mathematics

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The Research on Consistency Checking and Improvement of Probabilistic Linguistic Preference Relation Based on Similarity Measure and Minimum Adjustment Model

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Abstract: In the process of decision making, the probabilistic linguistic term set (PLTS) is a useful tool to express the evaluation information provided by decision makers (DMs). On the basis of PLTS, the probabilistic linguistic preference relation (PLPR) has been proposed, which can well describe the uncertainty of preferences when experts conduct pairwise comparison between any two alternatives. The consistency analysis is an essential process to check whether the preferences are reasonable and logical. For the consistency checking and improvement of PLPR, some methods have been developed to conduct the work. However, the previous methods seldom consider whether the information of original preferences is distorted after the adjustment of inconsistency preferences, and the adjustment processes are complicated in much of the literature. To overcome the defects of existing methods, we developed a novel PLPR consistency analysis model, and this paper mainly contains two sections. On the one hand, a new consistency index and the consistency checking method are proposed based on similarity measure, respectively. On the other hand, based on the idea of minimum adjustment, we constructed an optimization model to improve the consistency level and develop the process of decision making on the basis of consistency analysis. A numerical example about talent recruitment is given to verify the feasibility of the proposed method. We have a comparative analysis with Zhang's method from many aspects including the decision results, consistency checking and improvement, as well as adjusted preferences, adjustment costs and consistence threshold. At length, the conclusion of this research is that the proposed consistency analysis model is superior to the previous method on the determination of adjustment parameter, as well as the adjustment cost and the retention of original preferences. To show the effectiveness and superiority, we have a comparative analysis with other approaches. At length, the conclusion of this study is drawn.

Keywords: probabilistic linguistic preference relation; consistency checking; consistency improvement; similarity; decision making

MSC: 90B50; 94D05; 03B52

Citation: Xiao, H.; Wu, S.; Cui, C. The Research on Consistency Checking and Improvement of Probabilistic Linguistic Preference Relation Based on Similarity Measure and Minimum Adjustment Model. *Mathematics* **2022**, *10*, 1369. https://doi.org/10.3390/math10091369

Academic Editors: Fu-Gui Shi and Bin Pang

Received: 10 March 2022
Accepted: 14 April 2022
Published: 20 April 2022

1. Introduction

In real society, decision making is an important management domain, and the purpose of decision making is to obtain the optimal scheme among quite a lot of alternatives. Decision making exists in various fields and aspects, e.g., earthquake shelter selection in emergency decision [1], the evaluation of technology companies for a bank [2], sustainable supplier selection in modern enterprise production management [3] and assessing the growth of business [4]. Due to the complexity of decision problem and cognition of human beings, people find it hard to make a clear judgment over distinct alternatives, so there exists much fuzzy information in people's evaluations in most cases. In view of this situation, the fuzzy decision is developed and has been widely researched. In fuzzy decision making, experts can simply give their preferences over alternatives by using some fuzzy expression models.

Generally, when evaluating the alternatives, experts tend to adopt one of two expression forms, namely, numerical rating and pairwise comparison. For the former, it may be difficult to cope with the complex problem because of the limited knowledge of DMs. Thus, people are inclined to use the latter form, namely the pairwise comparison. By using this expression form, DMs only need to analyse the preference between two alternatives every time. Take a simple example: when people evaluate three cars, they only need to give their preference degree, e.g., car A is preferable to car B, or car A preferable to C. With respect to the preference relation, it has been widely researched in various decision applications [5,6]. There are two kinds of preference relations, numerical preference relations [7] and linguistic preference relations [8–10]. Comparing with the use of numerical preference relations, people tend to utilize the linguistic labels to describe their preferences owing to the simplicity of expression. For example, when people are evaluating a product, they can only use given linguistic terms, such as "terrible", "indifferent" or "good" [11]. Zadeh [12] first proposed the concept of linguistic variables (LV), which cater to the expression habits of human beings. However, to some extent, there exist fuzziness and hesitance when people evaluate the alternatives [13]. Thus, Zadeh [14] proposed the fuzzy sets (FSs) theory, and Torra [15] introduced the concept of hesitant fuzzy sets (HFSs). For FSs or HFSs, the membership degree well portrays the uncertainty of experts' preferences. Subsequently, Rodriguez et al. [16] developed the theory of hesitant fuzzy linguistic term sets (HFLTSs), which combines the HFSs and LV in order to describe the hesitance and fuzziness of linguistic variables. For a HFLTS, the importance or weight of every linguistic term is identical by default. Nevertheless, in reality, the importance of every linguistic term should be different because of the distinct preference degree over various alternatives and the cognitive complexity of humans [17]. To make up for this shortcoming, Pang et al. [18] proposed probabilistic linguistic term sets (PLTSs) in which every linguistic term has a different probability. Subsequently, to apply the PLTS to the preference relation, Zhang et al. [19] proposed the probabilistic linguistic preference relation (PLPR). PLPR well presents the fuzzy preference information in the form of PLTS, which has attracted great attention from scholars, e.g., the research on network consensus analysis of PLPR [20] and the application in financial products selection based on PLPR environment [21].

The consistency is a vital property in preference relation, which can guarantee that DMs provide the evaluation information without any self-contradiction. If we do not have a consistency analysis for preference relation, the inconsistency preferences may lead to illogical and unreasonable decision results [22,23]. Thus, consistency analysis is always an important research topic. In general, consistency analysis consists of two sections, namely consistency checking and consistency improvement [24]. There are many methods proposed to check and improve the numerical preference consistency [25,26] and linguistic preference consistency. Alonso et al. [27] explored the relation between additive consistency and transitivity, which was also extended to the multiplicative consistency and linguistic cases. However, it can not measure the consistency degree of inconsistency preference relations, so Dong et al. [22] proposed a consistency index that can measure the consistency degree and identify whether the consistency of preference relation is accepted. Since Zhang et al. [19] introduced the concept of PLPR and presented an accepted consistency index based on a distance measure between a preference matrix and a consistency matrix, the consistency measure methods of PLPR have been investigated in recent years, including individual consistency checking for PLPR [28–31] and a combination of individual consistency and group consensus for PLPR [17,32,33]. To better express the PLPR consistency, Tian et al. [34] proposed a new additive consistency by using expected value. Wang et al. [35] developed the expected multiplicative consistency of InPLPRs (incomplete PLPRs) and constructed a multi-stage consistency-improving optimization model to improve the consistency of the InPLPR. Xue et al. [36] conducted a comparative study between PLPR, distributed preference relations in consistency, and found that the consistency of PLPRs is derived by following mathematical equations. Liao et al. [37] proposed a method to check the consistency of PLPR and constructed a model to complete the InPLPR based on additive consistency and social strategy. Although there are many research endeavors related to consistency checking and improvement, few study consider the adjustment cost, including the number of adjusted elements, as well as the distortion of preference information provided to DMs. Thus, the defects of existing methods have two aspects. On the one hand, the adjustment cost is not considered, which often leads to the consumption of large costs in the process of consistency analysis. On the other hand, it does not consider the distortion of preference information of DMs.

To overcome the above shortcomings and motivated by Zhang's method [19] of a consistency measure for PLPR, this paper mainly proposes a novel approach to conduct consistency analysis for PLPR. In Zhang's research [19], the definition of additive consistency for PLPR was introduced and was used to obtain the consistency matrix. Besides, a consistency index was developed to measure the consistency degree and identify whether the preference matrix satisfies the accepted consistency. However, in Zhang's method [19], it needs to adjust all preferences if the preference matrix does not satisfy the consistency level. As a result, it leads to consuming much cost and changes the original preferences provided by DMs to a large extent. Hence, to improve the method above, the main work and contributions in this paper are as follows:

(1) A new method to check the consistency of PLPRs is proposed based on similarity, and the consistency index is given. We utilized the PLTS cosine similarity to construct the consistency index, which can measure consistency degrees of preferences and preference matrix. Additionally the consistency checking matrix was developed to identify the preferences with unaccepted consistency.
(2) Based on the idea of minimum adjustment, we constructed an optimization model to improve the consistency level. The main purpose was to retain the original preference information provided by DMs to the greatest extent.
(3) We developed the decision-making process based on the consistency analysis and used a numerical example to verify the feasibility of our methods.
(4) We included a comparative analysis with other methods on consistency checking and improvement in order to show the advantages and disadvantages of our methods.

The rest of the contents are arranged as follows. In Section 2, some basics are introduced, including the definitions of PLTS, PLPR and related operations. Subsequently, the new method to check consistency for PLPR based on similarity is proposed in Section 3. Moreover, we developed the consistency improvement optimization model based on the minimum adjustment and propose the process of decision making on the basis of consistency analysis in Section 4. In Section 5, a numerical example of talent recruitment is used to verified the feasibility of our methods. The comparative analysis and discussion are given in Section 6. Finally, we conduct the conclusion in Section 7.

2. Preliminaries

In this section, some basics about probabilistic linguistic term sets (PLTSs) and probabilistic linguistic preference relation (PLPR) are introduced.

2.1. PLTS and Basic Operations

To cater to the expression habits of people, Zadeh [12] introduced the LV, which can be used to describe people's preferences when facing the decision problem. A linguistic term set (LTS) contains a group of linguistic variables. For example, let S be an LTS, and $S = \{s_\alpha | \alpha = -1, 0, 1\}$, s_{-1} = "bad", s_0 = "medium" and s_1 = "good". In most cases, the weights or importance of terms in an LTS are different because of the cognitive uncertainty and the preferences over various alternatives. Thus, Pang et al. [18] proposed the probabilistic linguistic term sets (PLTSs) on the basis of HFLTSs. The concept and operation of PLTS are as follows.

Definition 1 [18]. *Let S be a LTS, $S = \{s_\alpha | \alpha = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$, where τ is an integer. Then the PLTS is given by*

$$L(p) = \{L^k(p^k) | L^k \in S, p^k \geq 0, k = 1, 2, \ldots, \#L(p), \textstyle\sum_{k=1}^{\#L(p)} p^k \leq 1\},$$

where L^k and p^k represent the k th term and its probability, respectively. #L(p) is the number of terms in L(p).

For a PLTS, the sum of probability may have two statuses, namely, complete and incomplete. When the sum of probability is equal to one, it is a complete PLTS. If the sum of probability is less than one, i.e., $\sum_{k=1}^{\#L(p)} p^k < 1$, it is an incomplete PLTS. For an incomplete PLTS, a normalized process should be conducted to make it be complete. According to Pang's normalized method [18], the normalized PLTS can be derived by

$$\dot{L}(p) = \{L^k(\dot{p}^k) | k = 1, 2, \ldots, \#L(p)\}, \tag{1}$$

where $\dot{p}^k = p^k / \sum_{k=1}^{\#L(p)} p^k$.

Besides, when processing two different PLTSs, if the number of terms in PLTS are distinct, we should unify the number. Namely, given any two PLTSs $L_1(p)$ and $L_2(p)$ where the numbers of linguistic terms in them are different, then,

(1) If $\#L_1(p) > \#L_2(p)$, add $\#L_1(p) - \#L_2(p)$ terms that are the smallest in $L_1(p)$. Additionally, the probabilities of added terms are zero.
(2) If $\#L_1(p) < \#L_2(p)$, add $\#L_2(p) - \#L_1(p)$ terms that are the smallest in $L_2(p)$. Additionally, the probabilities of added terms are zero.

Definition 2 [18]. *Let $L(p) = \{L^k(p^k) | k = 1, 2, \ldots, \#L(p)\}$ be a PLTS; then, the score function of PLTS is given by*

$$E(L(p)) = s_{\overline{\alpha}}, \overline{\alpha} = \textstyle\sum_{k=1}^{\#L(p)} I(L^k) p^k / \sum_{k=1}^{\#L(p)} p^k, \tag{2}$$

where $I(L^k)$ is an extract function that obtains the subscript of L^k. $\#L(p)$ is the number of terms in $L(p)$.

In order to model the deviation among terms in a PLTS, Pang [18] introduced the concept of deviation degree:

Definition 3 [18]. *Let $L(p) = \{L^k(p^k) | k = 1, 2, \ldots, \#L(p)\}$ be a PLTS; then, the deviation degree of $L(p)$ is given by*

$$\sigma(L(p)) = \frac{\sqrt{\sum_{k=1}^{\#L(p)} (p^k (I(L^k) - \overline{\alpha})^2)}}{\sum_{k=1}^{\#L(p)} p^k}, \tag{3}$$

where $\overline{\alpha} = \sum_{k=1}^{\#L(p)} I(L^k) p^k / \sum_{k=1}^{\#L(p)} p^k$, and $\#L(p)$ is the number of terms in $L(p)$.

To compare any two PLTSs, we firstly compare the score functions of two PLTSs. The bigger the score function, the bigger the PLTS. If the score functions are identical, we use the deviation degree to compare. However, the smaller the deviation, the bigger the PLTS. The comparison rules are summarized as follows:

(1) If $E(L_1(p)) > E(L_2(p))$, then $L_1(p) > L_2(p)$;
(2) If $E(L_1(p)) < E(L_2(p))$, then $L_1(p) < L_2(p)$;
(3) If $E(L_1(p)) = E(L_2(p))$, then:
    (1) If $\sigma(L_1(p)) > \sigma(L_2(p))$, then $L_1(p) > L_2(p)$;
    (2) If $\sigma(L_1(p)) < \sigma(L_2(p))$, then $L_1(p) < L_2(p)$;
    (3) If $\sigma(L_1(p)) = \sigma(L_2(p))$, then $L_1(p) = L_2(p)$.

To compute the PLTSs, some basic operations are given by Zhang [19], which include addition and scalar multiplication.

(1) The addition:

$$L_1(p) \oplus L_2(p) = \cup\{L_3^{k_3}(p_3^{k_3})|k_1 = 1,2,\ldots,\#L_1(p); k_2 = 1,2,\ldots,\#L_2(p)\},$$

where $L_3^{k_3} = L_1^{k_1} \oplus L_2^{k_2}, p_3^{k_3} = p_1^{k_1} p_2^{k_2}$.

(2) The scalar multiplication:

$$\lambda L_1(p) = \cup\{\lambda L_1^{k_1}(p_1^{k_1})|k_1 = 1,2,\ldots,\#L_1(p)\},$$

where λ is a real number.

In order to avoid that the subscript of PLT exceeds the interval range after operations, a transformation function is introduced to normalize the extreme values by Wang and Xu [38]:

$$f(s_\alpha) = \begin{cases} s_{-\tau}\ , if\ s_\alpha < s_{-\tau} \\ s_\tau\ , if\ s_\alpha > s_\tau \\ s_\alpha\ , otherwise \end{cases}$$

When integrating the information in the form of PLTSs, the aggregation operators are often used. Pang [18] proposed some aggregation operators such as the probabilistic linguistic averaging (PLA) operator, the probabilistic linguistic weighted averaging (PLWA) operator and so on.

Definition 4 [18]. *Suppose that* $L_i(p) = \{L_i^k(p_i^k)|k = 1,2,\ldots,\#L_i(p)\}(i = 1,2,\ldots,n)$ *are n PLTSs; then, the probabilistic linguistic averaging (PLA) operator is*

$$\begin{aligned} PLA(L_1(p), L_2(p), \ldots, L_n(p)) &= \tfrac{1}{n}(L_1(p) \oplus L_2(p) \oplus \cdots \oplus L_n(p)) \\ &= \tfrac{1}{n}(\cup_{L_1^k \in L_1(p), L_2^k \in L_2(p), \ldots, L_n^k \in L_n(p)}\{p_1^k L_1^k \oplus p_2^k L_2^k \oplus \cdots \oplus p_n^k L_n^k\})' \end{aligned} \tag{4}$$

where L_i^k*and* p_i^k *represent the kth term and its probability in*$L_i(p)$*, respectively.*

Definition 5 [18]. *Suppose that* $L_i(p) = \{L_i^k(p_i^k)|k = 1,2,\ldots,\#L_i(p)\}(i = 1,2,\ldots,n)$ *are n PLTSs; then, the probabilistic linguistic weighted averaging (PLWA) operator is*

$$\begin{aligned} PLWA(L_1(p), L_2(p), \ldots, L_n(p)) &= w_1 L_1(p) \oplus w_2 L_2(p) \oplus \ldots \oplus w_n L_n(p) \\ &= \cup_{L_1^k \in L_1(p)}\{w_1 p_1^k L_1^k\} \oplus \cup_{L_2^k \in L_2(p)}\{w_2 p_2^k L_2^k\} \oplus \ldots \oplus \cup_{L_n^k \in L_n(p)}\{w_n p_n^k L_n^k\}\ ' \end{aligned} \tag{5}$$

where L_i^k *and* p_i^k *represent the kth term and its probability in* $L_i(p)$*, respectively.*

2.2. PLPR

In the process of decision making, DMs make evaluation by comparing any two alternatives, which is called preference relation (PR). For example, when a DM gives the preference degree as 0.6 for an alternative A over B, it means that the alternative A is better than B. On the basis of PLTS, Zhang et al. [19] introduced the probabilistic linguistic preference relation (PLPR), in which the preference degree is described by PLTS.

Definition 6 [19]. *Let X be the set of alternatives,* $X = \{x_i|i = 1,2,\ldots,n\}$*. Then, the probabilistic linguistic preference relation (PLPR) on X given by DMs is* $P = (L_{ij}(p))_{n\times n} \subset X \times X$, $L_{ij}(p) = \{L_{ij}^k(p_{ij}^k)|k = 1,2,\ldots,\#L_{ij}(p)\}(i,j = 1,2,\ldots,n)$, $p_{ij}^k > 0, \sum_{k=1}^{\#L_{ij}(p)} p_{ij}^k \le 1$, *where* $L_{ij}(p)$ *represents the preference degree of alternative* x_i *to alternative* x_j *by PLTS.*

The matrix form of PLPR is

$$P = \begin{bmatrix} L_{11} & L_{12} & \dots & L_{1n} \\ L_{21} & L_{22} & \dots & L_{2n} \\ \dots & \dots & \dots & \dots \\ L_{n1} & L_{n2} & \dots & L_{nn} \end{bmatrix}$$

There are some properties that PLPR satisfies:

(1) $p_{ij}^k = p_{ji}^k$, $L_{ij}^k = neg(L_{ji}^k)$, $L_{ii}(p) = \{s_0(1)\}$, $\#L_{ij} = \#L_{ji}$;

(2) $L_{ij}^k p_{ij}^k \le L_{ij}^{k+1} p_{ij}^{k+1}$ $for\ i \le j$, $L_{ji}^k p_{ji}^k \ge L_{ji}^{k+1} p_{ji}^{k+1}$ $for\ i \ge j$.

When comparing two alternatives x_i and x_j, the PLPR is denoted as L_{ij}. If $i < j$, L_{ij} is in the upper triangular matrix. Additionally, if $i > j$, L_{ij} is in the lower triangular matrix. The preferences in the upper triangular matrix are symmetrical with ones in the lower triangular matrix. For two preferences, L_{ij} and L_{ji}, $i < j$, they are symmetrical about the diagonal, so $L_{ij}^k = neg(L_{ji}^k)$. However, their probabilities are identical. The preferences on the diagonal represent the self comparison of x_i, so $L_{ii}(p) = \{s_0(1)\}$.

3. The Consistency Checking Based on Similarity Measure

Consistency checking is an essential process for preferences provided by DMs. The most commonly used method is the transitivity of preference; i.e., if alternative x_i is better than x_j, and alternative x_j is better than x_k, then alternative x_i is better than x_k. For a preference matrix, when all elements satisfy the consistency, the matrix is called a complete consistency matrix. However, in most cases, it will consume a high cost and still be impossible to achieve the complete consistency matrix. Thus, in decision making, we are inclined to reach the accepted consistency, which can be more feasible and reasonable. The accepted consistency is usually measured by a consistency index, and it presets the threshold of accepted consistency. Once the consistency degree of a preference matrix reaches the threshold of accepted consistency, this matrix will be regarded as a consistency matrix. With respect to the consistency index, Zhang [19] utilized the distance measure between preference matrix and complete consistency matrix to check whether the preference matrix satisfies the accepted consistency. Inspired by Zhang [19], we introduce the similarity measure to define the consistency index. The illustration of symbols in this paper are shown in Table 1.

Table 1. The illustration of symbols.

Symbols	Illustration	Symbols	Illustration
$L(p)$	The PLTS	M	The consistency checking matrix
L_{ij}	PLPR of alternative x_i over x_j	CI_{ij}	The consistency degree of L_{ij}
P	The preference matrix	$CI(P)$	The consistency degree of matrix P
$\overline{P}$	The modified preference matrix	ξ	The threshold of consistency index
$\overline{P}^c$	The preference matrix with complete consistency		

Definition 7 [19]. *In a PLPR* $P = (L_{ij}(p))$, *it is an additive consistency PLPR if*

$$L_{ij}(p) = \begin{cases} \frac{1}{n}\left(\oplus_{e=1}^{n}\left(L_{ie}(p) \oplus L_{ej}(p)\right)\right) & i,j = 1,2,\dots,n, i \ne j \\ \{s_0(1)\} & otherwise \end{cases} \tag{6}$$

We denote the additive consistency PLPR L_{ij} as $\overline{L}_{ij}^c$. If all preferences L_{ij} in the preference matrix P are consistency matrices, the matrix P is called the complete consistency matrix $\overline{P}^c$:

$$\overline{P}^c = \begin{bmatrix} \overline{L}_{11}^c & \overline{L}_{12}^c & \cdots & \overline{L}_{1n}^c \\ \overline{L}_{21}^c & \overline{L}_{22}^c & \cdots & \overline{L}_{2n}^c \\ \cdots & \cdots & \cdots & \cdots \\ \overline{L}_{n1}^c & \overline{L}_{n2}^c & \cdots & \overline{L}_{nn}^c \end{bmatrix}$$

In Zhang's method [19], the consistency index CI is given by measuring the distance between P and $\overline{P}^c$. First, calculate the distance between L_{ij} in P and $\overline{L}_{ij}^c$ in $\overline{P}^c$; then obtain the general distance between P and $\overline{P}^c$. Thus, the consistency index can be described as $CI(P) = d(P, \overline{P}^c)$. Inspired by Zhang's method [19], we develop a novel method to define the consistency index for PLPR based on similarity measure. Luo et al. [39] proposed the cosine similarity of PLTS, which can well be used to measure the similarity between any two PLTSs.

Definition 8 [39]. *Let $L_1(p)$ and $L_2(p)$ be two PLPTs, $S = \{s_\alpha | \alpha = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$, L_1^k, and $L_2^k \in S$; the cosine similarity is as follows:*

$$\cos(L_1(p), L_2(p)) = \frac{\sum_{k=1}^{\#L(p)} \left(\tau^{(L_1^k p_1^k)/\tau} \times \tau^{(L_2^k p_2^k)/\tau}\right)}{\sqrt{\sum_{k=1}^{\#L(p)} \left(\tau^{(L_1^k p_1^k)/\tau}\right)^2} \times \sqrt{\sum_{k=1}^{\#L(p)} \left(\tau^{(L_2^k p_2^k)/\tau}\right)^2}}, \tag{7}$$

where L_1^k and p_1^k represent the kth term and its probability in $L_1(p)$. $\#L(p)$ represents the number of terms in $L(p)$.

Proposition 1. *In Equation (7), the cosine value is always equal to one when it only contains one term in $L_1(p)$ and $L_2(p)$. The proof is given below:*

Proof. In Equation (7), when there is only one element in $L_1(p)$ and $L_2(p)$, respectively, namely $\#L(p) = 1$, then the molecule is reduced to:

$$\sum_{k=1}^{1} \left(\tau^{(L_1^1 p_1^1)/\tau} \times \tau^{(L_2^1 p_2^1)/\tau}\right) = \tau^{(L_1^1 p_1^1 + L_2^1 p_2^1)/\tau}$$

For the denominator,

$$\sum_{k=1}^{1} \left(\tau^{(L_1^1 p_1^1)/\tau}\right)^2 = \left(\tau^{(L_1^1 p_1^1)/\tau}\right)^2, \ \sum_{k=1}^{1} \left(\tau^{(L_2^1 p_2^1)/\tau}\right)^2 = \left(\tau^{(L_2^1 p_2^1)/\tau}\right)^2$$

Because $\tau^{(L_1^k p_1^k)/\tau} > 0$ and $\tau^{(L_2^k p_2^k)/\tau} > 0$, then

$$\sqrt{\left(\tau^{(L_1^k p_1^k)/\tau}\right)^2} = \tau^{(L_1^k p_1^k)/\tau}, \ \sqrt{\left(\tau^{(L_2^k p_2^k)/\tau}\right)^2} = \tau^{(L_2^k p_2^k)/\tau}$$

Thus, when $\#L(p) = 1$, then

$$\begin{aligned} \cos(L_1(p), L_2(p)) &= \frac{\sum_{k=1}^{\#L(p)} \left(\tau^{(L_1^k p_1^k)/\tau} \times \tau^{(L_2^k p_2^k)/\tau}\right)}{\sqrt{\sum_{k=1}^{\#L(p)} \left(\tau^{(L_1^k p_1^k)/\tau}\right)^2} \times \sqrt{\sum_{k=1}^{\#L(p)} \left(\tau^{(L_2^k p_2^k)/\tau}\right)^2}} \\ &= \frac{\sum_{k=1}^{1} \left(\tau^{(L_1^1 p_1^1)/\tau} \times \tau^{(L_2^1 p_2^1)/\tau}\right)}{\sqrt{\sum_{k=1}^{1} \left(\tau^{(L_1^1 p_1^1)/\tau}\right)^2} \times \sqrt{\sum_{k=1}^{1} \left(\tau^{(L_2^1 p_2^1)/\tau}\right)^2}} \\ &= \frac{\tau^{(L_1^1 p_1^1 + L_2^1 p_2^1)/\tau}}{\tau^{(L_1^1 p_1^1)/\tau} \times \tau^{(L_2^1 p_2^1)/\tau}} = \frac{\tau^{(L_1^1 p_1^1 + L_2^1 p_2^1)/\tau}}{\tau^{(L_1^1 p_1^1 + L_2^1 p_2^1)/\tau}} \\ &= 1 \end{aligned}$$

The proof ends. □

If there is only one term in any two PLTSs, it is unreasonable to use the method proposed by Luo [39] to measure their similarities. Therefore, the cosine similarity in Equation (7) needs to be improved. When $\#L(p) = 1$, we adopt $\cos(L_1(p), L_2(p)) = \frac{2\tau - \left|L_1^k p_1^k - L_2^k p_2^k\right|}{2\tau}$. Then, the improved cosine similarity is

$$\cos(L_1(p), L_2(p)) = \begin{cases} \frac{2\tau - |L_1 p_1 - L_2 p_2|}{2\tau} & ,\#L(p) = 1 \\ \frac{\sum_{k=1}^{\#L(p)} \left(\tau^{(L_1^k p_1^k)/\tau} \times \tau^{(L_2^k p_2^k)/\tau}\right)}{\sqrt{\sum_{k=1}^{\#L(p)} \left(\tau^{(L_1^k p_1^k)/\tau}\right)^2} \times \sqrt{\sum_{k=1}^{\#L(p)} \left(\tau^{(L_2^k p_2^k)/\tau}\right)^2}} & ,\#L(p) \geq 2 \end{cases} \tag{8}$$

Definition 9. *With the aid of cosine similarity, the consistency index CI_{ij} of preference $L_{ij}(p)$ in P is*

$$CI_{ij} = CI(L_{ij}(p), \overline{L}_{ij}^c(p)) = \cos(L_{ij}(p), \overline{L}_{ij}^c(p)), \tag{9}$$

where $L_{ij}(p)$ represents the preference for x_i over x_j by PLTS in the preference matrix P, and $\overline{L}_{ij}^c(p)$ represents the preference by PLTS in the complete consistency matrix $\overline{P}^c$.

Evidently, the bigger the cosine value between $L_{ij}(p)$ and $\overline{L}_{ij}^c(p)$, the higher their similarity. Hence, the bigger the cosine value, the higher the consistency degree of preference $L_{ij}(p)$. In the process of consistency checking, the consistency degrees CI_{ij} of all preferences in P firstly are obtained. Subsequently, the consistency checking matrix containing all CI_{ij} is given by

$$M = \begin{bmatrix} CI_{11} & CI_{12} & \dots & CI_{1n} \\ CI_{21} & CI_{22} & \dots & CI_{2n} \\ \dots & \dots & \dots & \dots \\ CI_{n1} & CI_{n2} & \dots & CI_{nn} \end{bmatrix}$$

Through the consistency checking matrix M, we can intuitively find out the preferences that do not reach the level of accepted consistency. Then, the preferences with unaccepted consistency will be selected to adjust in order to satisfy the overall consistency of preference matrix P. To measure the consistency degree of preference matrix P, a novel consistency index is developed as follows.

Definition 10. *Let $P = (L_{ij}(p))_{n \times n} \subset X \times X$ be a PLPR matrix, and its complete consistency matrix is $\overline{P}^c$. Then the consistency degree of P is denoted*

$$CI(P) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \cos(L_{ij}(p), \overline{L}_{ij}^c(p)), \tag{10}$$

where $L_{ij}(p)$ is the preference by PLTS in P, and $\overline{L}_{ij}^c(p)$ represents the preference by PLTS in $\overline{P}^c$.

Remark 1. *From Equations (9) and (10), we see that there exists a close relation between CI_{ij} and $CI(P)$. Namely, CI_{ij} denotes the consistency degree of a single preference $L_{ij}(p)$, which is the element of preference matrix P. $CI(P)$ is the mean of all cosine similarities of $L_{ij}(p)$ and $\overline{L}_{ij}^c(p)$. Although the purpose of consistency checking is to measure whether $CI(P)$ satisfies the threshold of accepted consistency ξ, the CI_{ij} obtained by Equation (9) is also important because we can obtain the consistency matrix M to identify the preferences with unaccepted consistency. For the value of ξ, we preset it in advance in accordance with the actual situation of the decision-making problem.*

To complete the process of consistency checking, the consistency checking Algorithm 1 is proposed as follows:

Algorithm 1. Consistency Checking Algorithm

Input: The original preference matrix P and the threshold of accepted consistency ξ.
Output: The consistency checking matrix M and consistency degree of P, namely, $CI(P)$.
Step 1. Obtain the normalized preference matrix P according to Equation (1).
Step 2. By using Equation (6), construct the complete consistency matrix $\overline{P}^c$.
Step 3. According to Equations (8) and (9), calculate CI_{ij} of preferences in preference matrix P and obtain the consistency checking matrix M.
Step 4. According to Equation (10), compute the consistency degree of P, namely $CI(P)$.
Step 5. Check whether $CI(P) \geq \xi$. If $CI(P) \geq \xi$, then preference matrix P reaches the level of accepted consistency. If $CI(P) < \xi$, then the P should be adjusted.
Step 6. End.

For the convenience of understanding, a brief illustration about the consistency checking algorithm is conducted. Originally, the PLTSs provided DMs may be incomplete in the preference matrices, so the purpose of step one is to achieve all the complete PLTSs. In step two, we construct the complete consistency matrix $\overline{P}^c$, which is mainly regarded as a benchmark for achieving the consistency degree of preference matrix P. For step three and step four, the two steps are used to compute the key consistency indicators and attain the consistency degree of P. In step five, we compare the consistency degree of P with the accepted consistency threshold ξ. Thus, we can check whether a preference matrix P reaches the accepted consistency level in terms of Algorithm 1.

Accordingly, in the following, an example is given to illustrate the process of consistency checking.

Example 1. *Suppose that $P = (L_{ij}(p))$ is the PLPR matrix. The LTS is $S = \{s_\alpha | \alpha = -2, -1, 0, 1, 2\}$. The threshold of accepted consistency is $\xi = 0.95$.*

$$P = \begin{bmatrix} \{s_0(1)\} & \{s_{-1}(0.2), s_1(0.3)\} & \{s_2(1)\} \\ \{s_1(0.2), s_{-1}(0.3)\} & \{s_0(1)\} & \{s_0(0.1), s_1(0.4)\} \\ \{s_{-2}(1)\} & \{s_0(0.1), s_{-1}(0.4)\} & \{s_0(1)\} \end{bmatrix}$$

The normalized P is

$$\dot{P} = \begin{bmatrix} \{s_0(1)\} & \{s_{-1}(0.4), s_1(0.6)\} & \{s_2(1)\} \\ \{s_1(0.4), s_{-1}(0.6)\} & \{s_0(1)\} & \{s_0(0.2), s_1(0.8)\} \\ \{s_{-2}(1)\} & \{s_0(0.2), s_{-1}(0.8)\} & \{s_0(1)\} \end{bmatrix}$$

By using Equation (6), we obtain the complete consistency matrix $\overline{P}^c$:

$$\overline{P}^c = \begin{bmatrix} \{s_0(1)\} & \begin{matrix} \{s_{-0.33}(0.128), s_0(0.032), \\ s_{0.33}(0.384), s_{0.66}(0.456)\} \end{matrix} & \{s_{0.66}(1)\} \\ \begin{matrix} \{s_{0.33}(0.128), s_0(0.032), \\ s_{-0.33}(0.384), s_{-0.66}(0.456)\} \end{matrix} & \{s_0(1)\} & \{s_{0.33}(0.024), s_{0.66}(0.976)\} \\ \{s_{-0.66}(1)\} & \{s_{-0.33}(0.024), s_{-0.66}(0.976)\} & \{s_0(1)\} \end{bmatrix}$$

According to Equations (7) and (8), the consistency checking matrix M can be achieved:

$$M = \begin{bmatrix} 1 & 0.9905 & 0.6650 \\ 0.9905 & 1 & 0.9996 \\ 0.6650 & 0.9996 & 1 \end{bmatrix}$$

According to Equation (10), we obtain the consistency degree of preference matrix P:$CI(P) = 0.9234$.

Because $CI(P) < \xi$, the preference matrix P does not reach the level of accepted consistency.

4. The Consistency Improvement and the Decision-Making Process

In Section 3, we explore the process of consistency checking, and the original preference matrix P needs to be adjusted if its consistency level is unaccepted. Subsequently, in this section, a novel method of consistency improvement is developed to help the original preference matrix P to reach the level of accepted consistency. Besides, to solve actual problems by the proposed method, we develop the process of decision making based on consistency analysis.

4.1. Consistency Improvement

Above of all, when adjusting the preference matrix P, an issue that can not be ignored is how to avoid the loss of original preference information as much as possible. If preference information is changed on a large scale, it will distort actual aspiration of DMs, which will lead to unreasonable decision results. Zhang [19] adopted an adjustment parameter to control the magnitude of initial information modification, but the determination of this parameter is random, which will influence the number of iterations and lead to greater adjustment costs.

To retain the original preference information to the greatest extent, we constructed an optimization model to obtain the least change of preferences after consistency improvement, which can be realized by controlling the distance between the modified matrix $\overline{P}$ and the original matrix P. On measuring the distance of PLTS, the Hamming distance is an appropriate method.

Definition 11 [19]. *Let $L_1(p)$ and $L_2(p)$ be two PLPTs, $S = \{s_\alpha | \alpha = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$, L_1^k, and $L_2^k \in S$; the distance between $L_1(p)$ and $L_2(p)$ is given by*

$$d(L_1(p), L_2(p)) = \sum_{k=1}^{\#L(p)} \left(p_1^k p_2^k (|I(L_1^k) - I(L_2^k)|/2\tau)\right), \tag{11}$$

where $I(L_1^k)$ represents the subscript of the kth term in $L_1(p)$. p_1^k is the probability of the kth term in $L_1(p)$.

According to Equation (11), the distance between any two matrices can be obtained. Given two matrices P_1 and P_2, $L_{ij}^1(p)$ and $L_{ij}^2(p)$ are the elements in the matrices P_1 and P_2, respectively. Then, the distance between the matrix P_1 and P_2 is given by

$$d(P_1, P_2) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} d(L_{ij}^1(p), L_{ij}^2(p)) \tag{12}$$

In a consistency analysis or group consensus analysis, the minimum adjustment is an important idea, which has two meanings. One is that it consumes the least costs to reach the accepted level of consistency or group consensus. The costs contain time, the number of adjusted elements or experts and so on. The other one indicates that the change of original evaluation information is smallest after adjustment. In the light of the consistency checking algorithm in Section 3, we can obtain the consistency checking matrix M, which is used to identify whether the preferences satisfy the accepted consistency. In order to reduce the number and cost of adjustment elements, we only choose the preferences with unaccepted consistency to revise instead of all preferences in the original matrix P. Based on the idea of minimum adjustment, an optimization model is constructed as follows:

$$Min\ d\left(P, \overline{P}\right)$$

$$s.t.\begin{cases} d(P,\overline{P}) = \frac{1}{n^2}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} d(L_{ij}(p),\overline{L}_{ij}(p)) & (a) \\ \overline{L}_{ij} = \alpha L_{ij} + (1-\alpha)\overline{L}_{ij}^{c} & (b) \\ 0 < \alpha < 1 & (c) \\ CI(\overline{P}) \geq \xi & (d) \\ I(\overline{P}) = \cos(\overline{P},\overline{P}^{c}) & (e) \\ 0 < CI(\overline{P}) \leq 1 & (f) \\ 0 < \xi < 1 & (g) \end{cases} \quad (13)$$

In model (13), the objective function ensures that the distance between the original matrix P and the adjusted matrix $\overline{P}$ is the least. For constrains (a)~(c), we can obtain the distance between P and $\overline{P}$. The α in constrain (b) is an adjustment parameter to control the adjustment proportion of preferences, and the interval of α is (0,1), stated in constrain (c). In Zhang's research [19], the adjustment parameter is set randomly and blindly, while we can obtain the optimal value of α according to the optimization model. The constrain (d) guarantees that the modified matrix $\overline{P}$ satisfies the level of accepted consistency, and ξ is the threshold of accepted consistency. For constrains (e) and (f), the consistency degree of $\overline{P}$ can be achieved, and its range is (0,1). The final constrain (g) limits the value range of ξ.

In model (13), the decision variable is the adjustment parameter α. According to the constrain (c), (d) and (f), the ranges of α and $CI(\overline{P})$ are $0 < \alpha < 1$, $\xi \leq CI(\overline{P}) \leq 1$. Evidently, when α and $CI(\overline{P})$ satisfy the conditions, there exists a feasible solution that conforms to the objective function. By using the software LINGO, we can solve the model. After solving the optimization model, the α can be determined to adjust the preferences with unaccepted consistency. At the same time, the optimization model remains the original preference information of experts to the greatest extent. The process of consistency improvement is summarized in the consistency improvement Algorithm 2 below.

Algorithm 2. Consistency Improvement Algorithm

Input: The matrix with unaccepted consistency matrix P and the consistency checking matrix M.
Output: The matrix with accepted consistency $\overline{P}$.
Step 1. According to the consistency checking matrix M, find out all the preferences with unaccepted consistency in matrix P.
Step 2. By using Equations (11) and (12) and model (13), obtain the value of adjustment parame-ter α.
Step 3. By using the parameter α, revise the preferences with unaccepted consistency. The formula is $\overline{L}_{ij} = \alpha L_{ij} + (1-\alpha)\overline{L}_{ij}^{c}$.
Step 4. According to adjusted preferences in step 3, obtain the preference matrix with accepted consistency $\overline{P}$.
Step 5. End.

In the consistency improvement algorithm, the key input is the unaccepted consistency matrix P, which is selected by the consistency checking algorithm. The purpose of Algorithm 2 is to improve the consistency level of the matrix P. There are mainly four steps in the above algorithm. For step one, it is used to select the elements with unaccepted consistency in matrix P, which will be regarded as the adjusted objects. In step two and step three, the two steps are the main process to solve the optimization model and adjust the unaccepted preferences in terms of α. In step four, we can attain the revised matrix $\overline{P}$ with an accepted consistency.

To better understand the algorithm, we give an example below.

Example 2. *In Example 1, owing to $CI(P) < \xi$, the preferences in matrix P need to be adjusted. In light of the consistency checking matrix M, we see that the preferences L_{13} and L_{31} do not satisfy the consistency level. Besides, because L_{13} and L_{31} are symmetrical about the diagonal, their consistencies are identical, namely $CI_{13} = CI_{31}$. Hence, we only need to adjust one preference, L_{13} or L_{31}. By using the model (13), the optimal value of parameter α is 0.6421.*

According to the adjustment parameter α, the L_{13} should be adjusted as $\overline{L}_{13} = 0.6421L_{13} + 0.3579\overline{L}^c_{13}$, $\overline{L}_{13} = \{s_{1.52}(1)\}$. In light of preference symmetry, $\overline{L}_{31} = \{s_{-1.52}(1)\}$. Thus, the adjusted preference matrix $\overline{P}$ is

$$\overline{P} = \begin{bmatrix} \{s_0(1)\} & \{s_{-1}(0.4), s_1(0.6)\} & \{s_{1.52}(1)\} \\ \{s_1(0.4), s_{-1}(0.6)\} & \{s_0(1)\} & \{s_0(0.2), s_1(0.8)\} \\ \{s_{-1.52}(1)\} & \{s_0(0.2), s_{-1}(0.8)\} & \{s_0(1)\} \end{bmatrix}$$

Again, we calculate the consistency degree of $\overline{P}$, and the result is $CI(\overline{P}) = 0.95$. Because $CI(\overline{P}) \geq \xi$, the modified preference matrix $\overline{P}$ satisfies the level of accepted consistency.

4.2. The Process of Decision Making Based on Consistency Analysis

In this subsection, the consistency checking and consistency improvement will be applied in the process of decision making. Given that the decision-making problem becomes more and more complex, DMs are hard-pressed to give clear and logical judgement. Thus, it is essential to have a consistency analysis after experts give their evaluation information by PLPR.

Let X be the set of alternatives, $X = \{x_i | i = 1, 2, \ldots, n\}$. The linguistic term set is S, and $S = \{s_\alpha | \alpha = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$, where τ is a positive integer. L_{ij} is represented as the PLPR of alternative x_i over alternative x_j, $L_{ij}(p) = \{L^k_{ij}(p^k_{ij}) | k = 1, 2, \ldots, \#L_{ij}(p)\}$. Denote P as the preference matrix provided by DMs. The purpose of decision making is to select the optimal scheme among several alternatives. It contains totally four main steps in the process of decision making based on the consistency analysis. To intuitively understand the overall decision process, the decision flowchart is shown in Figure 1.

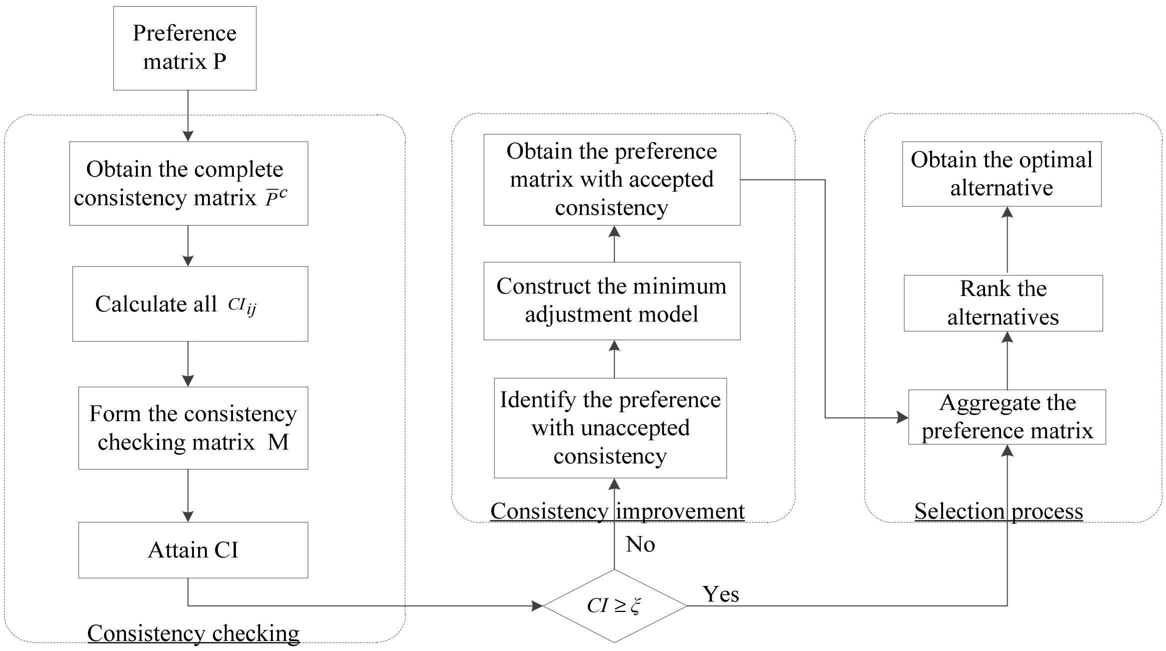

Figure 1. The flowchart of decision making.

The main steps in the process of decision making based on consistency analysis are as follows:

Step 1. Collect the evaluation information by preference matrix P. Above all, experts describe their preferences over alternatives by pairwise comparison in the form of PLPR. Owing to the existence of inconsistency preferences, conduct a consistency analysis in the next step.

Step 2. According to the consistency checking algorithm (in Section 3), we find out the preference matrix with an unaccepted consistency.

Step 3. By using the consistency improvement algorithm (in Section 4.1), we improve the consistency level of the matrix P with an unaccepted consistency.

Step 4. Aggregate the preferences of every alternative and rank the alternatives. By Equations (4) and (5), utilize the PLA or PLWA operator to obtain the synthetic preference of the alternative. According to the Equations (2) and (3), obtain the synthetic preference of alternatives and rank the alternatives. Finally, the optimal alternative is selected.

5. Numerical Example

In the above, we propose a decision method based on consistency analysis, which is helpful in checking whether the preferences provided by DMs are self-contradictory. In this section, the method will be applied to the case of talent recruitment. As is known, talent is a very important resource for enterprises that will affect the development of enterprises. Thus, when conducting personnel recruitment, the division manager or examiner is inclined to strictly investigate candidates from different aspects, such as education, ability, quality, age and so on. Owing to the limited time and the lack of sufficient information, the examiner tends to make a pairwise comparison between any two candidates. Suppose that there are four candidates who will apply for an Internet company, and the examiner evaluates them by using PLPR. The purpose of examiner is to select the optimal person from the four candidates. The four candidates are denoted as the set of alternatives $X = \{x_1, x_2, x_3, x_4\}$. The linguistic term set is $S = \{s_{-2}, s_{-1}, s_0, s_1, s_2\}$. The terms in S represent that "s_{-2} = terrible, s_{-1} = bad, s_0 = indifferent, s_1 = good, s_2 = perfect". After interviewing candidates, the examiner gives the preference matrix P in light of their performance.

$$P = \begin{bmatrix} \{s_0(1)\} & \{s_{-1}(0.3), s_1(0.6)\} & \{s_2(1)\} & \{s_0(0.5), s_1(0.5)\} \\ \{s_1(0.3), s_{-1}(0.6)\} & \{s_0(1)\} & \{s_1(1)\} & \{s_1(0.3), s_2(0.2)\} \\ \{s_{-2}(1)\} & \{s_{-1}(1)\} & \{s_0(1)\} & \{s_{-1}(1)\} \\ \{s_0(0.5), s_{-1}(0.5)\} & \{s_{-1}(0.3), s_{-2}(0.2)\} & \{s_1(1)\} & \{s_0(1)\} \end{bmatrix}$$

According to the decision-making process proposed in Section 4.2, we conduct the work of talent recruitment. The threshold of accepted consistency is preset to $\xi = 0.93$.

Step 1. The evaluation information has been presented in the preference matrix P above.

Step 2. According to the consistency checking algorithm (in Section 3), we have a consistency analysis for the preference matrix P.

According to Equation (1), obtain the normalized P:

$$\dot{P} = \begin{bmatrix} \{s_0(1)\} & \{s_{-1}(0.33), s_1(0.67)\} & \{s_2(1)\} & \{s_0(0.5), s_1(0.5)\} \\ \{s_1(0.33), s_{-1}(0.67)\} & \{s_0(1)\} & \{s_1(1)\} & \{s_1(0.6), s_2(0.4)\} \\ \{s_{-2}(1)\} & \{s_{-1}(1)\} & \{s_0(1)\} & \{s_{-1}(1)\} \\ \{s_0(0.5), s_{-1}(0.5)\} & \{s_{-1}(0.6), s_{-2}(0.4)\} & \{s_1(1)\} & \{s_0(1)\} \end{bmatrix}$$

By using Equation (6), construct the complete consistency matrix $\overline{P}^c$.

$$\overline{P}^c = \begin{bmatrix} \{s_0(1)\} & \begin{matrix}\{s_{-0.5}(0.0762), s_{-0.25}(0.1211), s_0(0.3109), \\ s_{0.25}(0.3572), s_{0.5}(0.1347)\}\end{matrix} & \{s_{0.5}(1)\} & \{s_{0.25}(0 .0495), s_{0.5}(0 .9505)\} \\ \begin{matrix}\{s_{0.5}(0.0762), s_{0.25}(0.1211), s_0(0.3109), \\ s_{-0.25}(0.3572), s_{-0.5}(0.1347)\}\end{matrix} & \{s_0(1)\} & \{s_{0.5}(1)\} & \{s_{0.25}(0.0594), s_{0.5}(0.9406)\} \\ \{s_{-0.5}(1)\} & \{s_{-0.5}(1)\} & \{s_0(1)\} & s_{-0.5}(1) \\ \{s_{-0.25}(0 .0495), s_{-0.5}(0 .9505)\} & \{s_{-0.25}(0.0594), s_{-0.5}(0.9406)\} & s_2(1) & \{s_0(1)\} \end{bmatrix}$$

According to Equations (8) and (9), calculate CI_{ij} of preferences in the P and obtain the consistency checking matrix M.

$$M = \begin{bmatrix} 1 & 0.9924 & 0.6250 & 1 \\ 0.9924 & 1 & 0.8750 & 0.9991 \\ 0.6250 & 0.8750 & 1 & 0.8750 \\ 1 & 0.9991 & 0.8750 & 1 \end{bmatrix}$$

According to Equation (10), we can obtain the consistency degree of preference matrix P: $CI(P) = 0.9208$. Because $CI(P) < \xi$, the original preferences need to be adjusted.

Step 3. By using the consistency improvement algorithm (in Section 4.1), conduct the process of consistency improvement. The most important thing is to determine the optimal value of adjustment parameter α. In light of model (13), we construct the objective function based on the minimum distance between P and $\overline{P}$, and it needs to satisfy all constrains in which the constrain (d) ensures that the adjusted matrix $\overline{P}$ is in accord with the level of accepted consistency.

According to the consistency checking matrix M, the preferences with unaccepted consistency are L_{13}, L_{23}, L_{31}, L_{32}, L_{34} and L_{43}. On account of the preference symmetry, we only need to obtain the adjusted preferences, L_{13}, L_{23} and L_{34}; then, L_{31}, L_{32} and L_{43} will be also known. By solving model (13), the adjustment parameter is $\alpha = 0.8040$. The adjusted matrix $\overline{P}$ can be carried out.

$$\overline{P} = \begin{bmatrix} \{s_0(1)\} & \{s_{-1}(0.33), s_1(0.67)\} & \{s_{1.71}(1)\} & \{s_0(0.5), s_1(0.5)\} \\ \{s_1(0.33), s_{-1}(0.67)\} & \{s_0(1)\} & \{s_{0.9}(1)\} & \{s_1(0.6), s_2(0.4)\} \\ \{s_{-1.71}(1)\} & \{s_{-0.9}(1)\} & \{s_0(1)\} & \{s_{0.1}(1)\} \\ \{s_0(0.5), s_{-1}(0.5)\} & \{s_{-1}(0.6), s_{-2}(0.4)\} & \{s_{-0.1}(1)\} & \{s_0(1)\} \end{bmatrix}$$

Calculate the consistency degree of $\overline{P}$; $CI(\overline{P}) = 0.93 \geq \xi$, which satisfies the accepted consistency.

Step 4. Aggregate the preferences of every alternative and rank the alternatives.

By Equation (4), utilize the PLA operator to obtain the synthetic preference of the alternative.

$$L_1 = \{s_{0.18}(0.165), s_{0.43}(0.165), s_{0.5}(0.670)\}$$

$$L_2 = \{s_{0.23}(0.402), s_{0.48}(0.268), s_{0.5}(0.330)\}$$

$$L_3 = \{S_{0.5}(l)\}$$

$$L_4 = \{s_{-0.28}(0.3), s_{-0.5}(0.7)\}$$

By Equation (2), the score functions of preferences are:

$$E(L_1) = s_{0.96},\ E(L_2) = s_{0.39},\ E(L_3) = s_{0.5},\ E(L_4) = s_{-0.43}$$

Thus, $L_1 > L_3 > L_2 > L_4$.

The ranking of alternatives is as follows:$x_1 \succ x_3 \succ x_2 \succ x_4$. Namely, candidate one is the best one.

6. Comparative Analysis and Discussion

In Section 5, the case about talent recruitment is given to illustrate the process of decision making based on consistency analysis. Subsequently, to analyse the effectiveness and superiority of this method, we have a comparison with Zhang's approach [19] and conduct a discussion.

6.1. Comparative Analysis

In this subsection, we have a comparison analysis with Zhang's method [19]. First of all, we review broadly the idea of Zhang's method with respect to the consistency checking and improvement. The additive consistency of PLPR (see Definition 5) is proposed to obtain the consistency preference matrix $\overline{P}^c$. To check the consistency, Zhang et al. [19]

developed a new consistency index based on distance measure between the original matrix P and the consistency matrix $\overline{P}^c$. The distance measure is given by

$$CI(P) = d(P, \overline{P}^c) = \frac{1}{T}\sqrt{\frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{\#L(p)}\left(p_{ij}^{(k)}\left(L_{ij}^{(k)} - \overline{L}_{ij}^{c(k)}\right)\right)\right)^2}. \tag{14}$$

Let $\varepsilon_{ij} = \sum_{k=1}^{\#L(p)}\left(p_{ij}^{(k)}\left(L_{ij}^{(k)} - \overline{L}_{ij}^{c(k)}\right)\right)$, where $\varepsilon_{ij}(i < j)$ is independent normally distributed with the mean 0 and the standard deviation σ, $\varepsilon_{ij} \sim N(0, \sigma^2)$. Because $\varepsilon_{ij} \sim N(0, \sigma^2)$, $\frac{n(n-1)}{2}\left(T \times \frac{1}{\sigma} \times CI(P)\right)^2$ is a chi-square distribution with $\frac{n(n-1)}{2}$ degrees of freedom, namely, $\frac{n(n-1)}{2}\left(T \times \frac{1}{\sigma} \times CI(P)\right)^2 \sim \chi^2\left(\frac{n(n-1)}{2}\right)$ [22]. According to a one-sided right-tailed test, at the significance level μ, the critical value of χ^2 is λ_μ; then, the threshold of accepted consistency ξ can be appropriately determined as follows:

$$\xi = \frac{\sigma}{T}\sqrt{\frac{2}{n(n-1)}\lambda_\mu} \tag{15}$$

When $\mu = 0.1$, $\sigma = 2$, $n = 4$ and $T = 5$, the value of ξ is 0.2424 [19]. Given that the consistency index is given by distance measure, the lower the $CI(P)$, the higher the consistency level of preference matrix P. Thus, if $CI(P) < \xi$, the preference matrix P satisfies the accepted consistency.

Then, by using Zhang's method [19], we solve the decision problem in Section 5. According to Equation (14), the consistency level of original matrix P is $CI(P) = 0.3602$. Because $CI(P) > \xi$, the preference matrix P needs to be adjusted through $\overline{L}_{ij} = \alpha L_{ij} + (1-\alpha)\overline{L}_{ij}^c$. In reference [18], α is set to 0.05. Thus, we can obtain the modified matrix $\overline{P}^{(1)}$ (where (1) represents the number of iterations):

$$\overline{P}^{(1)} = \begin{bmatrix} \{s_0(1)\} & \begin{gathered}\{s_{-0.97}(0.1677), s_{-0.93}(0.1624),\\ s_{0.93}(0.1322), s_{0.96}(0.5378)\}\end{gathered} & \{s_{1.93}(1)\} & \begin{gathered}\{s_{0.01}(0.0248), s_{0.03}(0.4753),\\ s_{0.96}(0.0248), s_{0.98}(0.4753)\}\end{gathered} \\ \begin{gathered}\{s_{0.97}(0.1677), s_{0.93}(0.1624),\\ s_{-0.93}(0.1322), s_{-0.96}(0.5378)\}\end{gathered} & \{s_0(1)\} & \{s_{0.98}(1)\} & \begin{gathered}\{s_{0.96}(0.0356), s_{0.98}(0.5644),\\ s_{1.91}(0.0238), s_{1.93}(0.3762)\}\end{gathered} \\ \{s_{-1.93}(1)\} & \{s_{-0.98}(1)\} & \{s_0(1)\} & \{s_{-0.98}(1)\} \\ \begin{gathered}\{s_{-0.01}(0.0248), s_{-0.03}(0.4753),\\ s_{-0.96}(0.0248), s_{-0.98}(0.4753)\}\end{gathered} & \begin{gathered}\{s_{-0.96}(0.0356), s_{-0.98}(0.5644),\\ s_{-1.91}(0.0238), s_{-1.93}(0.3762)\}\end{gathered} & \{s_{0.98}(1)\} & \{s_0(1)\} \end{bmatrix}$$

By using Equation (14), the consistency level is $CI(P) = 0.2860$. Because $CI(P) > \xi$, the preference matrix again needs to adjusted. The modified matrix $\overline{P}^{(2)}$ is

$$\overline{P}^{(2)} = \begin{bmatrix} \{s_0(1)\} & \begin{gathered}\{s_{-0.93}(0.0852), s_{-0.89}(0.245),\\ s_{0.88}(0.2382), s_{0.93}(0.4317)\}\end{gathered} & \{s_{1.86}(1)\} & \begin{gathered}\{s_{0.03}(0.0248), s_{0.05}(0.4753),\\ s_{0.93}(0.0248), s_{0.95}(0.4753)\}\end{gathered} \\ \begin{gathered}\{s_{0.93}(0.0852), s_{0.89}(0.245),\\ s_{-0.88}(0.2382), s_{-0.93}(0.4317)\}\end{gathered} & \{s_0(1)\} & \{s_{0.956}(1)\} & \begin{gathered}\{s_{0.93}(0.0356), s_{0.95}(0.5644),\\ s_{1.83}(0.0238), s_{1.85}(0.3762)\}\end{gathered} \\ \{s_{-1.86}(1)\} & \{s_{-0.956}(1)\} & \{s_0(1)\} & \{s_{-0.96}(1)\} \\ \begin{gathered}\{s_{-0.03}(0.0248), s_{-0.05}(0.4753),\\ s_{-0.93}(0.0248), s_{-0.95}(0.4753)\}\end{gathered} & \begin{gathered}\{s_{-0.93}(0.0356), s_{-0.95}(0.5644),\\ s_{-1.83}(0.0238), s_{-1.85}(0.3762)\}\end{gathered} & \{s_{0.96}(1)\} & \{s_0(1)\} \end{bmatrix}$$

Calculate the consistency degree: $CI(P) = 0.2014$, $CI(P) < \xi$. The modified matrix satisfies the accepted consistency. By Equation (5), the synthetic preference of alternative can be obtained:

$$\begin{aligned} L_1 &= \{s_{0.23}(0.0852), s_{0.24}(0.2450), s_{0.5}(0.6699)\} \\ L_2 &= \{s_{0.24}(0.2591), s_{0.25}(0.1429), s_{0.48}(0.2680), s_{0.5}(0.3302)\} \\ L_3 &= \{s_{0.94}(1)\} \\ L_4 &= \{s_{0.5}(1)\} \end{aligned}$$

The score functions of preferences are:

$$E(L_1) = s_{0.41},\ E(L_2) = s_{0.39},\ E(L_3) = s_{0.94},\ E(L_4) = s_{0.5}$$

Thus, $L_3 > L_4 > L_1 > L_2$.

The ranking of alternatives is $x_3 \succ x_4 \succ x_1 \succ x_2$. The candidate three is the optimal one.

6.2. Discussion

The decision result obtained by using Zhang's method [19] is different from ours. However, this is normal because the approaches of consistency analysis are distinct, which is shown in Table 2. The main differences are as follows:

Table 2. The comparison with other method.

The Method	Consistency Checking	Consistency Improvement	The Number of Modified Preferences	Whether it Considers the Cost of Adjustment	The Threshold of Accepted Consistency	The Number of Iteration
In Zhang's research [19]	Distance measure	Using the adjustment parameter	All preferences	No	The fix value by Probability distribution	2
In this paper	Similarity measure	The optimization model based on minimum adjustment	The preferences with unaccepted consistency	Yes	Preset in advance according to decision problem	1

(1) The methods of consistency checking are different. In Zhang's method [19], it adopts the distance measure to define the consistency index, which can obtain the consistency degree of preference matrix P. In terms of Zhang's method [19], if the consistency degree of preference matrix P does not reach the level of accepted consistency, all preferences need to be modified, which will generate significant adjustment costs. In this paper, we use a similarity measure to check the consistency level and calculate the consistency degree of a single preference CI_{ij} and consistency degree of preference matrix $CI(P)$. The consistency checking matrix M consisting of consistency degrees of a single preference CI_{ij} can be used to find out which preference is unaccepted. According to the consistency matrix, we only need to modify the preferences with an unaccepted consistency. Thus, it consumes less adjustment cost to improve the consistency by using our method, which is superior.

(2) The methods of consistency improvement are distinct. It uses the iteration algorithm to adjust the preference matrix in Zhang's approach [19] and the number of iterations is two in the decision process of Section 6.1, while the number of iteration is one by using our method. Additionally, the adjustment parameter α is set randomly and blindly. Different from Zhang's method [19], we constructed the minimum adjustment model to solve the optimal value of α, which ensures that the modified matrix satisfies the accepted consistency level, and the distance between the original preference matrix P and the modified matrix $\overline{P}$ is the least. Thus, it can retain the original preference information provided by experts, so our method is more reasonable and superior.

(3) In Zhang's research [19], the threshold of accepted consistency ξ is obtained by probability distribution, namely Equation (15). Thus, the value of ξ is stable if the number of alternatives and T are assured. Nevertheless, we preset the value of ξ in advance according to the situation of actual decision making, which is more flexible because the value of ξ is influenced by many subjective and objective factors. When solving the decision problem in Section 5, the value of ξ is preset to 0.93 while the fix value of ξ is 0.2424 in Section 6.1. Hence, in this research, it is different to determine the threshold of accepted consistency from Zhang's method [19].

7. Conclusions

In this paper, to solve the problem of consistency checking and improvement, a new consistency index is proposed based on a similarity measure between the original preference matrix and complete preference matrix. On the basis of consistency index, we calculate the consistency degree of a single preference and the preference matrix, constructing a consistency checking matrix to identify which preference is not able to satisfy the accepted consistency. Thus, an algorithm is designed to realize the consistency checking. Subsequently, based on the idea of minimum adjustment, we develop an optimization model to determine the optimal value of adjustment parameter so that it can assure that the distance between original preference matrix and modified preference matrix is the least under the condition that the modified preference matrix satisfies the accepted consistency. To conduct the consistency improvement, an algorithm is designed to reach the goal. Besides, the process of decision making is proposed based on consistency analysis in order to obtain a scientific and accurate result when using the PLPR. After the analysis of numerous examples and a comparative analysis, we see that it is feasible, efficient and superior to use the proposed methods in this paper. In short, this research is meaningful in theory and practice. In theory, we extend the method to conduct consistency analysis of PLPR. In practice, the proposed decision-making process can be applied to many decision fields, such as education evaluation, financial investment, supplier selection and so on.

However, there exist some limitations in this research. On the one hand, in the process of consistency improvement, the threshold of accepted consistency ξ is flexibly predetermined according to the actual situation of decision problem. However, it needs a more reasonable and scientific method to derive. On the other hand, we only explored how to apply consistency analysis to decision making instead of group decision making, so the application area of the proposed method is relatively small. In the future, we will solve the limitations in this research. In addition, we will conduct a further study of consistency checking and improvement for PLPR, and the emphasis will be placed on solving more practical decision-making problems.

Author Contributions: Conceptualization, H.X. and S.W.; Methodology, S.W.; Validation, C.C. and S.W.; Investigation, H.X., S.W. and C.C.; Data curation, S.W.; Writing-original and editing, S.W. and C.C.; Supervision, H.X. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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