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Article

Brauer Configuration Algebras Arising from Dyck Paths

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Abstract: The enumeration of Dyck paths is one of the most remarkable problems in Catalan combinatorics. Recently introduced categories of Dyck paths have allowed interactions between the theory of representation of algebras and cluster algebras theory. As another application of Dyck paths theory, we present Brauer configurations, whose polygons are defined by these types of paths. It is also proved that dimensions of the induced Brauer configuration algebras and the corresponding centers are given via some integer sequences related to Catalan triangle entries.

Keywords: Brauer configuration algebra; Catalan triangle; Dyck path; path algebra

MSC: 16G20; 16G30; 16G60

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1. Introduction

In the last few years, several combinatorial objects have allowed the research development of the theory of representation of algebras. For instance, the number of triangulations of a polygon with $n+3$ sides, or the number of *Dyck paths* of length $2n$ in the plane connecting the origin with a point $P=(2n,0)$. Edges in these paths are either *rises* (linking points (x,y) and $(x+1,y+1)$ in $\mathbb{N}^2$) or *falls* (connecting points (x,y) and $(x+1,y-1)$ in $\mathbb{N}^2$) [1]. Caldero, Chapoton, and Schiffler [2] proved that any triangulation T of an $(n+3)$-polygon defines a category $\mathrm{mod}_{\mathbb{F}}\ Q_T$ of finitely generated modules over a path algebra $\mathbb{F}Q_T$ induced by a quiver Q_T arisen from the triangulation T. They also proved that $\mathrm{mod}_{\mathbb{F}}\ Q_T$ can be identified with the category of diagonals defining the triangulation.

Following the ideas by Caldero et al., Cañadas et al. [3] introduced a categorical equivalence between a category $\mathcal{C}_{2n}$ of Dyck paths and a category of representations rep $\mathbb{A}_{n-1}$ of a quiver of Dynkin type $\mathbb{A}_{n-1}$. These approaches allow obtaining formulas for cluster variables of type $\mathbb{A}$ and frieze patterns in terms of Dyck paths and perfect matchings of some snake graphs. It is worth pointing out that finding formulas of these types is a significant problem in the cluster algebras theory.

On the other hand, Brauer configuration algebras (BCAs) are bound quiver algebras introduced by Green and Schroll in [4]. The structure of the indecomposable projective modules over these types of algebras is given by some combinatorial data, which are also used to determine their dimension. BCAs were used by Espinosa [5] to categorify integer sequences in the sense of Ringel and Fahr, i.e., the numbers in these sequences can be considered as invariants of objects in a category. They are also helpful in several fields of applied mathematics (cryptography, graph energy theory, algebraic combinatorics, etc. [6–9]).

Contributions

This paper establishes a connection between Dyck paths and Brauer configuration algebras researches by proving that Dyck paths define suitable words associated with polygons in a Brauer configuration. We introduce new Catalan–Brauer configurations

(CBCs) obtaining formulas for the dimensions of the corresponding BCAs and their centers. To do that, we introduce an integer sequence $S = \{\mathfrak{h}^{i}_{x,j}\}$ whose properties allow obtaining formulas relating its elements with entries of the Catalan triangle. Thus, the approach allows giving another manifestation of Catalan numbers via BCAs.

Figure 1 shows how Brauer configuration algebras and Dyck paths theories are related to the main results presented in this paper.

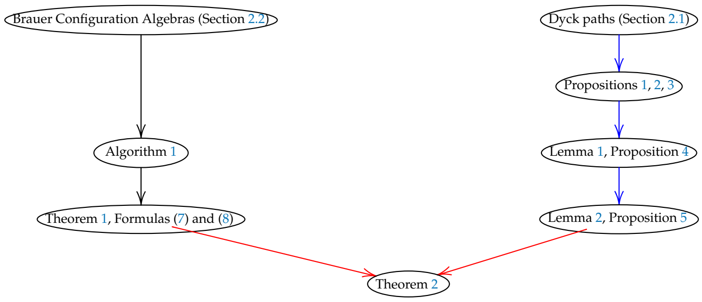

Figure 1. Main results presented in this paper (targets of blue and red arrows) allow establishing a connection between Brauer configuration algebras and Dyck paths theories. Propositions 1–3 give properties of a suitable integer sequence related to the Catalan triangle entries via Lemma 1 and Proposition 4. We introduce a Brauer configuration Γ^{x}, whose vertices occur in polygons according to entries of suitable matrices whose properties are given in Lemma 2. Proposition 5 proves that the Catalan triangle gives entries of such matrices. Theorem 2 gives a formula for the dimension of a Brauer configuration algebra (and its corresponding center) defined by Dyck paths.

The organization of this paper goes as follows; the main definitions and notation are given in Section 2, we recall the definitions and notation used throughout the document. In particular, we recall notions of Dyck paths, Catalan triangle, and Brauer configuration algebra (Section 2.2). In Section 3, we give our main results. We introduce an integer sequence whose elements are related to the entries of the Catalan triangle. The numbers in this sequence allow giving a formula to compute the dimension of some Brauer configuration algebras, whose polygons are defined by Dyck paths. The concluding remarks are given in Section 4.

2. Background and Related Work

This section is devoted to reminding the basic notation and results concerning Dyck paths, the Catalan triangle, and Brauer configuration algebras, which are helpful for a better understanding of the paper.

2.1. Dyck Paths and Catalan Triangle

In this section, we make a brief review on Dyck paths and the Catalan triangle [1,10].

Dyck paths, as defined in the introduction, were enumerated by Stanley [1], who proved that there are $C_n = \frac{1}{n+1}\binom{2n}{n}$ Dyck paths of length $2n$, where C_n denotes the nth Catalan number. $\mathfrak{p}_{2n} = \{P \mid P \text{ is a Dyck path}, |P| = 2n\}$.

For $u \geq 0$ and $0 \leq v \leq u$. The *Catalan's triangle* is an integer array whose entries $C(u,v)$ are given by the formula [10].

$$C(u,v) = \frac{u-v+1}{u+1}\binom{u+v}{v}, \quad u \geq 0, \quad \text{and} \quad 0 \leq v \leq u. \tag{1}$$

There are many ways of finding Catalan triangle entries in the specialized literature. For instance, the array (3) gives such numbers according to the recurrence (2). Such a recurrence was found by Cañadas et al. [3] via some seed vectors associated with positive integral diamonds of type $\mathbb{A}_n$ arising from the theory of integer friezes.

$$\mathfrak{p}_{x,y} = \begin{cases} \mathfrak{p}_{1,1} = \mathfrak{p}_{1,2} = 1, & \\ \sum\limits_{i=y-1}^{x} \mathfrak{p}_{x-1,i}, & \text{if } y > 1, \text{ and } x > 1, \\ \sum\limits_{i=1}^{x} \mathfrak{p}_{x-1,i}, & \text{if } y = 1, \text{ and } x > 1. \end{cases} \tag{2}$$

$$\begin{array}{lllll} \mathfrak{p}_{1,2} = 1 & \mathfrak{p}_{1,1} = 1 & & & \\ \mathfrak{p}_{2,3} & \mathfrak{p}_{2,2} & \mathfrak{p}_{2,1} & & \\ \mathfrak{p}_{3,4} & \mathfrak{p}_{3,3} & \mathfrak{p}_{3,2} & \mathfrak{p}_{3,1} & \\ \mathfrak{p}_{4,5} & \mathfrak{p}_{4,4} & \mathfrak{p}_{4,3} & \mathfrak{p}_{4,2} & \mathfrak{p}_{4,1} \\ \vdots & & & & \ddots \end{array} \tag{3}$$

2.2. Brauer Configuration Algebras

Green and Schroll introduced the notion of a BCA (Brauer Configuration Algebra). The authors refer the interested reader to [4,11] for a more detailed study of BGAs (Brauer Graph Algebras) and BCAs. In the sequel, we make a brief description of the structure of these algebras.

A BCA Λ_Γ is a bound quiver algebra induced by a Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ of sets, with the functions and orders satisfying the following conditions:

- Elements of the set Γ_0 are called *vertices*;
- Γ_1 consists of multisets called *polygons*, which consist of vertices that can be repeated. Moreover, if $U \in \Gamma_1$. Then $|U| > 1$;
- μ denotes a function from the set of vertices to the set of positive integers. Green and Schroll called this function *the multiplicity function*, associated with the Brauer configuration Γ;
- If a vertex h in a polygon W occurs t times. Then, we will write $occ(h, W) = t$. $val(h) = \sum\limits_{W \in \Gamma_1} occ(h, W)$ is said to be the *valency* of the vertex h, which is said to be non-truncated if $val(h)\mu(h) \neq 1$ (otherwise, it is *non-truncated*). We let S_h denote the maximal set of polygons containing a non-truncated vertex. If $S_h = \{U_{i_1}, U_{i_2}, \ldots, U_{i_m}\}$. Then S_h is endowed with a well-order $<$, which allows writing S_h in the following form:

$$U_{i_1}^{h_1} < U_{i_2}^{h_2} < \cdots < U_{i_m}^{h_m}, \quad h_{i_s} > 0. \tag{4}$$

The symbol $U_{i_j}^x$ is used to denote that $occ(h, U_{i_j}) = x$. In successor sequences U^x denotes a subsequence of length x with the form:

$$\underbrace{U < U \cdots < U}_{x-times} \tag{5}$$

The set $(S_h, <)$ is said to be the *successor sequence* associated with the vertex h. Note that if $U_i < U_j$ is a covering in S_h and $h' \in U_i \cap U_j$ with $h' \neq h$ then the relation $U_i < U_j$ also appears in the sequence $S_{h'}$.

In this work, it is assumed that each polygon $U_i \in \Gamma_1$ is given by a word w_{U_i} of the form

$$w_{U_i} = \alpha_1^{s_1} \alpha_2^{s_2} \ldots \alpha_{t-1}^{s_{t-1}} x_t^{s_t} \tag{6}$$

where for each i, $1 \leq i \leq t$, α_i is a vertex, and $s_i = occ(\alpha_i, U_i)$.

Algorithm 1 is a short version of an algorithm introduced in [6] by Cañadas et al. to construct a Brauer configuration algebra.

Algorithm 1: Building a BCA

1. **Input.** $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$.
2. **Output.** The BCA $\Lambda_\Gamma = \mathbb{F}Q_\Gamma / I_\Gamma$ induced by the BC Γ.
3. Remove truncated vertices.
4. Define the Brauer quiver $Q_\Gamma = (Q_0, Q_1, s, t)$, such that
   (a) $Q_0 = \Gamma_1$,
   (b) For each covering $U_i \leq U_{i+1}$ contained in a successor sequence S_α, define an arrow a^α for which $s(a^\alpha) = U_i$, and $t(a^\alpha) = U_{i+1}$,
   (c) To each ordered successor sequence S_α define a special cycle C_α associated with a vertex α by adding a relation r_α of the form $U_{\max\ S_\alpha} < U_{\max\ S_\alpha}$.
5. Define the path algebra $\mathbb{F}Q_\Gamma$.
6. Define an admissible ideal I_Γ generated by the following relations:
   (a) $a^\alpha a^{\alpha'}$, if $\alpha, \alpha' \in \Gamma_0$, $\alpha \neq \alpha'$, $a^\alpha a^{\alpha'} \in \mathbb{F}Q_\Gamma$,
   (b) $C_\alpha^{\mu(\alpha)} f$, if f is the first arrow of a special cycle C_α associated with α,
   (c) $(l^\alpha)^{\mu(\alpha)+1}$, if l is a loop associated with a non-truncated vertex $\alpha \in \Gamma_0$, and $val(\alpha) = 1$.
   (d) $C_\alpha^{\mu(\alpha)} - C_{\alpha'}^{\mu(\alpha')}$, for any pair of special cycles associated with vertices $\alpha, \alpha' \in U$, $U \in \Gamma_1$.
7. Define the Brauer configuration algebra $\Lambda_\Gamma = \mathbb{F}Q_\Gamma / I_\Gamma$.
8. The union of classes of proper prefixes of special cycles with classes of special cycles provide an $\mathbb{F}$ basis $\mathcal{B}$ of Λ_Γ.

Later on, if there is not possibility of confusion, we will assume notations Q (for a quiver), I (for an admissible ideal) and Λ (for a Brauer configuration algebra).

Theorem 1 provides algebraic properties of BCAs [4].

Theorem 1 ([4], Theorem B, Propositions 2.7, 3.2 and 3.5, Theorem 3.10, Corollary 3.12)**.** *Let $\Lambda = \mathbb{F}Q/I$ be a Brauer configuration algebra induced by a Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$.*

1. *There is a bijection between the set of indecomposable projective modules over Λ and Γ_1;*
2. *If P_V is an indecomposable projective module over a BCA Λ defined by a polygon V in Γ_1. Then* rad $P_V = \sum\limits_{i=1}^{r} U_i$*, where $U_i \cap U_j$ is a simple Λ-module for any $1 \leq i, j \leq r$, and r is the number of (non-truncated) vertices of V;*
3. *I is admissible, whereas Λ is a multiserial symmetric algebra. Moreover, if Γ is connected, then Λ is indecomposable as an algebra;*
4. *If* rad P *(*soc P*) denotes the radical (socle) of an indecomposable projective module P, and* $\mathrm{rad}^2\ P \neq 0$*. Then, the number of summands in the heart* rad $P/$soc P *of P equals the number of non-truncated vertices of the polygons in Γ corresponding to P counting repetitions;*
5. *If Λ_Γ and $\Lambda_{\Gamma'}$ are BCAs, induced by Brauer configurations $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$, and $\Gamma' = (\Gamma_0 \backslash \{h\}, \Gamma_1 \backslash V \cup V', \mu, \mathcal{O})$, where $V' = V \backslash \{h\}$, $|V| \geq 3$, and $val(h)\mu(h) = 1$. Then, Λ_Γ is isomorphic to $\Lambda_{\Gamma'}$.*

Green and Schroll in [4] proved that the dimension of a Brauer configuration algebra is given by the following formula.

$$\dim_{\mathbb{F}} \Lambda = 2|\Gamma_1| + \sum_{i \in \Gamma_0} val(i)(\mu(i)val(i) - 1). \tag{7}$$

Sierra [12] obtained the next formula for the dimension of the center of a connected Brauer configuration algebra Λ_Γ with radical square different from zero.

$$\dim_{\mathbb{F}} Z(\Lambda) = 1 + \sum_{\alpha \in \Gamma_0} \mu(\alpha) + |\Gamma_1| - |\Gamma_0| + \#(Loops\ Q) - |\mathcal{C}_\Gamma|, \tag{8}$$

where $|\mathcal{C}_\Gamma| = \{\alpha \in \Gamma_0 \mid val(\alpha) = 1, \text{ and } \mu(\alpha) > 1\}$.

As an example, we use compositions of the number 3 to define a Brauer configuration $\Delta = (\Delta_0, \Delta_1, \mu, \mathcal{O})$ for which;

- $\Delta_0 = \{1, 2, 3\}$;
- $\Delta_1 = \{U_1 = \{3, 1, 1, 1\}, U_2 = \{2, 1\}, U_3 = \{1, 2\}\}$;
- $\mu(1) = \mu(2) = 1, \quad \mu(3) = 2$;
- Successor sequences: $S_1 = U_1^{(3)} < U_2 < U_3, \quad S_2 = U_2 < U_3, \quad S_3 = U_1$;
- $val(1) = 5, \quad val(2) = 2, \quad val(3) = 1$;
- $|\Delta_0| = 3, \quad |\Delta_1| = 3, \quad |\mathcal{C}_\Delta| = 1$;
- $\dim_{\mathbb{F}} \Lambda_\Delta = 29$;
- $\dim_{\mathbb{F}} Z(\Lambda_\Delta) = 8$.

The following Figure 2 shows the Brauer quiver Q_Δ.

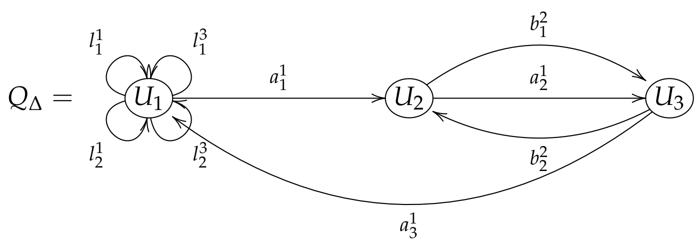

Figure 2. Brauer quiver associated Q_Δ induced by the Brauer configuration Δ.

The admissible ideal I_Δ is generated by the following relations (a and a' denote first arrows of special cycles C_α associated with a vertex α).

- $l_i^1 l_j^3, \quad (l_i^1)^2, \quad (l_i^3)^2, \quad$ for all possible values of i and j;
- $l_i^3 a_1^1, \quad a_1^1 b_1^2, \quad b_1^2 a_3^1, \quad C_1^1 a, \quad C_2^1 a', \quad C_1^1 \sim C_2^1$, for all possible special cycles associated with vertices 1 and 2.

3. Main Results

The results in this section allow establishing interactions between Catalan combinatorics via Dyck paths and Brauer configuration algebras. It is proved that indecomposable projective modules over some Brauer configuration algebras define Dyck paths. We compute the dimension of these algebras and their corresponding centers.

3.1. Dyck Paths Arising from Brauer Configuration Algebras

For a fixed integer x, and sets of letters

$$F_x = \{\mathfrak{a}_{x_1}^{x_2}\}_{0 \le x_1 \le x-1,\, x_1 < x_2 \le x}, \quad G_x = \{\mathfrak{b}_{x_1}^{x_2}\}_{0 \le x_1 \le x-1,\, x_1 < x_2 \le x}. \tag{9}$$

It is defined a Brauer configuration $\Gamma^x = (\Gamma_0^x, \Gamma_1^x, \mu^x, \mathcal{O}^x)$, where

$$\Gamma_0^x = F_x \cup G_x. \tag{10}$$

If a product or concatenation c is defined on the set Γ_0^x in the following fashion:

$$c(\delta) = \begin{cases} \mathfrak{a}_p^{q+1}, & \text{if } \delta = \mathfrak{a}_p^q, \\ \mathfrak{b}_p^q, & \text{if } \delta = \mathfrak{a}_p^q, \\ \mathfrak{a}_{p+1}^{q+1}, & \text{if } \delta = \mathfrak{b}_p^q, \\ \mathfrak{b}_{p+1}^q, & \text{if } \delta = \mathfrak{b}_p^q, \end{cases} \tag{11}$$

for suitable integers p and q.

Then the word w_V associated with a polygon $V \in \Gamma_1^x$ has the form $w_V = \delta_1 \dots \delta_{2n} = \prod_{h=1}^{2n} \delta_h$, where $\delta_1 = \mathfrak{a}_0^1$ and $\delta_q = c(\delta_{q-1})$.

The orientation $\mathcal{O}^x$ is defined by an order $<$ satisfying the following conditions: Successor sequences associated with vertices can be defined by adopting the following relation

$$\begin{aligned} V < V' \quad & \text{if and only if there exists a positive integer } r \text{ such that} \\ \mathfrak{N}_V^{r_g} = \mathfrak{N}_{V'}^{r_g}, \quad & \text{if} \quad 0 < r_g < r, \\ \mathfrak{N}_{V'}^{r_g} < \mathfrak{N}_V^{r_g} \quad & \text{if} \quad r_g = r. \end{aligned} \tag{12}$$

where $\mathfrak{N}_V^{r_g}$ is the number of $\mathfrak{a}$ words appearing before the first occurrence of an r_g-$\mathfrak{b}$ word in V.

If no confusion arises, henceforth, we will write polygons in terms of their corresponding words. In successor sequences polygons $V_i \in \Gamma_1^x$ are (linearly) ordered in the form $V_1 < V_2 < \dots$, where

$$V_1 = \prod_{q=1}^{n} \mathfrak{a}_0^q \prod_{p=0}^{n} \mathfrak{b}_p^x, \quad V_2 = \prod_{q=1}^{n-1} \mathfrak{a}_0^q (\mathfrak{b}_0^{n-1} \mathfrak{a}_1^n) \prod_{p=1}^{n-1} \mathfrak{b}_p^x, \quad \dots \tag{13}$$

Remark 1. *It is worth pointing out that under these circumstances, it is easy to prove that there is a bijection between words of type V_i and Dyck paths of type $\mathfrak{p}_{2n}$.*

The multiplicity function $\mu^x : \Gamma_0^x \to \mathbb{N}^+$ is defined in such a way that

$$\mu^x(\delta) = 2, \text{ if } val(\delta) = 1, \quad \mu^x(\delta) = 1, \text{ otherwise.} \tag{14}$$

Brauer configurations of type Γ^x are called *Catalan–Brauer* configurations. We note that, if $V \in \Gamma_1^x$, then $occ(\mathfrak{a}_{x_1}^{x_2}, V)$ is given by an entry of the $(\frac{x_1(2x-x_1)}{2} + x_2 - \frac{1}{2})$-row of the $\frac{x(x+1)}{2} \times \mathfrak{p}_{x,2}$-matrix $A_x = (a_{i,j}^x)$ shown in Figure 3 (see identities (2)). On the other hand, the same row in a matrix $B_x = (b_{i,j}^x)$ of the same size gives $occ(\mathfrak{b}_{x_1}^{x_2}, V)$. Note that, $A_1 = B_1 = (1)$.

Entries of Matrix $A_{x-1}^{p,\mathfrak{p}_{(x-1,p+1)}} = (a_{i,j}^{x-1,p})$ are given by the following identities:

$$a_{i,j}^{x-1,p} = \begin{cases} a_{i,j}^{x-1}, & \text{if } i > p, \\ 0, & \text{otherwise}, \end{cases} \tag{15}$$

where $0 \le p \le x-1$.

Entries $b_{i,j}^{x-1,p} \in B_{x-1}^{p,\mathfrak{p}_{(x-1,p+2)}}$ are given by the same formulas for $-1 \le p \le x-1$.

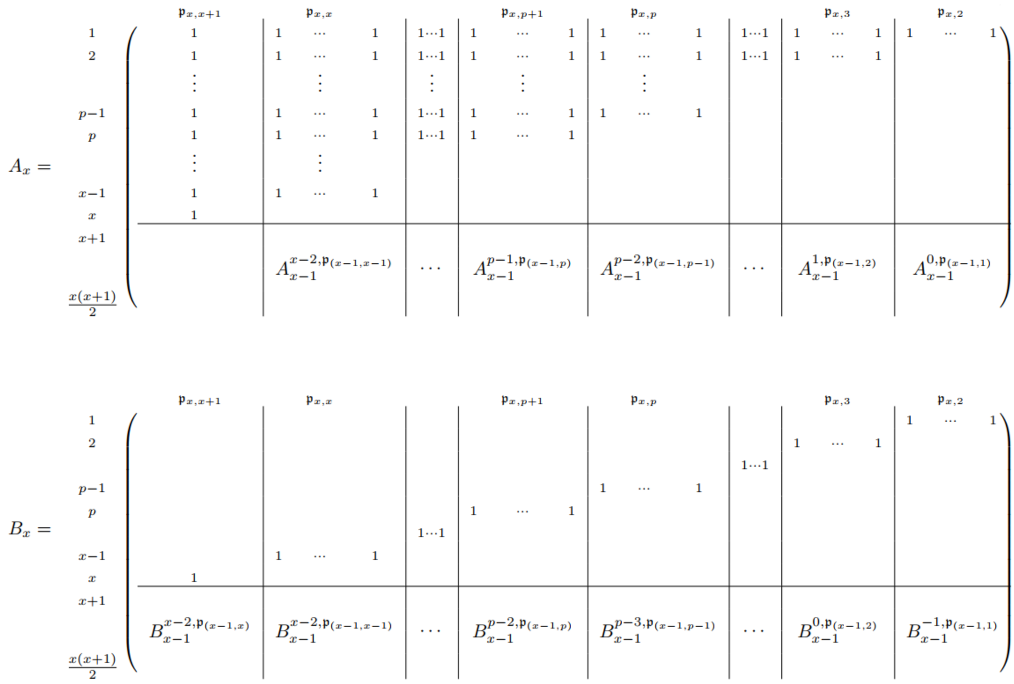

Figure 3. Matrices A_x and B_x whose entries are defined by identities (15).

Paths in $\mathbb{F}Q$ associated with the Brauer quiver induced by Γ^x are given by words of the form $\prod\limits_{h=1}^{k} a^{p}_{q_h}$ or $\prod\limits_{h=1}^{k'} b^{p'}_{q'_h}$. In this case, $a^x_{p,q_r} = b^x_{p',q'_{r'}} = 1$ with $q_r < q_{r+1}$, $q'_r < q'_{r'+1}$, $1 \leq q_r, q'_{r'} \leq \mathfrak{p}_{x,2}$. And $1 \leq p \leq \frac{x(x+1)}{2}$.

$(a^p_q)^2$ and $(b^{p'}_{q'})^2$ denote loops associated with vertices δ for which $val(\delta) = 1$.

Special cycles C_p associated with a given non-truncated vertex δ_p are given by the following identities:

$$C_p = \begin{cases} \prod\limits_{h=t}^{k} a^p_{q_h} \prod\limits_{h=1}^{t-1} a^p_{q_h}, & a^x_{p,q_t} = 1, \text{ and } \delta_p \in F_x, \\ \prod\limits_{h=t}^{k'} b^{p'}_{q'_h} \prod\limits_{h=1}^{t-1} b^{p'}_{q'_h}, & b^x_{p',q'_t} = 1, \text{ and } \delta_p \in G_x, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

The admissible ideal I associated with the Brauer configuration Γ^x is generated by the following set of relations.

- Relations of type I.

$$\begin{aligned} \prod_{h\in\{q_{s_1},\dots,q_{k_1}\}} a^{p_1}_h \prod_{h\in\{q_1,\dots,q_{s_1-1}\}} a^{p_1}_h &= \dots = \prod_{h\in\{q_{s_t},\dots,q_{k_t}\}} a^{p_t}_h \prod_{h\in\{q_1,\dots,q_{s_t-1}\}} a^{p_t}_h, \\ \prod_{h\in\{q_{s_t},\dots,q_{k_t}\}} a^{p_t}_h \prod_{h\in\{q_1,\dots,q_{s_t-1}\}} a^{p_t}_h &= \dots = \prod_{h\in\{q'_{n_1},\dots,q'_{k'_1}\}} b^{p'_1}_h \prod_{h\in\{q'_{n_f},\dots,q'_{k'_f}\}} b^{p'_1}_h, \\ \prod_{h\in\{q'_{n_1},\dots,q'_{k'_1}\}} b^{p'_1}_h \prod_{h\in\{q'_{n_f},\dots,q'_{k'_f}\}} b^{p'_1}_h &= \dots = \prod_{h\in\{q'_{n_1},\dots,q'_{k'_1}\}} b^{p'_f}_h \prod_{h\in\{q'_1,\dots,q'_{n_f-1}\}} b^{p'_f}_h. \end{aligned} \tag{17}$$

$q_{s_w} = q_{s_{w'}} = q'_{n_z}$, for any w, w' and z, $1 \leq q_{s_1} \leq \mathfrak{p}_{x,2}$, $p_1, \dots, p_t, p'_1, \dots, p'_h \in \{1, 2, \dots, \frac{x(x+1)}{2}\}$.

- Relations of type II.

$$\prod_{h\in\{q_{s_1},\dots,q_{k_1}\}} a_h^{p_r} \quad \prod_{h\in\{q_1,\dots,q_{s_1-1},q_{s_1}\}} a_h^{p_r},$$
$$\prod_{h\in\{q'_{n_1},\dots,q'_{k'_1}\}} b_h^{p'_f} \quad \prod_{h\in\{q'_1,\dots,q'_{n_f-1},q'_{n_1}\}} b_h^{p'_f}, \tag{18}$$

  for appropriated positive integers $p_r \in \{p_1,\dots,p_t\}$ and $p'_f \in \{p'_1,\dots,p'_{t'}\}$.
- Relations of type III.

$$a_{q_s}^{p_r} a_{q_n}^{p_h}, \quad a_{q_s}^{p_r} b_{q'_x}^{p'_h}, \quad b_{q'_n}^{p'_h} a_{q_s}^{p_r}, \quad b_{q'_s}^{p'_r} b_{q'_s}^{p'_h}, \tag{19}$$

  for all the possible products in Λ_{Γ^x}.

We let $\Lambda_{\Gamma^x} = \mathbb{F}Q_{\Gamma^x}/I^x$ denote the *Catalan–Brauer configuration algebra* induced by the Brauer configuration Γ^x.

As an example, we define the Catalan–Brauer configuration $\Gamma^2 = (\Gamma_0^2, \Gamma_1^2, \mu^2, \mathcal{O}^2)$, for which

$$\begin{aligned}
\Gamma_0^2 &= \{\mathfrak{a}_0^1, \mathfrak{a}_0^2, \mathfrak{a}_1^2, \mathfrak{b}_0^1, \mathfrak{b}_0^2, \mathfrak{b}_1^2\}.\\
\Gamma_1 &= \{V_1, V_2\}, \quad w_{V_1} = \mathfrak{a}_0^1\mathfrak{a}_0^2\mathfrak{b}_0^2\mathfrak{b}_1^2, \quad w_{V_2} = \mathfrak{a}_0^1\mathfrak{b}_0^1\mathfrak{a}_1^2\mathfrak{b}_1^2.\\
\mu^2(\delta) &= 1, \text{ if } \delta \in \{\mathfrak{a}_0^1, \mathfrak{b}_1^2\}, \quad \mu^2(\delta) = 2, \text{ otherwise.}\\
V_1 &< V_2, \text{ in any successor sequence.}
\end{aligned} \tag{20}$$

The following are the matrices A_2 and B_2.

$$A_2 = \begin{pmatrix} occ(\mathfrak{a}_0^1, V_1) & occ(\mathfrak{a}_0^1, V_2)\\ occ(\mathfrak{a}_0^2, V_1) & occ(\mathfrak{a}_0^2, V_2)\\ occ(\mathfrak{a}_1^2, V_1) & occ(\mathfrak{a}_1^2, V_2)\end{pmatrix},$$

$$B_2 = \begin{pmatrix} occ(\mathfrak{b}_0^1, V_1) & occ(\mathfrak{b}_0^1, V_2)\\ occ(\mathfrak{b}_0^2, V_1) & occ(\mathfrak{b}_0^2, V_2)\\ occ(\mathfrak{b}_1^2, V_1) & occ(\mathfrak{b}_1^2, V_2)\end{pmatrix}.$$

Figure 4 shows the Brauer quiver Q_{Γ^2}.

$Q_{\Gamma^2} =$

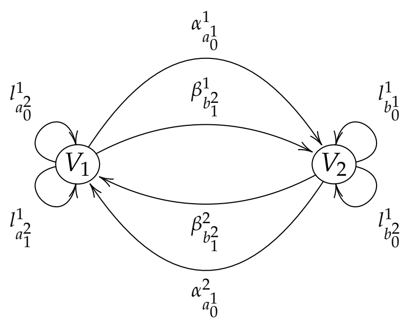

Figure 4. Brauer quiver associated with a Catalan–Brauer configuration algebra Λ_{Γ^2}.

Identities (17)–(19) induce the following set of relations ρ_{Γ^2}:

- $l_u^1 l_v^1$, $(l_u^1)^3$, $l^1_{a_i}\alpha^1_{a_0^1}$, $l^1_{a_i}\beta^1_{b_1^2}$. For all possible values of u and v;
- $l^1_{b_i}\alpha^2_{a_0^1}$, $l^1_{b_i}\beta^2_{b_1^2}$, $\alpha_u\beta_v$. For all possible values of i, u and v;
- $C_{u_i} \sim C_{v_i}$, for all special cycles associated with vertices $u_i, v_i \in V_i$, $i = 1,2$;
- $C_{u_i}a$, for any special cycle associated with a vertex $u_i \in \Gamma_0$, a is the first arrow of C_{u_i}.

The Catalan–Brauer configuration algebra Λ_{Γ^2} is defined in such a way that $\Lambda_{\Gamma^2} = \mathbb{F}Q_{\Gamma^2}/I_{\Gamma^2}$, where $I_{\Gamma^2} = \langle \rho_{\Gamma^2} \rangle$ is an admissible ideal generated by relations ρ_{Γ^2}. Figure 5 shows the indecomposable projective Λ_{Γ^2}-modules.

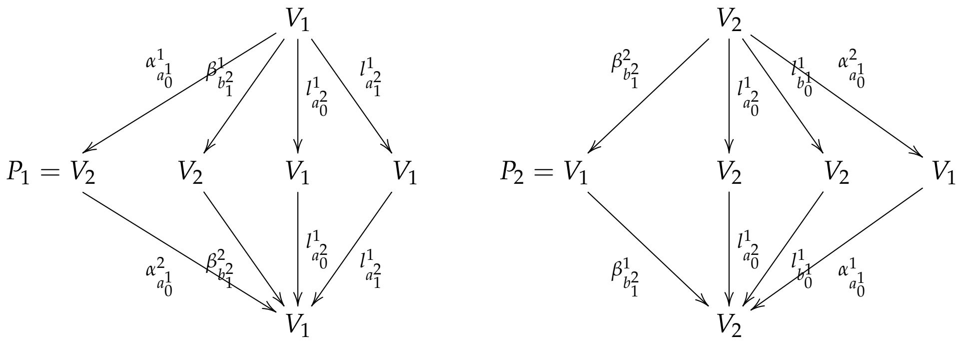

Figure 5. Indecomposable projective Λ_{Γ^2}-modules. Note that the number of composition series equals the number of non-truncated vertices in the corresponding polygon.

3.2. Dimension of a Catalan–Brauer Configuration Algebra and Its Corresponding Center

The dimensions of the Catalan–Brauer configuration algebras are given in this section based on a new family of integer sequences, whose elements are related to Catalan triangle entries.

Let $\mathfrak{h}^p_{x,q}$ be integer numbers, such that

$$\begin{aligned} \mathfrak{h}^p_{1,q} &= 1, \\ \mathfrak{h}^p_{x,q} &= \sum_{\substack{c-d=p-q \\ x-1\leq c\leq p}} \mathfrak{h}^c_{x-1,d} \text{ if } x > 1, \\ \mathfrak{h}^p_{x,q} &= 0 \text{ if } q \leq 0, \end{aligned} \tag{21}$$

$p \geq x-1 \quad q \leq p+1$.

Figure 6 shows integer sequences $\mathfrak{h}^p_{x,q}$ for $x = 2, \ldots, 5$.

x	$p\backslash q$	1	2	3	4	5	6	7	8	9	10	x	1	2	3	4	5	6	7	8	9	10
2	1	1	1									3										
	2	1	2	2									1	2	2							
	3	1	2	3	3								1	3	5	5						
	4	1	2	3	4	4							1	3	6	9	9					
	5	1	2	3	4	5	5						1	3	6	10	14	14				
	6	1	2	3	4	5	6	6					1	3	6	10	15	20	20			
	7	1	2	3	4	5	6	7	7				1	3	6	10	15	21	27	27		
	8	1	2	3	4	5	6	7	8	8			1	3	6	10	15	21	28	35	35	
	9	1	2	3	4	5	6	7	8	9	9		1	3	6	10	15	21	28	36	44	44
4	3	1	3	5	5							5										
	4	1	4	9	14	14							1	4	9	14	14					
	5	1	4	10	19	28	28						1	5	14	28	42	42				
	6	1	4	10	20	34	48	48					1	5	14	34	62	90	90			
	7	1	4	10	20	35	55	75	75				1	5	14	35	69	117	165	165		
	8	1	4	10	20	35	56	83	110	110			1	5	14	35	70	125	200	275	275	
	9	1	4	10	20	35	56	84	119	154	154		1	5	14	35	70	126	209	319	429	429

Figure 6. Numbers $\mathfrak{h}^p_{2\leq x\leq 5,\ q}$.

Arithmetic properties of numbers $\mathfrak{h}^p_{x,q}$ are given by the following Propositions 1–3.

Proposition 1. $\mathfrak{h}^p_{x,j} = \mathfrak{h}^p_{x,q-1} + \mathfrak{h}^{p-1}_{x-1,j}$, *for* $p \geq 1$, $q \leq p$, *and* $1 < x < p+1$.

Proof. If $p = 1$, $\mathfrak{h}^1_{2,1} = \mathfrak{h}^0_{1,1} + \mathfrak{h}^1_{2,0} = 1$.

We suppose that the statement holds true for $p = m$ and $1 < x < m+1$. Then, for $p = m+1$ and $x = 2$,

$$\mathfrak{h}^{m+1}_{2,j} = \sum_{\substack{m+1-q=c-d \\ 1\leq c\leq m+1}} \mathfrak{h}^{c}_{1,d} = (m-q)+1 = \sum_{\substack{m-q=c-d \\ 1\leq c\leq m+1}} \mathfrak{h}^{c}_{1,s} + \mathfrak{h}^{m}_{1,q} = \mathfrak{h}^{m+1}_{2,q-1} + \mathfrak{h}^{m}_{1,j},$$

if it is assumed that the proposition holds for $x = i-1 < m+1$. Then

$$\begin{aligned} \mathfrak{h}^{m+1}_{i,q} &= \sum_{\substack{m+1-q=c-d \\ i-1\leq c\leq m+1}} \mathfrak{h}^{c}_{i-1,d} \\ &= \sum_{\substack{m+2-q=c-d \\ i-1\leq c\leq m+1}} \mathfrak{h}^{c}_{i-1,d} + \sum_{\substack{m-q=c-d \\ i-2\leq c\leq m}} \mathfrak{h}^{c}_{i-2,d} \\ &= \mathfrak{h}^{m+1}_{i,j-1} + \mathfrak{h}^{m}_{i-1,j}. \end{aligned}$$

The result follows by induction. We are done. □

Proposition 2. $\mathfrak{h}^{p}_{x,p+1} = \mathfrak{h}^{p}_{x,p}$, *for* $p \geq 1$ *and* $1 < x \leq p+1$.

Proof. If $p = 1$ then $\mathfrak{h}^{1}_{2,2} = 1 = \mathfrak{h}^{1}_{2,1}$. If the proposition is valid for $p = \iota$ and $1 < x < \iota+1$, then the following identities hold for $p = \iota+1$ and $x = 2$.

$$\mathfrak{h}^{\iota+1}_{2,\iota+2} = \sum_{\substack{-1=c-d \\ 1\leq c\leq \iota+1}} \mathfrak{h}^{c}_{1,d} = \sum_{\substack{0=c-d \\ 1\leq c\leq \iota+1}} \mathfrak{h}^{c}_{1,d} = \mathfrak{h}^{\iota+1}_{2,\iota+1}.$$

If the validness of the proposition holds for $x = i-1 < \iota+2$, then

$$\mathfrak{h}^{\iota+1}_{i,\iota+2} = \sum_{\substack{1=c-d \\ i-1\leq c\leq \iota+1}} \mathfrak{h}^{c}_{i-1,d} = \sum_{\substack{0=c-d \\ i-1\leq c\leq \iota+1}} \mathfrak{h}^{c}_{i-1,d} = \mathfrak{h}^{\iota+1}_{i,\iota+1}.$$

Thus, the statement holds by induction. We are done. □

Proposition 3. *For* $p \geq k-1$, $1 \leq q \leq p+2-k$, *and* $k \geq 1$ *fixed, it holds that* $\mathfrak{h}^{p}_{k,q} = \binom{2k-p+q}{k-p+q+1}$.

Proof. To proceed by induction, we note that for any $q \geq 1$, $\mathfrak{h}^{3}_{1,q} = 1$, and $\mathfrak{h}^{4}_{4,q} = q$. Since $\mathfrak{h}^{p}_{k,q} = \mathfrak{h}^{p}_{k,q-1} + \mathfrak{h}^{p-1}_{k-1,q}$. It holds that $\mathfrak{h}^{p}_{k,q} = \binom{2k-p+q-1}{k-p+q-1+1} + \binom{2k-2-p+1+q}{k-p+1+q+1-1} = \binom{2k-p+q}{k-p+q+1}$. We are done. □

Lemma 1 and Proposition 4 give the relationships between integer numbers $\mathfrak{h}^{p}_{x,q}$ and entries of Catalan triangle $\mathfrak{p}_{x,y}$ (see identities (2)).

Lemma 1. $\mathfrak{p}_{x,x+1-m} = \sum\limits_{q=1}^{x} \mathfrak{h}^{x-1}_{m,q}$ *for* $x \geq 1$, $1 \leq m \leq x$.

Proof. If $x = 1$, $\mathfrak{p}_{1,1} = 1 = \mathfrak{h}^{0}_{1,1}$. If it is assumed that the result is valid for $x = \iota$ and $1 \leq m \leq \iota$. Thus, if $x = \iota+1$ and $m = 1$, it holds that $\sum\limits_{q=1}^{\iota+1} \mathfrak{h}^{\iota}_{1,q} = \iota+1 = \mathfrak{p}_{\iota+1,\iota+1}$.

If the lemma holds true for $m = i-1 < \iota+1$, then

$$
\begin{aligned}
\sum_{q=1}^{\iota+1} \mathfrak{h}^{\iota}_{i,q} &= \sum_{\substack{c-d=\iota-1\\ i-1\le c\le \iota}} \mathfrak{h}^{c}_{i-1,d} + \cdots + \sum_{\substack{c-d=1\\ i-1\le c\le \iota}} \mathfrak{h}^{c}_{i-1,d} \\
&= \sum_{\substack{c-d=\iota-1\\ i-1\le c\le \iota-1}} \mathfrak{h}^{c}_{i-1,d} + \cdots + \sum_{\substack{c-d=1\\ i-1\le c\le \iota-1}} \mathfrak{h}^{c}_{i-1,d} + \sum_{q=1}^{\iota+1} \mathfrak{h}^{\iota}_{i-1,q} \\
&= \sum_{q=1}^{\iota} \mathfrak{h}^{\iota-1}_{i,q} + \sum_{q=1}^{\iota+1} \mathfrak{h}^{\iota}_{i-1,q} \\
&= \mathfrak{p}_{\iota,\iota+1-i} + \mathfrak{p}_{\iota+1,\iota+3-i} \\
&= \mathfrak{p}_{\iota,\iota+1-i} + \sum_{p=\iota+2-i}^{\iota+1} \mathfrak{p}_{\iota,p} \\
&= \sum_{p=\iota+1-i}^{\iota+1} \mathfrak{p}_{\iota,p} = \mathfrak{p}_{\iota+1,\iota+2-i}.
\end{aligned}
$$

□

Proposition 4. $\mathfrak{p}_{x,x+1-m} = \mathfrak{h}^{x}_{1+m,x+1}$, *for* $x \ge 1$ *and* $1 \le m \le x$.

Proof. If $x = 1$, $\mathfrak{p}_{1,1} = 1 = \mathfrak{h}^{1}_{2,2}$. If the result holds for $x = \iota$ and $1 \le m \le \iota$, then if $x = \iota + 1$ and $s = 1$. Thus

$$\mathfrak{p}_{\iota+1,\iota+1} = \iota + 1 = \sum_{\substack{-1=c-d\\ 1\le c\le \iota+1}} \mathfrak{h}^{c}_{1,d} = \mathfrak{h}^{\iota+1}_{2,\iota+2},$$

If the proposition is valid for $s = i - 1 < \iota + 1$. Then

$$
\begin{aligned}
\mathfrak{p}_{\iota+1,\iota+2-i} &= \sum_{q=1}^{\iota+1} \mathfrak{h}^{\iota}_{i,q} \\
&= \sum_{\substack{c-d=\iota-1\\ i-1\le c\le \iota}} \mathfrak{h}^{c}_{i-1,d} + \cdots + \sum_{\substack{c-d=1\\ i-1\le c\le \iota}} \mathfrak{h}^{c}_{i-1,d} \\
&= \sum_{q=1}^{\iota+1} \mathfrak{h}^{\iota}_{i-1,q} + \cdots + \sum_{q=1}^{i} \mathfrak{h}^{i-1}_{i-1,q} \\
&= \mathfrak{p}_{\iota+1,\iota+1-i} + \cdots + \mathfrak{p}_{i,2} \\
&= \mathfrak{h}^{\iota+1}_{i,\iota+2} + \cdots + \mathfrak{h}^{i}_{i,i+1} \\
&= \mathfrak{h}^{\iota+1}_{i+1,\iota+2}.
\end{aligned}
$$

The result holds by induction. We are done. □

Each $m \times n$-matrix $M = (m_{pq})_{1\le p\le m,\ 1\le q\le n}$ with entries in a commutative ring $\mathbb{F}$ has associated a list of column vectors

$$\mathcal{L}^M = (\mathcal{V}^M, \mathcal{V}^{M,\nu}, \mathcal{D}^{M,\nu}, \mathcal{E}^{M,\nu}, \mathcal{F}^{M,\nu}). \tag{22}$$

where,

$$
\begin{aligned}
&\mathcal{V}^M = (v^M_{p,1})_{1\le p\le m}, \quad v^M_{p,1} = \sum_{q=1}^{n} m_{pq}, \\
&\mathcal{V}^{M,\nu} = (v^{M,\nu}_{p,1})_{1\le p\le m}, \quad v^{M,\nu}_{p,1} = v^M_{p,1}, \text{ if } 1 \le p \le \nu, \quad v^{M,\nu}_{p,1} = 0, \text{ otherwise}, \\
&\mathcal{D}^{M,\nu} = (d^{M,\nu}_{p,1})_{1\le p\le m}, \quad d^{M,\nu}_{p,1} = 0, \quad \text{if } 1 \le p \le \nu, \quad d^{M,\nu}_{p,1} = v^M_{p,1}, \text{ otherwise}, \\
&\mathcal{E}^{M,\nu} = (e^{M,\nu}_{p,1})_{1\le p\le m}, \ e^{M,\nu}_{p,1} = d^{M,\nu}_{p,1}, \text{ if } 1 \le p \le \nu, \ e^{M,\nu}_{p,1} = d^{M,\nu}_{\nu+k,1} - v^M_{k,1}, \ \nu + 1 \le p \le m, \\
&\mathcal{F}^{M,\nu} = \mathcal{V}^M - \mathcal{E}^{M,\nu}.
\end{aligned}
\tag{23}
$$

Remark 2. *Henceforth, we will assume the notation* $\phi_0(A_x^{p,\mathfrak{p}_{x,p+1}}) = (a_i^{x,p})_{1\leq i\leq \frac{x(x+1)}{2}}$ *and* $\phi_0(B_x^{p,\mathfrak{p}_{x,p+2}}) = (b_i^{x,p})_{1\leq i\leq \frac{x(x+1)}{2}}$.

The following result holds for maps $\phi_\nu : M_{m\times n}(\mathbb{F}) \to M_{m\times 1}(\mathbb{F})$, $\varphi_\nu : M_{m\times n}(\mathbb{F}) \to M_{m\times 1}(\mathbb{F})$, such that, $\phi_\nu(M) = \mathcal{D}^{M,\nu}$, and $\varphi_\nu(M) = \mathcal{F}^{M,\nu}$. For instance,

$$\phi_0(A_2) = \begin{pmatrix} 2\\ 1\\ 1 \end{pmatrix}, \phi_1(A_2) = \begin{pmatrix} 0\\ 1\\ 1 \end{pmatrix}, \text{and } \varphi_1(A_2) = \begin{pmatrix} 0\\ 2\\ 1 \end{pmatrix}. \tag{24}$$

Lemma 2. *Let A_x and B_x be the matrices given in Figure 3. Then*

1. $$\phi_0(A_x^{p,\mathfrak{p}_{x,p+1}}) = \begin{cases} \phi_0(A_x), & \text{if } p = 0,\\ \phi_1(A_x^{p-1,\mathfrak{p}_{x,p}}), & \text{if } p = 1,\\ \phi_p(A_x^{p-1,\mathfrak{p}_{x,p}}) + \varphi_x(A_{x-1}^{p-2,\mathfrak{p}_{(x-1,p-1)}}), & \text{if } 2 \leq p \leq x-1, \end{cases}$$
2. $$\phi_0(B_x^{p,\mathfrak{p}_{x,p+2}}) = \begin{cases} \phi_0(B_x), & \text{if } p = 0 \text{ or } p = -1,\\ \phi_p(B_x^{p-1,\mathfrak{p}_{x,p+1}}) + \varphi_x(B_{x-1}^{p-2,\mathfrak{p}_{x-1,p}}), & \text{if } 1 \leq p \leq x-1. \end{cases}$$

For $x > 0$.

Proof. (i) Let $A_x = (a_{i,j}^x)$ be an $\frac{x(x+1)}{2} \times \mathfrak{p}_{x,2}$ matrix, and $A_x^{p,\mathfrak{p}_{x,p+1}} = (a_{i,j}^{x,p})$ be an $\frac{x(x+1)}{2} \times \mathfrak{p}_{x,p+1}$ matrix, whose entries satisfy identities (15).

- If $p = 0$ then $A_x^{0,\mathfrak{p}_{x,1}}$ is a matrix with $\mathfrak{p}_{x,1}$ columns. Then $a_i^{x,0} = v_{i,1}^{A_x}$ for $1 \leq i \leq \frac{x(x+1)}{2}$, provided that, $\mathfrak{p}_{x,1} = \mathfrak{p}_{x,2}$, $a_{i,j}^{x,0} = a_{i,j}^x$;
- If $p = 1$ then $A_x^{1,\mathfrak{p}_{x,2}}$ is a matrix with $\mathfrak{p}_{x,2}$ columns, $a_{i,j}^{x,1} = a_{i,j}^x$ for $2 \leq i \leq \frac{x(x+1)}{2}$, i.e., $a_i^{x,1} = v_{i,1}^{A_x}$ for $2 \leq i \leq \frac{x(x+1)}{2}$, and $a_1^{x,1} = 0$;
- If $2 \leq p \leq x-1$, and $A_x^{p,\mathfrak{p}_{x,p+1}}$ is a matrix with $\mathfrak{p}_{x,p+1}$ columns. In this case, the entries of the matrix $(A_x^{p,\mathfrak{p}_{x,p+1}})'$ obtained from matrix $A_x^{p,\mathfrak{p}_{x,p+1}}$ by deleting the $p-1$th row and all columns C_j, $\mathfrak{p}_{x,p+1} \leq j \leq \mathfrak{p}_{x,p}$ equals $A_x^{p-1,\mathfrak{p}_{x,p}}$. Thus, the matrix $A_{x-1}^{p-2,\mathfrak{p}_{x-1,p-1}}$ provides entries in columns C_j, for which $x+1 \leq j \leq \frac{x(x+1)}{2}$, i.e.,

$$a_i^{x,p} = \sum_{q=1}^{\mathfrak{p}_{x,p}} a_{i,q}^{x,p-1} - \sum_{q=1}^{\mathfrak{p}_{x-1,p-1}} a_{i-x,q}^{x-1,p-2},$$

for $x+1 \leq i \leq \frac{x(x+1)}{2}$ and $a_i^{x,p} = 0$, otherwise.

The item 2 can be proved by using similar arguments as in the case 1. We are done. □

The following notation is assumed in the proof of Proposition 5.

- $a_s = 1 + w - (s+1) - \frac{s(s+1)-k(k+1)}{2}$;
- $b_s = s - k + w - (s+1) - (\frac{s(s+1)-k(k-1)}{2} + 1)$;
- $b_x = x - k + w - (\frac{x(x+1)-k(k+1)}{2} + 1)$;
- $c_s = s - k + w - (s+1) - (\frac{s(s+1)-k(k-1)}{2} + 1) - (v-1)$;
- $c_x = x - k + w - (\frac{x(x+1)-k(k+1)}{2} + 1) - (v-1)$;
- $d_s = 1 + w - \frac{(s+2)(s+1)-k(k+1)}{2}$;
- $d_x = 1 + w - \frac{x(x+1)-k(k+1)}{2}$;
- $e_s = s - k + w - (\frac{(s+2)(s+1)-k(k-1)}{2} + 1)$;
- $f_s = s - k + w - (\frac{(s+2)(s+1)-k(k-1)}{2} + 1) - (v-1)$;
- $g_s = s - k + w - \frac{(s+2)(s+1)-k(k+1)}{2}$;

- $h_s = w - \frac{(s+2)(s+1)-k(k+1)}{2}$;
- $j_s = s+1-k+w-(\frac{(s+2)(s+1)-k(k+1)}{2}+1)$;
- $k_s = v+w-(s+1)-\frac{s(s+1)-k(k+1)}{2}$.

Proposition 5. *Let A_x and B_x be the matrices shown in Figure 3. Then, the following identities hold.*

1. $a_w^{x,v} = \mathfrak{p}_{k,d_x}\mathfrak{h}_{x-k+1,c_x}^{b_x}$ *for* $0 \leq v \leq x-1$;
2. $b_w^{x,v} = \mathfrak{p}_{k-1,(d_x-1)}\mathfrak{h}_{(x-k+1),c_x-1}^{b_x+1}$ *for* $-1 \leq v \leq x-1$,

with $\frac{n(n+1)-k(k+1)}{2} \leq w \leq \frac{x(x+1)-k(k-1)}{2}$ *for* $1 \leq k \leq x$, *and* $x > 1$.

Proof. 1. If $x = 2$ and $v = 0$, then $\phi_0(A_2^{0,\mathfrak{p}_{2,1}}) = \phi_0(A_2)$. Thus,

$$\begin{aligned} a_1^{2,0} &= 2 = \mathfrak{p}_{2,2}\mathfrak{h}_{1,1}^0, \\ a_2^{2,0} &= 1 = \mathfrak{p}_{2,3}\mathfrak{h}_{1,2}^1, \\ a_3^{2,0} &= 1 = \mathfrak{p}_{1,2}\mathfrak{h}_{2,2}^1. \end{aligned}$$

If $v = 1$ then $\phi_0(A_2^{1,\mathfrak{p}_{2,2}}) = \phi_1(A_2)$, then

$$\begin{aligned} a_1^{2,1} &= 0 = \mathfrak{p}_{2,2}\mathfrak{h}_{1,0}^0, \\ a_2^{2,1} &= 1 = \mathfrak{p}_{1,2}\mathfrak{h}_{1,1}^1, \\ a_3^{2,1} &= 1 = \mathfrak{p}_{1,2}\mathfrak{h}_{2,1}^1. \end{aligned}$$

If the result is valid for $x = s$ and $0 \leq v \leq s$. Moreover, $\phi_0(A_{s+1}^{0,\mathfrak{p}_{s+1,1}}) = \phi_0(A_{s+1})$ if $x = s+1$ and $v = 0$ (see Lemma 2). $a_w^{s+1,0} = \mathfrak{p}_{s+1,w+v} = \mathfrak{p}_{s+1,v+w}\mathfrak{h}_{1,w}^{w-1}, \quad 0 < w < s+1$.

The rows between $s+2$ and $\frac{(s+1)(s+2)}{2}$ satisfy the following identities.
$a_w^{s+1,0} = \sum_{v=0}^{s-1}\mathfrak{p}_{k,a_s}\mathfrak{h}_{s+1-k,c_s}^{b_s} = \mathfrak{p}_{k,d_s}\sum_{v=0}^{s-1}\mathfrak{h}_{s+1-k,f_s}^{e_s} = \mathfrak{p}_{k,d_s}\mathfrak{p}_{g_s,h_s} = \mathfrak{p}_{k,d_s}\mathfrak{h}_{s+2-k,g_s+1}^{g_s}$, with $1 \leq k \leq m$.

If $v = 1$, then $\phi_0(A_{s+1}^{1,\mathfrak{p}_{s+1},2}) = \phi_1(A_{s+1}^{0,\mathfrak{p}_{s,1}})$. Thus, $a_1^{s+1,1} = 0 = \mathfrak{p}_{s+1,2}\mathfrak{h}_{1,0}^0$, and

$a_w^{s+1,1} = a_w^{s+1,0} = \mathfrak{p}_{k,d_s}\mathfrak{h}_{s+2-k,g_s+1}^{g_s} = \mathfrak{p}_{k,d_s}\mathfrak{h}_{s+2-k,g_s}^{g_s}$. For $\frac{(s+2)(s+1)-k(k+1)}{2} < w \leq \frac{(s+2)(s+1)-k(k-1)}{2}$ with $w \neq 1$, and $1 \leq k \leq s+1$.

If the proposition holds for $v = p-1 < s$. Therefore, it holds that $\phi_0(A_{s+1}^{p,\mathfrak{p}_{s+1,p+1}}) = \phi_p(A_{s+1}^{p-1,\mathfrak{p}_{s+1,p}}) + \varphi_{s+1}(A_s^{p-2,\mathfrak{p}_{(s,p-1)}})$ if $v = p$ (see Lemma 2).

$$a_w^{s+1,p} = 0 = \mathfrak{p}_{s+1,1+w}\mathfrak{h}_{1,w-p}^{w-1}, \quad \text{for } 1 \leq w \leq p.$$

$$a_w^{s+1,p} = a_w^{s+1,p-1} = \mathfrak{p}_{s+1,1+u}\mathfrak{h}_{1,w+1-p}^{w-1} = \mathfrak{p}_{s+1,1+w}\mathfrak{h}_{1,w-p}^{w-1}, \quad \text{for } p+1 \leq w \leq s+1.$$

$$a_w^{s+1,p} = \mathfrak{p}_{k,d_s}\mathfrak{h}_{s+2-k,j_s-(p-2)}^{j_s} - \mathfrak{p}_{k,l_s}t_{b_s+k,b_s+k-(p-3)}^{s+1-k}, \quad \text{for } w \geq s+1.$$

Proposition 1 implies that

$$a_w^{s+1,p} = \mathfrak{p}_{k,k_s}\mathfrak{h}_{s+2-k,j_s-(p-1)}^{j_s}, \quad \text{for } 1 \leq k \leq s.$$

The proof of case 2 requires similar arguments as those exposed in case 1. We are done. □

The following map $\delta : \mathbb{N} \to \{1, 2\}$ such that

$$\delta(x) = \begin{cases} 1, & \text{if } x \neq 2, \\ 2, & \text{if } x = 2. \end{cases}$$

is used to give dimension formulas for Catalan–Brauer algebras Γ^x and their centers.

Theorem 2. *Let Λ_{Γ^x} be a Catalan–Brauer configuration algebra. Then*

1. $\frac{\dim_{\mathbb{F}} \Lambda_{\Gamma^x}}{2} = \frac{1}{n+1}\binom{2x}{x} + \delta(x) + \sum\limits_{u=1}^{\frac{x(x+1)}{2}} (t_{(a_u^{x,0}-1)} + t_{(b_u^{x,0}-1)})$.
2. $\dim_{\mathbb{F}} Z(\Lambda_{\Gamma^x}) = 1 + 2\delta(x) + \frac{1}{x+1}\binom{2x}{x}$.

where, $x > 0$ and t_i denotes the ith triangular number.

Proof. 1. Firstly, we note that the number of vertices in the Brauer quiver Q_{Γ_x} is given by the xth Catalan number $C_x = \frac{1}{x+1}\binom{2x}{x}$. Secondly, we note that $val(\mathfrak{a}_{x_1}^{x_2})$ (resp. $val(\mathfrak{b}_{n_1}^{n_2})$) is given by $a_{m_a}^{x,0}$, $m_a = xx_1 + x_2 - \frac{x_1(x_1+1)}{2}$ (resp. $b_{m_b}^{x,0}$, $m_b = xx_1 + x_2 - \frac{x_1(x_1+1)}{2}$). As a consequence of Proposition 5, we have that $a_x^{x,0} = 1 = b_x^{x,0}$ for any x. In particular, for $x = 2$, it holds that $a_2^{0,1} = b_2^{0,1} = 1$.

2. $\#Loops(Q_{\Lambda_{\Gamma^x}}) = |\mathcal{C}_{\Lambda_{\Gamma^x}}|$ (see identity (8)).
□

As an example, the following are the dimensions associated with the Brauer configuration (20). We note that for $x = 2$, it holds that $C_2 = 2$, $a_1^{2,0} = 2 = b_3^{2,0}$, and $a_2^{2,0} = a_3^{2,0} = b_1^{2,0} = b_2^{2,0} = 1$. Thus

$$\begin{aligned} \dim_{\mathbb{F}} \Lambda_{\Gamma^2} &= 2(C_2 + \delta(2)) + 12 - 8 \\ &= 2(2+2) + 4 = 12, \end{aligned}$$

and

$$\dim_{\mathbb{F}} Z(\Lambda_{\Gamma^2}) = 1 + 2\delta(2) + C_2 = 7.$$

4. Concluding Remarks

Catalan–Brauer configuration algebras (CBCAs) is a way to relate Catalan combinatorics with the BCAs theory. Dimension formulas of such CBCAs and their centers can be obtained by using entries of the Catalan triangle. The procedure interprets such entries as numbers of some novel integer sequences $\mathfrak{h}_{x,j}^i$ dealing with binomial numbers.

It is an interesting task for the future to investigate additional relationships between sequences $\mathfrak{h}_{x,j}^i$ and different Catalan objects.

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Abbreviations

The following abbreviations are used in this manuscript:

BCA	Brauer Configuration Algebra
C_n	nth Catalan number
$C(n,k)$	Catalan triangle entry
$\dim_{\mathbb{F}} \Lambda_\Gamma$	Dimension of a Brauer configuration algebra
$\dim_{\mathbb{F}} Z(\Lambda_\Gamma)$	Dimension of the center of a Brauer configuration algebra
CBCA	Catalan-Brauer Configuration Algebra
$\mathbb{F}$	Field
Γ_0	Set of vertices of a Brauer configuration Γ
$\mathrm{occ}(\alpha, V)$	Number of occurrences of a vertex α in a polygon V
t_n	nth triangular number
$V_i^{(\alpha)}$	Ordered sequence of polygons
$val(\alpha)$	Valency of a vertex α
w_V	The word associated with a polygon V

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