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# New Sampling Expansion Related to Derivatives in Quaternion Fourier Transform Domain

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Abstract: The theory of quaternions has gained a firm ground in recent times and is being widely explored, with the field of signal and image processing being no exception. However, many important aspects of quaternionic signals are yet to be explored, particularly the formulation of Generalized Sampling Expansions (GSE). In the present article, our aim is to formulate the GSE in the realm of a one-dimensional quaternion Fourier transform. We have designed quaternion Fourier filters to reconstruct the signal, using the signal and its derivative. Since derivatives contain information about the edges and curves appearing in images, therefore, such a sampling formula is of substantial importance for image processing, particularly in image super-resolution procedures. Moreover, the presented sampling expansion can be applied in the field of image enhancement, color image processing, image restoration and compression and filtering, etc. Finally, an illustrative example is presented to demonstrate the efficacy of the proposed technique with vivid simulations in MATLAB.

Keywords: quaternion algebra; quaternionic signals; quaternion fourier transform; sampling expansion

MSC: 42A38; 26A33

### 1. Introduction

The Shanon's sampling theorem [1] in the Fourier domain is one of the remarkable, profound and elegant concepts of digital signal processing which serves as a bridge between the analog and digital signals. The theorem asserts that a bandlimited signal can be completely reconstructed from its values at regularly spaced times. Since the seminal work of Shannon, many generalizations of the classical sampling theorem have been developed; however, the most important ramification came in the form of the Generalized Sampling Expansion (GSE), which is often referred as the multi-channel sampling expansion, mainly for the reason that it relies on non-uniform or multi-channel data acquisition. The multichannel sampling procedure has attained a respectable status in the context of signal processing due to the fact that it can be employed in situations where the classical Shannon's sampling procedure is not applicable. For instance, the classical Shanon's sampling theorem is infeasible for broad-band or non-stationary signals in case the sampling rate is not chosen in accordance to the demand of a particular domain; however, the GSE proves to be handy in such situations as it primarily relies on multi-channel data acquisition. As of now, the GSE has been successfully applied in diverse aspects of signal and image processing, such as digital flight control, flexible analog-digital converters, data compression, image super-resolution and several other fields [2–4].

On the other hand, the quaternion algebra has flourished as one of the nicest alternatives to the familiar system of real and complex numbers. The quaternion algebra offers a simple and insightful approach for the efficient representation of signals, wherein several



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**Copyright:** © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). components are to be controlled simultaneously, for instance, in three-dimensional computer graphics, aerospace engineering, artificial intelligence and color image processing. Keeping in view the merits of the quaternion algebra, T.A. Ell [5] introduced the notion of the quaternion Fourier transform (QFT) as an extension of the classical Fourier transform to hyper-complex algebras. Since its advent, the QFT has proven to be a harbinger of new research trends and has been successfully applied in numerous aspects of signal and image processing, particularly in speech recognition, acoustics, three-dimensional color field processing, space color video processing, crystallography, aerospace engineering and for the solution of many types of quaternionic differential equations [6–20].

Owing to the prolificacy of the GSE, the concept has been extended to several new directions in the open literature. For instance, Hoskins and Pinto [21] extended the GSE to bandlimited distributions, Wei et al. [22] introduced the GSE to the fractional Fourier transform, Cheung [23] extended the GSE to multi-dimensional setting and Li and Xu [24] investigated the GSE in the linear canonical domain. Nevertheless, very recently, Shah and Tantary [25] formulated the lattice-based multi-channel sampling theorem in the multi-dimensional linear canonical domain and also demonstrated its applications for signal reconstruction and image super-resolution. In 2016, R. Roopkumar introduced the quaternionic one-dimensional fractional Fourier transform [26]. Bahri et al. defined a onedimensional quaternion Fourier Transform in 2019 [27]. Siddiqui et al. gave the definition of a quaternion one-dimensional linear canonical transform [28]. M.I. Khalil investigated a new steganography technique for hiding a textual message within a cover image using the quaternion Fourier transform [29]. Although the uniform sampling theorems for bandlimited signals in the quaternion domain have been derived in [30], to date, not a single attempt has been made to extend the GSE to the quaternion Fourier domain in one dimension. The aim of this paper is to fill this gap by formulating the GSE for one-dimensional quaternionic bandlimited signals in Fourier transform domain. Such a sampling expansion shall be of critical significance in diverse aspects of color image processing, image enhancement, image restoration and compression, filtering and so on.

The rest of the article is divided into five sections. In Section 2, we present the preliminaries including the fundamental notions of quaternion algebra and the quaternion Fourier transform abreast of their basic properties. Section 3 is completely devoted to the formulation of GSE for one-dimensional bandlimited quaternion signals. Section 4 incorporates the details about the reconstruction formula using the derivatives of the quaternionic signal. In addition, an illustrative example with simulation results describing the applicability of the proposed sampling expansion in signal reconstruction is presented in Section 5. Finally, a conclusion is extracted in Section 6.

## 2. The Quaternion Algebra and Quaternion Fourier Transform

In this section, we present the fundamental notions regarding the celebrated quaternion algebra and the associated Fourier transform, which shall be subsequently used while formulating the main results.

#### 2.1. Quaternion Algebra

The concept of quaternion algebra was introduced by W.R. Hamilton [31,32] in 1843. In the sequel, the quaternion algebra appeared to be designated by the letter H in honor of the Irish mathematician Sir W.R. Hamilton. The quaternion algebra provides an extension of a well-known complex number system to an associative and non-commutative four-dimensional algebra. In the hyper-complex plane, the quaternion algebra is written as follows:

$$H = \left\{ h = h_0 + h_1 \mathbf{i} + h_2 \mathbf{j} + h_3 \mathbf{k}; \quad h_0, h_1, h_2, h_3 \in \mathbb{R} \right\},\$$

where the three imaginary units *i*, *j* and *k* obey the Hamilton's multiplication laws as:

$$ijk = i^2 = j^2 = k^2 = -1$$

and

$$ij = -ji = k$$
,  $jk = -kj = i$ ,  $ki = -ik = j$ 

The addition of any pair of quaternions  $h = h_0 + h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k}$  and  $h' = h'_0 + h'_1\mathbf{i} + h'_2\mathbf{j} + h'_3\mathbf{k}$  is defined component-wise, whereas the multiplication is governed by the following law:

$$h h' = (h_0 h'_0 - h_1 h'_1 - h_2 h'_2 - h_3 h'_3) + (h_1 h'_0 + h_0 h'_1 + h_2 h'_3 - h_3 h'_2) \mathbf{i} + (h_0 h'_2 + h_2 h'_0 + h_3 h'_1 - h_1 h'_3) \mathbf{j} + (h_0 h'_3 + h_3 h'_0 + h_1 h'_2 - h_2 h'_1) \mathbf{k}.$$

Moreover, the conjugate of any quaternion *h* is given by  $\overline{h} = h_0 - h_1 \mathbf{i} - h_2 \mathbf{j} - h_3 \mathbf{k}$ , so that the multiplication of the quaternion *h* with its conjugate  $\overline{h}$  takes the form  $h\overline{h} = h_0^2 + h_1^2 + h_2^2 + h_3^2$ . In addition, the modulus of the quaternion *h* is given by  $|h| = \sqrt{h\overline{h}} = \sqrt{h_0^2 + h_1^2 + h_2^2 + h_3^2}$ . Finally, we note that the inverse of any non-zero quaternion *h* can be expressed as  $h^{-1} = \frac{\overline{h}}{|h|^2}$ .

#### 2.2. The Quaternion Fourier Transform

The quaternion Fourier transform (QFT) serves as a natural extension of the usual complex-valued Fourier transform; however, under such an extension, certain properties of the classical Fourier transform are lost, while most of the fundamental properties, such as the Parseval's and inversion formulae, are preserved under suitable conditions in the hypercomplex algebra. Primarily, the quaternion Fourier transforms can be broadly classified into two categories: the one-dimensional QFT and the two-dimensional QFT. Nonetheless, the extension beyond two dimensions is possible, but the same is best suited in the context of the well-known Clifford algebra: a generalization of both the Grassmann's exterior algebra and Hamilton's algebra of quaternions. One of the intrinsic features of the Clifford analysis is that it encompasses all dimensions at once, as opposed to a multi-dimensional tensorial approach with tensor products of one-dimensional phenomena. To work within the scope of this article, we shall omit any further reference to the Clifford algebra and the associated Fourier transform.

Due to the non-commutativity of the quaternions, there are several approaches of defining the two-dimensional quaternion Fourier transform. In fact, for a given two-dimensional quaternion signal, the two-sided, right-sided and left-sided quaternion Fourier transforms are defined as follows:

(i) Two-sided QFT: For a given two-dimensional quaternionic signal f(u, v), the twosided QFT is denoted by  $F_T(r, s)$  and is defined as:

$$F_{\mathcal{T}}(r,s) = \int_{\mathbb{R}^2} e^{-\mathbf{i}2\pi r u} f(u,v) e^{-\mathbf{j}2\pi s v} du \, dv. \tag{1}$$

(ii) Right-sided QFT: The right-sided QFT for two-dimensional quaternionic signal f(u, v), is denoted by  $F_R(r, s)$  and is defined as:

$$F_{\mathcal{R}}(r,s) = \int_{\mathbb{R}^2} f(u,v) \, e^{-\mathbf{i}2\pi r u} e^{-\mathbf{j}2\pi s v} du \, dv. \tag{2}$$

(iii) Left-sided QFT: For the quaternionic signal f(u, v), the left-sided QFT is denoted by  $F_L(r, s)$  and is defined as:

$$F_{\mathcal{L}}(r,s) = \int_{\mathbb{R}^2} e^{-\mathbf{i}2\pi r u} e^{-\mathbf{j}2\pi s v} f(u,v) \, du \, dv. \tag{3}$$

Not withstanding the notion of two-dimensional QFT, another important variant of the quaternion Fourier transform is the one-dimensional QFT, which is practically reliable in the sense that it is governed by a one-dimensional integral expression. Below, we present the formal definition of the one-dimensional quaternion Fourier transform. **Definition 1.** *One-dimensional QFT:* For a given quaternion-valued signal  $f(t) \in L^1(\mathbb{R}, H)$ , the one-dimensional quaternion Fourier transform is denoted by  $F_Q\{f\}(\omega)$  and is defined as [27]:

$$F_Q\{f\}(\omega) = \int_R f(t) e^{-j2\pi\omega t} dt.$$
(4)

The inversion formula corresponding to the one-dimensional QFT given in (4) is defined in [27]:

$$f(t) = F_Q^{-1} \Big[ F_Q \{f\} \Big](t) = \int_R F_Q \{f\}(\omega) e^{\mathbf{j} 2\pi \omega t} d\omega.$$
(5)

Moreover, if the input function  $f(t) \in L^1(R, H)$  is a continuously differentiable function, then the following differentiation property holds:

$$F_{Q}\left\{\frac{d^{n}f}{dt^{n}}\right\}(\omega) = \left(\mathbf{j}2\pi\omega\right)^{n}F_{Q}\left\{f\right\}(\omega), \quad n \in \mathbb{N}$$
(6)

In addition, other fundamental properties of the one-dimensional QFT given in (4) can be found in [27].

From Definition 1, it follows that, for the function f(t) to be a real value, we can interchange the position of the kernel  $e^{-2\pi j\omega t}$  to either the left or right side of the function. Otherwise, the same is not possible due to the non-commutativity of the quaternions.

In [30], the authors have derived GSE associated with QFT in two dimensions. They have used the matrix and vector approach for this. In the present article, we have proposed the GSE using the signal and its derivative as that of [22]. This GSE is performed for the first time in one-dimensional quaternionic signals.

#### 3. Generalized Sampling Expansion in the Quaternion Fourier Domain

This section constitutes the centerpiece of this article and is completely devoted to the formulation and validity of the GSE in the context of a one-dimensional QFT.

To begin with, we have the following definition:

**Definition 2.** *Bandlimited Signal:* A signal f(t) is said to be bandlimited in the quaternion Fourier domain if  $F_Q{f}(\omega) = 0$ , for  $|\omega| > B_{\alpha}$ . In that case, the scalar  $B_{\alpha}$  is called as the bandwidth of the quaternion-valued signal f(t).

In order to facilitate the formulation of the GSE in the quaternion Fourier domain, we choose a set of *M* linear filters:

$$H_1^{\alpha}(\omega), H_2^{\alpha}(\omega), \dots, H_M^{\alpha}(\omega), \tag{7}$$

which are bandlimited in the quaternion Fourier domain. Applying these M linear filters to the quaternionic bandlimited signal f(t) and invoking the inversion Formula (5) yields the following output:

$$g_k(t) = \int_R F_Q\{f\}(\omega) H_k^{\alpha}(\omega) e^{j2\pi\omega t} d\omega, \quad k = 1, 2, \cdots, M.$$
(8)

Then, our goal is to demonstrate that the bandlimited signal f(t) can be exactly reconstructed from the outputs

$$g_k(nT_0)$$
,  $k = 1, 2, \dots, M$  and  $n \in \mathbb{Z}$ ,

which are sampled at 1/M of the Nyquist rate of the quaternion Fourier domain with the sampling period  $T_0$ , satisfying  $T_0 = MT = M\pi/B_{\alpha}$ . To do so, we generate the following system of equations:

$$H_{1}^{\alpha}(\omega) Y_{1}(\omega, t) + \dots + H_{M}^{\alpha}(\omega) Y_{M}(\omega, t) = 1$$

$$H_{1}^{\alpha}(\omega + c) Y_{1}(\omega, t) + \dots + H_{M}^{\alpha}(\omega + c) Y_{M}(\omega, t) = e^{\mathbf{j}2\pi ct}$$

$$\vdots$$

$$H_{1}^{\alpha}(\omega + (M-1)c) Y_{1}(\omega, t) + \dots + H_{M}^{\alpha}(\omega + (M-1)c) Y_{M}(\omega, t) = e^{\mathbf{j}2\pi(M-1)ct}$$
(9)

where  $-B_{\alpha} \leq \omega \leq -B_{\alpha} + c$  and  $c = 2B_{\alpha}/M$  is the sub-bandwidth parameter in the sense that it divides the total band of the given signal in the quaternion Fourier domain into Mequal parts. The above system of equations gives rise to a set of M unknown functions  $Y_1(\omega, t), \ldots, Y_M(\omega, t)$ , and the necessary condition for this system to have a solution is that the determinant of its coefficients is non-zero for every  $\omega \in [-B_{\alpha}, -B_{\alpha} + c]$ . Moreover, another reasonable assumption, which ought to be mentioned, is that each of these Munknown functions can be expanded via the Fourier series over the interval  $[-B_{\alpha}, -B_{\alpha} + c]$ . Consequently, we infer that the filter functions  $H_k^{\alpha}(\omega)$  are not completely arbitrary as they have to satisfy the system of Equation (9).

**Theorem 1.** Suppose that a quaternionic bandlimited signal f(t) with bandwidth  $B_{\alpha}$  is passed through M quaternion Fourier filters  $H_1^{\alpha}(\omega), H_2^{\alpha}(\omega), \dots, H_M^{\alpha}(\omega)$ . Then, we have:

$$f(t) = \sum_{n=-\infty}^{\infty} \left[ g_1(nT_0) \, y_1(t-nT_0) + g_2(nT_0) \, y_2(t-nT_0) + \dots + g_M(nT_0) \, y_M(t-nT_0) \right], \tag{10}$$

where  $g_k(t), k = 1, 2, ..., M$  are the M outputs of the given set of quaternion Fourier filters (7) and

$$T_0=\frac{2\pi}{c}=\frac{M\pi}{B_{\alpha}},$$

with the synthesis functions being given by:

$$y_k(t) = \frac{1}{c} \int_{-B_{\alpha}}^{-B_{\alpha}+c} Y_k(\omega, t) e^{j2\pi\omega t} d\omega.$$
(11)

**Proof.** Firstly, note that:

$$c(t+nT_0) = ct + cnT_0 = ct + \left(\frac{2B_\alpha}{M}\right)n\left(\frac{M\pi}{B_\alpha}\right) = ct + 2n\pi,$$
(12)

which demonstrates that the right side of the system of Equation (9) is periodic with period  $T_0$ . In addition, since each of the filter functions appearing in (7) are independent of t, therefore, we conclude that each of the functions  $Y_1(\omega, t), Y_2(\omega, t), \ldots, Y_M(\omega, t)$  must be periodic in t with the same period  $T_0$ . Thus, we have:

$$Y_k(\omega, t + nT_0) = Y_k(\omega, t), \qquad k = 1, 2, \dots, M.$$
 (13)

Consequently, in view of (11), we obtain:

$$y_{k}(t - nT_{0}) = \frac{1}{c} \int_{-B_{\alpha}}^{-B_{\alpha} + c} Y_{k}(\omega, (t - nT_{0})) e^{j2\pi\omega(t - nT_{0})} d\omega$$
$$= \frac{1}{c} \int_{-B_{\alpha}}^{-B_{\alpha} + c} Y_{k}(\omega, t) e^{j2\pi\omega t} e^{-j2\pi\omega nT_{0}} d\omega, \qquad k = 1, 2, \dots, M.$$
(14)

The expression (14) clearly indicates that  $y_k(t - nT_0)$  is the *n*th Fourier coefficient of the Fourier series expansion of the function  $Y_k(\omega, t) e^{j2\pi\omega t}$  over the interval  $[-B_{\alpha}, -B_{\alpha} + c]$ . Hence, employing the formal definition of Fourier series yields the following:

$$Y_k(\omega,t) e^{j2\pi\omega t} = \sum_{n=-\infty}^{\infty} y_k(t-nT_0) e^{j2\pi\omega nT_0}, \qquad k = 1, 2, \dots, M.$$
(15)

Upon multiplying both sides of the system (9) with  $e^{j2\pi\omega t}$  and then plugging (15) into (9), we obtain:

$$H_{1}^{\alpha}(\omega) \sum_{n=-\infty}^{\infty} y_{1}(t-nT_{0}) e^{\mathbf{j}2\pi\omega nT_{0}} + \dots + H_{M}^{\alpha}(\omega) \sum_{n=-\infty}^{\infty} y_{M}(t-nT_{0}) e^{\mathbf{j}2\pi\omega nT_{0}} = e^{\mathbf{j}2\pi\omega t}$$

$$H_{1}^{\alpha}(\omega+c) \sum_{n=-\infty}^{\infty} y_{1}(t-nT_{0}) e^{\mathbf{j}2\pi\omega nT_{0}} + \dots + H_{M}^{\alpha}(\omega+c) \sum_{n=-\infty}^{\infty} y_{M}(t-nT_{0}) e^{\mathbf{j}2\pi\omega nT_{0}} = e^{\mathbf{j}2\pi(c+\omega)t}$$

$$\vdots \qquad (16)$$

$$H_{1}^{\alpha}(\omega + (M-1)c) \sum_{n=-\infty}^{\infty} y_{1}(t-nT_{0}) e^{j2\pi\omega nT_{0}} + \dots + H_{M}^{\alpha}(\omega + (M-1)c) \sum_{n=-\infty}^{\infty} y_{M}(t-nT_{0}) e^{j2\pi\omega nT_{0}} = e^{j2\pi(c(M-1)+\omega)t}$$

where  $-B_{\alpha} \leq \omega \leq -B_{\alpha} + c$ . In addition, by virtue of (12), we have  $e^{j2\pi\omega nT_0} = e^{j2\pi(\omega+c)nT_0}$ , so the system (16) can be unified as follows:

$$H_1^{\alpha}(\omega) \sum_{n=-\infty}^{\infty} y_1(t-nT_0) e^{\mathbf{j}2\pi\omega nT_0} + \dots + H_M^{\alpha}(\omega) \sum_{n=-\infty}^{\infty} y_M(t-nT_0) e^{\mathbf{j}2\pi\omega nT_0} = e^{\mathbf{j}2\pi\omega t}$$
(17)

Invoking the inversion formula for the one-dimensional QFT, we obtain:

$$f(t) = \int_{R} F_{Q}\{f\}(\omega) e^{j2\pi\omega t} d\omega$$
  
=  $\int_{R} F_{Q}\{f\}(\omega) [H_{1}^{\alpha}(\omega) \sum_{n=-\infty}^{\infty} y_{1}(t-nT_{0}) e^{j2\pi\omega nT_{0}}$   
+  $\cdots$  +  $H_{M}^{\alpha}(\omega) \sum_{n=-\infty}^{\infty} y_{M}(t-nT_{0}) e^{j2\pi\omega nT_{0}}]d\omega.$ 

Finally, using the expression (8), we obtain the desired GSE as:

$$f(t) = \sum_{n=-\infty}^{\infty} \Big[ g_1(nT_0) \, y_1(t-nT_0) + g_2(nT_0) \, y_2(t-nT_0) + \dots + g_M(nT_0) \, y_M(t-nT_0) \Big].$$

This evidently completes the proof.  $\Box$ 

Here, it is important to mention that the above-obtained GSE reveals that a quaternionic bandlimited signal can be exactly reconstructed from a given set of *M* quaternion Fourier filters; that is, if the original signal is not directly accessible, we still can reconstruct the signal using the GSE.

## 4. Sampling Using the Signal and Its Derivative

Having formulated the GSE pertaining to the quaternion Fourier transform, we shall next obtain another reconstruction formula using the derivatives of the quaternionic signal. Since derivatives contain information about the edges and curves appearing in images, therefore, such a sampling formula is of substantial importance for image processing, particularly in image super-resolution procedures.

**Theorem 2.** If  $B_{\alpha}$  is the bandwidth of a continuously differentiable quaternion-valued function f, then we have:

$$f(t) = \sum_{n=-\infty}^{\infty} \left[ f(nT_0) \left( \frac{\sin^2(\pi B_{\alpha}(t - nT_0))}{\pi^2 B_{\alpha}^2(t - nT_0)^2} \right) + f'(nT_0) \left( \frac{\sin^2(\pi B_{\alpha}(t - nT_0))}{\pi^2 B_{\alpha}^2(t - nT_0)} \right) \right].$$
 (18)

**Proof.** In view of the differentiation property (6) of the QFT, we choose the quaternionic filters as:

$$H_1^{\alpha}(\omega) = 1, H_2^{\alpha}(\omega) = (\mathbf{j}2\pi\omega), H_3^{\alpha}(\omega) = (\mathbf{j}2\pi\omega)^2, \dots, H_M^{\alpha}(\omega) = (\mathbf{j}2\pi\omega)^{M-1}.$$
 (19)

In addition, for M = 2, the sub-bandwidth parameter is given by  $c = B_{\alpha}$ . Consequently, the system of Equation (9) becomes:

$$Y_1(\omega, t) + (\mathbf{j}2\pi\omega) Y_2(\omega, t) = 1$$
  

$$Y_1(\omega, t) + (\mathbf{j}2\pi(\omega + B_\alpha)) Y_M(\omega, t) = e^{\mathbf{j}2\pi B_\alpha t}$$
(20)

Upon solving the above pair of equations, we obtain:

$$Y_1(\omega,t) = 1 - \frac{\omega(e^{\mathbf{j}2\pi B_{\alpha}t} - 1)}{B_{\alpha}}, \quad \text{and} \quad Y_2(\omega,t) = \frac{e^{\mathbf{j}2\pi B_{\alpha}t} - 1}{\mathbf{j}2\pi B_{\alpha}t}$$
(21)

Therefore, plugging the explicit expressions for the functions  $Y_1(\omega, t)$  and  $Y_2(\omega, t)$  into (11) yields:

$$y_1(t) = \frac{1}{B_{\alpha}} \int_{-B_{\alpha}}^{0} \left[ 1 - \frac{\omega(e^{j2\pi B_{\alpha}t} - 1)}{B_{\alpha}} \right] e^{j2\pi\omega t} d\omega = \frac{\sin^2(\pi B_{\alpha}t)}{\pi^2 B_{\alpha}^2 t^2}$$
(22)

and

$$y_2(t) = \frac{1}{B_{\alpha}} \int_{-B_{\alpha}}^{0} \left( \frac{e^{\mathbf{j}2\pi B_{\alpha}t} - 1}{\mathbf{j}2\pi B_{\alpha}t} \right) e^{\mathbf{j}2\pi\omega t} d\omega = \frac{\sin^2(\pi B_{\alpha}t)}{\pi^2 B_{\alpha}^2 t}.$$
 (23)

Now, it remains to obtain the respective outputs of the filters applied to the quaternionic bandlimited signal. In this direction, we shall invoke (8), so that:

$$g_1(t) = \int_R F_Q\{f\}(\omega) H_1^{\alpha}(\omega) e^{\mathbf{j}2\pi\omega t} d\omega = \int_R F_Q\{f\}(\omega) e^{\mathbf{j}2\pi\omega t} d\omega = f(t)$$
(24)

and

$$g_{2}(t) = \int_{R} F_{Q}\{f\}(\omega) H_{2}^{\alpha}(\omega) e^{j2\pi\omega t} d\omega$$
  

$$= \int_{R} (j2\pi\omega) F_{Q}\{f\}(\omega) e^{j2\pi\omega t} d\omega$$
  

$$= \int_{R} F_{Q}\left\{\frac{df}{dt}\right\}(\omega) e^{j2\pi\omega t} d\omega$$
  

$$= \frac{df}{dt} = f'(t).$$
(25)

Finally, implementing (23)–(25) into (10), we obtain the desired reconstruction formula involving the derivatives of the quaternion-valued signal as:

$$f(t) = \sum_{n=-\infty}^{\infty} \left[ f(nT_0) \left( \frac{\sin^2(\pi B_{\alpha}(t-nT_0))}{\pi^2 B_{\alpha}^2(t-nT_0)^2} \right) + f'(nT_0) \left( \frac{\sin^2(\pi B_{\alpha}(t-nT_0))}{\pi^2 B_{\alpha}^2(t-nT_0)} \right) \right]$$

This completes the proof.  $\Box$ 

By choosing the values of  $M \ge 2$  and noting that  $g_k(t) = f^{k-1}(t)$ , k = 1, 2, ..., M, we can obtain a reconstruction formula involving the higher-order derivatives of the quaternion-valued signal f(t).

# 5. Simulation Results

In order to show the correctness and effectiveness of the derived results, we have used simulations in MATLAB. Consider the quaternion-valued signal  $f(t) = h \sin c(t)$ , where  $h = h_0 + h_1 \mathbf{i} + h_2 \mathbf{j} + h_3 \mathbf{k} \in H$ . In order to carry out the numerical simulations, we choose  $h_0 = 1$ ,  $h_1 = 2$ ,  $h_2 = 3$  and  $h_3 = 4$  and plot the corresponding function f(t) in Figure 1. Firstly, we claim that f is bandlimited in the quaternion Fourier domain. To do so, we proceed as follows:

$$F_{Q}{f}(\omega) = \int_{R} f(t) e^{-j2\pi\omega t} dt$$
  
=  $\int_{R} \sin ch(t) e^{-j2\pi\omega t} dt$   
=  $\frac{h}{2} \chi_{[-1,1]}(-2\omega)$   
=  $\frac{h}{2} \chi_{[-1/2,1/2]}(-2\omega),$  (26)

where  $\chi$  denotes the usual characteristic function. From expression (26), it is clear that indeed the quaternion Fourier spectrum of the signal f lives onto the interval [-1/2, 1/2] and is zero outside. Hence, we conclude that f is bandlimited in the quaternion Fourier domain with bandwidth  $B_{\alpha} = 1/2$ . The quaternion Fourier transform of the signal f(t) is plotted in Figure 2. Consequently, we have  $c = 2B_{\alpha}/M = 1/M$  and  $T_0 = MT = M\pi/B_{\alpha} = 2\pi$ . Hence, we choose M = 1 and  $H_1^{\alpha}(\omega) = 1$ , yielding  $Y_k(\omega, t) = 1$  and  $g_1(t) = f(t)$ , so that  $g_1(nT_0) = g_1(2n\pi) = h \sin c(2n\pi)$ . In addition, note that the synthesis function is given by:

$$y_1(t) = \frac{1}{c} \int_{-B_{\alpha}}^{-B_{\alpha}+c} Y_k(\omega, t) e^{j2\pi\omega t} d\omega$$
$$= \int_{-1/2}^{1/2} e^{j2\pi\omega t} d\omega$$
$$= \left[ \frac{e^{j2\pi\omega t}}{j2\pi t} \right]_{-1/2}^{1/2}$$
$$= \frac{1}{\pi t} \left( \frac{e^{j\pi t} - e^{-j\pi t}}{2j} \right)$$
$$= \frac{\sin(\pi t)}{\pi t}$$
$$= \sin c(t).$$

Finally, using the reconstruction Formula (10) with M = 1, we obtain:

$$f(t) = \sum_{n=-\infty}^{\infty} \left[ g_1(nT_0) \, y_1(t-nT_0) \right] = h \sum_{n=-\infty}^{\infty} \left[ \sin c(2n\pi) \, \sin c(t-nT_0) \right]. \tag{27}$$



**Figure 1.** The quaternion-valued signal  $f(t) = (1 + 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) \sin c(t)$ .



**Figure 2.** The quaternion Fourier transform of the quaternion-valued signal f(t).

The sampled signal  $f(nT_0)$  is plotted in Figure 3, whereas the reconstructed signal (27) is plotted in Figure 4.



**Figure 3.** The sampled parts of the quaternion-valued signal given by  $f(nT_0)$ .



Figure 4. The reconstructed quaternionic signal.

#### 6. Conclusions

Focusing on the signals in one dimension and its derivatives in the QFT domain, this paper investigates several versions of the Generalized (multi-channel) Sampling Expansion. Firstly, we obtained the most general form of the sampling expansion involving general quaternionic filter and synthesis functions. Secondly, we deduced the GSE involving the derivatives of the input signal as derivatives are significant where information is required about the edges and curves appearing in images; therefore, such a sampling formula is of critical importance for image processing, which stimulates interest for future work on the subject. Moreover, this work can lead researchers to focus on the different aspects of one-dimensional signals in the quaternion domain. Finally, the applicability of the proposed multi-channel sampling procedure is demonstrated via an illustrative example

on the quaternion signal reconstruction. The results in the simulation part clearly show the effectiveness of the proposed scheme to reconstruct the signal from its derivatives.

Our future work about the sampling of quaternionic one-dimensional LCT is in progress.

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