mathematics

Article

Controllability and Hyers–Ulam Stability of Differential Systems with Pure Delay

Ahmed M. Elshenhab *,† and Xingtao Wang †

School of Mathematics, Harbin Institute of Technology, Harbin 150001, China; xingtao@hit.edu.cn
* Correspondence: ahmedelshenhab@stu.hit.edu.cn
† These authors contributed equally to this work.

Abstract: Dynamic systems of linear and nonlinear differential equations with pure delay are considered in this study. As an application, the representation of solutions of these systems with the help of their delayed Mittag–Leffler matrix functions is used to obtain the controllability and Hyers–Ulam stability results. By introducing a delay Gramian matrix, we establish some sufficient and necessary conditions for the controllability of linear delay differential systems. In addition, by applying Krasnoselskii's fixed point theorem, we establish some sufficient conditions of controllability and Hyers–Ulam stability of nonlinear delay differential systems. Our results improve, extend, and complement some existing ones. Finally, two examples are given to illustrate the main results.

Keywords: controllability; delay differential system; delayed matrix function; Hyers–Ulam stability; delay Gramian matrix; Krasnoselskii's fixed point theorem

MSC: 93B05; 93C23; 93D05

Citation: Elshenhab, A.M.; Wang, X. Controllability and Hyers–Ulam Stability of Differential Systems with Pure Delay. *Mathematics* **2022**, *10*, 1248. https://doi.org/10.3390/math10081248

Academic Editors: José A. Tenreiro Machado, Jan Awrejcewicz, Hamed Farokhi, Roman Starosta, José M. Vega, Hari Mohan Srivastava and Ying-Cheng Lai

Received: 28 March 2022
Accepted: 7 April 2022
Published: 11 April 2022

1. Introduction

Numerous processes in mechanical and technological systems were described using fractional delay differential equations. These systems are frequently utilized in the modelling of phenomena in technological and scientific problems. These models have applications in diffusion processes [1], viscoelastic systems [2,3], modeling disease [4], forced oscillations, signal analysis, control theory, biology, computer engineering, finance, and population dynamics; see for instance [5–7]. On the other hand, in 2003, Khusainov and Shuklin [8] constructed a novel notion of a delayed exponential matrix function to represent the solutions of linear delay differential equations. In 2008, Khusainov et al. [9] used this method to express the solutions of an oscillating system with pure delay by constructing a delayed matrix sine and a delayed matrix cosine. This pioneering research yielded plenty of novel results on the representation of solutions [10–14], which are applied in the stability analysis [15,16], and control problems [17,18] of time-delay systems. The controllability of systems is one of the most fundamental and significant concepts in modern control theory, which consists of determining the control parameters that steer the solutions of a control system from its initial state to its final state using the set of admissible controls, where initial and final states may vary over the entire space. In recent decades, the controllability of differential delay systems has been studied by many authors. There are a few recent studies in the literature on control theory [19–24] and Ulam stability [25–28] for delay differential equations.

However, to the best of our knowledge, no study exists dealing with the controllability of the linear delay differential equations

$$\begin{aligned} &y''(x) + \mathbb{A}y(x-h) = \mathbb{B}u(x), \ \ x \in \Omega := [0, x_1], \\ &y(x) \equiv \psi(x), \ y'(x) \equiv \psi'(x), \ \ -h \le x \le 0, \end{aligned} \tag{1}$$

and the controllability and Hyers–Ulam stability of the corresponding nonlinear delay differential equations

$$\begin{aligned} y''(x) + \mathbb{A}y(x-h) &= f(x, y(x)) + \mathbb{B}u(x), \quad x \in \Omega, \\ y(x) \equiv \psi(x),\ y'(x) &\equiv \psi'(x), \quad -h \le x \le 0, \end{aligned} \tag{2}$$

where $h > 0$ is a delay; $x_1 > (n-1)h$, $y(x) \in \mathbb{R}^n$, $\psi \in C([-h, 0], \mathbb{R}^n)$, $\mathbb{A} \in \mathbb{R}^{n\times n}$, and $\mathbb{B} \in \mathbb{R}^{n\times m}$ are matrices; $u(x) \in \mathbb{R}^m$ shows the control vector; and $f \in C(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ is a given function.

Very recently, Elshenhab and Wang [11] gave a new representation of solutions of the linear differential equations with pure delay

$$\begin{aligned} y''(x) + \mathbb{A}y(x-h) &= f(x), \quad x \ge 0, \\ y(x) \equiv \psi(x),\ y'(x) &\equiv \psi'(x), \quad -h \le x \le 0, \end{aligned} \tag{3}$$

as follows:

$$\begin{aligned} y(x) = {} & \mathcal{H}_h(\mathbb{A}(x-h))\psi(0) + \mathcal{M}_h(\mathbb{A}(x-h))\psi'(0) \\ & - \mathbb{A}\int_{-h}^{0} \mathcal{M}_h(\mathbb{A}(x-2h-\vartheta))\psi(\vartheta)d\vartheta \\ & + \int_0^x \mathcal{M}_h(\mathbb{A}(x-h-\vartheta))f(\vartheta)d\vartheta, \end{aligned} \tag{4}$$

where $\mathcal{H}_h(\mathbb{A}(x))$ and $\mathcal{M}_h(\mathbb{A}(x))$ are called the delayed matrix functions formulated by

$$\mathcal{H}_h(\mathbb{A}(x)) := \begin{cases} \Theta, & -\infty < x < -h, \\ \mathbb{I}, & -h \le x < 0, \\ \mathbb{I} - \mathbb{A}\frac{x^2}{2!}, & 0 \le x < h, \\ \vdots & \vdots \\ \mathbb{I} - \mathbb{A}\frac{x^2}{2!} + \mathbb{A}^2\frac{(x-h)^4}{4!} \\ + \cdots + (-1)^r\mathbb{A}^r\frac{(x-(r-1)h)^{2r}}{(2r)!}, & (r-1)h \le x < rh, \end{cases} \tag{5}$$

and

$$\mathcal{M}_h(\mathbb{A}(x)) := \begin{cases} \Theta, & -\infty < x < -h, \\ \mathbb{I}(x+h), & -h \le x < 0, \\ \mathbb{I}(x+h) - \mathbb{A}\frac{x^3}{3!}, & 0 \le x < h, \\ \vdots & \vdots \\ \mathbb{I}(x+h) - \mathbb{A}\frac{x^3}{3!} + \mathbb{A}^2\frac{(x-h)^5}{5!} \\ + \cdots + (-1)^r\mathbb{A}^r\frac{(x-(r-1)h)^{2r+1}}{(2r+1)!}, & (r-1)h \le x < rh, \end{cases} \tag{6}$$

respectively, where $r = 0, 1, 2, \ldots$, and the notations $\mathbb{I}$ is the $n \times n$ identity matrix and Θ is the $n \times n$ null matrix.

Applying Formula (4), the solution of (2) can be expressed as

$$\begin{aligned} y(x) = {} & \mathcal{H}_h(\mathbb{A}(x-h))\psi(0) + \mathcal{M}_h(\mathbb{A}(x-h))\psi'(0) \\ & - \mathbb{A}\int_{-h}^{0} \mathcal{M}_h(\mathbb{A}(x-2h-\vartheta))\psi(\vartheta)d\vartheta \\ & + \int_0^x \mathcal{M}_h(\mathbb{A}(x-h-\vartheta))f(\vartheta, y(\vartheta))d\vartheta \\ & + \int_0^x \mathcal{M}_h(\mathbb{A}(x-h-\vartheta))\mathbb{B}u(\vartheta)d\vartheta, \end{aligned} \tag{7}$$

Motivated by [11,17], as an application, the explicit formula of solutions (7) of (3) and the delayed matrix functions are used to obtain controllability results on $\Omega = [0, x_1]$.

The rest of this paper is arranged as follows: In Section 2, we give some preliminaries, basic notations and fundamental definitions, and some lemmas. Furthermore, we give two very important lemmas, which provide estimations of norms for the delayed matrix functions, which are used while discussing controllability and Hyers–Ulam stability. In Section 3, we give sufficient and necessary conditions of the controllability of (1) by introducing a delay Gramian matrix. In Section 4, we establish sufficient conditions of the controllability of (2) by applying Krasnoselskii's fixed point theorem. In Section 5, we discuss the Hyers–Ulam stability of (2) on the finite time interval $[0, x_1]$. Finally, we give two examples to illustrate the main results.

2. Preliminaries

Throughout the paper, we refer to $C(\Omega, \mathbb{R}^n)$ as the Banach space of vector-valued continuous function from $\Omega \to \mathbb{R}^n$ endowed with the norm $\|y\|_{C(\Omega)} = \max_{x\in\Omega}\|y(x)\|$ for a norm $\|\cdot\|$ on $\mathbb{R}^n$, and the matrix norm as $\|\mathbb{A}\| = \max_{\|y\|=1}\|Ay\|$, where $\mathbb{A} : \mathbb{R}^n \to \mathbb{R}^n$. We define a space $C^1(\Omega, \mathbb{R}^n) = \{y \in C(\Omega, \mathbb{R}^n) : y' \in C(\Omega, \mathbb{R}^n)\}$. Let X, Y be two Banach spaces and $L_b(X, Y)$ be the space of bounded linear operators from X to Y. Now, $L^p(\Omega, Y)$ indicates the Banach space of functions $f : \Omega \to Y$ that are Bochner integrable normed by $\|f\|_{L^p(\Omega,Y)}$ for some $1 < p < \infty$. Furthermore, we let $\|\psi\|_C = \max_{s\in[-h,0]}\|\psi(s)\|$ and $\|\psi'\|_C = \max_{s\in[-h,0]}\|\psi'(s)\|$.

We recall some basic notations and fundamental definitions used throughout this paper.

Definition 1 ([6])**.** *The Mittag–Leffler function with two parameters is given by*

$$\mathbb{E}_{\alpha,\gamma}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + \gamma)}, \quad \alpha,\ \gamma > 0,\ z \in \mathbb{C},$$

where Γ *is a gamma function. Especially, if* $\gamma = 1$*, then*

$$\mathbb{E}_{\alpha,1}(z) = \mathbb{E}_{\alpha}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + 1)}, \quad \alpha > 0.$$

Definition 2 ([28])**.** *The systems (1) or (2) are controllable on* $\Omega = [0, x_1]$ *if there exists a control function* $u \in L^2(\Omega, \mathbb{R}^m)$ *such that (1) or (2) have a solution* $y : [-h, x_1] \to \mathbb{R}^n$ *with* $y(0) = y_0$*,* $y'(0) = y'_0$ *satisfies* $y(x_1) = y_1$ *for all* $y_0, y'_0, y_1 \in \mathbb{R}^n$*.*

Definition 3 ([27])**.** *The system (2) is Hyers—Ulam stable on* $[0, x_1]$ *if there exists, for a given constant* $\varepsilon > 0$*, a function* $\varphi \in C(\Omega, \mathbb{R}^n)$ *satisfying the inequality*

$$\left\|\varphi''(x) + \mathbb{A}\varphi(x-h) - f(x, \varphi(x)) - \mathbb{B}u(x)\right\| \le \varepsilon, \quad x \in [0, x_1], \tag{8}$$

and there exists a solution $y \in C(\Omega, \mathbb{R}^n)$ *of (2) and a constant* $M > 0$ *such that*

$$\|\varphi(x) - y(x)\| \le M\varepsilon, \quad \text{for all } x \in [0, x_1].$$

Remark 1 ([27])**.** *A function* $\varphi \in C(\Omega, \mathbb{R}^n)$ *is a solution of the inequality (8) if and only if there exists a function* $g \in C(\Omega, \mathbb{R}^n)$ *such that*

(i) $\|g(x)\| \le \varepsilon$, $x \in \Omega$.
(ii) $\varphi''(x) = -\mathbb{A}\varphi(x-h) + f(x, \varphi(x)) + \mathbb{B}u(x) + g(x)$, $x \in \Omega$.

Lemma 1. *For any* $x \in [(m-1)h, mh]$*,* $m = 1, 2, \ldots,$ *we have*

$$\|\mathcal{H}_h(\mathbb{A}(x))\| \le \mathbb{E}_2\left(\|\mathbb{A}\|x^2\right).$$

Proof. Using (5), we obtain the following

$$\begin{aligned}\|\mathcal{H}_h(\mathbb{A}(x))\| &\le 1 + \|\mathbb{A}\|\frac{x^2}{2!} + \|\mathbb{A}\|^2\frac{(x-h)^4}{4!} \\ &+ \cdots + \|\mathbb{A}\|^m\frac{(x-(m-1)h)^{2m}}{(2m)!} \\ &\le 1 + \|\mathbb{A}\|\frac{x^2}{2!} + \|\mathbb{A}\|^2\frac{x^4}{4!} + \cdots + \|\mathbb{A}\|^m\frac{x^{2m}}{(2m)!} \\ &\le \sum_{k=0}^{\infty}\frac{\left(\|\mathbb{A}\|x^2\right)^k}{(2k)!} = \mathbb{E}_2\left(\|\mathbb{A}\|x^2\right).\end{aligned}$$

This completes the proof. □

Lemma 2. *For any $x \in [(m-1)h, mh]$, $m = 1, 2, \ldots$, we have*

$$\|\mathcal{M}_h(\mathbb{A}(x))\| \le (x+h)\mathbb{E}_{2,2}\left(\|\mathbb{A}\|(x+h)^2\right).$$

Proof. Using (6), we obtain the following

$$\begin{aligned}\|\mathcal{M}_h(\mathbb{A}(x))\| &\le (x+h) + \|\mathbb{A}\|\frac{x^3}{3!} + \|\mathbb{A}\|^2\frac{(x-h)^5}{5!} \\ &+ \cdots + \|\mathbb{A}\|^m\frac{(x-(m-1)h)^{2m+1}}{(2m+1)!} \\ &\le (x+h) + \|\mathbb{A}\|\frac{(x+h)^3}{3!} + \|\mathbb{A}\|^2\frac{(x+h)^5}{5!} \\ &+ \cdots + \|\mathbb{A}\|^m\frac{(x+h)^{2m+1}}{(2m+1)!} \\ &\le \sum_{k=0}^{\infty}\frac{\left[\|\mathbb{A}\|(x+h)^2\right]^k(x+h)}{(2k+1)!} = (x+h)\mathbb{E}_{2,2}\left(\|\mathbb{A}\|(x+h)^2\right).\end{aligned}$$

This completes the proof. □

Lemma 3. *Let $\varphi \in C(\Omega, \mathbb{R}^n)$ be a solution of the inequality (8). Then, φ is a solution of the inequality*

$$\|\varphi(x) - \varphi^*(x)\| \le \frac{x^2\varepsilon}{2}\mathbb{E}_{2,2}\left(\|\mathbb{A}\|x^2\right),$$

where

$$\begin{aligned}\varphi^*(x) &= \mathcal{H}_h(\mathbb{A}(x-h))\psi(0) + \mathcal{M}_h(\mathbb{A}(x-h))\psi'(0) \\ &- \mathbb{A}\int_{-h}^{0}\mathcal{M}_h(\mathbb{A}(x-2h-\vartheta))\psi(\vartheta)d\vartheta \\ &+ \int_0^x \mathcal{M}_h(\mathbb{A}(x-h-\vartheta))f(\vartheta,\varphi(\vartheta))d\vartheta \\ &+ \int_0^x \mathcal{M}_h(\mathbb{A}(x-h-\vartheta))Bu_\varphi(\vartheta)d\vartheta.\end{aligned}$$

Proof. From Remark 1, the solution of the equation

$$\varphi''(x) = -\mathbb{A}\varphi(x-h) + f(x,\varphi(x)) + \mathbb{B}u(x) + g(x), \quad x \in \Omega,$$

can be written as

$$\begin{aligned}
\varphi(x) = {} & \mathcal{H}_h(\mathbb{A}(x-h))\psi(0) + \mathcal{M}_h(\mathbb{A}(x-h))\psi'(0) \\
& - \mathbb{A}\int_{-h}^{0} \mathcal{M}_h(\mathbb{A}(x-2h-\vartheta))\psi(\vartheta)d\vartheta \\
& + \int_0^x \mathcal{M}_h(\mathbb{A}(x-h-\vartheta))f(\vartheta,\varphi(\vartheta))d\vartheta \\
& + \int_0^x \mathcal{M}_h(\mathbb{A}(x-h-\vartheta))\mathbb{B}u_\varphi(\vartheta)d\vartheta \\
& + \int_0^x \mathcal{M}_h(\mathbb{A}(x-h-\vartheta))g(\vartheta)d\vartheta.
\end{aligned}$$

From Lemma 2, we obtain

$$\begin{aligned}
\|\varphi(x)-\varphi^*(x)\| & \le \int_0^x \|\mathcal{M}_h(\mathbb{A}(x-h-\vartheta))\|\|g(\vartheta)\|d\vartheta \\
& \le \varepsilon\int_0^x (x-\vartheta)\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|(x-\vartheta)^2\Big)d\vartheta \\
& \le \frac{x^2\varepsilon}{2}\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x^2\Big),
\end{aligned}$$

for all $x \in \Omega$. This ends the proof. □

Lemma 4 (Krasnoselskii's fixed point theorem, [29])**.** *Let C be a closed, convex, and non-empty subset of a Banach space X. Suppose that the operators A and B be maps from C into X such that $Ax + By \in C$ for every pair $x, y \in C$. If A is compact and continuous and B is a contraction mapping, then there exists $z \in C$ such that $z = Az + Bz$.*

3. Controllability of Linear Delay Differential System

In this section, we establish some sufficient and necessary conditions for controllability of (1) by introducing a delay Gramian matrix defined by

$$W_h^{\mathcal{M}}[0,x_1] = \int_0^{x_1} \mathcal{M}_h(\mathbb{A}(x_1-h-\vartheta))\mathbb{B}\mathbb{B}^T\mathcal{M}_h\Big(\mathbb{A}^T(x_1-h-\vartheta)\Big)d\vartheta. \tag{9}$$

It follows from the definition of the matrix $W_h^{\mathcal{M}}[0,x_1]$ that it is always positive semidefinite for $x \ge 0$.

Theorem 1. *The linear system (1) is controllable if and only if $W_h^{\mathcal{M}}[0,x_1]$ is positive definite.*

Proof. Sufficiency. Let $W_h^{\mathcal{M}}[0,x_1]$ be positive definite; then, it is non-singular and its inverse is well-defined. As a result, we can derive the associated control input $u(x)$, for any finite terminal conditions $y_1, y_1' \in \mathbb{R}^n$, as

$$u(x) = \mathbb{B}^T\mathcal{M}_h\Big(\mathbb{A}^T(x_1-h-x)\Big)\Big(W_h^{\mathcal{M}}\Big)^{-1}[0,x_1]\beta, \tag{10}$$

where

$$\begin{aligned}
\beta = {} & y_1 - \mathcal{H}_h(\mathbb{A}(x_1-h))\psi(0) - \mathcal{M}_h(\mathbb{A}(x_1-h))\psi'(0) \\
& + \mathbb{A}\int_{-h}^{0} \mathcal{M}_h(\mathbb{A}(x_1-2h-\vartheta))\psi(\vartheta)d\vartheta.
\end{aligned} \tag{11}$$

From (7), the solution $y(x_1)$ of (1) can be formulated as:

$$\begin{aligned} y(x_1) &= \mathcal{H}_h(\mathbb{A}(x_1-h))\psi(0) + \mathcal{M}_h(\mathbb{A}(x_1-h))\psi'(0) \\ &- \mathbb{A}\int_{-h}^{0} \mathcal{M}_h(\mathbb{A}(x_1-2h-\vartheta))\psi(\vartheta)d\vartheta \\ &+ \int_0^{x_1} \mathcal{M}_h(\mathbb{A}(x_1-h-\vartheta))\mathbb{B}u(\vartheta)d\vartheta. \end{aligned} \tag{12}$$

Substituting (10) into (12), we obtain the following:

$$\begin{aligned} &y(x_1) \\ &= \mathcal{H}_h(\mathbb{A}(x_1-h))\psi(0) + \mathcal{M}_h(\mathbb{A}(x_1-h))\psi'(0) \\ &- \mathbb{A}\int_{-h}^{0} \mathcal{M}_h(\mathbb{A}(x_1-2h-\vartheta))\psi(\vartheta)d\vartheta \\ &+ \int_0^{x_1} \mathcal{M}_h(\mathbb{A}(x_1-h-\vartheta))\mathbb{B}\mathbb{B}^T\mathcal{M}_h\Big(\mathbb{A}^T(x_1-h-\vartheta)\Big)d\vartheta\Big(W_h^{\mathcal{M}}\Big)^{-1}[0,x_1]\beta. \end{aligned} \tag{13}$$

Using (9) and (11) in (13), we obtain

$$\begin{aligned} y(x_1) &= \mathcal{H}_h(\mathbb{A}(x_1-h))\psi(0) + \mathcal{M}_h(\mathbb{A}(x_1-h))\psi'(0) \\ &- \mathbb{A}\int_{-h}^{0} \mathcal{M}_h(\mathbb{A}(x_1-2h-\vartheta))\psi(\vartheta)d\vartheta + \beta \\ &= y_1. \end{aligned}$$

We can see from (3) and (4) that the boundary conditions hold. Thus, (1) is controllable.

Necessity. Assume that (1) is controllable. For the sake of a contradiction, suppose that $W_h^{\mathcal{M}}[0,x_1]$ is not positive definite; there exists at least a nonzero vector $z \in \mathbb{R}^n$ such that $z^T W_h^{\mathcal{M}}[0,x_1]z = 0$, which implies that

$$\begin{aligned} 0 &= z^T W_h^{\mathcal{M}}[0,x_1]z \\ &= \int_0^{x_1} z^T\mathcal{M}_h(\mathbb{A}(x_1-h-\vartheta))\mathbb{B}\mathbb{B}^T\mathcal{M}_h\Big(\mathbb{A}^T(x_1-h-\vartheta)\Big)zd\vartheta \\ &= \int_0^{x_1}\Big[z^T\mathcal{M}_h(\mathbb{A}(x_1-h-\vartheta))\mathbb{B}\Big]\Big[z^T\mathcal{M}_h(\mathbb{A}(x_1-h-\vartheta))\mathbb{B}\Big]^T d\vartheta \\ &= \int_0^{x_1}\Big\|z^T\mathcal{M}_h(\mathbb{A}(x_1-h-\vartheta))\mathbb{B}\Big\|d\vartheta. \end{aligned}$$

Hence,

$$z^T\mathcal{M}_h(\mathbb{A}(x_1-h-\vartheta))\mathbb{B} = (0,\ldots,0) := \mathbf{0}^T, \quad \text{for all } \vartheta \in \Omega, \tag{14}$$

where $\mathbf{0}$ denotes the n dimensional zero vector. Consider the initial points $y_0 = y_0' = \mathbf{0}$ and the final point $y_1 = z$ at $x = x_1$. Since (1) is controllable, from Definition 2, there exists a control function $u_1(x)$ that steers the response from $\mathbf{0}$ to $y_1 = z$ at $x = x_1$. Then,

$$\begin{aligned} y_1 = z &= -\mathbb{A}\int_{-h}^{0} \mathcal{M}_h(\mathbb{A}(x_1-2h-\vartheta))\psi(\vartheta)d\vartheta \\ &+ \int_0^{x_1} \mathcal{M}_h(\mathbb{A}(x_1-h-\vartheta))\mathbb{B}u_1(\vartheta)d\vartheta. \end{aligned} \tag{15}$$

Multiplying (15) by z^T and using (14), we obtain $z^T z = 0$. This is a contradiction to $z \neq \mathbf{0}$. Thus, $W_h[0,x_1]$ is positive definite. This ends the proof. □

Corollary 1. *Let $\mathbb{A} = \mathbb{A}^2$ in (1). Then, Theorem 1 holds.*

Corollary 2. *Let $\mathbb{A} = \mathbb{A}^2$ in (1) such that $\mathbb{A}$ is a nonsingular $n \times n$ matrix. Then, the linear system (1) is controllable if and only if $W_h[0, x_1]$ is nonsingular, where $W_h[0, x_1]$ is defined as*

$$W_h[0, x_1] = \mathbb{A}^{-1} \int_0^{x_1} \sin_h(\mathbb{A}(x_1 - h - \vartheta)) BB^T \sin_h\left(\mathbb{A}^T (x_1 - h - \vartheta)\right),$$

and $\sin_h(Ax)$ and $\cos_h(Ax)$ are called the delayed matrix of sine and cosine type, respectively, defined in [9].

Proof. From the definition of $\mathcal{H}_h(\mathbb{A}(x))$ and $\mathcal{M}_h(\mathbb{A}(x))$ in the case of the matrix $\mathbb{A} = \mathbb{A}^2$, we find that

$$\mathcal{H}_h\left(\mathbb{A}^2(x)\right) = \cos_h(\mathbb{A}(x)),\ \mathcal{M}_h\left(\mathbb{A}^2(x)\right) = \mathbb{A}^{-1} \sin_h(\mathbb{A}(x)),$$

which implies that

$$\begin{aligned}
W_h^{\mathcal{M}}[0, x_1] &= \int_0^{x_1} \mathcal{M}_h\left(\mathbb{A}^2(x_1 - h - \vartheta)\right) \mathbb{B}\mathbb{B}^T \mathcal{M}_h\left(\left(\mathbb{A}^2\right)^T (x_1 - h - \vartheta)\right) d\vartheta \\
&= \int_0^{x_1} \mathbb{A}^{-1} \sin_h(\mathbb{A}(x_1 - h - \vartheta)) \mathbb{B}\mathbb{B}^T \sin_h\left(\mathbb{A}^T (x_1 - h - \vartheta)\right) \left(\mathbb{A}^{-1}\right)^T d\vartheta \\
&= \mathbb{A}^{-1} \int_0^{x_1} \sin_h(\mathbb{A}(x_1 - h - \vartheta)) \mathbb{B}\mathbb{B}^T \sin_h\left(\mathbb{A}^T (x_1 - h - \vartheta)\right) d\vartheta \left(\mathbb{A}^T\right)^{-1} \\
&= W_h[0, x_1] \left(\mathbb{A}^T\right)^{-1}.
\end{aligned}$$

Hence,

$$W_h[0, x_1] = W_h^{\mathcal{M}}[0, x_1] \mathbb{A}^T. \tag{16}$$

From the conclusion of Theorem 1, we have that $W_h^{\mathcal{M}}[0, x_1]$ is nonsingular. Thus, from (16), we find that $W_h[0, x_1]$ is also nonsingular. This completes the proof. □

4. Controllability of Nonlinear Delay Differential System

In this section, we establish the sufficient conditions of controllability of (2) using Krasnoselskii's fixed point theorem.

We impose the following assumptions:

(G1) The function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and there exists a constant $L_f \in L^q(\Omega, \mathbb{R}^+)$ and $q > 1$ such that

$$\|f(x, y_1) - f(x, y_2)\| \le L_f(x) \|y_1 - y_2\|, \quad \text{for all } x \in \Omega,\ y_1, y_2 \in \mathbb{R}^n.$$

Let $\sup_{x \in \Omega} f(x, 0) = M_f < \infty$.

(G2) The linear operator $Q : L^2(\Omega, \mathbb{R}^m) \to \mathbb{R}^n$ is defined by

$$Q = \int_0^{x_1} \mathcal{M}_h(\mathbb{A}(x_1 - h - \vartheta)) \mathbb{B} u(\vartheta) d\vartheta.$$

Suppose that Q^{-1} exists and takes values in $L^2(\Omega, \mathbb{R}^m) / \ker Q$, and there exists a constant $M_1 > 0$ such that $\|Q^{-1}\| \le M_1$.

To establish our result, we now employ Krasnoselskii's fixed point theorem.

Theorem 2. *Let* (**G1**) *and* (**G2**) *hold. Then, the nonlinear system (2) is controllable if*

$$M_2 \left[1 + \frac{M_1 x_1^2}{2} \mathbb{E}_{2,2}\left(\|\mathbb{A}\| x_1^2\right) \|\mathbb{B}\|\right] < 1, \tag{17}$$

where

$$M_2 = \frac{x_1^{1+\frac{1}{p}}}{(p+1)^{\frac{1}{p}}} \mathbb{E}_{2,2}\left(\|\mathbb{A}\|x_1^2\right)\left\|L_f\right\|_{L^q(\Omega,\mathbb{R}^+)} \text{ and } \frac{1}{p}+\frac{1}{q}=1,\ p,q>1.$$

Proof. Before we start to prove this theorem, we shall use the following assumptions and estimates: we consider the set

$$B_\epsilon = \left\{ y \in C([-h,x_1],\mathbb{R}^n) : \|y\|_{C[-h,x_1]} = \sup_{x\in[-h,x_1]} \|y(x)\| \le \epsilon \right\}.$$

Let $x \in [0,x_1]$. From (**G1**) and Hölder inequality, we obtain

$$\begin{aligned}
&\int_0^x (x-\vartheta)\mathbb{E}_{2,2}\left(\|\mathbb{A}\|(x-\vartheta)^2\right) L_f(\vartheta)d\vartheta \\
&\le \left(\int_0^x \left((x-\vartheta)\mathbb{E}_{2,2}\left(\|\mathbb{A}\|(x-\vartheta)^2\right)\right)^p d\vartheta\right)^{\frac{1}{p}} \left(\int_0^x L_f^q(\vartheta)d\vartheta\right)^{\frac{1}{q}} \\
&\le \mathbb{E}_{2,2}\left(\|\mathbb{A}\|x^2\right)\left(\int_0^x (x-\vartheta)^p d\vartheta\right)^{\frac{1}{p}} \left(\int_0^x L_f^q(\vartheta)d\vartheta\right)^{\frac{1}{q}} \\
&= \frac{x^{2-\frac{1}{q}}}{(p+1)^{\frac{1}{p}}}\mathbb{E}_{2,2}\left(\|\mathbb{A}\|x^2\right)\left\|L_f\right\|_{L^q(\Omega,\mathbb{R}^+)}.
\end{aligned} \tag{18}$$

Furthermore, consider the following control function u_y:

$$\begin{aligned}
u_y(x) = \ & Q^{-1}\big[y_1 - \mathcal{H}_h(\mathbb{A}(x_1-h))\psi(0) - \mathcal{M}_h(\mathbb{A}(x_1-h))\psi'(0) \\
& +\mathbb{A}\int_{-h}^0 \mathcal{M}_h(\mathbb{A}(x_1-2h-\vartheta))\psi(\vartheta)d\vartheta \\
& -\int_0^{x_1} \mathcal{M}_h(\mathbb{A}(x_1-h-\vartheta))f(\vartheta,y(\vartheta))d\vartheta\big](x),
\end{aligned} \tag{19}$$

for $x \in \Omega$. From (18), (19), (**G1**), and (**G2**) and Lemmas 1 and 2, we obtain

$$
\begin{aligned}
\|u_y(x)\| \leq{} & \left\|Q^{-1}\right\|(\|y_1\| + \|\mathcal{H}_h(\mathbb{A}(x_1-h))\|\|\psi(0)\| \\
& + \|\mathcal{M}_h(\mathbb{A}(x_1-h))\|\|\psi'(0)\| \\
& + \|\mathbb{A}\| \int_{-h}^{0} \|\mathcal{M}_h(\mathbb{A}(x_1-2h-\vartheta))\|\|\psi(\vartheta)\|d\vartheta \\
& + \int_0^{x_1} \|\mathcal{M}_h(\mathbb{A}(x_1-h-\vartheta))\|\|f(\vartheta,y(\vartheta))\|d\vartheta) \\
& \leq M_1\|y_1\| + M_1\mathbb{E}_2\left(\|\mathbb{A}\|(x_1-h)^2\right)\|\psi\|_C \\
& + M_1 x_1 \mathbb{E}_{2,2}\left(\|\mathbb{A}\|x_1^2\right)\|\psi'\|_C \\
& + M_1\|\mathbb{A}\|\|\psi\|_C \int_{-h}^{0}(x_1-h-\vartheta)\mathbb{E}_{2,2}\left(\|\mathbb{A}\|(x_1-h-\vartheta)^2\right)d\vartheta \\
& + M_1 \int_0^{x_1}(x_1-\vartheta)\mathbb{E}_{2,2}\left(\|\mathbb{A}\|(x_1-\vartheta)^2\right)L_f(\vartheta)\|y(\vartheta)\|d\vartheta \\
& + M_1 \int_0^{x_1}(x_1-\vartheta)\mathbb{E}_{2,2}\left(\|\mathbb{A}\|(x_1-\vartheta)^2\right)\|f(\vartheta,0)\|d\vartheta \\
& \leq M_1\|y_1\| + M_1\mathbb{E}_2\left(\|\mathbb{A}\|(x_1-h)^2\right)\|\psi\|_C \\
& + M_1 x_1 \mathbb{E}_{2,2}\left(\|\mathbb{A}\|x_1^2\right)\|\psi'\|_C + \frac{M_1\|\mathbb{A}\|\|\psi\|_C x_1^2}{2}\mathbb{E}_{2,2}\left(\|\mathbb{A}\|x_1^2\right) \\
& + \frac{M_1 x^{2-\frac{1}{q}}}{(p+1)^{\frac{1}{p}}}\mathbb{E}_{2,2}\left(\|\mathbb{A}\|x^2\right)\left\|L_f\right\|_{L^q(\Omega,\mathbb{R}^+)}\|y\|_{C(\Omega)} \\
& + \frac{M_1 M_f x_1^2}{2}\mathbb{E}_{2,2}\left(\|\mathbb{A}\|x_1^2\right) \\
& \leq M_1\|y_1\| + M_1 M_2 \epsilon + M_1\theta(x_1),
\end{aligned} \tag{20}
$$

where

$$
\begin{aligned}
\theta(x) ={} & \mathbb{E}_2\left(\|\mathbb{A}\|(x-h)^2\right)\|\psi\|_C + x\mathbb{E}_{2,2}\left(\|\mathbb{A}\|x^2\right)\|\psi'\|_C \\
& + \frac{x^2\left(\|\mathbb{A}\|\|\psi\|_C + M_f\right)}{2}\mathbb{E}_{2,2}\left(\|\mathbb{A}\|x^2\right).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
& \|u_y(x) - u_z(x)\| \\
& \leq M_1 \int_0^{x_1} \|\mathcal{M}_h(\mathbb{A}(x_1-h-\vartheta))\|\|f(\vartheta,y(\vartheta)) - f(\vartheta,z(\vartheta))\|d\vartheta \\
& \leq M_1 \int_0^{x_1} \|\mathcal{M}_h(\mathbb{A}(x_1-h-\vartheta))\|L_f(\vartheta)\|y(\vartheta) - z(\vartheta)\|d\vartheta \\
& \leq M_1 M_2 \|y-z\|_{C(\Omega)}.
\end{aligned} \tag{21}
$$

We also define the operators $\mathcal{L}_1$, $\mathcal{L}_2$ on $\mathbb{B}_\epsilon$ as follows:

$$
\begin{aligned}
(\mathcal{L}_1 y)(x) ={} & \mathcal{H}_h(\mathbb{A}(x-h))\psi(0) + \mathcal{M}_h(\mathbb{A}(x-h))\psi'(0) \\
& - \mathbb{A}\int_{-h}^{0} \mathcal{M}_h(\mathbb{A}(x-2h-\vartheta))\psi(\vartheta)d\vartheta \\
& + \int_0^x \mathcal{M}_h(\mathbb{A}(x-h-\vartheta))\mathbb{B}u_y(\vartheta)d\vartheta,
\end{aligned} \tag{22}
$$

$$
(\mathcal{L}_2 y)(x) = \int_0^x \mathcal{M}_h(\mathbb{A}(x-h-\vartheta))f(\vartheta,y(\vartheta))d\vartheta. \tag{23}
$$

Now, we see that B_ϵ is a closed, bounded, and convex set of $C([-h,x_1],\mathbb{R}^n)$. Therefore, our proof is divided into three main steps.

Step 1. We prove $\mathcal{L}_1 y + \mathcal{L}_2 z \in B_\epsilon$ for all $y,z \in B_\epsilon$.

For each $x \in \Omega$ and $y,z \in B_\epsilon$, using (20), we obtain

$$
\begin{aligned}
&\|\mathcal{L}_1 y + \mathcal{L}_2 z\|_{C[-h,x_1]} \\
&= \sup_{x\in[-h,x_1]} \|(\mathcal{L}_1 y + \mathcal{L}_2 z)(x)\| \\
&\leq \sup_{x\in[-h,x_1]} \{\|\mathcal{H}_h(\mathbb{A}(x-h))\|\|\psi(0)\| + \|\mathcal{M}_h(\mathbb{A}(x-h))\|\|\psi'(0)\| \\
&+\|\mathbb{A}\| \int_{-h}^{0} \|\mathcal{M}_h(\mathbb{A}(x-2h-\vartheta))\|\|\psi(\vartheta)\|d\vartheta \\
&+ \int_0^x \|\mathcal{M}_h(\mathbb{A}(x-h-\vartheta))\|\|\mathbb{B}\|\|u_y(\vartheta)\|d\vartheta \\
&+ \int_0^x \|\mathcal{M}_h(\mathbb{A}(x-h-\vartheta))\|\|f(\vartheta,z(\vartheta))\|d\vartheta\} \\
&\leq \mathbb{E}_2\Big(\|\mathbb{A}\|(x-h)^2\Big)\|\psi\|_C + x\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x^2\Big)\|\psi'\|_C \\
&+ \frac{x^2\|\mathbb{A}\|\|\psi\|_C}{2}\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x^2\Big) + \frac{M_f x^2}{2}\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x^2\Big) \\
&+ \frac{x^2}{2}\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x^2\Big)\|\mathbb{B}\|(M_1\|y_1\| + M_1M_2\epsilon + M_1\theta(x_1))d\vartheta \\
&+ \frac{x^{2-\frac{1}{q}}}{(p+1)^{\frac{1}{p}}}\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x^2\Big)\Big\|L_f\Big\|_{L^q(\Omega,\mathbb{R}^+)}\|z\|_{C(\Omega)} \\
&\leq \theta(x_1) + M_2\epsilon + \frac{M_1x^2}{2}\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x_1^2\Big)\|\mathbb{B}\|\|y_1\| \\
&+ \frac{M_1M_2\epsilon x^2}{2}\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x_1^2\Big)\|\mathbb{B}\| + \frac{M_1\theta(x_1)x^2}{2}\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x_1^2\Big)\|\mathbb{B}\| \\
&\leq \theta(x_1)\left[1 + \frac{M_1x_1^2}{2}\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x_1^2\Big)\|\mathbb{B}\|\right] + \frac{M_1x_1^2}{2}\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x_1^2\Big)\|\mathbb{B}\|\|y_1\| \\
&+ M_2\left[1 + \frac{M_1x_1^2}{2}\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x_1^2\Big)\|\mathbb{B}\|\right]\epsilon.
\end{aligned}
$$

Thus, for some ϵ sufficiently large and from (17), we have $\mathcal{L}_1 y + \mathcal{L}_2 z \in B_\epsilon$.

Step 2. We prove that $\mathcal{L}_1 : B_\epsilon \to C([-h,x_1],\mathbb{R}^n)$ is a contraction.

For each $x \in \Omega$ and $y,z \in B_\epsilon$, using (21), we obtain

$$
\begin{aligned}
\|(\mathcal{L}_1 y)(x) - (\mathcal{L}_1 z)(x)\| &\leq \int_0^x \|\mathcal{M}_h(\mathbb{A}(x-h-\vartheta))\|\|\mathbb{B}\|\|u_y(\vartheta) - u_z(\vartheta)\|d\vartheta \\
&\leq \|\mathbb{B}\|M_1M_2\|y-z\|_{C(\Omega)} \int_0^x \|\mathcal{M}_h(\mathbb{A}(x-h-\vartheta))\|d\vartheta \\
&\leq \frac{x_1^2\|\mathbb{B}\|M_1M_2}{2}\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x_1^2\Big)\|y-z\|_{C(\Omega)} \\
&\leq \mu\|y-z\|_{C(\Omega)},
\end{aligned}
$$

where $\mu := \frac{x_1^2\|\mathbb{B}\|M_1M_2}{2}\mathbb{E}_{2,2}(\|\mathbb{A}\|x_1^2)$. From (17), note $\mu < 1$, we conclude that $\mathcal{L}_1$ is a contraction mapping.

Step 3. We prove $\mathcal{L}_2 : B_\epsilon \to C([-h,x_1],\mathbb{R}^n)$ is a continuous compact operator.

Firstly, we show that $\mathcal{L}_2$ is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ as $n \to \infty$ in B_ϵ. Thus, for each $x \in \Omega$, using (23) and Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}
&\|(\mathcal{L}_2 y_n)(x) - (\mathcal{L}_2 y)(x)\| \\
&\leq \int_0^x \|\mathcal{M}_h(\mathbb{A}(x-h-\vartheta))\| \|f(\vartheta, y_n(\vartheta)) - f(\vartheta, y(\vartheta))\| d\vartheta \\
&\leq \int_0^x (x-\vartheta)\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|(x-\vartheta)^2\Big) L_f(\vartheta) \|y_n(\vartheta) - y(\vartheta)\| d\vartheta \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Hence, $\mathcal{L}_2 : B_\epsilon \to C([-h, x_1], \mathbb{R}^n)$ is continuous.

Next, we prove that $\mathcal{L}_2$ is uniformly bounded on B_ϵ. For each $x \in \Omega$, $y \in B_\epsilon$, we have

$$\begin{aligned}
\|\mathcal{L}_2 y\| &= \sup_{x\in\Omega} \|(\mathcal{L}_2 y)(x)\| \\
&\leq \sup_{x\in\Omega} \left\{ \int_0^x \|\mathcal{M}_h(\mathbb{A}(x-h-\vartheta))\| \|f(\vartheta, y(\vartheta))\| d\vartheta \right\} \\
&\leq \frac{x^{2-\frac{1}{q}}}{(p+1)^{\frac{1}{p}}} \mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x^2\Big) \Big\|L_f\Big\|_{L^q(\Omega,\mathbb{R}^+)} \|y\|_{C(\Omega)} \\
&+ \frac{M_f x^2}{2} \mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x^2\Big) \\
&\leq M_2 \epsilon + \frac{M_f x_1^2}{2} \mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x_1^2\Big),
\end{aligned}$$

which implies that $\mathcal{L}_2$ is uniformly bounded on B_ϵ.

It remains to show that $\mathcal{L}_2$ is equicontinuous. For each $x_2, x_3 \in \Omega$, $0 < x_2 < x_3 \leq x_1$ and $y \in B_\epsilon$, using (23), we obtain

$$\begin{aligned}
&(\mathcal{L}_2 y)(x_3) - (\mathcal{L}_2 y)(x_2) \\
&\leq \int_0^{x_3} \mathcal{M}_h(\mathbb{A}(x_3-h-\vartheta)) f(\vartheta, y(\vartheta)) d\vartheta \\
&- \int_0^{x_2} \mathcal{M}_h(\mathbb{A}(x_2-h-\vartheta)) f(\vartheta, y(\vartheta)) d\vartheta \\
&= K_1 + K_2,
\end{aligned}$$

where

$$\Psi_1 = \int_{x_2}^{x_3} \mathcal{M}_h(\mathbb{A}(x_3-h-\vartheta)) f(\vartheta, y(\vartheta)) d\vartheta,$$

and

$$\Psi_2 = \int_0^{x_2} [\mathcal{M}_h(\mathbb{A}(x_3-h-\vartheta)) - \mathcal{M}_h(\mathbb{A}(x_2-h-\vartheta))] f(\vartheta, y(\vartheta)) d\vartheta.$$

Thus,

$$\|(\mathcal{L}_2 y)(x_3) - (\mathcal{L}_2 y)(x_2)\| \leq \|\Psi_1\| + \|\Psi_2\|. \tag{24}$$

Now, we can check $\|\Psi_i\| \to 0$ as $x_2 \to x_3$, $i = 1, 2$. For Ψ_1, we obtain

$$\begin{aligned}\Psi_1 &\leq \int_{x_2}^{x_3}(x_3-\vartheta)\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|(x_3-\vartheta)^2\Big)L_f(\vartheta)\|y(\vartheta)\|d\vartheta \\ &+\int_{x_2}^{x_3}(x_3-\vartheta)\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|(x_3-\vartheta)^2\Big)\|f(\vartheta,0)\|d\vartheta \\ &\leq \frac{(x_3-x_2)^{2-\frac{1}{q}}}{(p+1)^{\frac{1}{p}}}\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x_3^2\Big)\Big\|L_f\Big\|_{L^q(\Omega,\mathbb{R}^+)}\|y\|_{C(\Omega)} \\ &+\frac{M_f(x_3-x_2)^2}{2}\mathbb{E}_{2,2}\Big(\|\mathbb{A}\|x_3^2\Big)\to 0, \text{ as } x_2\to x_3.\end{aligned}$$

For Ψ_2, we obtain

$$\begin{aligned}\|\Psi_2\| &\leq \epsilon\int_0^{x_2}\|\mathcal{M}_h(\mathbb{A}(x_3-h-\vartheta))-\mathcal{M}_h(\mathbb{A}(x_2-h-\vartheta))\|L_f(\vartheta)d\vartheta \\ &+M_f\int_0^{x_2}\|\mathcal{M}_h(\mathbb{A}(x_3-h-\vartheta))-\mathcal{M}_h(\mathbb{A}(x_2-h-\vartheta))\|d\vartheta.\end{aligned}$$

From (6), we know that $\mathcal{M}_h(Ax)$ is uniformly continuous for $x\in\Omega$. Hence,

$$\|\mathcal{M}_h(\mathbb{A}(x_3-h-\vartheta))-\mathcal{M}_h(\mathbb{A}(x_2-h-\vartheta))\|\to 0, \text{ as } x_2\to x_3.$$

Therefore, we have $\|\Psi_i\|\to 0$ as $x_2\to x_3$, $i=1,2$, which implies that, using (24),

$$\|(\mathcal{L}_2y)(x_3)-(\mathcal{L}_2y)(x_2)\|\to 0, \quad \text{as } x_2\to x_3,$$

for all $y\in B_\epsilon$. Thus, the Arzelà-Ascoli theorem tells us that $\mathcal{L}_2$ is compact on B_ϵ.

Therefore, according to Krasnoselskii's fixed point theorem (Lemma 4), $\mathcal{L}_1+\mathcal{L}_2$ has a fixed point y on B_ϵ. In addition, y is also a solution of (2) and $(\mathcal{L}_1y+\mathcal{L}_2y)(x_1)=y_1$. This means that u_y steers the system (2) from y_0 to y_1 in finite time x_1, which implies that (2) is controllable on Ω. This completes the proof. □

Corollary 3. *Let $\mathbb{A}=\mathbb{A}^2$ in (2). Then, Theorem 2 holds.*

Corollary 4. *Let $\mathbb{A}=\mathbb{A}^2$ in (2) such that $\mathbb{A}$ is a nonsingular $n\times n$ matrix. Then, Theorem 2 coincides with Theorem 4.1 in [17].*

Proof. Since $\mathcal{M}_h\big(\mathbb{A}^2(x)\big)=\mathbb{A}^{-1}\sin_h(\mathbb{A}x)$. From (**G1**) and Hölder inequality, we obtain

$$\begin{aligned}&\int_0^x\Big\|\mathcal{M}_h\Big(\mathbb{A}^2(x-h-\vartheta)\Big)\Big\|L_f(\vartheta)d\vartheta \\ &=\Big\|\mathbb{A}^{-1}\Big\|\int_0^x\Big\|\mathbb{A}\mathcal{M}_h\Big(\mathbb{A}^2(x-h-\vartheta)\Big)\Big\|L_f(\vartheta)d\vartheta \\ &\leq\Big\|\mathbb{A}^{-1}\Big\|\int_0^x\|\sin_h(\mathbb{A}(x-h-\vartheta))\|L_f(\vartheta)d\vartheta \\ &\leq\Big\|\mathbb{A}^{-1}\Big\|\int_0^x\|\sinh(\|\mathbb{A}\|(x-\vartheta))\|L_f(\vartheta)d\vartheta \\ &\leq\Big\|\mathbb{A}^{-1}\Big\|\bigg(\int_0^x(\sinh(\|\mathbb{A}\|(x-\vartheta)))^pd\vartheta\bigg)^{\frac{1}{p}}\bigg(\int_0^x L_f^q(\vartheta)d\vartheta\bigg)^{\frac{1}{q}} \\ &=\Big\|\mathbb{A}^{-1}\Big\|\bigg(\int_0^x\frac{\exp(\|\mathbb{A}\|p(x-\vartheta))}{2^p}d\vartheta\bigg)^{\frac{1}{p}}\bigg(\int_0^x L_f^q(\vartheta)d\vartheta\bigg)^{\frac{1}{q}} \\ &=\Big\|\mathbb{A}^{-1}\Big\|\bigg(\frac{1}{2^p\|\mathbb{A}\|p}(\exp(\|\mathbb{A}\|px-1))\bigg)^{\frac{1}{p}}\Big\|L_f\Big\|_{L^q(\Omega,\mathbb{R}^+)}.\end{aligned} \tag{25}$$

and

$$\begin{aligned}
&\int_0^x \left\| \mathcal{M}_h\left(\mathbb{A}^2(x-h-\vartheta)\right) \right\| \|f(\vartheta,0)\| d\vartheta \\
&= \left\| \mathbb{A}^{-1} \right\| \int_0^x \|\sin_h(\mathbb{A}(x-h-\vartheta))\| \|f(\vartheta,0)\| d\vartheta \\
&\leq M_f \left\| \mathbb{A}^{-1} \right\| \int_0^x \|\sinh(\|\mathbb{A}\|(x-\vartheta))\| d\vartheta \\
&= \frac{M_f \|\mathbb{A}^{-1}\|}{\|\mathbb{A}\|} \|\cosh(\|\mathbb{A}\|x) - 1\|.
\end{aligned} \tag{26}$$

By a similar way in the proof of Theorem 2 at $\mathbb{A} = \mathbb{A}^2$ and by virtue of (25) and (26), we obtain the same conclusion in Theorem 4.1 in [17]. This ends the proof. □

Remark 2. *We note that Corollary 1 extends Theorems 3.1 and 4.1 in [17] by choosing the matrix $\mathbb{A}$ as an arbitrary, not necessarily squared matrix, and Corollaries 2 and 4 coincide with Theorems 3.1 and 4.1 in [17]. Therefore, our results in Corollaries 1–4 extend and improve Theorems 3.1 and 4.1 in [17] by removing the condition that $\mathbb{A}$ is a nonsingular matrix.*

5. Hyers–Ulam Stability of Nonlinear Delay Differential System

In this section, we discuss the Hyers–Ulam stability of (2) on the finite time interval $[0, x_1]$.

Theorem 3. *Let* (**G1**), (**G2**) *and (17) be satisfied. Then, the system (2) is Hyers–Ulam stable.*

Proof. With the help of Theorem 2, let $z \in C(\Omega, \mathbb{R}^n)$ be a solution of the inequality (8) and y be the unique solution of (2), that is,

$$\begin{aligned}
y(x) = &\mathcal{H}_h(\mathbb{A}(x-h))\psi(0) + \mathcal{M}_h(\mathbb{A}(x-h))\psi'(0) \\
&- \mathbb{A} \int_{-h}^0 \mathcal{M}_h(\mathbb{A}(x-2h-\vartheta))\psi(\vartheta) d\vartheta \\
&+ \int_0^x \mathcal{M}_h(\mathbb{A}(x-h-\vartheta)) f(\vartheta, y(\vartheta)) d\vartheta \\
&+ \int_0^x \mathcal{M}_h(\mathbb{A}(x-h-\vartheta)) \mathbb{B} u_y(\vartheta) d\vartheta.
\end{aligned}$$

From Lemma 3, in a similar way to the proof of Theorem 2, and by virtue of (21), we obtain

$$\begin{aligned}
\|z(x) - y(x)\| &\leq \|z(x) - z^*(x)\| + \|z^*(x) - y(x)\| \\
&\leq \frac{x^2 \varepsilon}{2} \mathbb{E}_{2,2}\left(\|\mathbb{A}\| x^2\right) \\
&+ \int_0^x \|\mathcal{M}_h(\mathbb{A}(x-h-\vartheta))\| \|\mathbb{B}\| \left\| u_z(\vartheta) - u_y(\vartheta) \right\| d\vartheta \\
&+ \int_0^x \|\mathcal{M}_h(\mathbb{A}(x-h-\vartheta))\| \|f(\vartheta, z(\vartheta)) - f(\vartheta, y(\vartheta))\| d\vartheta \\
&\leq \frac{x_1^2 \varepsilon}{2} \mathbb{E}_{2,2}\left(\|\mathbb{A}\| x_1^2\right) \\
&+ \frac{x_1^2 \|\mathbb{B}\| M_1 M_2}{2} \mathbb{E}_{2,2}\left(\|\mathbb{A}\| x_1^2\right) \|z - y\|_{C(\Omega)} \\
&+ M_2 \|z - y\|_{C(\Omega)} \\
&= \frac{x_1^2 \varepsilon}{2} \mathbb{E}_{2,2}\left(\|\mathbb{A}\| x_1^2\right) \\
&+ M_2 \left(1 + \frac{x_1^2 \|\mathbb{B}\| M_1}{2} \mathbb{E}_{2,2}\left(\|\mathbb{A}\| x_1^2\right)\right) \|z - y\|_{C(\Omega)}.
\end{aligned}$$

Therefore,

$$\|z-y\|_{C(\Omega)} \leq \frac{x_1^2\varepsilon}{2(1-\rho)}\mathbb{E}_{2,2}\left(\|\mathbb{A}\|x_1^2\right),$$

where

$$\rho := M_2\left(1+\frac{x_1^2\|\mathbb{B}\|M_1}{2}\mathbb{E}_{2,2}\left(\|\mathbb{A}\|x_1^2\right)\right).$$

Thus,

$$\|z(x)-y(x)\| \leq N\varepsilon, \quad N = \frac{x_1^2}{2(1-\rho)}\mathbb{E}_{2,2}\left(\|\mathbb{A}\|x_1^2\right).$$

This completes the proof. □

6. Examples

In this section, we present applications of the results derived.

Example 1. *Consider the following linear delay differential controlled system:*

$$\begin{aligned} &y''(x)+\mathbb{A}y(x-0.5)=\mathbb{B}u(x), \quad \text{for } x\in\Omega:=[0,1],\\ &y(x)\equiv\psi(x),\ y'(x)\equiv\psi'(x) \quad \text{for } -0.5\leq x\leq 0, \end{aligned} \tag{27}$$

where

$$\mathbb{A}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ \mathbb{B}=\begin{pmatrix}1\\2\end{pmatrix},\ \psi(x)=\begin{pmatrix}2x\\x\end{pmatrix},\ \psi'(x)=\begin{pmatrix}2\\1\end{pmatrix}.$$

We note that $\mathbb{B}\in\mathbb{R}^{2\times1}$ *and* $u(x)\in\mathbb{R}$ *shows the control vector. Constructing the corresponding delay Gramian matrix of (27) via (9), we obtain*

$$\begin{aligned} W_{0.5}^{\mathcal{M}}[0,1] &= \int_0^1 \mathcal{M}_{0.5}(\mathbb{A}(0.5-\vartheta))\mathbb{B}\mathbb{B}^T\mathcal{M}_{0.5}\left(\mathbb{A}^T(0.5-\vartheta)\right)d\vartheta\\ &=: O_1+O_2, \end{aligned}$$

where

$$O_1 = \int_0^{0.5}\mathcal{M}_{0.5}(\mathbb{A}(0.5-\vartheta))\mathbb{B}\mathbb{B}^T\mathcal{M}_{0.5}\left(\mathbb{A}^T(0.5-\vartheta)\right)d\vartheta,$$

for $(0.5-\vartheta)\in(0,0.5)$,

$$O_2 = \int_{0.5}^{1}\mathcal{M}_{0.5}(\mathbb{A}(0.5-\vartheta))\mathbb{B}\mathbb{B}^T\mathcal{M}_{0.5}\left(\mathbb{A}^T(0.5-\vartheta)\right)d\vartheta,$$

for $(0.5-\vartheta)\in(-0.5,0)$, *where*

$$\mathcal{H}_{0.5}(\mathbb{A}(x)) := \begin{cases} \Theta, & -\infty<x<-0.5,\\ \mathbb{I}, & -0.5\leq x<0,\\ \mathbb{I}-\mathbb{A}\frac{x^2}{2}, & 0\leq x<0.5,\\ \mathbb{I}-\mathbb{A}\frac{x^2}{2}+\mathbb{A}^2\frac{(x-0.5)^4}{4!}, & 0.5\leq x<1, \end{cases}$$

and

$$\mathcal{M}_{0.5}(\mathbb{A}(x)) := \begin{cases} \Theta, & -\infty<x<-0.5,\\ \mathbb{I}(x+0.5), & -0.5\leq x<0,\\ \mathbb{I}(x+0.5)-\mathbb{A}\frac{x^3}{3!}, & 0\leq x<0.5,\\ \mathbb{I}(x+0.5)-\mathbb{A}\frac{x^3}{3!}+\mathbb{A}^2\frac{(x-0.5)^5}{5!}, & 0.5\leq x<1. \end{cases}$$

Next, we can calculate that

$$O_1=\begin{pmatrix}0.28242&0.57396\\0.57396&1.1667\end{pmatrix}, \qquad O_2=\begin{pmatrix}4.1667\times10^{-2}&8.3333\times10^{-2}\\8.3333\times10^{-2}&0.16667\end{pmatrix}.$$

Then, we obtain

$$W_{0.5}^{\mathcal{M}}[0,1] = O_1 + O_2 = \begin{pmatrix} 0.32409 & 0.65729 \\ 0.65729 & 1.3334 \end{pmatrix},$$

and

$$\left(W_{0.5}^{\mathcal{M}}\right)^{-1}[0,1] = \begin{pmatrix} 11962.865 & -5897.01 \\ -5897.01 & 2907.638 \end{pmatrix}.$$

Therefore, we see that $W_{0.5}^{\mathcal{M}}[0,1]$ *is positive definite. Furthermore, for any finite terminal conditions* $y_1, y_1' \in \mathbb{R}^2$ *such that* $y(x_1) = y_1 = (y_{11}, y_{12})^T$, $y'(x_1) = y_1' = (y_{11}', y_{12}')^T$*; as a result, we can establish the corresponding control as follows:*

$$u(x) = \mathbb{B}^T \mathcal{M}_{0.5}\left(\mathbb{A}^T(0.5-x)\right)\left(W_{0.5}^{\mathcal{M}}\right)^{-1}[0,1]\beta,$$

where

$$\begin{aligned} \beta &= y_1 - \mathcal{M}_{0.5}(\mathbb{A}(0.5))\psi'(0) + \mathbb{A}\int_{-0.5}^{0} \mathcal{M}_{0.5}(\mathbb{A}(-\vartheta))\psi(\vartheta)d\vartheta \\ &= \begin{pmatrix} y_{11} - 2.1042 \\ y_{12} - 1 \end{pmatrix}. \end{aligned}$$

Hence, the system (27) is controllable on $[0,1]$ *by Theorem 1.*

Example 2. *Consider the following nonlinear delay differential controlled system:*

$$\begin{gathered} y''(x) + \mathbb{A}y(x-0.6) = f(x, y(x)) + \mathbb{B}u(x), \text{ for } x \in \Omega_1 := [0,1.2], \\ y(x) \equiv \psi(x),\ y'(x) \equiv \psi'(x) \text{ for } -0.6 \le x \le 0, \end{gathered} \tag{28}$$

where

$$\mathbb{A} = \mathbb{B} = \mathbb{I}^{2\times 2},\ \psi(x) = \begin{pmatrix} 3x+1 \\ x^2 \end{pmatrix},\ \psi'(x) = \begin{pmatrix} 3 \\ 2x \end{pmatrix},$$

$$f(x, y(x)) = \begin{pmatrix} 0.5(x-0.6)\cos[y_1(x)] \\ 0.5(x-0.6)\cos[y_2(x)] \end{pmatrix}.$$

Now, we set $u(x) = \widetilde{y}$*, where* $\widetilde{y} \in \mathbb{R}^2$*. From the definition of* Q *in* (**G2**)*, we obtain*

$$\begin{aligned} Q &= \int_0^{1.2} \mathcal{M}_{0.6}(\mathbb{A}(0.6-\vartheta))\mathbb{B}d\vartheta\widetilde{y} \\ &= \int_0^{0.6} \mathcal{M}_{0.6}(\mathbb{A}(0.6-\vartheta))d\vartheta\widetilde{y} + \int_{0.6}^{1.2} \mathcal{M}_{0.6}(\mathbb{A}(0.6-\vartheta))d\vartheta\widetilde{y} \\ &= \int_0^{0.6}\left[\mathbb{I}(1.2-\vartheta) - \mathbb{I}\frac{(0.6-\vartheta)^3}{3!}\right]d\vartheta\widetilde{y} + \int_{0.6}^{1.2}\mathbb{I}(1.2-\vartheta)d\vartheta\widetilde{y} \\ &= \begin{pmatrix} 0.5346 & 0 \\ 0 & 0.5346 \end{pmatrix}\widetilde{y} + \begin{pmatrix} 0.18 & 0 \\ 0 & 0.18 \end{pmatrix}\widetilde{y} \\ &= \begin{pmatrix} 0.7146 & 0 \\ 0 & 0.7146 \end{pmatrix}\widetilde{y}, \end{aligned}$$

where

$$\mathcal{M}_{0.6}(\mathbb{A}(x)) := \begin{cases} \Theta, & -\infty < x < -0.6, \\ \mathbb{I}(x+0.6), & -0.6 \le x < 0, \\ \mathbb{I}(x+0.6) - \mathbb{A}\frac{x^3}{3!}, & 0 \le x < 0.6, \\ \mathbb{I}(x+0.6) - \mathbb{A}\frac{x^3}{3!} + \mathbb{A}^2\frac{(x-0.6)^5}{5!}, & 0.6 \le x < 1. \end{cases}$$

Define the inverse $Q^{-1} : \mathbb{R}^2 \to L^2(\Omega_1, \mathbb{R}^2)$ *by*

$$\left(Q^{-1}\widetilde{y}\right)(x) := \begin{pmatrix} 1.3994 & 0 \\ 0 & 1.3994 \end{pmatrix}\widetilde{y}.$$

Then, we obtain

$$\left\|\left(Q^{-1}\widetilde{y}\right)(x)\right\| \leq \left\|\begin{pmatrix} 1.3994 & 0 \\ 0 & 1.3994 \end{pmatrix}\right\| \|\widetilde{y}\| = 1.3994\|\widetilde{y}\|,$$

and thus, we obtain $\|Q^{-1}\| \leq 1.3994 =: M_1$. *Hence, the assumption* (**G2**) *is satisfied by* Q.

Next, keep in mind that $|\cos\lambda - \cos\delta| \leq |\lambda - \delta|$, *for all* $\lambda, \delta \in \mathbb{R}$, *we have*

$$\begin{aligned}
&\|f(x,y) - f(x,z)\| \\
&= |0.5(x-0.6)|\sqrt{(\cos[y_1(x)] - \cos[z_1(x)])^2 + (\cos[y_2(x)] - \cos[z_2(x)])^2} \\
&\leq |0.5(x-0.6)|\sqrt{(y_1(x) - z_1(x))^2 + (y_2(x) - z_2(x))^2} \\
&= |0.5(x-0.6)|\|y - z\|,
\end{aligned}$$

for all $x \in \Omega_1$, *and* $y(x), z(x) \in \mathbb{R}^2$. *We set* $L_f(x) = |0.5(x-0.6)|$ *such that* $L_f \in L^q(\Omega_1, \mathbb{R}^+)$ *in* (**G1**). *By choosing* $p = q = 2$, *we have*

$$\left\|L_f\right\|_{L^2(\Omega_1,\mathbb{R}^+)} = \left(\int_0^{1.2} [0.5(\vartheta - 0.6)]^2 d\vartheta\right)^{\frac{1}{2}} = 0.18974.$$

Then, we obtain

$$M_2 = \frac{(1.2)^{1+\frac{1}{2}}}{(3)^{\frac{1}{2}}}\mathbb{E}_{2,2}\left((1.2)^2\right)\left\|L_f\right\|_{L^q(\Omega,\mathbb{R}^+)} = 0.18114.$$

Finally, we calculate that

$$M_2\left[1 + \frac{M_1(1.2)^{1.8}}{1.8}\mathbb{E}_{1.8,1.8}\left(\|\mathbb{A}\|(1.2)^2\right)\|\mathbb{B}\|\right] = 0.41072 < 1,$$

which implies that all the conditions of Theorems 2 and 3 are satisfied. Therefore, the system (28) is controllable and Hyers–Ulam stable.

Remark 3. *It is worth noting that Theorems 3.1 and 4.1 in [17] are not applicable to ascertaining the controllability of the systems (27) and (28) because the square of matrix* $\mathbb{A}$ *is used in [17] rather than* $\mathbb{A}$, *and the systems (27) and (28) are considered with matrix* $\mathbb{A}$ *rather than* $\mathbb{A}^2$. *That is, the term* $\mathbb{A}^2 y(x-h)$ *is replaced by* $\mathbb{A}y(x-h)$; *then, the definition of* $\sin_h \mathbb{A}x$ *and* $\cos_h \mathbb{A}x$ *must be modified by using the square root* $\sqrt{\mathbb{A}}$ *instead of* $\mathbb{A}$. *However,* $\sqrt{\mathbb{A}}$, *in the general case, does not exist as in Example 1 or may not be unique (including the possibility of infinitely many different square roots as in Example 2). Therefore, these two examples demonstrate the effectiveness of the obtained results.*

7. Conclusions

In this work, we established some sufficient and necessary conditions for the controllability of linear delay differential systems by using a delay Gramian matrix and the representation of solutions of these systems with the help of their delayed matrix functions. Furthermore, we established some sufficient conditions of controllability and Hyers–Ulam stability of nonlinear delay differential systems by applying Krasnoselskii's fixed point theorem and the representation of solutions of these systems. Finally, we gave two examples to demonstrate the effectiveness of the obtained results. The results are applicable to all singular, non-singular and arbitrary matrices, not necessarily squared. As a result, our results improve, extend, and complement the existing ones in [17].

One possible direction in which to extend the results of this paper is toward fractional differential and conformable fractional differential systems of order $\alpha \in (1, 2]$. Another challenge is to find out if similar results can be derived in the case of variable delays in (1) and (2).

Author Contributions: All authors contributed equally in this research paper. All authors have read and agreed to the published version of the manuscript.

Funding: The research was supported Partially by National Natural Science Foundation of China (Grant Nos. 10871056 and 10971150).

Acknowledgments: The authors sincerely appreciate the editors and anonymous referees for their carefully reading and helpful comments for improving this paper. The research was supported Partially by National Natural Science Foundation of China (Grant Nos. 10871056 and 10971150).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Obembe, A.D.; Hossain, M.E.; Abu-Khamsin, S.A. Variable-order derivative time fractional diffusion model for heterogeneous porous media. *J. Pet. Sci. Eng.* **2017**, *152*, 391–405. [CrossRef]
2. Coimbra, C.F.M. Mechanics with variable-order differential operators. *Ann. Phys.* **2003**, *12*, 692–703. [CrossRef]
3. Heymans, N.; Podlubny, I. Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. *Rheol. Acta* **2006**, *45*, 765–771. [CrossRef]
4. Sweilam, N.H.; Al-Mekhlafi, S.M. Numerical study for multi-strain tuberculosis (TB) model of variable-order fractional derivatives. *J. Adv. Res.* **2016**, *7*, 271–283. [CrossRef] [PubMed]
5. Diethelm, K. *The Analysis of Fractional Differential Equations*; Springer: Berlin, Germany, 2010.
6. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science BV: Amsterdam, The Netherlands, 2006.
7. Tarasov, V. *Handbook of Fractional Calculus with Applications*; de Gruyter: Berlin, Germany, 2019.
8. Khusainov, D.Y.; Shuklin, G.V. Linear autonomous time-delay system with permutation matrices solving. *Stud. Univ. Zilina Math. Ser.* **2003**, *17*, 101–108.
9. Khusainov, D.Y.; Diblík, J.; Růžičková, M.; Lukáčová, J. Representation of a solution of the Cauchy problem for an oscillating system with pure delay. *Nonlinear Oscil.* **2008**, *11*, 276–285. [CrossRef]
10. Elshenhab, A.M.; Wang, X.T. Representation of solutions for linear fractional systems with pure delay and multiple delays. *Math. Meth. Appl. Sci.* **2021**, *44*, 12835–12850. [CrossRef]
11. Elshenhab, A.M.; Wang, X.T. Representation of solutions of linear differential systems with pure delay and multiple delays with linear parts given by non-permutable matrices. *Appl. Math. Comput.* **2021**, *410*, 126443. [CrossRef]
12. Elshenhab, A.M.; Wang, X.T. Representation of solutions of delayed linear discrete systems with permutable or nonpermutable matrices and second-order differences. *Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **2022**, *116*, 58. [CrossRef]
13. Diblík, J.; Fečkan, M.; Pospíšil, M. Representation of a solution of the Cauchy problem for an oscillating system with multiple delays and pairwise permutable matrices. *Abstr. Appl. Anal.* **2013**, *2013*, 931493. [CrossRef]
14. Diblík, J.; Mencáková, K. Representation of solutions to delayed linear discrete systems with constant coefficients and with second-order differences. *Appl. Math. Lett.* **2020**, *105*, 106309. [CrossRef]
15. Liu, L.; Dong, Q.; Li, G. Exact solutions and Hyers–Ulam stability for fractional oscillation equations with pure delay. *Appl. Math. Lett.* **2021**, *112*, 106666. [CrossRef]
16. Diblík, J.; Khusainov, D.Y.; Baštinec, J.; Sirenko, A.S. Exponential stability of linear discrete systems with constant coefficients and single delay. *Appl. Math. Lett.* **2016**, *51*, 68–73. [CrossRef]
17. Liang,C.; Wang, J.; O'Regan, D. Controllability of nonlinear delay oscillating systems. *Electron. J. Qual. Theory Differ. Equ.* **2017**, *2017*, 1–18. [CrossRef]
18. Diblík, J.; Fečkan, M.; Pospíšil, M. On the new control functions for linear discrete delay systems. *SIAM J. Control Optim.* **2014**, *52*, 1745–1760. [CrossRef]
19. Yi, S.; Nelson, P.W.; Ulsoy, A.G. Controllability and observability of systems of linear delay differential equation via the matrix Lambert W function. *IEEE Trans. Automat. Control* **2008**, *53*, 854–860. [CrossRef]
20. Wang, J.; Luo, Z.; Fečkan, M. Relative controllability of semilinear delay differential systems with linear parts defined by permutable matrices. *Eur. J. Control* **2017**, *38*, 39–46. [CrossRef]
21. Khusainov, D.Y.; Shuklin, G.V. Relative controllability in systems with pure delay. *Int. J. Appl. Math.* **2005**, *2*, 210–221. [CrossRef]

22. Diblík, J.; Khusainov, D.Y.; Lukáčová, J.; Růžičková, M. Control of oscillating systems with a single delay. *Adv. Differ. Equ.* **2010**, *2010*, 108218. [CrossRef]
23. Karthikeyan, K.; Tamizharasan, D.; Nieto, J.J.; Nisar, K.S. Controllability of second-order differential equations with state-dependent delay. *IMA J. Math. Control Inform.* **2021**, *38*, 1072–1083. [CrossRef]
24. Klamka, J. *Controllability of Dynamical Systems*; Kluwer Academic: Dordrecht, The Netherlands, 1993.
25. Jung, S.M. *Ulam–Hyers-Rassias Stability of Functional Equations in Mathematical Analysis*; Hadronic Press: Palm Harbor, FL, USA, 2001.
26. Aruldass, A.R.; Pachaiyappan, D.; Park, C. Hyers–Ulam stabilityof second-order differential equations using Mahgoub transform. *Adv. Differ. Equ.* **2021**, *2021*, 23. [CrossRef]
27. RusI, A. Ulam stability of ordinary differential equations. *Stud. Univ. Babeş-Bolyai Math.* **2009**, *54*, 125–133.
28. Sharma, J.P.; George, R.K. Controllability of matrix second order systems: A trigonometric matrix approach. *Electron. J. Differ. Equ.* **2007**, *80*, 1–14.
29. Smart, D.R. *Fixed Point Theorems*; University Press: Cambridge, UK, 1980.