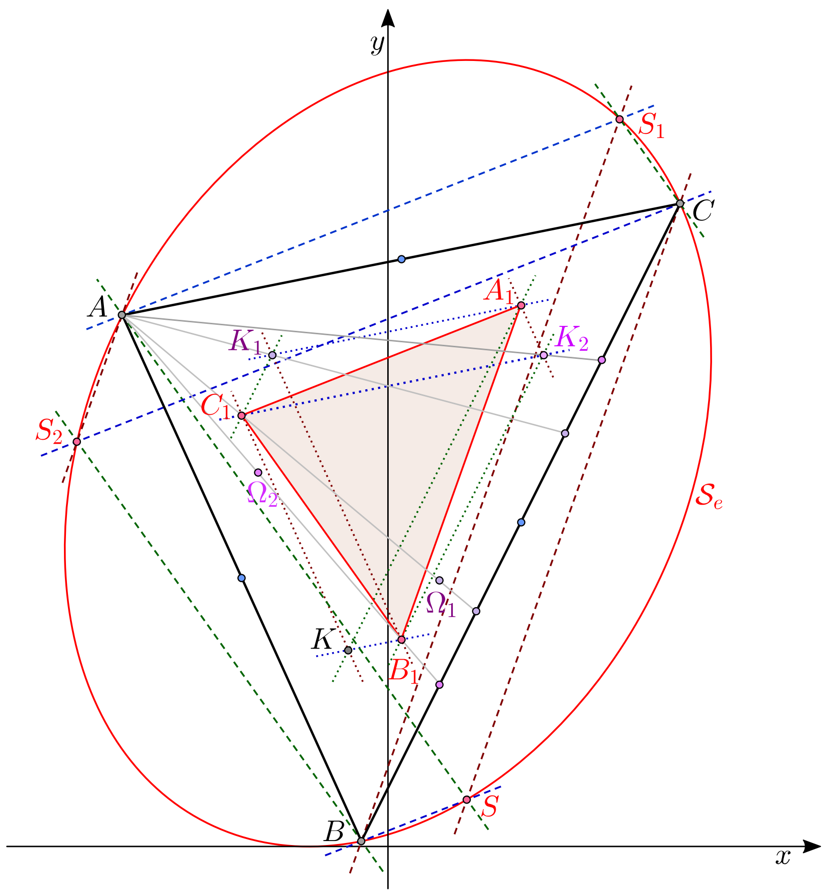

Figure 3. The circumscribed Steiner ellipse $\mathcal{S}_e$ of the triangle ABC. Visualization of statements of Theorems 8–11.

Theorem 9. *Let $A_1B_1C_1$ be the first Brocard triangle of an allowable triangle ABC. Then the lines parallel to B_1C_1, C_1A_1, and A_1B_1 through C, A, and B, through B, C, and A, pass through a common point S_1, respectively S_2. In addition, the lines parallel to BC, CA, and AB through C_1, A_1, and B_1, respectively B_1, C_1, and A_1, pass through a common point K_1, respectively K_2 (see Figure 3). In the case of the standard triangle ABC these points are given by*

$$S_1 = \Big(-\frac{3p_1}{q}, \frac{9pp_1}{q^2} - q\Big), \qquad S_2 = \Big(-\frac{3p_2}{q}, \frac{9pp_2}{q^2} - q\Big), \tag{8}$$

$$K_1 = \Big(\frac{3p_1}{2q}, -\frac{5}{6}q\Big), \qquad K_2 = \Big(\frac{3p_2}{2q}, -\frac{5}{6}q\Big). \tag{9}$$

Proof. According to (5), the slope of the line B_1C_1 is $a - \frac{3p}{q}$, and the line through C parallel to it is given by $y - c^2 = \big(a - \frac{3p}{q}\big)(x - c)$, i.e., $qy = (aq - 3p)x + (c^2 - ca)q + 3cp$. The point S_1 lies on this line because

$$\begin{aligned}
qy &- (aq - 3p)x - (c^2 - ca)q - 3cp \\
&= \tfrac{9pp_1}{q} - q^2 + 3ap_1 - \tfrac{9pp_1}{q} + (ca - c^2)q - 3cp \\
&= (ca - c^2 - q)q + a(bc^2 + ca^2 + ab^2) - 3abc^2 \\
&= (ca - ab)q - 2abc^2 + ca^3 + a^2b^2 \\
&= -a(a + 2c)(a^2 + ac + c^2) + 2ac^2(a + c) + a^3c + a^2(a + c)^2 = 0.
\end{aligned}$$

Substitutions $B \leftrightarrow C$ and $b \leftrightarrow c$ imply the substitution $p_1 \to p_2$, and using this substitution, the previous proof shows that the line through B, parallel to B_1C_1, passes through S_2. Cyclic permutations $a \to b \to c \to a$ imply $C_1A_1 \parallel AS_1 \parallel CS_2$, and $A_1B_1 \parallel BS_1 \parallel AS_2$. The line

$$y = -ax + \tfrac{1}{6q}(9cp + 3c^2q - 3caq - 2q^2)$$

is parallel to BC, and it passes through C_1. Let us show that this line also passes through K_1, i.e., that

$$-\tfrac{5}{6}q = -a \cdot \tfrac{3p_1}{2q} + \tfrac{1}{6q}(9cp + 3c^2q - 3caq - 2q^2),$$

i.e.,

$$3cp - 3ap_1 + c^2q - caq + q^2 = 0,$$

which, because $q + c^2 = ab$ and $p = abc$, after dividing by a, becomes equivalent to

$$3bc^2 - 3p_1 + (b - c)q = 0.$$

However, we have

$$\begin{aligned}
3bc^2 - 3p_1 + (b - c)q &= 3bc^2 - bc^2 - ca^2 - ab^2 + (b - c)(bc + ca + ab) \\
&= bc^2 + b^2c - a^2c - ac^2 \\
&= bc(b + c) - ac(a + c) = -bca + acb = 0.
\end{aligned}$$

Other statements of the theorem are proved in a similar way. □

Using the obvious term *three-parallelogy*, statements of Theorems 8 and 9 can be briefly summarized as the following corollary.

Corollary 1. *The allowable triangle ABC and its first Brocard triangle $A_1B_1C_1$ are three-parallelogic, and in the case of a standard triangle ABC, centers of parallelogy are the Steiner point S of the triangle ABC and points S_1, S_2, and the symmedian center K of the triangle ABC and points K_1, K_2 (see Figure 3).*

In the Euclidean case, the statement about three-parallelogy, without proof, can be found in [10].

Theorem 10. *Points S, S_1, and S_2 from Theorem 9 lie on the circumscribed Steiner ellipse of the triangle ABC (see Figure 3).*

Proof. The statement for Steiner point S is proved in [7]. For the point $S_1 = (x, y)$ we have

$$\begin{aligned} &q^2x^2 - 9pxy - 3qy^2 - 6pqx - 4q^2y + 9p^2 \\ &\quad = 9p_1^2 + 243\tfrac{p^2p_1^2}{q^3} - 27pp_1 - 3q\big(\tfrac{9pp_1}{q^2} - q\big)^2 + 18pp_1 - 36pp_1 + 4q^3 + 9p^2 \\ &\quad = 9p^2 + 9pp_1 + 9p_1^2 + q^3 = 0, \end{aligned}$$

and this point lies on the circumscribed Steiner ellipse of the triangle ABC, which, by [7], has the equation $q^2x^2 - 9pxy - 3qy^2 - 6pqx - 4q^2y + 9p^2 = 0$. Analogous proof is for the point S_2. $\square$

Theorem 11. *Points K_1 and K_2 from Theorems 9 are reciprocal to Crelle–Brocard points Ω_1 and Ω_2 of that triangle (see Figure 3).*

In the Euclidean case this statement, without proof, can be found in [11].

Proof. For the point $K_1 = (x, y)$ we get

$$\begin{aligned} &q^2x^2 - 9pxy - 3qy^2 - 6pqx - 4q^2y + 9p^2 \\ &\quad = \tfrac{9}{4}p_1^2 + \tfrac{45}{4}pp_1 - \tfrac{25}{12}q^3 - 9pp_1 + \tfrac{10}{3}q^3 + 9p^2 \\ &\quad = 9p^2 + \tfrac{9}{4}pp_1 + \tfrac{9}{4}p_1^2 + \tfrac{15}{12}q^3 \\ &\quad = \tfrac{27}{4}p^2 - \tfrac{1}{4}q^3 + \tfrac{5}{4}q^3 \\ &\quad = \tfrac{1}{4}(27p^2 + 4q^3), \end{aligned}$$

$$\begin{aligned} &3pqx^2 + 4q^2xy - 9py^2 + (9p^2 + 4q^3)x - 12pqy - 4pq^2 \\ &\quad = \tfrac{27}{4} \cdot \tfrac{pp_1^2}{q} - 5p_1q^2 - \tfrac{25}{4}pq^2 + \tfrac{27}{2} \cdot \tfrac{p^2p_1}{q} + 6p_1q^2 + 10pq^2 - 4pq^2 \\ &\quad = \tfrac{27}{2} \cdot \tfrac{p^2p_1}{q} + \tfrac{27}{4} \cdot \tfrac{pp_1^2}{q} - \tfrac{1}{4}pq^2 + p_1q^2 \\ &\quad = \tfrac{1}{4q}(54p^2p_1 + 27pp_1^2 + 3pq^3) - (p - p_1)q^2 \\ &\quad = \tfrac{3p}{4q}(18pp_1 + 9p_1^2 - 9p^2 - 9pp_1 - 9p_1^2) - (p - p_1)q^2 \\ &\quad = -\tfrac{3p}{4q}(9p^2 - 9pp_1) - (p - p_1)q^2 \\ &\quad = -\tfrac{1}{4q}(p - p_1)(27p^2 + 4q^3), \end{aligned}$$

$$\begin{aligned} &9p^2x^2 + 12pqxy + 4q^2y^2 + 8pq^2x - (9p^2 - 4q^3)y - 12p^2q \\ &\quad = \tfrac{81}{4} \cdot \tfrac{p^2p_1^2}{q^2} - 15pp_1q + \tfrac{25}{9}q^4 + 12pp_1q + \tfrac{15}{2}p^2q - \tfrac{10}{3}q^4 - 12p^2q \\ &\quad = \tfrac{81}{4} \cdot \tfrac{p^2p_1^2}{q^2} - \tfrac{9}{2}p^2q - 3pp_1q - \tfrac{5}{9}q^4 \\ &\quad = \tfrac{1}{36q^2}(729p^2p_1^2 - 162p^2q^3 - 108pp_1q^3 - 20q^6) \\ &\quad = \tfrac{1}{36q^2}(729p^2p_1^2 - 162p^2q^3 - 108pp_1q^3 - 8q^6 + 12q^3(9p^2 + 9pp_1 + 9p_1^2)) \\ &\quad = \tfrac{1}{36q^2}(729p^2p_1^2 - 54p^2q^3 + 108p_1^2q^3 - 8q^6) \\ &\quad = \tfrac{1}{36q^2}(27p^2 + 4q^3)(27p_1^2 - 2q^3), \end{aligned}$$

and, its reciprocal point has coordinates

$$\frac{\frac{1}{4q}(p-p_1)(27p^2+4q^3)}{\frac{1}{4}(27p^2+4q^3)}=\frac{p-p_1}{q}$$

and

$$\frac{\frac{1}{36q^2}(27p^2+4q^3)(27p_1^2-2q^3)}{\frac{1}{4}(27p^2+4q^3)}=\frac{1}{9q^2}(27p_1^2-2q^3)$$

which is the point Ω_1. The proof for K_2 is analogous. □

Let us consider the indirect similarity given by

$$x'=-\frac{1}{2}x, \qquad y'=\frac{3p}{2q}x+\frac{1}{2}y-\frac{q}{3} \tag{10}$$

and vice versa

$$x=-2x', \qquad y=\frac{6p}{q}x'+2y'+\frac{2}{3}q. \tag{11}$$

The similarity (10) obviously maps the point $A=(a,a^2)$ to the point A_1, and points B and C to points B_1 and C_1. This similarity maps the Steiner point S, which is, according to [7], given by $S=(-\frac{3p}{q},\frac{9p^2}{q^2})$, to the point with abscissa $\frac{3p}{2q}$ and ordinate equal to $\frac{3p}{2q}\cdot(-\frac{3p}{q})+\frac{1}{2}\cdot\frac{9p^2}{q^2}-\frac{q}{3}=-\frac{q}{3}$, which is the point K. The point S_1 is mapped to the point with abscissa $\frac{3p_1}{2q}$ and ordinate $\frac{3p}{2q}(-\frac{3p_1}{q})+\frac{1}{2}(\frac{9pp_1}{q^2}-q)-\frac{q}{3}=-\frac{5}{6}q$, which is the point K_1. The centroid $G=(0,-\frac{2}{3}q)$ is under the similarity (10) mapped onto itself.

The circumscribed circle and the circumscribed Steiner ellipse of the triangle ABC have, according to [3,7], equations

$$y=x^2$$

and

$$q^2x^2-9pxy-3qy^2-6pqx-4q^2y+9p^2=0,$$

which, after substitutions (11), become

$$\tfrac{6p}{q}x'+2y'+\tfrac{2}{3}q=4x'^2$$

and

$$q^2\cdot 4x'^2+18px'\big(\tfrac{6p}{q}x'+2y'+\tfrac{2}{3}q\big)-3q\big(\tfrac{6p}{q}x'+2y'+\tfrac{2}{3}q\big)^2+$$
$$+12pqx'-4q^2\big(\tfrac{6p}{q}x'+2y'+\tfrac{2}{3}q\big)+9p^2=0.$$

After rearrangements and replacements $x'\rightsquigarrow x$, $y'\rightsquigarrow y$ this can be written as

$$y=2x^2-\frac{3p}{q}x-\frac{q}{3}$$

and

$$4q^2x^2-36pxy-12qy^2-24pqx-16qy+9p^2-4q^3=0,$$

which are, according to [5,7], equations of the Brocard circle and the inscribed Steiner ellipse of the triangle ABC.

Let us find the equation of the fixed line, i.e., of the axis of similarity (10). The transformation $x\to-2x$, $y\to\frac{6p}{q}x+2y+\frac{2}{3}q$ maps the line

$$y=kx+l \tag{12}$$

to the line

$$\frac{6p}{q}x+2y+\frac{2}{3}q=-2kx+l,$$

i.e., to the line

$$y=-(k+\frac{3p}{q})x+\frac{l}{2}-\frac{q}{3},$$

which coincides with the line (12) if and only if

$$-(k+\frac{3p}{q})=k,\quad \frac{l}{2}-\frac{q}{3}=l.$$

Therefore,

$$k=-\frac{3p}{2q},\quad l=-\frac{2}{3}q,$$

and the required axis is the line

$$y=-\frac{3p}{2q}x-\frac{2}{3}q,$$

which is, according to [7], the Steiner axis of the triangle ABC. The last result is in accordance with Theorem 4.

Thus, we have proved:

Theorem 12. *An allowable triangle ABC and its first Brocard triangle $A_1B_1C_1$ are indirectly similar. This similarity has the center at the common centroid of these two triangles, its coefficient equals $-\frac{1}{2}$, its axis is the Steiner axis of the triangle ABC, and it maps the points S, S_1, and S_2 from Theorem 9 to points K, K_1, and K_2 from the same theorem.*

Corollary 2. *The symmedian center of the allowable triangle ABC is the Steiner point of its Brocard triangle $A_1B_1C_1$.*

For the Euclidean case see [9].

Corollary 3. *The circumscribed Steiner ellipse of the first Brocard triangle of a given allowable triangle is the inscribed Steiner ellipse of that triangle.*

Corollary 4. *Segments GS and GK, where G is the centroid of the allowable triangle ABC, S its Steiner point, and K its symmedian center, have the same perpendicular bisector as pairs of lines from Theorem 4.*

For the Euclidean case see [12].

Theorem 13. *An allowable triangle is homologous with the complementary triangle of its first Brocard triangle.*

Proof. Points B_1 and C_1 have the midpoint $A_{1m}=\left(\frac{a}{4},-\frac{1}{12q}(9ap+3bcq+7q^2)\right)$ because

$$9bp+3b^2q-2q^2+9cp+3c^2q-2q^2=-9ap+3(-q-bc)q-4q^2=-(9ap+3bcq+7q^2).$$

The line

$$9aqy=(9b^2c^2+6bcq-5q^2)x+3pq-9bcp-4aq^2$$

passes through $A=(a,a^2)$ and A_1 because

$$(9b^2c^2+6bcq-5q^2)a+3pq-9bcp-4aq^2=9abcq-9aq^2=9a^3q$$

and

$$
\begin{aligned}
(9b^2c^2+6bcq-5q^2)\frac{a}{4}+3pq-9bcp-4aq^2 &= -\tfrac{27}{4}ab^2c^2+\tfrac{9}{2}abcq-\tfrac{21}{4}aq^2\\
&= -\tfrac{3a}{4}(9bc(q+a^2)-6bcq+7q^2)\\
&= -\tfrac{1}{12q}(9ap+3bcq+7q^2)\cdot 9aq,
\end{aligned}
$$

which is the line AA_{1m}. This line also passes through the point $\big(\frac{3p}{5q}, \frac{1}{45q^2}(27p^2-20q^3)\big)$ because

$$
\begin{aligned}
(9b^2c^2+6bcq-5q^2)\cdot\tfrac{3p}{5q}+3pq-9bcp-4aq^2 &= \tfrac{27}{5}\cdot\tfrac{b^2c^2p}{q}+\tfrac{18}{5}bcp-9bcp-4aq^2\\
&= \tfrac{1}{5q}(27b^2c^2p-27bcpq-20aq^3)\\
&= \tfrac{1}{5q}(27a^2bcp-20aq^3)\\
&= \tfrac{1}{45q^2}(27p^2-20q^3)\cdot 9aq. \qquad \square
\end{aligned}
$$

Theorem 14. *If $A_1B_1C_1$ is the first Brocard triangle of the allowable triangle ABC, then its Crelle–Brocard points Ω_1 and Ω_2 divide in equal proportions the pairs of segments CB_1, BC_1; AC_1, CA_1; BA_1, AB_1.*

For the Euclidean case, without proof, see [10].

Proof. According to Theorem 7 and its proof, points Ω_1 and Ω_2 lie on lines CB_1, AC_1, BA_1, and BC_1, CA_1, AB_1. From (1) and (3) we get the ratios

$$
\frac{d(B_1,\Omega_1)}{d(C,\Omega_1)}=\frac{\frac{p-p_1}{q}+\frac{b}{2}}{\frac{p-p_1}{q}-c}=\frac{2p-2p_1+bq}{2(p-p_1-cq)},\quad \frac{d(C_1,\Omega_2)}{d(B,\Omega_2)}=\frac{\frac{p-p_2}{q}+\frac{c}{2}}{\frac{p-p_2}{q}-b}=\frac{2p-2p_2+cq}{2(p-p_2-bq)},
$$

so we have to prove the equality

$$
\big(2(p-p_1)+bq\big)(p-p_2-bq)=\big(2(p-p_2)+cq\big)(p-p_1-cq),
$$

which is, after dividing by q, equivalent to

$$
b(p-p_2-2p+2p_1-bq)=c(p-p_1-2p+2p_2-cq),
$$

and since $p+p_1+p_2=0$, equivalent to

$$
3bp_1-b^2q=3cp_2-c^2q.
$$

This last equality holds true because

$$
\begin{aligned}
3bp_1-3cp_2-(b^2-c^2)q &= b(bc^2+ca^2+ab^2)-c(b^2c+c^2a+a^2b)-(b+c)(b-c)q\\
&= a(b^3-c^3)+a(b-c)q\\
&= a(b-c)(b^2+bc+c^2+q)=0. \qquad \square
\end{aligned}
$$

Theorem 15. *If $A_1B_1C_1$ is the first Brocard triangle of the allowable triangle ABC with the Brocard angle ω, then BCA_1, CAB_1, and ABC_1 are isosceles triangles, which have angles at sides BC, CA, and AB equal to ω.*

For the Euclidean case see [9].

Proof. According to the proof of Theorem 7, lines $A\Omega_1$ and $A\Omega_2$, containing points C_1 and B_1, have slopes $\omega-c$ and $-(\omega+b)$. By cyclic permutations it follows that lines $B\Omega_1$ and $C\Omega_2$, containing the point A_1, have slopes $\omega-a$ and $-(\omega+a)$. So we get

$$
\begin{aligned}
\angle(BC,BA_1)&=\omega-a-(-a)=\omega,\\
\angle(CA_1,BC)&=-a+\omega+a=\omega. \qquad \square
\end{aligned}
$$

Theorem 16. *If P is the center of homology of the triangle ABC and its first Brocard triangle, and if D, E, and F are points symmetrical to P with respect to the midpoints of sides BC, CA, and AB, then the triangles ABC and DEF are symmetrical with respect to the midpoint S of the Crelle–Brocard points Ω_1 and Ω_2.*

For the Euclidean case see [13].

Proof. $P + D = B + C$ implies $D = B + C - P$. By Theorem 7, $P + \Omega_1 + \Omega_2 = 3G$, where G is the centroid of ABC. We get $P + 2S = 3G$, i.e., $P + 2S = A + B + C$ or $2S = A + D$. □

4. Conclusions

In this paper we introduce the first Brocard triangle of a triangle in the isotropic plane, and study its connections with some other objects related to the given triangle. Some of the most important statements that we prove are the following: the allowable triangle ABC and its first Brocard triangle are homologous, where the centers of homologies are the point reciprocal to the symmedian center of the triangle and the Crelle–Brocard points of that triangle; the allowable triangle ABC and its first Brocard triangle are three-parallelogic and indirectly similar, and the allowable triangle is homologous with the complementary triangle of its first Brocard triangle. The analytical approach used in this paper was introduced and developed in [3]. The obtained results are compared with similar results in the Euclidean plane.

Author Contributions: Conceptualization, V.V., Z.K.-B. and R.K.-Š.; validation; writing—original draft preparation, V.V., Z.K.-B. and R.K.-Š. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the referees for their valuable comments which helped to improve the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Sachs, H. *Ebene Isotrope Geometrie*; Vieweg-Verlag: Braunschweig, Germany; Wiesbaden, Germany, 1987.
2. Strubecker, K. Geometrie in einer isotropen Ebene I–III. *Math. Naturw. Unterr.* **1962**, *15*, 297–306, 343–351, 385–394.
3. Kolar-Šuper, R.; Kolar-Begović, Z.; Volenec, V.; Beban-Brkić, J. Metrical relationships in a standard triangle in an isotropic plane. *Math. Commun.* **2005**, *10*, 149–157.
4. Volenec, V.; Kolar-Begović, Z.; Kolar-Šuper, R. Crelle-Brocard points of the triangle in an isotropic plane. *Math. Pannon.* **2013**, *24*, 167–181.
5. Kolar-Begović, Z.; Kolar-Šuper, R.; Volenec, V. Brocard circle of the triangle in an isotropic plane. *Math. Pannon.* **2017/2018**, *26*, 103–113.
6. Volenec, V.; Kolar-Begović, Z. On Some Properties of Kiepert Parabola in the Isotropic Plane. *Rad Hrvat. Akad. Znan. Umjet. Mat. Znan.* **2022**, *accepted for publication*.
7. Volenec, V.; Kolar-Begović, Z.; Kolar-Šuper, R. Steiner's ellipses of the triangle in an isotropic plane. *Math. Pannon.* **2010**, *21*, 229–238.
8. Alasia, C. *La Recente Geometria del Triangolo*; Nabu Press: Città di Castello, Italy, 1900.
9. Neuberg, J. Note III, Sur la Géométrie récente du triangle. In *E. Rouché et Ch. de Comberousse, Traité de Géométrie, I*; Gauthier-Villars: Paris, France, 1922.
10. Casey, J. Géométrie élémentaire récente. *Mathesis* **1889**, *9*, 5–70.
11. Gabriel-Marie, F. *Exercices de Géométrie*, 5th ed.; Maison A. Mame et Fils: Tours, France, 1912.
12. Cesáro, E. Remarques sur la géométrie du triangle. *Nouv. Ann. Math.* **1887**, *6*, 215–242.
13. Schwatt, J.F. Aufgabe 1405. *Z. Math. Naturwiss. Unterr.* **1895**, *26*, 279.