

Article

An Exponentiated Multivariate Extension for the Birnbaum-Saunders Log-Linear Model

Guillermo Martínez-Flórez ^{1,†} , Rafael Bráz Azevedo-Farias ^{2,†} and Roger Tovar-Falón ^{1,*,†} ¹ Departamento de Matemáticas y Estadística, Facultad de Ciencias Básicas, Universidad de Córdoba, Montería 230002, Colombia; guillermomartinez@correo.unicordoba.edu.co² Department of Statistics and Applied Mathematics, Federal University of Ceara, Fortaleza 60455-670, Brazil; rafa@ufc.br

* Correspondence: rjtoovar@correo.unicordoba.edu.co

† These authors contributed equally to this work.

Abstract: In this work, a bivariate extension of the univariate exponentiated sinh-normal distribution is proposed. The properties of the new distribution, which is called the bivariate exponentiated sinh-normal distribution, are studied in detail, and the maximum likelihood method is considered to estimate the unknown model parameters. In addition, the extension of the new distribution to the case of regression models is proposed. Monte Carlo simulation experiments are carried out to investigate the performance of the used estimation method, and two applications to real datasets are presented for illustrative purposes.

Keywords: sinh-normal distribution; exponentiated distribution; maximum likelihood estimation; two-stage estimation procedure

MSC: 60E05; 62H05



Citation: Martínez-Flórez, G.; Azevedo-Farias, R.B.; Tovar-Falón, R. An Exponentiated Multivariate Extension for the Birnbaum-Saunders Log-Linear Model. *Mathematics* **2022**, *10*, 1299. <https://doi.org/10.3390/math10081299>

Academic Editor: Christophe Chesneau

Received: 15 March 2022

Accepted: 11 April 2022

Published: 14 April 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The univariate sinh-normal (SHN) model was introduced by Rieck and Nedelman [1] as a non-linear transformation of the normal distribution. A random variable (RV) Z , is said to have a three-parameter SHN model, denoted by $Z \sim \text{SHN}(\lambda, \xi, \eta)$, if,

$$\frac{2}{\lambda} \sinh\left(\frac{Z - \xi}{\eta}\right) \sim N(0, 1),$$

where $\lambda > 0$ is a shape parameter, $\xi \in \mathbb{R}$ is a location parameter and, $\eta > 0$ is a scale parameter. The probability density function (PDF) of an RV following the $\text{SHN}(\lambda, \xi, \eta)$ distribution, is given by:

$$f(z) = b'_z \phi(b_z), \quad (1)$$

where $b_z = b(z, \lambda, \xi, \eta) = (2/\lambda) \sinh((z - \xi)/\eta)$ and $b'_z = (2/\lambda\eta) \cosh((z - \xi)/\eta)$ is the derivative of b_z regarding the Z variable. The SHN distribution is also known as the log-Birnbaum-Saunders (LBS) model. Notice that if $Z \sim \text{SHN}(\lambda, \xi, \eta = 2)$, and $T = \exp(Z)$, then it follows a Birnbaum-Saunders (BS) distribution with parameters λ and $\beta = \exp(\xi)$, which is denoted by $\text{BS}(\lambda, \beta)$.

The BS distribution was proposed by Birnbaum and Saunders [2] to explain the survival time and the stress produced in the material because of the law of occurred cumulative damage. The BS distribution can also be obtained by applying a non-linear transformation to the normal model, and more details about this distribution can be found in Birnbaum and Saunders [2,3]. Extensions of the BS model to the family of elliptical distributions, or symmetric family of distributions were introduced by Díaz-García and Leiva-Sánchez [4], which are known in the literature as generalized BS (GBS) distribution.

Several authors have considered other extensions. For example, Vilca-Labra and Leiva-Sánchez [5] extended the GBS model to situations of skew-elliptical distributions, while Martínez-Flórez et al. [6] studied the BS distribution by supposing that Z follows a power-normal (PN) distribution. The resulting model is called exponentiated BS or alpha-power BS distribution.

The PN distribution, also known as the exponentiated normal model, has been studied by Durrans [7], Gupta and Gupta [8] and Pewsey et al. [9], among others. It is said that the RV Z follows a PN distribution with parameter α , which is denoted by $Z \sim \text{PN}(\alpha)$, if its PDF is given by:

$$\varphi(z; \alpha) = \alpha \phi(z) \{\Phi(z)\}^{\alpha-1}, \quad z \in \mathbb{R}, \quad (2)$$

where $\alpha \in \mathbb{R}^+$ is a parameter that controls the shape of the distribution. The cumulative distribution function (CDF) of the PN model is $F_\Phi(z) = \{\Phi(z)\}^\alpha$. On the other hand, the PDF of the RV T , following an alpha-power Birnbaum–Saunders distribution, is given by:

$$\varphi_F(t; \lambda, \tau, \alpha) = \alpha f(a_t) \{F(a_t)\}^{\alpha-1} \frac{t^{-3/2}(t+\tau)}{2\lambda\tau^{1/2}}, \quad t > 0, \quad (3)$$

where a_t is:

$$a_t = a_t(\lambda, \tau) = \frac{1}{\lambda} \left(\sqrt{\frac{t}{\tau}} - \sqrt{\frac{\tau}{t}} \right). \quad (4)$$

which is denoted by $\text{BSPN}(\lambda, \beta, \alpha)$. More information about the BSPN model can be found in Martínez-Flórez et al. [6]. A generalization to the asymmetric models of the SHN distribution was proposed by Leiva et al. [10]. These authors considered that RV Z follows a skew-normal (SN) distribution with parameter λ , see Azzalini [11]. Thus, the resulting PDF is:

$$f(z) = 2b'_z \phi(b_z) \Phi(\lambda b_z). \quad (5)$$

The model in (5) is called log-skew-Birnbaum–Saunders, and is denoted by $\text{SSN}(\alpha, \xi, \eta)$. In addition, Martínez-Flórez et al. [12] proposed an asymmetric extension of the SHN model, by supposing that $(2/\lambda) \sinh((Z - \xi)/\eta) \sim \text{PN}(\alpha)$, resulting in the following alpha-power sinh-normal model,

$$f(z) = \alpha b'_z \phi(b_z) \{\Phi(b_z)\}^{\alpha-1}, \quad (6)$$

which is denoted by $\text{SHPN}(\lambda, \xi, \eta, \alpha)$ and, its CDF is given by:

$$\mathcal{F}(z) = \left\{ \Phi \left(\frac{2}{\lambda} \sinh \left(\frac{z - \xi}{\eta} \right) \right) \right\}^\alpha = \{\Phi(b(z, \lambda, \xi, \eta))\}^\alpha.$$

One can show that if $T \sim \text{BSPN}(\lambda, \tau, \alpha)$, then $Z = \log(T) \sim \text{SHPN}(\lambda, \log(\tau), 2, \alpha)$. An extension of the log-Birnbaum–Saunders (LBS) distribution to regression models has been considered in Rieck and Nedelman [1], which became known as the log-linear BS regression model and can be written as:

$$Y_i = \log(T_i) = \mathbf{x}_i^\top \boldsymbol{\beta} + \epsilon_i, \quad i = 1, 2, \dots, n, \quad (7)$$

where Y_i is the log-survival time corresponding to the i th experimental unit, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^\top$ is an unknown parameter vector and $\mathbf{x}_i = (x_{i1}, x_{i2}, \dots, x_{ip})^\top$ is a p -dimensional vector with known values of the explanatory variables and, ϵ_i are independent and identically distributed RVs with $\epsilon_i \sim \text{SHN}(\lambda, 0, 2)$, $i = 1, 2, \dots, n$.

Univariate asymmetric extensions for the log-BS linear regression model have been given by Leiva et al. [10], Lemonte [13] and Martínez-Flórez et al. [12]. The first two proposals assume that the errors follow an asymmetric sinh-normal distribution, while the latter proposal provides an alpha-power sinh-normal distribution. This latest model is

called the log-Birnbaum–Saunders alpha-power normal regression model, which is denoted by $\text{RLBSPN}(\lambda, \mathbf{x}_i^\top \boldsymbol{\beta}, 2, \alpha)$.

Some extensions for the bivariate or multivariate situations of the BS and SHN models have been treated in the literature for the cases of symmetric and asymmetric distributions. Díaz-García and Domínguez-Molina [14] for example, introduced a multivariate version of the sinh-normal distribution. The authors defined a multivariate vector $\mathbf{Y}$ with multivariate sinh-normal distribution, denoted by $\mathbf{Y} \sim \text{SN}_q(\alpha \mathbf{1}_q, \boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma} = 2\mathbf{I}_q$ is the identity matrix of size $q \times q$. The PDF of $\mathbf{Y}$ is:

$$\pi(\mathbf{y}) = (2\pi\alpha^2)^{-q/2} \left(\prod_{j=1}^q \cosh\left(\frac{y_j - \mu_j}{2}\right) \right) \exp\left\{ -\frac{2}{\alpha^2} \sum_{j=1}^q \sinh^2\left(\frac{y_j - \mu_j}{2}\right) \right\}, \quad \mathbf{y} \in \mathbb{R}^q.$$

Lemonte [15] introduced a version of the multivariate BS regression model, which generalizes the univariate regression model introduced by Rieck and Nedelman [1]. This model is supposed to have n multivariate RVs $\mathbf{y}_1, \dots, \mathbf{y}_n$ and, each one measures a number of responses q_i . Thus, in the multivariate log-linear regression model $\mathbf{y}_i = \mathbf{X}_i \boldsymbol{\beta} + \boldsymbol{\epsilon}_i$, for $i = 1, 2, \dots, n$, it is assumed that $\boldsymbol{\epsilon}_i \sim \text{SN}_{q_i}(\alpha \mathbf{1}_{q_i}, \mathbf{0}_{q_i}, 2\mathbf{I}_{q_i})$, where $\mathbf{1}_{q_i}$ and $\mathbf{0}_{q_i}$ are vectors of ones and zero, respectively, of size q_i . Martínez-Flórez et al. [16] proposed the multivariate log-BS regression model, and Romeiro et al. [17] studied a robust multivariate BS regression model. Marchant et al. [18] considered a multivariate log-linear model for BS distributions. As can be seen, few works have been published where the multivariate log-BS regression model is studied, so new extensions are welcome for this important and applicable case of regression models in survival analyses and other areas of knowledge.

In other cases, multivariate distributions can be built based on the theory of copula for distributions where the marginal distributions are known by using the Sklar's Theorem, see [19,20]. The way to construct a multivariate distribution from Sklar's Theorem is as follows: Let $X_1, \dots, X_p$ be p RVs with continuous CDFs $F_{X_1}(x_1), \dots, F_{X_p}(x_p)$, respectively, then, according to Sklar's theorem, $F_{X_1, \dots, X_p}(x_1, \dots, x_p)$ has a unique copula representation:

$$F_{X_1, \dots, X_p}(x_1, \dots, x_p) = C(F_{X_1}(x_1), \dots, F_{X_p}(x_p)).$$

Moreover, it is well known that many dependence properties of a multivariate distribution depend only on the corresponding copula. Therefore, many dependence properties of a multivariate distribution can be obtained by studying the corresponding copula. Particularly in the case of marginal distributions with distribution SHN, based on the bivariate normal copula, Kundu [21] proposed the bivariate sinh-normal distribution. Specifically, the random vector (X_1, X_2) is said to have a bivariate sinh-normal distribution, which is denoted by $\text{BSHN}(\alpha_1, \alpha_2, \mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$, if for $\alpha_1, \alpha_2, \sigma_1, \sigma_2 > 0$, $\mu_1, \mu_2 \in \mathbb{R}$ and $-1 < \rho < 1$, (X_1, X_2) , its joint CDF is:

$$F_{X_1, X_2}(x_1, x_2) = \Phi_2(b(x_1, \alpha_1, \mu_1, \sigma_1), b(x_2, \alpha_2, \mu_2, \sigma_2), \rho), \quad (8)$$

where $\Phi_2(\cdot)$ denotes the CDF of the bivariate normal distribution and ρ is the parameter that controls the dependency relationship in the bivariate normal copula. Kundu [21] studied the main properties of the BSHN model. Kundu and Gupta [22] proposed a bivariate extension for the power-normal model and studied the main properties of the model. This proposal was made by using the Clayton copula. The Clayton copula is usually attributed to Clayton [23]; however, it has been used by Kimeldorf and Sampson [24], and it is defined for $(u, v) \in [0, 1] \times [0, 1]$ as:

$$C_\delta(u, v) = (u^{-1/\delta} + v^{-1/\delta} - 1)^{-\delta}, \quad \delta > 0.$$

The first systematic study of this model was made by Cook and Johnson [25], who interpreted the parameter δ as a measure of dependence between u and v . Therefore, independence between variables is obtained when δ approaches zero. This dependency

model can be extended to the multivariate case by considering a p -variate Clayton copula, which is defined by:

$$C_\delta(u_1, u_2, \dots, u_p) = \left(\sum_{j=1}^p \{u_j\}^{-1/\delta} - (p-1) \right)^{-\delta}, \quad (9)$$

where $0 < u_j < 1$ for $j = 1, 2, \dots, p$. Using Equation (9), Kundu and Gupta [22] considered that for $j = 1, 2$, $X_j \sim \text{PN}(\alpha_j)$ with respective CDF $F_{X_j}(x_j)\{\Phi(x_j)\}^{\alpha_j}$, and defined the bivariate PN distribution, which is denoted by $\text{BPN}(\alpha_1, \alpha_2, \delta)$ for $\alpha_1, \alpha_2 > 0$ and $\alpha_2 > 0$. The joint CDF is:

$$F_{X_1, X_2}(x_1, x_2) = \left(\{\Phi(x_1)\}^{-\alpha_1/\delta} + \{\Phi(x_2)\}^{-\alpha_2/\delta} - 1 \right)^{-\delta}, \quad (10)$$

where $\delta > 0$ is the parameter that controls the dependency in the Clayton copula.

Lemonte et al. [26] introduced the bivariate Birnbaum–Saunders power-normal (BVBS-PN) distribution, which is defined below.

Definition 1. A bivariate random vector $\mathbf{T} = (T_1, T_2)$ has a BVBS-PN distribution, if its joint PDF is given by:

$$f_{T_1 T_2}(t_1, t_2) = \frac{\delta + 1}{\delta} \frac{\alpha_1 \alpha_2 \phi(a_{t_1}) \{\Phi(a_{t_1})\}^{-\alpha_1/\delta - 1} \phi(a_{t_2}) \{\Phi(a_{t_2})\}^{-\alpha_2/\delta - 1} A_{t_1} A_{t_2}}{\left(\{\Phi(a_{t_1})\}^{-\alpha_1/\delta} + \{\Phi(a_{t_2})\}^{-\alpha_2/\delta} - 1 \right)^{\delta + 2}}, \quad (11)$$

where for $j = 1, 2$, $a_{t_j} = \frac{1}{\lambda_j} \left(\sqrt{t_j/\tau_j} - \sqrt{\tau_j/t_j} \right)$ and $A_{t_j} = [t_j^{-3/2}(t_j + \tau_j)]/2\lambda_j\tau_j^{1/2}$. We use the notation $\mathbf{T} = (T_1, T_2) \sim \text{BVBS-PN}(\lambda_1, \lambda_2, \tau_1, \tau_2, \alpha_1, \alpha_2, \delta)$.

Properties of the BVBS-PN distribution as well as the estimation process of the parameters can be seen in Lemonte et al. [26].

In this paper, the goal is to build a skewed bivariate version of the sinh-normal model by assuming that the marginal distributions follow a $X_j \sim \text{SHPN}(\lambda_j, \xi_j, \eta_j, \alpha_j)$ model and then to study the bivariate sinh-normal regression model. The rest of the paper is organized as follows: In Section 2, we consider the bivariate extension of the univariate SHN power-normal distribution. We study this bivariate extension in detail, derive several of its main properties and perform the estimation process of the model parameters by using the maximum likelihood (ML) method. Section 3 deals with the extension to the case of regression models through the incorporation of covariates. Section 4 presents a Monte Carlo simulation study that was carried out to evaluate the performance of the direct approach to estimate the unknown parameters. Two applications of real datasets for illustrative purposes are presented in Section 5. Finally, Section 6 closes the paper with some concluding remarks.

2. Bivariate Sinh-Normal Exponentiated Distribution

Extending the definition of a sinh-normal RV as a non-linear transformation of a normal RV, we define a vector with bivariate exponentiated sinh-normal distribution.

Definition 2. A bivariate random vector (Y_1, Y_2) has a bivariate exponentiated sinh-normal standard distribution, denoted by $\text{BSHPN}(\lambda_1, \lambda_2, \xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2, \delta)$, if its joint CDF is given by:

$$F_{Y_1 Y_2}(y_1, y_2) = \left(\{\Phi(b_1)\}^{-\alpha_1/\delta} + \{\Phi(b_2)\}^{-\alpha_2/\delta} - 1 \right)^{-\delta}, \quad (12)$$

where $\delta > 0$ is the parameter that controls the dependency in the Clayton copula and $b_j = b(y_j, \lambda_j, \xi_j, \eta_j) = (2/\lambda_j) \times \sinh((y_j - \xi_j)/\eta_j)$, for $j = 1, 2$.

The joint PDF of $(Y_1, Y_2) \sim \text{BSHPN}(\lambda_1, \lambda_2, \xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2, \delta)$ can be written as:

$$f_{Y_1 Y_2}(y_1, y_2) = \frac{\delta + 1}{\delta} \frac{\alpha_1 \alpha_2 \phi(b_1) \{\Phi(b_1)\}^{-\alpha_1/\delta-1} \phi(b_2) \{\Phi(b_2)\}^{-\alpha_2/\delta-1} b'_1 b'_2}{\left(\{\Phi(b_1)\}^{-\alpha_1/\delta} + \{\Phi(b_2)\}^{-\alpha_2/\delta} - 1\right)^{\delta+2}}, \quad (13)$$

where $b'_j = (2/\lambda_j \eta_j) \times \cosh((y_j - \xi_j)/\eta_j)$, for $j = 1, 2$. The joint survival function of (Y_1, Y_2) is then:

$$S_{Y_1 Y_2}(y_1, y_2) = 1 - \{\Phi(b_1)\}^{\alpha_1} + \{\Phi(b_2)\}^{\alpha_2} + \left(\{\Phi(b_1)\}^{-\alpha_1/\delta} + \{\Phi(b_2)\}^{-\alpha_2/\delta} - 1\right)^{-\delta}.$$

Theorem 1. If $(Y_1, Y_2) \sim \text{BSHPN}(\lambda_1, \lambda_2, \xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2, \delta)$ then:

- (i) $Y_j \sim \text{SHPN}(\lambda_j, \xi_j, \eta_j, \alpha_j)$, for $j = 1, 2$.
- (ii) The CDF of Y_1 given $Y_2 = y_2$ is:

$$F_{Y_1|Y_2}(y_1|Y_2 = y_2) = \frac{\{\Phi(b_2)\}^{-\alpha_2(\delta+1)/\delta}}{\left(\{\Phi(b_1)\}^{-\alpha_1/\delta} + \{\Phi(b_2)\}^{-\alpha_2/\delta} - 1\right)^{\delta+1}}.$$

- (iii) $(Z_1, Z_2) = (b(Y_1, \lambda_1, \xi_1, \eta_1), b(Y_2, \lambda_2, \xi_2, \eta_2)) \sim \text{BPN}(\alpha_1, \alpha_2, \delta)$, and consequently, $Y_j = \text{arcsinh}(\lambda_j Z_j / 2) \eta_j + \xi_j$, for $j = 1, 2$.

Proposition 1. Let $(Y_1, Y_2) \sim \text{BSHPN}(\lambda_1, \lambda_2, \xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2, \delta)$,

- (i) If $a_1, a_2 > 0$, it follows that $(a_1 Y_1, a_2 Y_2) \sim \text{BSHPN}(\lambda_1, \lambda_2, a_1 \xi_1, a_2 \xi_2, a_1 \eta_1, a_2 \eta_2, \alpha_1, \alpha_2, \delta)$.
- (ii) $F_{-Y_1, Y_2}(y_1, y_2) = \left(\{1 - \Phi(b_1^*)\}^{-\alpha_1/\delta} + \{\Phi(b_2)\}^{-\alpha_2/\delta} - 1\right)^{-\delta}$.
- (iii) $F_{Y_1, -Y_2}(y_1, y_2) = \left(\{\Phi(b_1)\}^{-\alpha_1/\delta} + \{1 - \Phi(b_2^*)\}^{-\alpha_2/\delta} - 1\right)^{-\delta}$.
- (iv) $F_{-Y_1, -Y_2}(y_1, y_2) = \left(\{1 - \Phi(b_1^*)\}^{-\alpha_1/\delta} + \{1 - \Phi(b_2^*)\}^{-\alpha_2/\delta} - 1\right)^{-\delta}$

where $b_j^* = \frac{2}{\lambda_j} \sinh\left(\frac{y_j - (-\xi_j)}{\eta_j}\right)$ for $j = 1, 2$.

The following proposition relates the bivariate BS and the bivariate sinh-normal models. This proposition is an extension of the property of the univariate case where the BS distribution and the sinh-normal model are related and by which the model sinh-normal regression is called log-Birnbaum–Saunders (LBS).

Proposition 2. Let $\mathbf{T} = (T_1, T_2) \sim \text{BVBSPN}(\lambda_1, \lambda_2, \tau_1, \tau_2, \alpha_1, \alpha_2, \delta)$. If $\mathbf{Y} = (Y_1, Y_2) = (\log(T_1), \log(T_2))$, then $\mathbf{Y} \sim \text{BSHPN}(\lambda_1, \lambda_2, \log(\tau_1), \log(\tau_2), 2, 2, \alpha_1, \alpha_2, \delta)$.

Proof. Given that, for $k = 1, 2$, $Y_k = g(T_k) = \log(T_k)$, then $\sqrt{T_k/\tau_k} = \exp\left(\frac{1}{2}(Y_k - \log(\tau_k))\right)$ and $\sqrt{\tau_k/T_k} = \exp\left(-\frac{1}{2}(Y_k - \log(\tau_k))\right)$, thus,

$$\begin{aligned} a_{t_k} &= \frac{1}{\lambda_k} \left(\sqrt{\frac{t_k}{\tau_k}} - \sqrt{\frac{\tau_k}{t_k}} \right) \\ &= \frac{2}{\lambda_k} \left(\frac{\exp\left(\frac{1}{2}(y_k - \log(\tau_k))\right) - \exp\left(-\frac{1}{2}(y_k - \log(\tau_k))\right)}{2} \right) \\ &= \frac{2}{\lambda_k} \sinh\left(\frac{y_k - \log(\tau_k)}{2}\right) = b_{y_k}. \end{aligned}$$

In addition, since $g_k^{-1}(y_k) = \exp(y_k)$, it follows that the Jacobian of the inverse transformation is $\|J_{g^{-1}(y)}\| = \exp(y_1) \exp(y_2) = \frac{\partial g_1^{-1}(y_1)}{\partial t_1} \frac{\partial g_2^{-1}(y_2)}{\partial t_2}$. By using a similar argument to that of a_{T_k} , it has that:

$$A_1 A_2 = \frac{1}{2} \frac{2}{\lambda_1} \cosh\left(\frac{y_1 - \log(\tau_1)}{2}\right) \frac{1}{2} \frac{2}{\lambda_2} \cosh\left(\frac{y_2 - \log(\tau_2)}{2}\right) \|J_{g^{-1}(y)}\| = b'_{y_1} b'_{y_2}.$$

By the transformation method for bivariate functions, the result follows. $\square$

One can use Proposition 2 to generate samples from the standard bivariate exponentiated sinh-normal distribution presented in Definition 2. In this case, to generate the bivariate vector with a BSHPN distribution, the algorithm described in Lemonte et al. [26] to generate a BVBSNP is used joint the Proposition 2 with scale parameters $\exp(\tau_1)$ and $\exp(\tau_2)$, respectively. To obtain scale η_1 and η_2 , it is enough to consider the transformed variables, say W_1 and W_2 , as $Z_1 = \eta_1 W_1/2$ and $Z_2 = \eta_2 W_2/2$.

Definition 3. A function g defined in $\mathbb{R}^2$ is said to be totally positive of order 2, denoted by TP_2 , if for all $x_1 < x_2$ and $y_1 < y_2$, with $x, y \in \mathbb{R}$,

$$g(x_1, y_1)g(x_2, y_2) \geq g(x_2, y_1)g(x_1, y_2).$$

Definition 4. If $(Y_1, Y_2) > 0 \sim BSHPN(\lambda_1, \lambda_2, \xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2, \delta)$, then $f_{Y_1, Y_2}(y_1, y_2)$ is TP_2 .

Proposition 3. Let $(Y_1, Y_2) \sim BSHPN(\lambda_1, \lambda_2, \xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2, \delta)$, then Y_1 is stochastically non-decreasing in Y_2 and vice versa, for any value of $\lambda_1, \lambda_2, \xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2$ and δ .

Found by Kundu and Gupta [22] from the Clayton copula for the BPN model, it follows that for any value of $\lambda_1, \lambda_2, \xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2$, and δ , the (i) Kendall's tau and (ii) median correlation coefficients become, respectively:

$$\tau = \frac{1}{1 + 2\delta} \quad \text{and} \quad M_{T_1, T_2} = 4C_\delta(1/2, 1/2) = 4(2^{(\delta+1)/\delta} - 1)^{-\delta}.$$

Figure 1 shows the behavior of the $BSHPN(0.75, 0.75, 0, 0, 1, 1, 1.75, 0.5, \delta)$ distribution, for some selected values of δ . The influence of the parameter δ when the marginal distributions have a dependency structure can be seen.

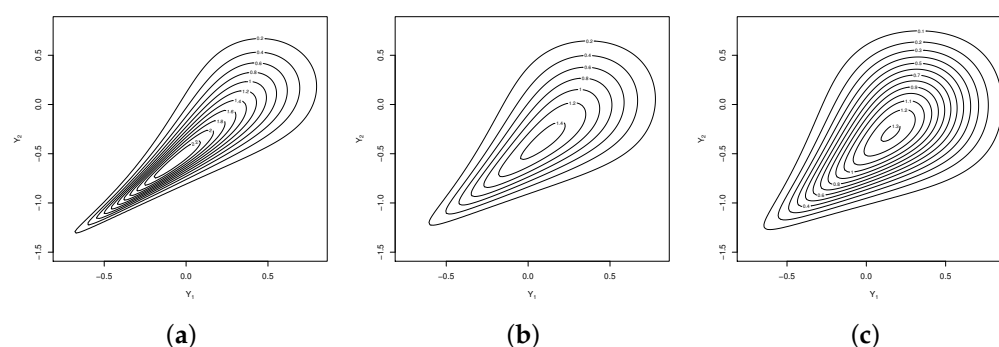


Figure 1. Density plots. (a) $\delta = 0.35$, (b) $\delta = 0.70$ and (c) $\delta = 1.0$.

Statistical Inference

Given the dependence structure induced by Clayton's copula for the estimating the process of the model parameters $BSHPN(\lambda_1, \lambda_2, \xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2, \delta)$, it is used for the maximum likelihood estimation (MLE) and the two-stage estimation method of Joe [27]. For the MLE, we perform a parameterization on $\alpha_1 = \delta\gamma_1, \alpha_2\delta\gamma_2, \dots, \alpha_p = \delta\gamma_p$, then; our focus is

on the estimated parameter vector $\theta_1 = (\lambda_1, \lambda_2, \xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \delta)^\top$. Thus, for a sample of size n from a BSHPN($\lambda_1, \lambda_2, \xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2, \delta$) distribution, let $(y_{11}, y_{21}), \dots, (y_{1n}, y_{2n})$, the log-likelihood function is given by:

$$\begin{aligned} \ell(\theta_1) = & n \log(\gamma_1 \gamma_2 \delta (\delta + 1)) - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^2 b_{ji}^2 - (\gamma_j + 1) \sum_{i=1}^n \sum_{j=1}^2 \log[\Phi(b_{ji})] \\ & - (\delta + 2) \sum_{i=1}^n \log \left(\sum_{j=1}^2 \{\Phi(b_{ji})\}^{-\gamma_j} - 1 \right) + \sum_{i=1}^n \sum_{j=1}^2 \log(b'_{ji}), \end{aligned} \quad (14)$$

which, upon maximization regarding $\lambda_j, \xi_j, \eta_j, \gamma_j$, for $j = 1, 2$ and δ , leads to the following equations:

$$\begin{aligned} \frac{\partial \ell(\theta)}{\partial \lambda_j} &= -\frac{1}{\lambda_j} \left[n - \sum_{i=1}^n b_{ji}^2 - (\gamma_j + 1) \sum_{i=1}^n b_{ji} w_{ji} - (\delta + 2) \sum_{i=1}^n b_{ji} k_{ji} \right] = 0, \\ \frac{\partial \ell(\theta)}{\partial \xi_j} &= -\frac{1}{\eta_j^2} \sum_{i=1}^n \frac{b_{ji}}{b'_{ji}} + \sum_{i=1}^n b'_{ji} [b_{ji} + (\gamma_j + 1) w_{ji} - (\delta + 2) k_{ji}] = 0, \\ \frac{\partial \ell(\theta)}{\partial \eta_j} &= -\frac{1}{\eta_j} \sum_{i=1}^n b'_{ji} - \frac{1}{\eta_j^2} \sum_{i=1}^n z_{ji} \frac{b_{ji}}{b'_{ji}} + \sum_{i=1}^n z_{ji} b'_{ji} [b_{ji} + (\gamma_j + 1) w_{ji} - (\delta + 2) k_{ji}] = 0, \\ \frac{\partial \ell(\theta)}{\partial \gamma_j} &= \frac{n}{\gamma_j} - \sum_{i=1}^n u_{ji} + (\delta + 2) \sum_{i=1}^n \frac{\{\Phi(b_{ji})\}^{-\gamma_j} \log[\Phi(b_{ji})]}{\sum_{j=1}^2 \{\Phi(b_{ji})\}^{-\gamma_j} - 1} = 0 \\ \frac{\partial \ell(\theta)}{\partial \delta} &= \frac{n}{\delta} + \frac{n}{\delta + 1} - \sum_{i=1}^n \log \left(\sum_{j=1}^2 \{\Phi(b_{ji})\}^{-\gamma_j} - 1 \right) = 0, \end{aligned}$$

where $k_j = \gamma_j \phi(b_j) \{\Phi(b_j)\}^{-\gamma_j-1} / \left(\sum_{j=1}^2 \{\Phi(b_j)\}^{-\gamma_j} - 1 \right)$, $u_j = \log\{\Phi(b_j)\}$ and $w_j = \phi(b_j) / \Phi(b_j)$ for $j = 1, 2$. Numerical approaches are required for solving the above system of equations. The observed information matrix follows from minus the second derivatives of the log-likelihood function, this is,

$$\mathbf{J}(\theta_1) = -\frac{\partial^2 \ell(\theta_1)}{\partial \theta_1 \partial \theta_1^\top}.$$

The observed information matrices for both parametrizations θ_1 and θ obey the relation $\mathbf{J}(\theta_1) = \mathbf{D}^\top \mathbf{J}(\theta) \mathbf{D}$, where $\mathbf{D}$ is a matrix containing derivatives of elements of the vector θ_1 respect to elements of the vector $\theta = (\lambda_1, \lambda_2, \xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2, \delta)$. From this relation, we can write $\mathbf{J}(\theta) = \mathbf{D}^{-1} \mathbf{J}(\theta_1) (\mathbf{D}^{-1})^\top$, and $\mathbf{I}(\theta) = \mathbf{D}^{-1} \mathbf{I}(\theta_1) (\mathbf{D}^{-1})^\top$. Therefore,

$$\mathbf{D} = \begin{pmatrix} \mathbf{I}_{6 \times 6} & \mathbf{0}_{6 \times 3} \\ \mathbf{0}_{3 \times 6} & \mathbf{D}_{\alpha\delta} \end{pmatrix},$$

where $\alpha = (\alpha_1, \alpha_2)^\top$ and, $\mathbf{D}_{\alpha\delta}$ is of the size 3×3 defined by:

$$\mathbf{D}_{\alpha\delta} = \begin{pmatrix} \delta \mathbf{I}_2 & \frac{1}{\delta} \alpha \\ -\delta^2 \mathbf{1}'_2 \text{diag}\{\alpha_j^{-1}\}_{j=1}^2 & 1 \end{pmatrix}.$$

That is, $\mathbf{D}$ is a block-diagonal matrix and it can be shown to be non-singular. Assuming the existence of the inverse of the $\mathbf{I}(\theta_1)$ matrix, given the regularity of the PN family, it follows that for large samples, the MLE $\hat{\theta}$ of the parameter vector θ , is asymptotically normal, that is,

$$\hat{\theta} \xrightarrow{D} N_9(\theta, \Sigma_{\hat{\theta}}),$$

resulting that the asymptotic variance of the MLE $\hat{\theta}$, is the inverse of $\mathbf{I}(\theta)$, which we denote by $\Sigma_{\hat{\theta}} = \mathbf{D}^{\top} \mathbf{I}^{-1}(\theta_1) \mathbf{D}$.

Joe [27] proposed a method for estimating copula models which consists of a two-step process as follows: the first step is based on the marginal distributions of the variables which the original model parameters are estimated, in this case, we have the model SHPN($\lambda_j, \xi_j, \eta_j, \alpha_j$) for $j = 1, 2$. Then, assuming that $Y_j \sim \text{SHPN}(\lambda_j, \xi_j, \eta_j, \alpha_j)$ for each $j = 1, 2$, the estimating equations that lead to MLE of the parameter vector $\theta_j = (\lambda_j, \xi_j, \eta_j, \alpha_j)^{\top}$ are:

$$\begin{aligned} U(\lambda_j) &= -\frac{n}{\lambda_j} + \frac{1}{\lambda_j} \sum_{i=1}^n b_{ji}^2 - \frac{\alpha_j - 1}{\lambda_j} \sum_{i=1}^n w_{ji} b_{ji}, \\ U(\xi_j) &= \sum_{i=1}^n \left(b_{ji} b'_{ji} - \frac{1}{\eta_j^2} \frac{b_{ji}}{b'_{ji}} \right) - (\alpha_j - 1) \sum_{i=1}^n w_{ji} b'_{ji}, \\ U(\eta_j) &= -\frac{n}{\eta_j} + \sum_{i=1}^n z_{ji} \left(b_{ji} b'_{ji} - \frac{1}{\eta_j^2} \frac{b_{ji}}{b'_{ji}} \right) - (\alpha_j - 1) \sum_{i=1}^n z_{ji} w_{ji} b'_{ji}, \\ U(\alpha_j) &= \frac{n}{\alpha_j} + \sum_{i=1}^n \log(b_{ji}), \end{aligned}$$

where $w_{ji} = \phi(b_{ji})/\Phi(b_{ji})$. Then, the MLE for the parameters λ_j , ξ_j , η_j and α_j are the solutions to the equations $U(\lambda_j) = 0$, $U(\xi_j) = 0$, $U(\eta_j) = 0$ and $U(\alpha_j) = 0$, which require numerical procedures.

In the second step, we replace the estimates found in the log-likelihood function step, so everything is limited in maximizing the resulting respective function log-likelihood, i.e., to maximize the function:

$$\ell(\delta) = n \log\left(\frac{\delta+1}{\delta}\right) - \frac{\delta+1}{\delta} \sum_{i=1}^n \sum_{j=1}^2 \log(u_j) - (\delta+2) \sum_{i=1}^n \log\left(\sum_{j=1}^2 u_j^{-1/\delta} - 1\right),$$

where $u_j = \{\Phi(b_{ji})\}^{\alpha_j}$.

The elements of the information matrix are given by $\mathbf{V} = (-\mathbf{C}^{-1})\mathbf{M}(-\mathbf{C}^{-1})'$, with,

$$-\mathbf{C} = \begin{pmatrix} \mathbf{C}_{11} & \mathbf{0}_{3p} \\ \mathbf{C}_{\theta_j\delta} & i_{\delta\delta} \end{pmatrix}.$$

Here,

$$\mathbf{C}_{\theta_j\delta} = -\mathbb{E}\left(\frac{\partial^2 \ell(\theta)}{\partial \lambda_1 \partial \delta}, \frac{\partial^2 \ell(\theta)}{\partial \xi_1 \partial \delta}, \frac{\partial^2 \ell(\theta)}{\partial \eta_1 \partial \delta}, \frac{\partial^2 \ell(\theta)}{\partial \alpha_1 \partial \delta}, \frac{\partial^2 \ell(\theta)}{\partial \lambda_2 \partial \delta}, \frac{\partial^2 \ell(\theta)}{\partial \xi_2 \partial \delta}, \frac{\partial^2 \ell(\theta)}{\partial \eta_2 \partial \delta}, \frac{\partial^2 \ell(\theta)}{\partial \alpha_2 \partial \delta}\right).$$

and $i_{\delta\delta} = -\mathbb{E}(\partial^2 \ell(\theta)/\partial \delta^2)$. The second derivatives are calculated from the log-likelihood function defined in (14) and expectations are calculated numerically by taking the corresponding elements $-\mathbb{E}(\partial^2 \ell(\theta)/\partial \theta_j \partial \delta)$ of the matrix $\mathbf{D}^{-1} \mathbf{I}(\theta_1) (\mathbf{D}^{-1})^{\top}$.

Finally, $\mathbf{C}_{11} = \text{block-diag}\{i_{\theta_j\theta_j}\}$, where $i_{\theta_j\theta_j} = -\mathbb{E}(\partial \ell_j(\theta_j)/\partial \theta_j \partial \theta_j^{\top})$ with $\ell_j(\theta_j)$ being the log-likelihood function of the marginal distributions for the parameter vector $\theta_j = (\lambda_j, \xi_j, \eta_j, \alpha_j)^{\top}$, and where expectations are calculating regarding the marginal distributions. On the other hand, according to Joe [27], $\mathbf{M} = \mathbb{E}[g g^{\top}]$ where:

$$g = (g_{\lambda_1}, g_{\xi_1}, g_{\eta_1}, g_{\alpha_1}, g_{\lambda_2}, g_{\xi_2}, g_{\eta_2}, g_{\alpha_2}, g_{\delta})^{\top},$$

for $j = 1, 2$, with:

$$\begin{aligned} g_{\lambda_j} &= -\frac{n}{\lambda_j} + \frac{n}{\lambda_j} b_{ji}^2 - n \frac{\alpha_j - 1}{\lambda_j} w_{ji} b_{ji}, \\ g_{\xi_j} &= n \left(b_{ji} b'_{ji} - \frac{1}{\eta_j^2} \frac{b_{ji}}{b'_{ji}} \right) - n(\alpha_j - 1) w_{ji} b'_{ji}, \\ g_{\eta_j} &= -\frac{n}{\eta_j} + n z_{ji} \left(b_{ji} b'_{ji} - \frac{1}{\eta_j^2} \frac{b_{ji}}{b'_{ji}} \right) - n(\alpha_j - 1) z_{ji} w_{ji} b'_{ji}, \\ g_{\alpha_j} &= \frac{n}{\alpha_j} + n \log(b_{ji}), \\ g_{\delta} &= \frac{n}{\delta^2(1+\delta)^2} \sum_{j=1}^2 \alpha_j \log\{\Phi(b_{ji})\} - n \log \left(\sum_{j=1}^2 \{\Phi(b_{ji})\}^{-\alpha_j/\delta} - 1 \right) + \\ &\quad \left(-\frac{n}{\delta} + \frac{n}{\delta+1} \right) - n \frac{\delta+2}{\delta^2} \frac{\sum_{j=1}^2 \alpha_j \{\Phi(b_{ji})\}^{-\alpha_j/\delta} \log\{\Phi(b_{ji})\}}{\sum_{j=1}^2 \{\Phi(b_{ji})\}^{-\alpha_j/\delta} - 1}. \end{aligned}$$

The elements of $\mathbf{M}$ can be obtained numerically by using the BSHPN($\lambda_1, \lambda_2, \xi_1, \xi_2, \eta_1, \eta_2, \alpha_1, \alpha_2, \delta$) model from its respective integrals.

3. Log-BSHPN Regression Model

Regression analysis, despite being a powerful tool in studies where you want to explain a set of dependent variables according to another set of variables, explanatory or dependent, merely assumes a distribution for that models errors, which generally contains at least one parameter that accounts for the correlation among the variables of interest. We now propose to use a distribution that not only contains the parameters of the proposed model, but also explain the structure of marginal dependence of the model errors, from the copula model BSHPN. It stands to reason that if the proposed distribution can explain the structure of dependence between marginal distributions, the best fit is obtained with conventional models that only take a distribution for the errors, since the correlation between variables is implicit in the dependence structure.

3.1. New Model

On the log-Birnbaum–Saunders power-normal (LBSPN) model proposed by Martínez-Flórez et al. [12], it is assumed that the errors follow a SHPN($\lambda, 0, 2, \alpha$) distribution and, therefore, the RV $Y = \log(T)$, where T represents the survival time, follows the SHPN($\lambda, \mathbf{x}_i^\top \boldsymbol{\beta}, 2, \alpha$) distribution with $\boldsymbol{\beta}$ being the parameters vector and, $\mathbf{x}_i$ a p -dimensional vector with values of the set of explanatory variables.

We extend this univariate exponentiated log-linear Birnbaum–Saunders regression model to the case of a bivariate vector. We considered the situation where the bivariate random vector (Y_1, Y_2) is collected, and the number of measured responses in each observation is n , with $\mathbf{y}_k = (y_{k1}, y_{k2}, \dots, y_{kn})^\top$ denoting a vector of observed dependent variables. Then, by defining $\mathbf{y} = \text{vec}(\mathbf{y}_1, \mathbf{y}_2)$, a vector of size $2n \times 1$, and $\mathbf{X} = \text{diag}(\mathbf{X}_1, \mathbf{X}_2)$ a matrix of size $2n \times (p_1 + p_2)$, with $\mathbf{X}_1$ of size $n \times p_1$ and $\mathbf{X}_2$ of size $n \times p_1$, and $\text{vec}(\cdot)$ being the vec operator which transforms a matrix in a column vector. Then, the bivariate log-linear Birnbaum–Saunders power-normal regression model, previously defined can be represented as:

$$\mathbf{y} = \mathbf{X}^\top \boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad (15)$$

where $\boldsymbol{\beta} = \text{vec}(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ a vector of size $(p_1 + p_2) \times 1$ and $\boldsymbol{\epsilon} = \text{vec}(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2)$ a vector of size $2n \times 1$, with $\boldsymbol{\epsilon}_i = (\epsilon_{1i}, \epsilon_{2i})^\top \sim \text{BSHPN}(\lambda_1, \lambda_2, 0, 0, 2, 2, \alpha_1, \alpha_2, \delta)$. That is:

$$(Y_{1i}, Y_{2i}) = (\log(T_{1i}), \log(T_{2i})) \sim \text{BSHPN}(\lambda_1, \lambda_2, \mathbf{x}_{1i}^\top \boldsymbol{\beta}, \mathbf{x}_{2i}^\top \boldsymbol{\beta}, 2, 2, \alpha_1, \alpha_2, \delta).$$

The regression model given in (15) is denoted by $\text{RBSHPN}(\lambda, \beta, 2 \times \mathbf{1}^\top, \alpha, \delta)$, where $\lambda = (\lambda_1, \lambda_2)$, $\beta = (\beta_1^\top, \beta_2^\top)^\top$, $\alpha = (\alpha_1, \alpha_2)$ and $\mathbf{1}$ is a vector of numbers one of size 2.

For Theorem 1, it follows that $Y_{ik} \sim \text{SHPN}(\lambda_k, \mathbf{X}_{ik}^\top \beta_k, 2, \alpha_k)$, satisfying for $i = 1, \dots, n$,

1. $\gamma_k = \exp(\mathbf{X}_{ki}^\top \beta_k)$ where β_k is a vector of unknown parameters.
2. The shape and skewness parameters in the model do not involve $\mathbf{X}_{ki}$, that is, for $k = 1, 2$, $\tau_{ki} = \tau_k$, $\lambda_{ki} = \lambda_k$ and $\alpha_{ki} = \alpha_k$.

3.2. Inference for the Log-BSHPN Regression Model

In order to estimate the parameters of the $\text{RBSHPN}(\lambda, \beta, 2\mathbf{1}^\top, \alpha, \delta)$ model, for a sample $(y_{11}, y_{21}), \dots, (y_{1n}, y_{2n})$, where $(Y_{1i}, Y_{2i}) \sim \text{BSHPN}(\lambda_1, \lambda_2, \mathbf{x}_{1i}\beta_1, \mathbf{x}_{2i}\beta_2, 2, 2, \alpha_1, \alpha_2, \delta)$, we define the following expressions: $\xi_1 = \xi_1(\theta) = \text{vec}(\xi_{11}, \xi_{12})$ and $\xi_2 = \xi_2(\theta) = \text{vec}(\xi_{21}, \xi_{22})$, with $\xi_{1k} = (\xi_{1k1}, \xi_{1k2}, \dots, \xi_{1kn})^\top$ and $\xi_{2k} = (\xi_{2k1}, \xi_{2k2}, \dots, \xi_{2kn})^\top$, where $\xi_{1ki} = \xi_{1ki}(\theta) = \frac{2}{\lambda_k} \cosh((y_{ki} - \mathbf{x}_{ki}^\top \beta_k)/2)$ and $\xi_{2ki} = \xi_{2ki}(\theta_k) = \frac{2}{\lambda_k} \sinh((y_{ki} - \mathbf{x}_{ki}^\top \beta_k)/2)$.

To obtain MLEs, we proceed with the same re-parameterization as in the case of the BSHPN model. Then, the log-likelihood function can be written as:

$$\begin{aligned} \ell(\theta_1) = & n \log(\gamma_1 \gamma_2 \delta (\delta + 1)/4) - \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^2 \xi_{2ki}^2 - \sum_{i=1}^n \sum_{k=1}^2 (\gamma_k + 1) \log[\Phi(\xi_{2ki})] \\ & - (\delta + 2) \sum_{i=1}^n \log \left(\sum_{k=1}^2 \{\Phi(\xi_{2ki})\}^{-\gamma_k} - 1 \right) + \sum_{i=1}^n \sum_{k=1}^2 \log(\xi_{1ki}). \end{aligned} \quad (16)$$

The score functions for the parameter vector θ_1 are:

$$\begin{aligned} \frac{\partial \ell(\theta)}{\partial \lambda_k} &= -\frac{1}{\lambda_k} \left[n - \sum_{i=1}^n \xi_{2ki}^2 - (\gamma_k + 1) \sum_{i=1}^n w_{2ki} \xi_{2ki} - (\delta + 2) \sum_{i=1}^n \kappa_{2ki} \xi_{2ki} \right], \\ \frac{\partial \ell(\theta)}{\partial \beta_{kj}} &= -\frac{1}{\eta_k^2} \sum_{i=1}^n x_{kij} \frac{\xi_{2ki}}{\xi_{1ki}} + \sum_{i=1}^n x_{kij} \xi_{1ki} [\xi_{2ki} + (\gamma_k + 1) w_{2ki} - (\delta + 2) \kappa_{2ki}], \text{ for } j = 1, \dots, p_1, p_2 \\ \frac{\partial \ell(\theta)}{\partial \gamma_k} &= \frac{n}{\gamma_k} - \sum_{i=1}^n u_{2ki} + (\delta + 2) \sum_{i=1}^n \frac{\{\Phi(\xi_{2ki})\}^{-\gamma_k} \log[\Phi(\xi_{2ki})]}{\sum_{k=1}^2 \{\Phi(\xi_{2ki})\}^{-\gamma_k} - 1} \\ \frac{\partial \ell(\theta)}{\partial \delta} &= \frac{n}{\delta} + \frac{n}{\delta + 1} - \sum_{i=1}^n \log \left(\sum_{k=1}^2 \{\Phi(\xi_{2ki})\}^{-\gamma_k} - 1 \right), \end{aligned}$$

where $\kappa_{2k} = \gamma_k \phi(\xi_{2k}) \{\Phi(\xi_{2k})\}^{-\gamma_k - 1} / (\sum_{k=1}^2 \{\Phi(\xi_{2k})\}^{-\gamma_k} - 1)$, $u_{2k} = \log\{\Phi(\xi_{2k})\}$ and $w_{2k} = \phi(\xi_{2k}) / \Phi(\xi_{2k})$ for $k = 1, 2$. Numerical approaches are required for solving the above system of equations. Again, as in the case of BSHPN model, the information matrices can be obtained from the relationship $\mathbf{J}(\theta) = \mathbf{D}^{-1} \mathbf{J}(\theta_1) (\mathbf{D}^{-1})^\top$ and $\mathbf{I}(\theta) = \mathbf{D}^{-1} \mathbf{I}(\theta_1) (\mathbf{D}^{-1})^\top$. Therefore,

$$\mathbf{D} = \begin{pmatrix} \mathbf{I}_{(p+2) \times (p+2)} & \mathbf{0}_{(p+2) \times 3} \\ \mathbf{0}_{3 \times (p+2)} & \mathbf{D}_{\alpha\delta} \end{pmatrix},$$

where $p = p_1 + p_2$ and $\mathbf{D}_{\alpha\delta}$ is defined as in the BSHPN model. In the same way of the model BSHPN, the two-step algorithm can be implemented by maximizing the log-likelihood function (17) for each response variable.

$$\ell(\lambda_k, \beta_k, \alpha_k) = -n \log(2\sqrt{2\pi}) + n \log(\alpha_k) + \sum_{i=1}^n \log(\xi_{1ki}) - \frac{1}{2} \sum_{i=1}^n \xi_{2ki}^2 + (\alpha_k - 1) \sum_{i=1}^n \log(\xi_{2ki}). \quad (17)$$

We have that the score function is given by:

$$U(\beta_{kj}) = \frac{1}{2} \sum_{i=1}^n x_{kij} \left(\xi_{1ki} \xi_{2ki} - \frac{\xi_{2ki}}{\xi_{1ki}} \right) - \frac{\alpha_k - 1}{2} \sum_{i=1}^n x_{kij} w_{2ki} \xi_{1ki}, \text{ for } j = 1, 2, \dots, p,$$

$$U(\lambda_k) = -\frac{n}{\lambda_k} + \frac{1}{\lambda_k} \sum_{i=1}^n \xi_{2ki}^2 - \frac{\alpha_k - 1}{\lambda_k} \sum_{i=1}^n w_{2ki} \xi_{2ki}, \quad U(\alpha_k) = \frac{n}{\alpha_k} + \sum_{i=1}^n \log(\xi_{1ki}).$$

The MLEs for the regression parameters and parameters α_k and λ_k are the solutions of the equations $U(\beta_{kj}) = 0$ ($j = 1, 2, \dots, p$), $U(\lambda_k) = 0$ and $U(\alpha_k) = 0$, which require numerical procedures. The information matrix can be obtained by proceeding as in the BSHPN model case.

4. Simulation Studies

To investigate the asymptotic properties of the MLEs of the parameters in the introduced models, we conducted two Monte Carlo simulation studies.

4.1. Model Without Covariates

In the first simulation, we studied the performance of the MLE $\hat{\theta} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\xi}_1, \hat{\xi}_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\delta})$ of the BSHPN model. We considered the values of the parameters $\lambda_1 = 0.65$, $\lambda_2 = 1.5$, $\eta_1 = 1.25$, $\eta_2 = 0.5$, $\alpha_1 = 0.75$ and $\alpha_2 = 1.75$, and $\delta = 0.2, 0.5, 0.8, 1.2$ and 1.6 . Without loss of generality, we took $\xi_1 = \xi_2 = 0$ and random sample sizes of $n = 30, 60$ and 100 . For each combination of parameters and sample size, 5000 random samples of the BSHPN model were generated. We used the direct MLE method and, to evaluate the performance of the estimators, the absolute value of the relative bias (RB) and the empirical standard deviation (ESD) were considered. To calculate the MLEs of the parameters, the optim function of the R Development Core Team [28] was used, in joint with iterative methods based on the Newton–Raphson algorithm to optimize the likelihood function. The results are given in Table 1.

Table 1. RB and ESD for the BSHPN model.

δ	n	$\hat{\lambda}_1$		$\hat{\lambda}_2$		$\hat{\eta}_1$		$\hat{\eta}_2$		$\hat{\alpha}_1$		$\hat{\alpha}_2$		$\hat{\delta}$	
		RB	ESD	RB	ESD	RB	ESD	RB	ESD	RB	ESD	RB	ESD	RB	ESD
0.2	30	0.0565	0.1049	0.1375	0.1032	0.1410	0.1529	1.1195	0.1628	0.0745	0.1586	0.0577	0.1233	0.6236	0.0973
	60	0.0273	0.0987	0.1359	0.0370	0.1388	0.0898	0.9753	0.1137	0.0509	0.1384	0.0336	0.0936	0.5204	0.0877
	100	0.0119	0.0987	0.1338	0.0308	0.1271	0.0804	0.8901	0.1067	0.0043	0.1267	0.0132	0.0856	0.4029	0.0849
0.5	30	0.0747	0.1189	0.1582	0.1321	0.1326	0.1563	1.1588	0.1761	0.0541	0.1486	0.0439	0.1267	0.2524	0.0956
	60	0.0251	0.1063	0.1407	0.0635	0.1478	0.0890	0.9922	0.1262	0.0369	0.1364	0.0287	0.0969	0.2046	0.0901
	100	0.0154	0.1000	0.1371	0.0326	0.1431	0.0853	0.8883	0.1086	0.0061	0.1251	0.0116	0.0853	0.1616	0.0848
0.8	30	0.0744	0.1239	0.1582	0.1354	0.1321	0.1575	1.1597	0.1771	0.0513	0.1448	0.0431	0.1379	0.1597	0.1044
	60	0.0254	0.1057	0.1406	0.0722	0.1478	0.0965	0.9849	0.1252	0.0374	0.1356	0.0281	0.1027	0.1279	0.0896
	100	0.0167	0.0997	0.1370	0.0359	0.1415	0.0893	0.8884	0.1088	0.0064	0.1248	0.0117	0.0855	0.1005	0.0848
1.2	30	0.0777	0.1214	0.1585	0.1382	0.1290	0.1690	1.1705	0.1829	0.0477	0.1698	0.0441	0.1350	0.1064	0.1049
	60	0.0227	0.1061	0.1404	0.0650	0.1471	0.0894	0.9827	0.1252	0.0336	0.1351	0.0269	0.0952	0.0844	0.0885
	100	0.0160	0.1000	0.1374	0.0352	0.1431	0.0883	0.8893	0.1094	0.0074	0.1251	0.0116	0.0854	0.0671	0.0848
1.6	30	0.0740	0.1196	0.1564	0.1378	0.1294	0.1630	1.1611	0.1851	0.0517	0.1551	0.0442	0.1369	0.0798	0.1026
	60	0.0219	0.1055	0.1395	0.0639	0.1470	0.0893	0.9858	0.1232	0.0357	0.1369	0.0281	0.0943	0.0631	0.0890
	100	0.0156	0.0999	0.1366	0.0353	0.1440	0.0873	0.8941	0.1084	0.0055	0.1265	0.0128	0.0853	0.0506	0.0850

In general, it can be seen that for the considered sample sizes in the simulation, the RB of the parameter estimators gradually decreases as the sample size increases. This decreasing behavior occurs for some estimators more quickly than for others. In particular, the RB of the $\hat{\lambda}_2$, $\hat{\eta}_1$ and $\hat{\delta}$ estimators decreases slowly for values of the parameter $\delta < 0$. In addition, the ESD has a similar behavior to the RB, with a tendency to decrease. This guarantees the asymptotic convergence of the MLEs of the BSHPN model parameters.

4.2. Model with Covariates

In the second simulation study, we analyzed the behavior of the MLEs of the parameters in the RBSHPN model presented in Section 3. For $i = 1, \dots, n$ we generated the

covariates $X_{1i} \sim U(0, 1)$ and $X_{2i} \sim N(0, 1)$, and we took the values of the β coefficients as $\beta_{10} = 1.5$, $\beta_{11} = 0.75$, $\beta_{20} = 0.15$, $\beta_{21} = 1.25$. We also took the parameter values $\lambda_1 = 0.25$, $\lambda_2 = 1.75$, $\alpha_1 = 2.5$, $\alpha_2 = 3.75$, and $\delta = 5.74, 7.75, 9.75, 11.75$ and 13.75 . For each scenario considered, we generate 5,000 Monte Carlo samples of sizes $n = 30, 60, 90$ and 120 . Again, we used the absolute value of the RB and ESD to assess the goodness of the MLEs.

Similar to the case without covariates, the results in Tables 2 and 3 show that the RB of some estimators decreases faster than others. For the $\hat{\beta}_{10}$, $\hat{\beta}_{11}$, $\hat{\beta}_{21}$ and $\hat{\lambda}_1$ estimators, for example, it can be seen that the RB tends to zero when the sample size increases, while the estimator $\hat{\beta}_{20}$ presented higher RB relating to the previous estimators, and they decrease more slow. For the $\hat{\lambda}_2$, $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\delta}$ estimators, the RB decreases slowly when the value of n increases. Relating to the ESD, Tables 2 and 3 show that the δ estimator presented the highest values of ESD; however, it can be seen that when going from $n = 30$ to $n = 120$, the value of the ESD decreased. On the other hand, the estimator with the lowest ESD values was $\hat{\lambda}_1$. The previous discussion suggests considering moderate or large sample sizes to achieve a desirable quality in the estimation of the parameters of the fitted models.

Table 2. RB and ESD for the BSHPN regression model.

δ	n	$\hat{\beta}_{10}$		$\hat{\beta}_{11}$		$\hat{\beta}_{20}$		$\hat{\beta}_{21}$	
		RB	ESD	RB	ESD	RB	ESD	RB	ESD
5.75	30	0.0513	0.2029	0.0075	0.1411	4.8722	1.2754	0.0045	0.1927
	60	0.0235	0.1722	0.0039	0.1017	2.2312	0.8706	0.0040	0.1370
	90	0.0128	0.1470	0.0031	0.0874	1.3396	0.7094	0.0024	0.1200
	120	0.0050	0.1358	0.0016	0.0806	1.0797	0.6080	0.0005	0.1041
7.75	30	0.0516	0.2005	0.0030	0.1370	4.6682	1.2665	0.0031	0.1948
	60	0.0238	0.1692	0.0020	0.1034	2.0594	0.9110	0.0020	0.1364
	90	0.0103	0.1493	0.0014	0.0914	1.4443	0.6958	0.0018	0.1195
	120	0.0081	0.1307	0.0009	0.0780	0.8668	0.5864	0.0009	0.1085
9.75	30	0.0552	0.1967	0.0037	0.1397	4.6550	1.2792	0.0010	0.1959
	60	0.0214	0.1698	0.0027	0.0990	2.2959	0.8782	0.0010	0.1371
	90	0.0107	0.1482	0.0004	0.0898	1.1475	0.6891	0.0007	0.1156
	120	0.0024	0.1376	0.0001	0.0795	0.7927	0.5867	0.0004	0.1061
11.75	30	0.0489	0.2033	0.0013	0.1420	4.8472	1.2829	0.0042	0.1962
	60	0.0256	0.1658	0.0011	0.1014	2.1831	0.8828	0.0028	0.1372
	90	0.0094	0.1471	0.0032	0.0880	1.3099	0.6926	0.0013	0.1158
	120	0.0020	0.1321	0.0002	0.0780	0.9535	0.5881	0.0012	0.1027
13.75	30	0.0484	0.2053	0.0029	0.1404	4.6512	1.2601	0.0088	0.1888
	60	0.0238	0.1740	0.0023	0.1004	2.2601	0.9043	0.0015	0.1369
	90	0.0086	0.1474	0.0014	0.0865	1.2911	0.6858	0.0005	0.1158
	120	0.0018	0.1294	0.0011	0.0797	0.8648	0.5625	0.0004	0.1030

Table 3. RB and ESD for the BSHPN regression model.

δ	n	$\hat{\lambda}_1$		$\hat{\lambda}_2$		$\hat{\alpha}_1$		$\hat{\alpha}_2$		$\hat{\delta}$	
		RB	ESD	RB	ESD	RB	ESD	RB	ESD	RB	ESD
5.75	30	0.0638	0.0718	0.6537	1.8735	0.7292	2.8177	0.3941	2.4540	2.5258	3.7024
	60	0.0242	0.0586	0.2580	0.9677	0.3857	2.1031	0.2026	1.8087	2.1814	2.6604
	90	0.0148	0.0486	0.1568	0.7171	0.2330	1.6496	0.1207	1.4997	1.9727	2.1305
	120	0.0062	0.0452	0.1218	0.5791	0.1652	1.4853	0.0935	1.3099	1.7918	1.8225
7.75	30	0.0667	0.0701	0.6231	1.8390	0.7584	2.8602	0.3772	2.4885	1.7040	3.6246
	60	0.0333	0.0580	0.2543	1.0159	0.3783	2.0259	0.1837	1.8490	1.4827	2.6877
	90	0.0138	0.0500	0.1613	0.7030	0.2263	1.6494	0.1282	1.4718	1.2929	2.1443
	120	0.0096	0.0433	0.1026	0.5506	0.1676	1.4143	0.0767	1.2660	1.1778	1.8179

Table 3. Cont.

δ	n	$\hat{\lambda}_1$		$\hat{\lambda}_2$		$\hat{\alpha}_1$		$\hat{\alpha}_2$		$\hat{\delta}$	
		RB	ESD	RB	ESD	RB	ESD	RB	ESD	RB	ESD
9.75	30	0.0731	0.0703	0.6356	1.9262	0.7557	2.7999	0.3690	2.4340	1.2504	3.7046
	60	0.0280	0.0573	0.2667	0.9890	0.3677	2.0128	0.1909	1.7801	1.0624	2.5474
	90	0.0149	0.0497	0.1423	0.6916	0.2282	1.307	0.0967	1.4230	0.9288	2.1922
	120	0.0004	0.0452	0.0972	0.5542	0.1411	1.4536	0.0710	1.2781	0.8241	1.7621
11.75	30	0.0650	0.0716	0.6617	1.9883	0.7465	2.9621	0.3785	2.4202	0.9468	3.6679
	60	0.0351	0.0577	0.2622	0.9928	0.3934	2.0125	0.1806	1.7887	0.8000	2.6014
	90	0.0060	0.0495	0.1552	0.7175	0.2097	1.5969	0.1072	1.4373	0.6889	2.0601
	120	0.0009	0.0433	0.1073	0.5652	0.1266	1.3654	0.0793	1.2502	0.6093	1.7579
13.75	30	0.0610	0.0743	0.6177	1.8426	0.7123	2.8455	0.3652	2.4398	0.7442	3.6734
	60	0.0345	0.0591	0.2728	1.0363	0.3800	2.0527	0.1878	1.8155	0.6255	2.6684
	90	0.0100	0.0492	0.1461	0.6716	0.2088	1.6615	0.1094	1.4306	0.5381	2.0308
	120	0.0004	0.0431	0.0986	0.5253	0.1187	1.3428	0.0809	1.2390	0.4738	1.7460

5. Illustrations

5.1. First Illustration

To illustrate the applicability of the BSHPN model, we used a dataset available in Johnson and Wichern [29], which corresponds to environmental pollution measures in some areas of the city of Los Angeles, which were taken at noon for several days. It is well known that the concentration of average air pollutants has been used in epidemiological surveillance as an indicator of the level of atmospheric contamination and its associated adverse effects in humans, for example, causing diseases such as bronchitis.

For air pollutant concentration, it is usually assumed that the data are uncorrelated and independent and thus, it does not require the diurnal or cyclic trend analysis; see Gokhale and Khare [30]. The data correspond to measures of carbon dioxide (CO₂) and, ozone concentration (O₃), in the atmosphere.

Initially, we estimate univariate SHN and SHPN models for each variable. The parameter estimates (standard error) for the SHN model for the variable CO₂ are $\hat{\lambda}_1 = 0.0089(0.0042)$, $\hat{\xi}_1 = 10.0537(0.5178)$ and $\hat{\eta}_1 = 752.1337(376.1523)$, while for the variable O₃ we obtained the estimates $\hat{\lambda}_2 = 0.5448(0.1725)$, $\hat{\xi}_2 = 9.4885(0.8484)$ and $\hat{\eta}_2 = 20.4544(5.7857)$. As found in Kundu [21], the marginal distributions follow a SHN(λ_j, ξ_j, η_j), $j = 1, 2$, distributions. The Kolmogorow–Smirnov test, with respective p -values given in parentheses, are given by $D = 0.1903(0.4313)$ and $D = 0.2381(0.1848)$.

For the SHPN($\lambda_j, \xi_j, \eta_j, \alpha_j$) model with $j = 1, 2$, we obtained the estimates $\hat{\lambda}_1 = 0.0425(0.0582)$, $\hat{\xi}_1 = 5.1809(1.9182)$, $\hat{\eta}_1 = 228.5995(314.1994)$ and $3.2453(1.2263)$ while for the variable O₃, the obtained estimates were $\hat{\lambda}_2 = 2.1792(1.2092)$, $\hat{\xi}_2 = 18.4771(2.6950)$, $\hat{\eta}_2 = 5.9601(1.2677)$ and $\hat{\alpha}_2 = 0.1309(0.1065)$. The Kolmogorow–Smirnov values were $D = 0.1667(0.6041)$ and $D = 0.1190(0.9272)$, respectively.

From the Kolmogorow–Smirnov distances and their respective p -values, it is clear that the SHPN($\lambda_j, \xi_j, \eta_j, \alpha_j$), $j = 1, 2$, models fit better the data than the SHN(λ_j, ξ_j, η_j), $j = 1, 2$, models. Rieck and Nedelman [1] showed that $b_z = b(z, \lambda, \xi, \eta) = \frac{2}{\lambda} \sinh\left(\frac{z-\xi}{\eta}\right) \sim N(0, 1)$ for the SHN model; while Martínez-Flórez et al. [16] showed that $b_z = b(z, \lambda, \xi, \eta) = \frac{2}{\lambda} \sinh\left(\frac{z-\xi}{\eta}\right) \sim \text{PN}(0, 1, \alpha)$ for the SHPN model.

The Figures 2 and 3 show the QQ-plots of the variable b_z for both variables CO₂ and O₃ under SHN and SHNP models. Therefore, it is clear that the SHPN model achieves a much better fit for the air pollution data than the SHN model.

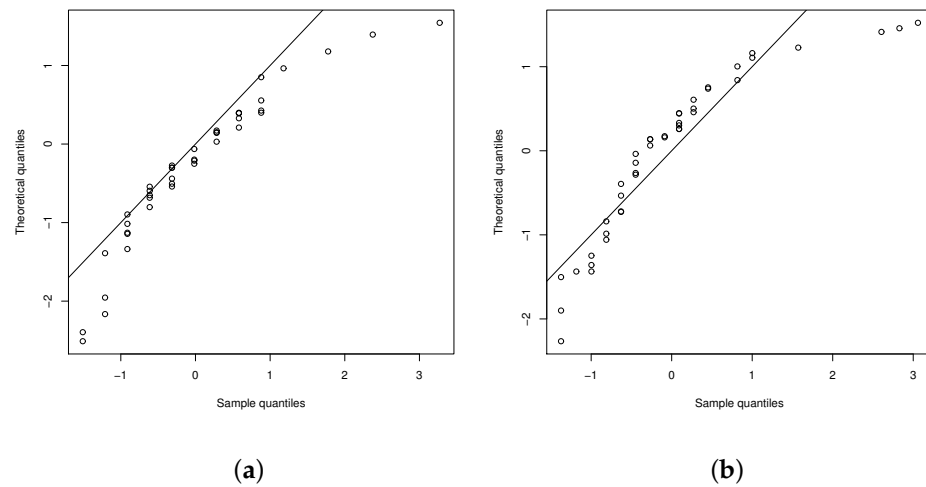


Figure 2. QQ-plots for SHN models. (a) CO₂ and (b) O₃.

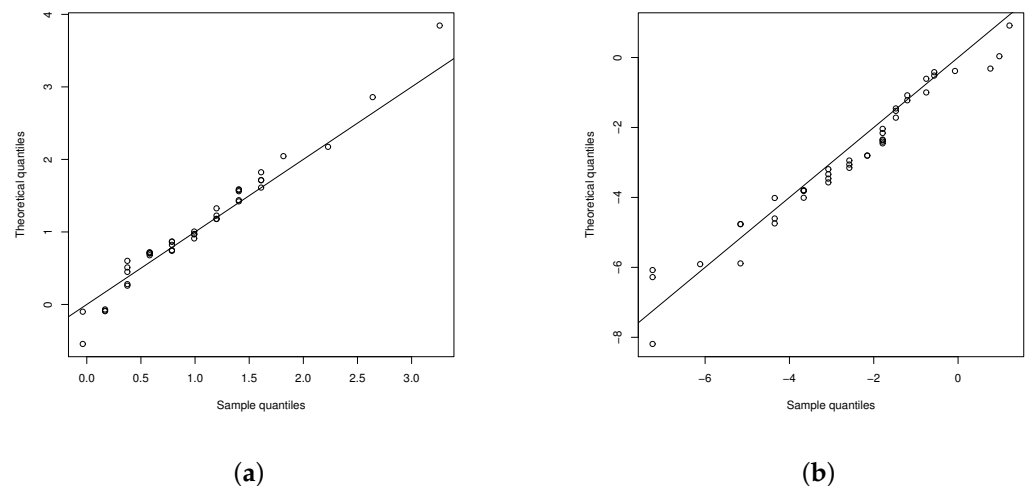


Figure 3. QQ-plots for SHPN models. (a) CO₂ and (b) O₃.

Thus, a bivariate model with skewed marginal distributions fit the air pollution dataset better. Using these parameter estimates as true values, the MLE for parameter δ was computed under the Clayton copula, leading to $\hat{\delta} = 2.7878$. Bootstrap 95% confidence intervals were obtained for parameter δ , given by $(0.2189, 2.8075)$. Finally, we estimated the bivariate sinh-normal models under a bivariate copula (BSHN), see Kundu [21], and the BSHPN model. The estimates of the parameters are given in Table 4.

Table 4. Parameter estimates for BSHN and BSHPN distributions.

Model	λ_1	ξ_1	η_1	α_1	λ_2	ξ_2	η_2	α_2	δ
BSHN	0.0103	10.0499	752.1	—	0.6314	9.4811	20.4529	—	0.5791
BSHPN	0.1047	4.0729	228.5	2.7821	5.4919	17.0693	9.1341	1.0908	1.9774

To compare the BSHN and BSHPN models, we used the AIC criteria, see Akaike [31], namely $AIC = -2 \times \hat{\ell}(\cdot) + 2k$, where k is the number of parameters for the considered model. The best model is the one with the smallest AIC. Thus, for the bivariate SHN model, we obtained $AIC = 507.2515$, whereas for the bivariate SHN power-normal model, $AIC = 161.5224$, leading to the conclusion that the BSHPN model fits the air pollution concentration densities better. Figure 4 presents the graphics of the fitted models, where a better fit is clearly seen in the BSHPN model.

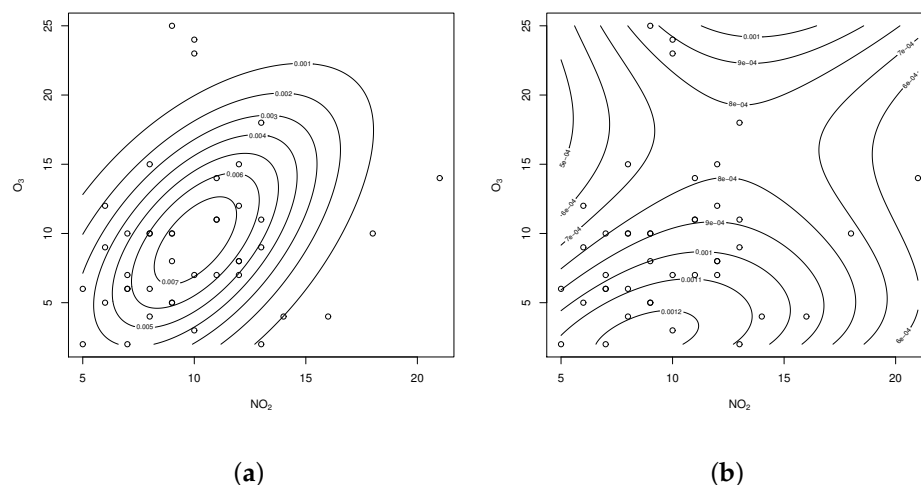


Figure 4. Contour plots. (a) BSHN model and (b) BSHPN model.

5.2. Second Illustration

Our second illustration considers a set of real data available in Lepadatu et al. [32]. The data are from a study where the main objective was to optimize the lifetime of dies and other responses (Von Misses stress, manufacturing force, etc.) in the metal forming process to improve the quality of a product or process, minimizing the effects of variation without removing the causes (because they are too difficult and costly to control).

The log-BS distribution appears in this context since, according to the authors, “the die of fatigue cracks are caused by repeat application of loads, which individually would be too small to cause failure.” The considered models here are: $y_{ik} = \log(T_{ik}) = \beta_1 + \beta_2 x_{i2k} + \epsilon_{ik}$, for $i = 1, 2, \dots, 15, k = 1, 2$ and where T_1 is the Von Misses stress, T_2 is the lifetime of the die and x_{i2k} is the work temperature.

Initially, we assume the independent bivariate model proposed by Lemonte [15], RBSHN, where it is assumed that, $\epsilon_{ik} \sim \text{SHN}(\tau, 0, 2)$, for $k = 1, 2$. For this model, we found that the MLEs were $\hat{\tau} = 2.7253(0.351)$, $\hat{\beta}_1 = 13.1437(1.3091)$ and $\hat{\beta}_2 = -0.0056(0.0016)$, with $AIC = 120.1228$. We checked the assumption of homogeneity of the shape parameter τ . Then, as suggested by Cook and Weisberg [33], the power model is usually used in practical situations. Thus, we assume for simplicity that $k_i = x_i^{\rho}$, where x_i is the work temperature. Then, $\rho = 0$ implies that $k_i = 1$; therefore, $\alpha_i = \alpha$ for all i . Therefore, the test of homogeneity of the shape parameter $\tau_i = \tau$ was equivalent to the hypothesis test $H_0 : \rho = 0$. The statistic of this test was, p -value = 0.9804, which leads to not rejecting the null hypothesis, concluding that the assumption of homogeneity of τ was acceptable.

Finally, we analyzed the assumption of independence of the errors in the proposed models. The Cramer-Von Mises test statistic yielded a calculated value of 0.1413, with p -value = 0.0004, which leads to the rejection of supposition that independence model fits the data well in the study. Therefore, a model that explains the dependence between the proposed model errors can better fit the dataset. Thus, we fit the RBSHPN regression model. The MLE estimates of the bivariate proposed model were: $\hat{\lambda}_1 = 0.0706(0.0230)$, $\hat{\lambda}_2 = 0.1779(0.0718)$, $\hat{\beta}_{11} = 10.8948(0.3882)$, $\hat{\beta}_{21} = -0.0052(0.0006)$, $\hat{\beta}_{12} = 15.2496(0.4659)$, $\hat{\beta}_{22} = -0.0054(0.0006)$, $\hat{\alpha}_1 = 0.0564(0.0088)$, $\hat{\alpha}_2 = 0.5080(0.2397)$ and $\hat{\delta} = 0.1148(0.0509)$.

It can be seen that, in any case, the hypothesis of no influence of the parameter α_j , for $j = 1, 2$ namely $H_0 : \alpha_j = 1$ is rejected. Therefore, with the hypothesis $H_0 : \delta = 0$.

For this model, we obtained $AIC = -35.2966$, which means that the regression model with dependence among the errors of the models better fit the dataset than the independence model.

In Figure 5a, we found contour plots of the errors of the RBSHPN fitted model. The dependency or relationship between the errors of the two proposed models was clearly observed in this graph, which are best explained, according to the contour plot, by the models RBSHPN. For the marginal distributions, we present the envelope plots for the

errors of the regression models of each marginal from the MLEs of the RBSHPN model, where it can be seen that these marginal models also present a good fit, see Figure 5b,c.

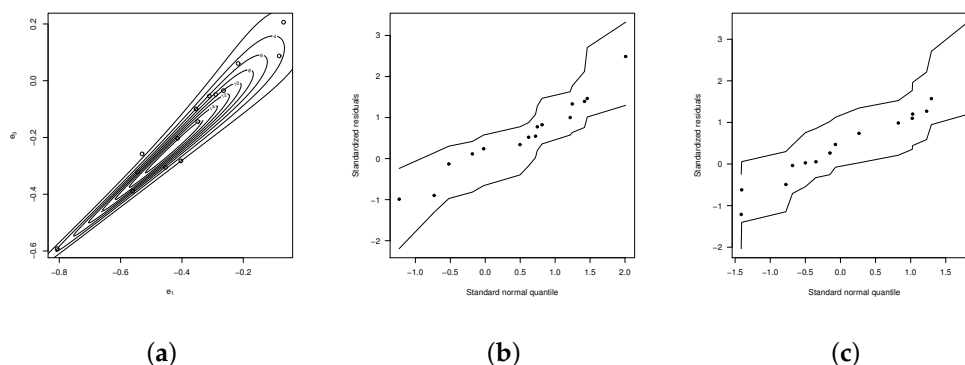


Figure 5. (a) Contour plot RBSHPN model. (b) Marginal envelope e_1 SHN model. (c) Marginal envelope e_2 SHN model.

6. Concluding Remarks

In this work, we have introduced a new multivariate distribution from the sinh-normal univariate distribution and the family of power-normal distributions. The new distribution, which can also be called exponential log-Birnbaum–Saunders, is simple and can be used to model positive multivariate data. The new multivariate exponential sinh-normal distribution is an absolutely continuous distribution whose marginals are exponentiated sinh-normal distributions. Using the frequentist approach, the estimation of the parameters of the multivariate exponential sinh-normal model is carried out using the MLE, as well as a two-stage estimation procedure (Joe [20]). We have also presented the extension of the new multivariate distribution to the case of regression models and for the two cases considered, we present the explicit form of the information matrices. Monte Carlo simulations carried out indicate that the estimation method is quite effective to estimate the unknown parameters of the proposed models. We illustrate the methodology developed in this work through two applications to real data. Through the applications of real data, we verify that the bivariate sinh-normal exponentiated and the corresponding extension to the proposed regression models are superior to some existing proposals in the literature and therefore constitute viable alternatives in the analysis of multivariate positive data.

Author Contributions: Conceptualization, G.M.-F., R.B.A.-F. and R.T.-F.; Methodology, G.M.-F., R.B.A.-F. and R.T.-F.; Data curation, G.M.-F. and R.B.A.-F.; Formal analysis, G.M.-F., R.B.A.-F. and R.T.-F.; Investigation, G.M.-F., R.B.A.-F. and R.T.-F.; Resources, G.M.-F., R.B.A.-F. and R.T.-F.; Software, G.M.-F. and R.B.A.-F.; Supervision, G.M.-F.; Validation, G.M.-F. and R.B.A.-F.; Visualization, G.M.-F., R.B.A.-F. and R.T.-F.; Writing—original draft, G.M.-F.; Writing—review and editing, G.M.-F., R.B.A.-F. and R.T.-F. All authors have read and agreed to the published version of the manuscript.

Funding: The research of G. Martínez-Flórez and R. Tovar-Falón was supported by project: Resolución de Problemas de Situaciones Reales Usando Análisis Estadístico a través del Modelamiento Multidimensional de Tasas y Proporciones; Esquemas de Monitoreamiento para Datos Asimétricos no Normales y una Estrategia Didáctica para el Desarrollo del Pensamiento Lógico-Matemático. Universidad de Córdoba, Colombia, Acta de Compromiso Número: FCB-05-19.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Details about data available are given in Section 5.

Acknowledgments: The authors are very thankful for the constructive comments made by the anonymous reviewers. The authors G. Martínez-Flórez and R. Tovar-Falón acknowledges the support given by Universidad de Córdoba, Montería, Colombia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Rieck, J.R.; Nedelman, J.R. A log-linear model for the Birnbaum-Saunders distribution. *Technometrics* **1991**, *33*, 51–60.
2. Birnbaum, Z.W.; Saunders, S.C. A new family of life distributions. *J. Appl. Probabil.* **1969**, *6*, 319–327. [\[CrossRef\]](#)
3. Birnbaum, Z.W.; Saunders, S.C. Estimation for a family of life distributions with applications to fatigue. *J. Appl. Probabil.* **1969**, *6*, 328–347. [\[CrossRef\]](#)
4. Díaz-García, J.A.; Leiva-Sánchez, V. A new family of life distributions based on the elliptically contoured distributions. *J. Stat. Plann. Inference* **2005**, *128*, 445–457. [\[CrossRef\]](#)
5. Vilca-Labra, F.; Leiva-Sánchez, V. A new fatigue life model based on the family of skew-elliptical distributions. *Commun. Stat.-Theory Methods* **2006**, *35*, 229–244. [\[CrossRef\]](#)
6. Martínez-Flórez, G.; Bolfarine, H.; Gómez, H.W. An alpha-power extension for the Birnbaum-Saunders distribution. *Statistics* **2014**, *48*, 896–912. [\[CrossRef\]](#)
7. Durrans, S.R. Distributions of fractional order statistics in hydrology. *Water Resour. Res.* **1992**, *28*, 1649–1655. [\[CrossRef\]](#)
8. Gupta, R.S.; Gupta, R.D. Generalized skew normal model. *Test* **2004**, *13*, 501–524. [\[CrossRef\]](#)
9. Pewsey, A.; Gómez, H.; Bolfarine, H. Likelihood-based inference for power distributions. *Test* **2012**, *21*, 775–789. [\[CrossRef\]](#)
10. Leiva, V.; Vilca, F.; Balakrishnan, N.; Sanhueza, A. A skewed sinh-normal distribution and its properties and application to air pollution. *Commun. Stat. Theory Methods* **2010**, *39*, 426–443. [\[CrossRef\]](#)
11. Azzalini, A. A class of distributions which includes the normal ones. *Scand. J. Stat.* **1985**, *12*, 171–178.
12. Martínez-Flórez, G.; Bolfarine, H.; Gómez, H.W. The log-linear Birnbaum-Saunders power model. *Methodol. Comput. Appl. Probabil.* **2017**, *19*, 913–933. [\[CrossRef\]](#)
13. Lemonte, A.J. A log-Birnbaum-Saunders regression model with asymmetric errors. *J. Stat. Comput. Simul.* **2012**, *82*, 17751785. [\[CrossRef\]](#)
14. Díaz-García, J.; Domínguez-Molina, J.R. Some generalisations of Birnbaum-Saunders and sinh-normal distributions. *Int. Math. Forum* **2006**, *1*, 1709–1727. [\[CrossRef\]](#)
15. Lemonte, A. A Multivariate Birnbaum-Saunders regression model. *J. Stat. Comput. Simul.* **2013**, *12*, 2244–2257. [\[CrossRef\]](#)
16. Martínez-Flórez, G.; Azevedo-Farias, R.; Moreno-Arenas, G. Multivariate log-Birnbaum-Saunders regression models. *Commun. Stat. Theory Methods* **2017**, *46*, 10166–10178. [\[CrossRef\]](#)
17. Romeiro, R.G.; Vilca, F.; Balakrishnan, N.; Borelli-Zeller, C. A robust multivariate Birnbaum-Saunders regression model. *Statistics* **2020**, *54*, 1094–1123. [\[CrossRef\]](#)
18. Marchant, C.; Leiva, V.; Cysneiros, F. A multivariate log-linear model for Birnbaum-Saunders distributions. *IEEE Trans. Reliab.* **2016**, *65*, 816–864. [\[CrossRef\]](#)
19. Nelsen, R.B. *An Introduction to Copulas*; Springer Series in Statistics; Springer: New York, NY, USA, 2006.
20. Joe, H. *Multivariate Models and Dependence Concepts*; Chapman and Hall: London, UK, 1997.
21. Kundu, D. Bivariate sinh-normal distribution and a related model. *Braz. J. Probabil. Stat.* **2015**, *29*, 590–607. [\[CrossRef\]](#)
22. Kundu, D.; Gupta, R.D. Power normal distribution. *Statistics* **2013**, *47*, 110–125. [\[CrossRef\]](#)
23. Clayton, D.G. A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. *Biometrika* **1978**, *65*, 141–151. [\[CrossRef\]](#)
24. Kimeldorf, G.; Sampson, A. Uniform representations of bivariate distributions. *Commun. Stat.* **1975**, *4*, 617–627. [\[CrossRef\]](#)
25. Cook, R.D.; Johnson, M.E. A family of distributions for modelling non-elliptically symmetric multivariate data. *J. R. Stat. Soc. Ser. B.* **1981**, *43*, 210–218.
26. Lemonte, A.; Martínez-Flórez, G.; Moreno-Arenas, G. Multivariate Birnbaum-Saunders power-normal model and associated inference. *Commun. Stat. Simul. Comput.* **2021**. [\[CrossRef\]](#)
27. Joe, H. Asymptotic efficiency of two-stage estimation method for copula-based models. *J. Multivariate Anal.* **2005**, *94*, 401–419. [\[CrossRef\]](#)
28. R Development Core Team. *R: A Language and Environment for Statistical Computing*; R Foundation for Statistical Computing: Vienna, Austria, 2021. Available online: <http://www.R-project.org> (accessed on 31 January 2022).
29. Johnson, R.A.; Wichern, D.W. *Applied Multivariate Analysis*, 4th ed.; Prentice-Hall: Hoboken, NJ, USA, 1999.
30. Gokhale, S.; Khare, M. Vehicle wake factor for heterogeneous traffic in urban environments. *Int. J. Environ. Pollut.* **2007**, *30*, 97–105. [\[CrossRef\]](#)
31. Akaike, H. A new look at statistical model identification. *IEEE Trans. Autom. Control* **1974**, *AU-19*, 716–722. [\[CrossRef\]](#)
32. Lepadatu, D.; Kobi, A.; Hambli, R.; Barreau, A. Lifetime multiple response optimization of metal extrusion die. In Proceedings of the Annual Reliability and Maintainability Symposium, Alexandria, VA, USA, 24–27 January 2005; Institute of Electrical and Electronics Engineers: Piscataway, NJ, USA, 2005; pp. 37–42.
33. Cook, R.D.; Weisberg, S. Diagnostics for heteroscedasticity in regression. *Biometrika* **1983**, *70*, 1–10. [\[CrossRef\]](#)