mathematics

MDPI

Article

Series of Floor and Ceiling Function—Part I: Partial Summations

Dhairya Shah [1], Manoj Sahni [2,*], Ritu Sahni [1], Ernesto León-Castro [3] and Maricruz Olazabal-Lugo [4]

[1] School of Liberal Studies, Pandit Deendayal Energy University, Gandhinagar 382426, India; dhairya.sbsc17@sls.pdpu.ac.in (D.S.); ritu.sahniv@sls.pdpu.ac.in (R.S.)
[2] Department of Mathematics, School of Technology, Pandit Deendayal Energy University, Gandhinagar 382426, India
[3] Faculty of Economics and Administrative Sciences, Universidad Católica de la Santísima Concepción, Concepción 4090541, Chile; eleon@ucsc.cl
[4] Department of Economics and Administrative, Universidad Autónoma de Occidente, Culiacan 80139, Mexico; maricruz.olazabal@uadeo.mx
* Correspondence: manoj.sahni@sot.pdpu.ac.in

Abstract: In this paper, we develop two new theorems relating to the series of floor and ceiling functions. We then use these two theorems to develop more than forty distinct novel results. Furthermore, we provide specific cases for the theorems and corollaries which show that our results constitute a generalisation of the currently available results such as the summation of first n Fibonacci numbers and Pascal's identity. Finally, we provide three miscellaneous examples to showcase the vast scope of our developed theorems.

Keywords: ceiling function; floor function; Faulhaber's formula; Fibonacci numbers; geometric series; partial summations

MSC: 11B68; 11A99

check for updates

Citation: Shah, D.; Sahni, M.; Sahni, R.; León-Castro, E.; Olazabal-Lugo, M. Series of Floor and Ceiling Function—Part I: Partial Summations. *Mathematics* **2022**, *10*, 1178. https://doi.org/10.3390/math10071178

Academic Editor: Jan L. Cieśliński

Received: 16 March 2022
Accepted: 31 March 2022
Published: 4 April 2022

1. Introduction

The concept of an integer part of x $(x \in \mathbb{R})$ was introduced by Legendre in the late 18th century, and Gauss introduced the "[x]" notation for the same. Following these discoveries, in the second half of 20th century, the terms "floor" and "ceiling" functions [1] were coined. The "floor of x" is mathematically defined as $\lfloor x \rfloor = max\{w \in \mathbb{Z} \mid w \leq x\}$, whereas the "ceiling of x" is defined as $\lceil x \rceil = min\{k \in \mathbb{Z} \mid k \geq x\}$. These two functions and their respective series have a wide range of applications in computer science [1]. Along with them, two of the other most famous findings in the theory of numbers are a partial sum—"the Faulhaber's Formula" [2] and a sequence—"the Fibonacci Numbers", both of which have very significant implication in different fields of mathematics and other sciences. The Fibonacci sequence, as it is widely known, frequently occurs in mathematics as well as across different patterns in nature. However, more than that, it is applied at different instances in computer science [1]. Moreover, as recently as 2020, the applications of Faulhaber's formula and its extensions [3] are being found in different areas of advanced sciences such as quantum gravity [4].

Due to such significances of partial summations, researchers have studied their properties of such partial sums and finite sequences and their relations in recent decades [3,5–14]. McGown and Parks [6] generalised Faulhaber's formula to non-integer powers using the Zeta function [15]. Schumacher [3] extended the formula for real values $x \in \mathbb{R}_0^+$, whereas Merca [7] provided an alternative of Faulhaber's formula as a finite discrete convolution. Moreover, Orosi [9] provided a simple derivation of the formula.

Outline of the Article

In this series of two papers, we aim to generalise the scope of infinite series and partial summations using the floor and ceiling functions.

"Part I" specifically deals with the finite summations and is structured as follows. Section 2 provides the list of preliminary results useful for our study. Section 3 consists of lemmas and theorems which lay the foundations for the results discussed in Sections 4–6. Section 6 is followed by Section 7 which is devoted to the corollaries of the results of the previous sections. Section 8 gives different results related to the Fibonacci numbers. Section 9 provides results on specific values—especially Section 9.3.2, which gives two alternate formulae for Faulhaber's formula. Section 10 provides proof for two of the corollaries using the simple yet powerful principle of mathematical induction. Finally, Section 11 concludes the article with a discussion of the scope for future studies.

2. Preliminaries

The following results along with the results discussed in the previous section are useful for our study and hence we explicitly provide them together in this section.

2.1. Faulhaber's (Bernoulli's) Formula

An expression of the sum of the q powers of the first n positive integers (Faulhaber's Formula [16]) can be equated as:

$$F^q(n) = \sum_{i=1}^{n} i^q = \frac{1}{q+1} \cdot \sum_{k=0}^{q} \binom{q+1}{k} \cdot B_k \cdot n^{q+1-k} = \frac{B_{q+1}(n+1) - B_{q+1}(1)}{q+1} \tag{1}$$

where B_j is Bernoulli's number of the second kind, defined as:

$$B_j = \sum_{k=0}^{j} \frac{1}{k+1} \cdot \sum_{t=0}^{k} (-1)^t \cdot (t+1)^j \cdot \binom{k}{t}$$

and $B_n(x)$ is a Bernoulli polynomial of order n.

2.2. Floor and Ceiling Functions

The floor function [17] of any real number x (denoted by $\lfloor x \rfloor$) gives the greatest integer not greater than x, i.e., $\lfloor x \rfloor = max\{w \in \mathbb{Z} \mid w \leq x\}$. For example, $\lfloor 1.4 \rfloor = 1, \lfloor 2 \rfloor = 2, \lfloor -3.4 \rfloor = -4$ and $\lfloor -2 \rfloor = -2$.

In the same way, the ceiling function [17] (denoted by $\lceil x \rceil$) gives the smallest integer not smaller than x, i.e., $\lceil x \rceil = \min\{k \in \mathbb{Z} \mid k \geq x\}$. For example, $\lceil 1.4 \rceil = 2, \lceil 2 \rceil = 2, \lceil -3.4 \rceil = -3$ and $\lceil -2 \rceil = -2$.

From above, we can see that $\lceil x \rceil = \lfloor x \rfloor = x$ if and only if $x \in \mathbb{Z}$.

2.3. Finite Lower-Order Polylogarithm

An important finite summation formula [18] of the form for values of $m \in \mathbb{N}$ and $z \in \mathbb{C}$ is given as:

$$\sum_{k=1}^{n} k^m z^k = \left(z\frac{d}{dz}\right)^m \frac{1-z^{n+1}}{1-z} \tag{2}$$

For example, for $m = 1$, the equation reduces to:

$$\sum_{k=1}^{n} kz^k = z \cdot \frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2} \tag{3}$$

2.4. Fibonacci Number

The n^{th} Fibonacci number [19] is given by the formula:

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} \ \textit{where}\ \varphi = \frac{1+\sqrt{5}}{2}$$

Furthermore, the summation of first n Fibonacci numbers—"Fibonacci Series"—can be obtained with the following formula:

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 \tag{4}$$

2.5. Pascal's Identity

Pascal's identity [5] and its alternate form can be written as follows:

$$(n+1)^{q+1} - 1 = \sum_{m=1}^{n} \left((m+1)^{q+1} - m^{q+1}\right) = \sum_{p=0}^{q} \binom{q+1}{p} (1^p + 2^p + \cdots + n^p), \tag{5}$$

$$n^{q+1} = \sum_{m=1}^{n} \left(m^{q+1} - (m-1)^{q+1}\right) = \sum_{p=0}^{q} (-1)^{q+p} \binom{q+1}{p} (1^p + 2^p + \cdots + n^p). \tag{6}$$

3. Foundations

Lemma 1 (Floor–Ceiling Lemma). *Let $m, n \in \mathbb{N}$, $a, b \in \mathbb{R}^+$ such that $\lfloor (bn)^a \rfloor = m$ and $\lfloor (b(n+1))^a \rfloor = m+1$ (i.e., n is the largest integer such that $\lfloor (bn)^a \rfloor = m$). Furthermore, let $y \in \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ then*

$$\sum_{i=1}^{n} f((\lfloor (bi)^a \rfloor + y)x) = \sum_{t=1}^{\lfloor (bn)^a \rfloor} \left[\left\lceil \frac{(t+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil \right] f((t+y)x) \tag{7}$$

Proof. Let $g_1(m)$ denote the function that yields the number of consecutive integers, k, for which $\lfloor (bn)^a \rfloor = m$ (or $\lfloor (b(n-k))^a \rfloor = \ldots = \lfloor (b(n-1))^a \rfloor = \lfloor (bn)^a \rfloor = m$), i.e., $g_1(m) = k$ gives the number of repetitions of m for a particular n. Then, we have

$$\sum_{i=1}^{n} f((\lfloor (bi)^a \rfloor + y)x) = \sum_{t=1}^{\lfloor (bn)^a \rfloor} g_1(t) f((t+y)x) \tag{8}$$

Now, we know that:

$$m = \lfloor (bn)^a \rfloor \Rightarrow m \le (bn)^a < m+1 \Rightarrow m^{\frac{1}{a}} \le (bn) < (m+1)^{\frac{1}{a}} (\because a \in \mathbb{R}^+)$$
$$\Rightarrow \frac{m^{\frac{1}{a}}}{b} \le n < \frac{(m+1)^{\frac{1}{a}}}{b} \Rightarrow \left\lceil \frac{m^{\frac{1}{a}}}{b} \right\rceil \le n < \left\lceil \frac{(m+1)^{\frac{1}{a}}}{b} \right\rceil (\because n \in \mathbb{N})$$

It follows that n is at least $\left\lceil \frac{m^{\frac{1}{a}}}{b} \right\rceil$ and at most strictly less than $\left\lceil \frac{(m+1)^{\frac{1}{a}}}{b} \right\rceil$.

Therefore, the number of the consecutive integer n is such that $m = \lfloor (bn)^a \rfloor$ is $\left\lceil \frac{(m+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{m^{\frac{1}{a}}}{b} \right\rceil$. Hence:

$$g_1(m) = \left\lceil \frac{(m+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{m^{\frac{1}{a}}}{b} \right\rceil \tag{9}$$

and the result follows from Equations (8) and (9). □

Theorem 1 (Floor–Ceiling Theorem)**.** *Let $m, n \in \mathbb{N}$, $a, b \in \mathbb{R}^+$ such that $\lfloor (bn)^a \rfloor = m$ and $\lfloor (b(n+1))^a \rfloor \neq m+1$ (i.e., n is NOT the largest integer such that $\lfloor (bn)^a \rfloor = m$). Furthermore, let $y \in \mathbb{C}$ and $\psi_1, f : \mathbb{C} \to \mathbb{C}$ then*

$$\psi_1(x) = \sum_{i=1}^{n} f((\lfloor (bi)^a \rfloor + y)x) = \sum_{t=1}^{\lfloor (bn)^a \rfloor} \left[\left\lceil \frac{(t+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil \right] f((t+y)x) - \left(\left\lceil \frac{(\lfloor (bn)^a \rfloor + 1)^{\frac{1}{a}}}{b} \right\rceil - (n+1) \right) f((\lfloor (bn)^a \rfloor + y)x) \quad (10)$$

Or equivalently

$$\psi_1(x) = \sum_{i=1}^{n} f((\lfloor (bi)^a \rfloor + y)x) = (n+1) f((\lfloor (bn)^a \rfloor + y)x) - \left\lceil \frac{1}{b} \right\rceil f(xy) - \sum_{t=1}^{\lfloor (bn)^a \rfloor} [f((t+y)x) - f((t+y-1)x)] \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil. \quad (11)$$

Proof. Let $\lfloor (b(n-k))^a \rfloor = \ldots = \lfloor (b(n-1))^a \rfloor = \lfloor (bn)^a \rfloor = m$ for some $k \in \mathbb{N}$ such that $0 < k < \left(\left\lceil \frac{(\lfloor (bn)^a \rfloor + 1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{(\lfloor (bn)^a \rfloor)^{\frac{1}{a}}}{b} \right\rceil \right)$ and $\lfloor (b(n-k-1))^a \rfloor = m-1$; then, from Lemma 1, we have:

$$\psi_1(x) = \sum_{i=1}^{n} f((\lfloor (bi)^a \rfloor + y)x) = \sum_{t=1}^{\lfloor (bn)^a \rfloor - 1} \left[\left\lceil \frac{(t+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil \right] f((t+y)x) + k \cdot f((\lfloor (bn)^a \rfloor + y)x)$$

Furthermore, let $m = \lfloor (bn)^a \rfloor = \lfloor (b(n+1))^a \rfloor = \ldots = \lfloor (b(n+z))^a \rfloor$ for smallest $z \in \mathbb{N}$ such that $\lfloor (b(n+z+1))^a \rfloor = m+1$ then, from the previous equation, we have

$$\psi_1(x) = \sum_{t=1}^{\lfloor (bn)^a \rfloor - 1} \left[\left\lceil \frac{(t+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil \right] f((t+y)x) + k \cdot f((\lfloor (bn)^a \rfloor + y)x) + z \cdot f((\lfloor (bn)^a \rfloor + y)x) - z \cdot f((\lfloor (bn)^a \rfloor + y)x)$$

Here, it can be easily seen that $g_1(m) = k + z$ for $m = \lfloor (bn)^a \rfloor$ as $m = \lfloor (b(n-k))^a \rfloor = \ldots = \lfloor (b(n-1))^a \rfloor = \lfloor (bn)^a \rfloor = \lfloor (b(n+1))^a \rfloor = \ldots = \lfloor (b(n+z))^a \rfloor$.

Therefore, $k + z = \left(\left\lceil \frac{(\lfloor (bn)^a \rfloor + 1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{(\lfloor (bn)^a \rfloor)^{\frac{1}{a}}}{b} \right\rceil \right)$.

Hence, by simplifying the previous equation, we obtain:

$$\psi_1(x) = \sum_{t=1}^{\lfloor (bn)^a \rfloor} \left[\left\lceil \frac{(t+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil \right] f((t+y)x) - z * f((\lfloor (bn)^a \rfloor + y)x)$$

Again, we have:

$$m+1 = \left\lfloor (b(n+z+1))^a \right\rfloor \Rightarrow m+1 \leq (b(n+z+1))^a < m+2$$
$$\Rightarrow (m+1)^{\frac{1}{a}} \leq (b(n+z+1)) < (m+2)^{\frac{1}{a}} \quad (\because a \in \mathbb{R}^+)$$
$$\Rightarrow \frac{(m+1)^{\frac{1}{a}}}{b} \leq n+z+1 < \frac{(m+2)^{\frac{1}{a}}}{b} \Rightarrow \left\lceil \frac{(m+1)^{\frac{1}{a}}}{b} \right\rceil \leq n+z+1 < \left\lceil \frac{(m+2)^{\frac{1}{a}}}{b} \right\rceil$$

$$\Rightarrow \left\lceil \frac{(m+1)^{\frac{1}{a}}}{b} \right\rceil - (n+1) \le z < \left\lceil \frac{(m+2)^{\frac{1}{a}}}{b} \right\rceil - (n+1)$$

Now, as it is known that $\lfloor (b(n+z+1))^a \rfloor = m+1$ for the smallest $z \in \mathbb{N}$, we obtain $z = \left\lceil \frac{(m+1)^{\frac{1}{a}}}{b} \right\rceil - (n+1) = \left\lceil \frac{(\lfloor (bn)^a \rfloor + 1)^{\frac{1}{a}}}{b} \right\rceil - (n+1)$ and hence Equation (10) is obtained.

Furthermore, consider the right-hand side of Equation (10):

$$\sum_{t=1}^{\lfloor (bn)^a \rfloor} \left[\left\lceil \frac{(t+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil \right] f((t+y)x) - \left(\left\lceil \frac{(\lfloor (bn)^a \rfloor + 1)^{\frac{1}{a}}}{b} \right\rceil - (n+1) \right) f((\lfloor (bn)^a \rfloor + y)x)$$

$$= \sum_{t=1}^{\lfloor (bn)^a \rfloor} \left\lceil \frac{(t+1)^{\frac{1}{a}}}{b} \right\rceil f((t+y)x) - \sum_{t=1}^{\lfloor (bn)^a \rfloor} \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil f((t+y)x) - \left(\left\lceil \frac{(\lfloor (bn)^a \rfloor + 1)^{\frac{1}{a}}}{b} \right\rceil - (n+1) \right) f((\lfloor (bn)^a \rfloor + y)x)$$

$$= \sum_{t=2}^{\lfloor (bn)^a \rfloor + 1} \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil f((t+y-1)x) - \sum_{t=1}^{\lfloor (bn)^a \rfloor} \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil f((t+y)x) - \left(\left\lceil \frac{(\lfloor (bn)^a \rfloor + 1)^{\frac{1}{a}}}{b} \right\rceil - (n+1) \right) f((\lfloor (bn)^a \rfloor + y)x)$$

$$= \sum_{t=1}^{\lfloor (bn)^a \rfloor} \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil f((t+y-1)x) - \left\lceil \frac{1^{\frac{1}{a}}}{b} \right\rceil f(yx) + \left\lceil \frac{(\lfloor (bn)^a \rfloor + 1)^{\frac{1}{a}}}{b} \right\rceil f((\lfloor (bn)^a \rfloor + y)x) - \sum_{t=1}^{\lfloor (bn)^a \rfloor} \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil f((t+y)x) - \left(\left\lceil \frac{(\lfloor (bn)^a \rfloor + 1)^{\frac{1}{a}}}{b} \right\rceil \right) f((\lfloor (bn)^a \rfloor + y)x) + (n+1) f((\lfloor (bn)^a \rfloor + y)x)$$

$$= (n+1) f((\lfloor (bn)^a \rfloor + y)x) - \left\lceil \frac{1}{b} \right\rceil f(yx) - \sum_{t=1}^{\lfloor (bn)^a \rfloor} \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil [f((t+y)x) - f((t+y-1)x)]$$

which is the right-hand side of the Equation (11). □

Remark 1. *Alternatively, using* $\lfloor (b(n+z))^a \rfloor = m+1$*, one can obtain* $z = \left\lceil \frac{(\lfloor (bn)^a \rfloor)^{\frac{1}{a}}}{b} \right\rceil - n$ *which leads to an alternate version of Equation (10) as*

$$\psi_1(x) = \sum_{i=1}^{n} f((\lfloor (bi)^a \rfloor + y)x) = \sum_{t=1}^{\lfloor (bn)^a \rfloor} \left[\left\lceil \frac{(t+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil \right] f((t+y)x) - \left(\left\lceil \frac{(\lfloor (bn)^a \rfloor)^{\frac{1}{a}}}{b} \right\rceil - n \right) f((\lfloor (bn)^a \rfloor + y)x) \quad (12)$$

Lemma 2 (Ceiling–Floor Lemma)**.** *Let $m, n \in \mathbb{N}$, $a, b \in \mathbb{R}^+$ such that $\lceil (bn)^a \rceil = m$ and $\lceil (b(n+1))^a \rceil = m+1$ (i.e., n is the largest integer such that $\lceil (bn)^a \rceil = m$). Furthermore, let $y \in \mathbb{C}$, and $f : \mathbb{C} \to \mathbb{C}$ then*

$$\sum_{i=1}^{n} f((\lceil (bi)^a \rceil + y)x) = \sum_{t=1}^{\lceil (bn)^a \rceil} \left[\left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(t-1)^{\frac{1}{a}}}{b} \right\rfloor \right] f((t+y)x) \tag{13}$$

Proof. Let $g_2(m)$ denote the function that yields the number of consecutive integers, k, for which $\lceil (bn)^a \rceil = m$ (or $\lceil (b(n-k))^a \rceil = \ldots = \lceil (b(n-1))^a \rceil = \lceil (bn)^a \rceil = m$), i.e., $g_2(m)$ gives the number of repetitions of m for particular n. Then, we have

$$\sum_{i=1}^{n} f((\lceil (bi)^a \rceil + y)x) = \sum_{t=1}^{\lceil (bn)^a \rceil} g_2(t) f((t+y)x) \tag{14}$$

Now, we know that

$$m = \lceil (bn)^a \rceil \Rightarrow m - 1 < (bn)^a \leq m \Rightarrow (m-1)^{\frac{1}{a}} < (bn) \leq m^{\frac{1}{a}} \quad (\because a \in \mathbb{R}^+)$$
$$\Rightarrow \frac{(m-1)^{\frac{1}{a}}}{b} < n \leq \frac{m^{\frac{1}{a}}}{b} \Rightarrow \left\lfloor \frac{(m-1)^{\frac{1}{a}}}{b} \right\rfloor < n \leq \left\lfloor \frac{m^{\frac{1}{a}}}{b} \right\rfloor \quad (\because n \in \mathbb{N})$$

It follows that n is at least strictly greater than $\left\lfloor \frac{(m-1)^{\frac{1}{a}}}{b} \right\rfloor$ and at most $\left\lfloor \frac{m^{\frac{1}{a}}}{b} \right\rfloor$. Therefore, the number of consecutive integers n is such that $m = \lceil (bn)^a \rceil$ is $\left\lfloor \frac{m^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(m-1)^{\frac{1}{a}}}{b} \right\rfloor$. Hence,

$$g_2(m) = \left\lfloor \frac{m^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(m-1)^{\frac{1}{a}}}{b} \right\rfloor \tag{15}$$

and the result follows from Equations (14) and (15). □

Theorem 2 (Ceiling–Floor Theorem)**.** *Let $m, n \in \mathbb{N}$, $a, b \in \mathbb{R}^+$ such that $\lceil (bn)^a \rceil = m$ and $\lceil (b(n+1))^a \rceil \neq m+1$ (i.e., n is NOT the largest integer such that $\lceil (bn)^a \rceil = m$). Furthermore, let $y \in \mathbb{C}$ and $\psi_2, f : \mathbb{C} \to \mathbb{C}$ then*

$$\begin{aligned} \psi_2(x) = \sum_{i=1}^{n} f((\lceil (bi)^a \rceil + y)x) &= \sum_{t=1}^{\lceil (bn)^a \rceil} \left[\left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(t-1)^{\frac{1}{a}}}{b} \right\rfloor \right] f((t+y)x) \\ &\quad - \left(\left\lfloor \frac{(\lceil (bn)^a \rceil)^{\frac{1}{a}}}{b} \right\rfloor - n \right) f((\lceil (bn)^a \rceil + y)x) \end{aligned} \tag{16}$$

Or equivalently

$$\begin{aligned} \psi_2(x) = \sum_{i=1}^{n} f((\lceil (bi)^a \rceil + y)x) &= n f((\lceil (bn)^a \rceil + y)x) \\ &\quad - \sum_{t=1}^{\lceil (bn)^a \rceil - 1} [f((t+y+1)x) - f((t+y)x)] \left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor. \end{aligned} \tag{17}$$

Proof. Let $\lceil (b(n-k))^a \rceil = \ldots = \lceil (b(n-1))^a \rceil = \lceil (bn)^a \rceil = m$ for some $k \in \mathbb{N}$ such that $0 < k < \left(\left\lfloor \frac{\lceil (bn)^a \rceil^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(\lceil (bn)^a \rceil - 1)^{\frac{1}{a}}}{b} \right\rfloor \right)$ and $\lceil (b(n-k-1))^a \rceil = m-1$; then, from Lemma 2, we have:

$$\psi_2(x) = \sum_{i=1}^{n} f((\lceil (bi)^a \rceil + y)x) = \sum_{t=1}^{\lceil (bn)^a \rceil - 1} \left[\left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(t-1)^{\frac{1}{a}}}{b} \right\rfloor \right] f((t+y)x) + k \cdot f((\lceil (bn)^a \rceil + y)x)$$

Furthermore, let $m = \lceil (bn)^a \rceil = \lceil (b(n+1))^a \rceil = \ldots = \lceil (b(n+z))^a \rceil$ for the largest possible $z \in \mathbb{N}$ such that $\lceil (b(n+z+1))^a \rceil = m+1$; then, from the previous equation, we have

$$\psi_2(x) = \sum_{t=1}^{\lceil (bn)^a \rceil - 1} \left[\left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(t-1)^{\frac{1}{a}}}{b} \right\rfloor \right] f((t+y)x) + k \cdot f((\lceil (bn)^a \rceil + y)x) + z \cdot f((\lceil (bn)^a \rceil + y)x) - z \cdot f((\lceil (bn)^a \rceil + y)x)$$

Here, it can be easily seen that $g_2(m) = k + z$ for $m = \lceil (bn)^a \rceil$ as $m = \lceil (b(n-k))^a \rceil = \ldots = \lceil (b(n-1))^a \rceil = \lceil (bn)^a \rceil = \lceil (b(n+1))^a \rceil = \ldots = \lceil (b(n+z))^a \rceil$.

Therefore, k + z = $\left(\left\lfloor \frac{(\lceil (bn)^a \rceil)^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(\lceil (bn)^a \rceil - 1)^{\frac{1}{a}}}{b} \right\rfloor \right)$.

Hence, by simplifying the previous equation, we obtain:

$$\psi_2(x) = \sum_{t=1}^{\lceil (bn)^a \rceil} \left[\left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(t-1)^{\frac{1}{a}}}{b} \right\rfloor \right] f((t+y)x) - z \cdot f((\lceil (bn)^a \rceil + y)x)$$

Again, we have:

$$m = \lceil (b(n+z))^a \rceil \Rightarrow m - 1 < (b(n+z))^a \leq m \Rightarrow (m-1)^{\frac{1}{a}} < (b(n+z)) \leq m^{\frac{1}{a}}$$
$$\Rightarrow \frac{(m-1)^{\frac{1}{a}}}{b} < (n+z) \leq \frac{m^{\frac{1}{a}}}{b} \Rightarrow \left\lfloor \frac{(m-1)^{\frac{1}{a}}}{b} \right\rfloor < (n+z) \leq \left\lfloor \frac{m^{\frac{1}{a}}}{b} \right\rfloor$$
$$\Rightarrow \left\lfloor \frac{(m-1)^{\frac{1}{a}}}{b} \right\rfloor - n < z \leq \left\lfloor \frac{m^{\frac{1}{a}}}{b} \right\rfloor - n$$

Now, as it is known that $\lceil (b(n+z))^a \rceil = m$ for the largest $z \in \mathbb{N}$, we obtain $z = \left\lfloor \frac{m^{\frac{1}{a}}}{b} \right\rfloor - n$ and hence Equation (16) is obtained.

Furthermore, from the right-hand side of Equation (16), we have:

$$\sum_{t=1}^{\lceil (bn)^a \rceil} \left[\left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(t-1)^{\frac{1}{a}}}{b} \right\rfloor \right] f((t+y)x) - \left(\left\lfloor \frac{(\lceil (bn)^a \rceil)^{\frac{1}{a}}}{b} \right\rfloor - n \right) f((\lceil (bn)^a \rceil + y)x)$$

$$= \sum_{t=1}^{\lceil (bn)^a \rceil} \left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor f((t+y)x) - \sum_{t=1}^{\lceil (bn)^a \rceil} \left\lfloor \frac{(t-1)^{\frac{1}{a}}}{b} \right\rfloor f((t+y)x) - \left(\left\lfloor \frac{(\lceil (bn)^a \rceil)^{\frac{1}{a}}}{b} \right\rfloor - n \right) f((\lceil (bn)^a \rceil + y)x)$$

$$= \sum_{t=2}^{\lceil (bn)^a \rceil +1} \left\lfloor \frac{(t-1)^{\frac{1}{a}}}{b} \right\rfloor f((t+y-1)x) - \sum_{t=1}^{\lceil (bn)^a \rceil} \left\lfloor \frac{(t-1)^{\frac{1}{a}}}{b} \right\rfloor f((t+y)x) - \left(\left\lfloor \frac{(\lceil (bn)^a \rceil)^{\frac{1}{a}}}{b} \right\rfloor - n \right) f((\lceil (bn)^a \rceil + y)x)$$

$$= \sum_{t=1}^{\lceil (bn)^a \rceil} \left\lfloor \frac{(t-1)^{\frac{1}{a}}}{b} \right\rfloor f((t+y-1)x) - \sum_{t=1}^{\lceil (bn)^a \rceil} \left\lfloor \frac{(t-1)^{\frac{1}{a}}}{b} \right\rfloor f((t+y)x) + nf((\lceil (bn)^a \rceil + y)x)$$

$$= nf((\lceil (bn)^a \rceil + y)x) - \sum_{t=1}^{\lceil (bn)^a \rceil} \left\lfloor \frac{(t-1)^{\frac{1}{a}}}{b} \right\rfloor [f((t+y)x) - f((t+y-1)x)]$$

$$= nf((\lceil (bn)^a \rceil + y)x) - \sum_{t=1}^{\lceil (bn)^a \rceil -1} [f((t+y+1)x) - f((t+y)x)] \left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor$$

which is the right-hand side of (17). □

4. Shah Formulae

Theorem 3 (F-Shah Formula)**.** *Let $a, b \in \mathbb{R}^+$ and $p, n \in \mathbb{N}$ then, the following equation holds true:*

$$^{F}S_b^{a,p}(n) = \sum_{i=1}^{n} \lfloor (bi)^a \rfloor^p = (n+1)\lfloor (bn)^a \rfloor^p - \sum_{t=1}^{\lfloor (bn)^a \rfloor} \sum_{k=0}^{p-1} (-1)^{p-k+1} \binom{p}{k} t^k \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil \tag{18}$$

Proof. Consider Equation (11) with $f(x) = x^p$ and $y = 0$; then, we have

$$\psi_1(x) = \sum_{i=1}^{n} ((\lfloor (bi)^a \rfloor)x)^p = (n+1)((\lfloor (bn)^a \rfloor)x)^p - \sum_{t=1}^{\lfloor (bn)^a \rfloor} \left[(tx)^p - ((t-1)x)^p \right] \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil$$

Hence, for $x = 1, \psi_1(x)$ reduces as:

$$\psi_1(1) = \sum_{i=1}^{n} \lfloor (bi)^a \rfloor^p = (n+1)\lfloor (bn)^a \rfloor^p - \sum_{t=1}^{\lfloor (bn)^a \rfloor} \left[t^p - (t-1)^p \right] \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil$$

Now, as it is known that $t^p - (t-1)^p = \sum_{k=0}^{p-1} (-1)^{p-k+1} \binom{p}{k} t^k$; then, by substitution, the previous equation reduces to Equation (18). □

Theorem 4 (C-Shah Formula)**.** *Let $a, b \in \mathbb{R}^+$ and $p, n \in \mathbb{N}$; then, the following equation holds true:*

$$^{C}S_b^{a,p}(n) = \sum_{i=1}^{n} \lceil (bi)^a \rceil^p = n\lceil (bn)^a \rceil^p - \sum_{t=1}^{\lceil (bn)^a \rceil -1} \sum_{k=0}^{p-1} \binom{p}{k} t^k \left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor \tag{19}$$

Proof. Consider Equation (17) with $f(x) = x^p$ and $y = 0$ then, we have

$$\psi_2(x) = \sum_{i=1}^{n} ((\lceil (bi)^a \rceil)x)^p = n((\lceil (bn)^a \rceil)x)^p - \sum_{t=1}^{\lceil (bn)^a \rceil -1} \left[((t+1)x)^p - ((t)x)^p \right] \left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor$$

Hence, for $x = 1, \psi_2(x)$ reduces as:

$$\psi_2(1) = \sum_{i=1}^{n} \lceil (bi)^a \rceil^p = n\lceil (bn)^a \rceil^p - \sum_{t=1}^{\lceil (bn)^a \rceil - 1} \left[(t+1)^p - t^p \right] \left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor$$

Again, as we have $(t+1)^p - t^p = \sum_{k=0}^{p-1} \binom{p}{k} t^k$; then, by substitution, the previous equation reduces to Equation (19). □

5. Floor and Ceiling Geometric Series

Theorem 5 (Floor Geometric Series)**.** *Let $z, x \in \mathbb{C}, a, b \in \mathbb{R}^+, n \in \mathbb{N}$; then, the following equation holds true:*

$$\sum_{i=1}^{n} z^{\lfloor (bi)^a \rfloor x} = (n+1) z^{\lfloor (bn)^a \rfloor x} - \left\lceil \frac{1}{b} \right\rceil - (1 - z^{-x}) \sum_{t=1}^{\lfloor (bn)^a \rfloor} z^{tx} \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil \tag{20}$$

Proof. By substituting $y = 0$ *and* $f(x) = z^x$ in Equation (11), we obtain:

$$\sum_{i=1}^{n} z^{\lfloor (bi)^a \rfloor x} = (n+1) z^{\lfloor (bn)^a \rfloor x} - \left\lceil \frac{1}{b} \right\rceil - \sum_{t=1}^{\lfloor (bn)^a \rfloor} \left[z^{tx} - z^{(t-1)x} \right] \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil$$

Furthermore, with a basic manipulation, we arrive at Equation (20). □

Theorem 6 (Ceiling Geometric Series)**.** *Let $z, x \in \mathbb{C}, a, b \in \mathbb{R}^+, n \in \mathbb{N}$; then, the following equation holds true:*

$$\sum_{i=1}^{n} z^{\lceil (bi)^a \rceil x} = n z^{\lceil (bn)^a \rceil x} - (z^x - 1) \sum_{t=1}^{\lceil (bn)^a \rceil - 1} z^{tx} \left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor \tag{21}$$

Proof. By substituting $y = 0$ *and* $f(x) = z^x$ in Equation (17), we obtain:

$$\sum_{i=1}^{n} z^{\lceil (bi)^a \rceil x} = n z^{\lceil (bn)^a \rceil x} - \sum_{t=1}^{\lceil (bn)^a \rceil - 1} \left[z^{(t+1)x} - z^{tx} \right] \left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor$$

Furthermore, with a few basic manipulations, we arrive at Equation (21). □

6. Floor and Ceiling Telescoping Equivalent Formulae

Theorem 7 (C-Telescoping Identity)**.** *Let $a, b \in \mathbb{R}^+, n \in \mathbb{N}$ and a sequence $k_n (\forall n \in \mathbb{N} \cup \{0\})$; then, the following equation holds true:*

$$\sum_{i=1}^{n} (k_i - k_{i-1}) \left\lceil \frac{i^{\frac{1}{a}}}{b} \right\rceil = \left\lceil \frac{(n+1)^{\frac{1}{a}}}{b} \right\rceil k_n - \left\lceil \frac{1}{b} \right\rceil k_0 - \sum_{i=1}^{n} \left[\left\lceil \frac{(i+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{i^{\frac{1}{a}}}{b} \right\rceil \right] k_i \tag{22}$$

Proof. Equation (22) is obtained by substituting $y = 0$ *and* $f(x) = k_x$ for some sequence k_t in Equations (10) and (11) and subtracting them. □

Theorem 8 (F-Telescoping Identity)**.** *Let $a, b \in \mathbb{R}^+, n \in \mathbb{N}$ and a sequence $k_n (\forall n \in \mathbb{N} \cup \{0\})$; then, the following equation holds true:*

$$\sum_{i=1}^{n-1} (k_{i+1} - k_i) \left\lfloor \frac{i^{\frac{1}{a}}}{b} \right\rfloor = \left\lfloor \frac{n^{\frac{1}{a}}}{b} \right\rfloor k_n - \sum_{i=1}^{n} \left[\left\lfloor \frac{i^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(i-1)^{\frac{1}{a}}}{b} \right\rfloor \right] k_i \tag{23}$$

Proof. Equation (23) is obtained by substituting $y = 0$ *and* $f(x) = k_x$ for some sequence k_t in Equations (16) and (17) and subtracting them. □

7. Corollaries

7.1. Corollaries of Section 4

Corollary 1. *For any $a, b \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the following equation holds true:*

$$^{F}S_{b}^{a,1}(n) = \sum_{i=1}^{n} \lfloor (bi)^a \rfloor = (n+1)\lfloor (bn)^a \rfloor - \sum_{t=1}^{\lfloor (bn)^a \rfloor} \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil \tag{24}$$

Proof. By substituting $p = 1$ in Equation (18), we arrive at Equation (24). □

Corollary 2. *For any $a, b \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the following equation holds true:*

$$^{C}S_{b}^{a,1}(n) = \sum_{i=1}^{n} \lceil (bi)^a \rceil = n\lceil (bn)^a \rceil - \sum_{t=1}^{\lceil (bn)^a \rceil - 1} \left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor \tag{25}$$

Proof. By substituting $p = 1$ in Equation (19), we arrive at Equation (25). □

Corollary 3. *For any $a \in \mathbb{R}^+$ and $p, n \in \mathbb{N}$, the following equation holds true:*

$$^{F}S^{a,p}(n) = \sum_{i=1}^{n} \lfloor i^a \rfloor^p = (n+1)\lfloor n^a \rfloor^p - \sum_{t=1}^{\lfloor n^a \rfloor} \sum_{k=0}^{p-1} (-1)^{p-k+1} \binom{p}{k} t^k \left\lceil t^{\frac{1}{a}} \right\rceil \tag{26}$$

Proof. By substituting $b = 1$ in Equation (18), we arrive at Equation (26). □

Corollary 4. *For any $a \in \mathbb{R}^+$ and $p, n \in \mathbb{N}$, the following equation holds true:*

$$^{C}S^{a,p}(n) = \sum_{i=1}^{n} \lceil i^a \rceil^p = n\lceil n^a \rceil^p - \sum_{t=1}^{\lceil n^a \rceil - 1} \sum_{k=0}^{p-1} \binom{p}{k} t^k \left\lfloor t^{\frac{1}{a}} \right\rfloor \tag{27}$$

Proof. By substituting $b = 1$ in Equation (19), we arrive at Equation (27). □

Corollary 5. *For any $a \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the following equation holds true:*

$$^{F}S^{a,1}(n) = \sum_{i=1}^{n} \lfloor i^a \rfloor = (n+1)\lfloor n^a \rfloor - \sum_{t=1}^{\lfloor n^a \rfloor} \left\lceil t^{\frac{1}{a}} \right\rceil \tag{28}$$

Proof. Equation (28) can be obtained by simply substituting $b = 1$ in Equation (24) or $p = 1$ in Equation (26). □

Corollary 6. *For any $a \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the following equation holds true:*

$$^{C}S^{a,1}(n) = \sum_{i=1}^{n} \lceil i^a \rceil = n\lceil n^a \rceil - \sum_{t=1}^{\lceil n^a \rceil - 1} \left\lfloor t^{\frac{1}{a}} \right\rfloor \tag{29}$$

Proof. Equation (29) can be obtained by simply substituting $b = 1$ in Equation (25) or $p = 1$ in Equation (27). □

Remark 2. *Corollaries 5 and 6 can be independently proven using the principle of mathematical induction (see Section 10).*

Hypothesis 1. *No result from Theorems 3 and 4 and Corollaries 1–6 can be proven using the principle of mathematical induction other than Corollaries 5 and 6.*

Corollary 7. *For any a of the form $a = \frac{1}{q}$ where $q \in \mathbb{N}$ and $p, n \in \mathbb{N}$, the following equation holds true:*

$$ {}^{F}S^{\frac{1}{q},p}(n) = \sum_{i=1}^{n} \left\lfloor \sqrt[q]{i} \right\rfloor^{p} = (n+1)\left\lfloor \sqrt[q]{n} \right\rfloor^{p} - \sum_{k=0}^{p-1} (-1)^{p-k+1} \binom{p}{k} F^{q+k}\left(\left\lfloor \sqrt[q]{n} \right\rfloor\right) \tag{30} $$

where $F^{q}(n)$ denotes Faulhaber's formula (see Section 2.1).

Proof. By substituting $a = \frac{1}{q}$, $q \in \mathbb{N}$ in Equation (26) assumes the form:

$$ \begin{aligned} {}^{F}S^{\frac{1}{q},p}(n) = \sum_{i=1}^{n} \left\lfloor \sqrt[q]{i} \right\rfloor^{p} &= (n+1)\left\lfloor \sqrt[q]{n} \right\rfloor^{p} - \sum_{t=1}^{\lfloor \sqrt[q]{n} \rfloor} \sum_{k=0}^{p-1} (-1)^{p-k+1} \binom{p}{k} t^{k} \left\lceil t^{\frac{1}{\left(\frac{1}{q}\right)}} \right\rceil \\ &= (n+1)\left\lfloor \sqrt[q]{n} \right\rfloor^{p} - \sum_{t=1}^{\lfloor \sqrt[q]{n} \rfloor} \sum_{k=0}^{p-1} (-1)^{p-k+1} \binom{p}{k} t^{k} \left\lceil t^{q} \right\rceil \end{aligned} $$

Now, as both $t, q \in \mathbb{N} \Rightarrow \lceil t^{q} \rceil = t^{q}$,

$$ \Rightarrow {}^{F}S^{\frac{1}{q},p}(n) = \sum_{i=1}^{n} \left\lfloor \sqrt[q]{i} \right\rfloor^{p} = (n+1)\left\lfloor \sqrt[q]{n} \right\rfloor^{p} - \sum_{k=0}^{p-1} (-1)^{p-k+1} \binom{p}{k} \left[\sum_{t=1}^{\lfloor \sqrt[q]{n} \rfloor} t^{q+k} \right] $$

Again, using Faulhaber's formula for $\sum_{t=1}^{\lfloor \sqrt[q]{n} \rfloor} t^{q+k}$, the previous equation reduces to Equation (30). □

Corollary 8. *For any a of the form $a = \frac{1}{q}$ where $q \in \mathbb{N}$ and $p, n \in \mathbb{N}$, the following equation holds true:*

$$ {}^{C}S^{\frac{1}{q},p}(n) = \sum_{i=1}^{n} \left\lceil \sqrt[q]{i} \right\rceil^{p} = n\left\lceil \sqrt[q]{n} \right\rceil^{p} - \sum_{k=0}^{p-1} \binom{p}{k} F^{q+k}\left(\left\lceil \sqrt[q]{n} \right\rceil\right) \tag{31} $$

Proof. By substituting $a = \frac{1}{q}$, $q \in \mathbb{N}$ in Equation (27) assumes the form:

$$ \begin{aligned} {}^{C}S^{\frac{1}{q},p}(n) = \sum_{i=1}^{n} \left\lceil \sqrt[q]{i} \right\rceil^{p} &= n\left\lceil \sqrt[q]{n} \right\rceil^{p} - \sum_{t=1}^{\lceil \sqrt[q]{n} \rceil - 1} \sum_{k=0}^{p-1} \binom{p}{k} t^{k} \left\lfloor t^{\frac{1}{\left(\frac{1}{q}\right)}} \right\rfloor \\ &= n\left\lceil \sqrt[q]{n} \right\rceil^{p} - \sum_{t=1}^{\lceil \sqrt[q]{n} \rceil - 1} \sum_{k=0}^{p-1} \binom{p}{k} t^{k} \left\lfloor t^{q} \right\rfloor \end{aligned} $$

Now, as both $t, q \in \mathbb{N} \Rightarrow \lceil t^{q} \rceil = t^{q}$,

$$ \Rightarrow {}^{C}S^{\frac{1}{q},p}(n) = \sum_{i=1}^{n} \left\lceil \sqrt[q]{i} \right\rceil^{p} = n\left\lceil \sqrt[q]{n} \right\rceil^{p} - \sum_{k=0}^{p-1} \binom{p}{k} \left[\sum_{t=1}^{\lceil \sqrt[q]{n} \rceil - 1} t^{q+k} \right] $$

Again, using Faulhaber's formula for $\sum_{t=1}^{(\lceil \sqrt[q]{n} \rceil - 1)} t^{q+k}$, the previous equation reduces to Equation (31). □

Corollary 9. *For any a of the form $a = \frac{1}{q}$ where $q \in \mathbb{N}$ and $n \in \mathbb{N}$, the following equation holds true:*

$$ {}^{F}S^{\frac{1}{q},1}(n) = \sum_{i=1}^{n} \left\lfloor \sqrt[q]{i} \right\rfloor = (n+1)\left\lfloor \sqrt[q]{n} \right\rfloor - F^{q}\left(\left\lfloor \sqrt[q]{n} \right\rfloor\right) \tag{32} $$

Proof. By substituting $p = 1$ in Equation (30), we arrive at Equation (32). □

Corollary 10. *For any a of the form $a = \frac{1}{q}$ where $q \in \mathbb{N}$ and $n \in \mathbb{N}$, the following equation holds true:*

$$ {}^{C}S^{\frac{1}{q},1}(n) = \sum_{i=1}^{n} \left\lceil \sqrt[q]{i} \right\rceil = (n)\lceil \sqrt[q]{n} \rceil - F^{q}(\lceil \sqrt[q]{n} \rceil - 1) \tag{33} $$

Proof. By substituting $p = 1$ in Equation (31), we arrive at Equation (33). □

7.2. Corollaries of Section 5

Corollary 11. *For any a of the form $a = \frac{1}{q}$ where $q \in \mathbb{N}$, $z \in \mathbb{C}$ and $n \in \mathbb{N}$, the following equation holds true:*

$$ \sum_{i=1}^{n} z^{\lfloor \sqrt[q]{i} \rfloor} = (n+1)z^{\lfloor \sqrt[q]{n} \rfloor} - 1 - (1 - z^{-1}) \cdot \left(z\frac{d}{dz} \right)^{q} \frac{1 - z^{\lfloor \sqrt[q]{n} \rfloor + 1}}{1 - z} \tag{34} $$

Proof. By substituting $a = \frac{1}{q}, q \in \mathbb{N}, b = x = 1$ in Equation (20), we obtain

$$ \sum_{i=1}^{n} z^{\lfloor \sqrt[q]{i} \rfloor} = (n+1)z^{\lfloor \sqrt[q]{n} \rfloor} - 1 - (1 - z^{-1}) \sum_{i=1}^{\lfloor \sqrt[q]{n} \rfloor} i^{q} z^{i} $$

Finally, substituting Equation (2) in previous equation, we obtain Equation (34). □

Corollary 12. *For any a of the form $a = \frac{1}{q}$ where $q \in \mathbb{N}$, $z \in \mathbb{C}$ and, $n \in \mathbb{N}$, the following equation holds true:*

$$ \sum_{i=1}^{n} z^{\lceil \sqrt[q]{i} \rceil} = nz^{\lceil \sqrt[q]{n} \rceil} - (z - 1) \cdot \left(z\frac{d}{dz} \right)^{q} \frac{1 - z^{\lceil \sqrt[q]{n} \rceil}}{1 - z} \tag{35} $$

Proof. By substituting $a = \frac{1}{q}, q \in \mathbb{N}, b = x = 1$ in Equation (21) we get,

$$ \sum_{i=1}^{n} z^{\lceil \sqrt[q]{i} \rceil} = nz^{\lceil \sqrt[q]{n} \rceil} - (z - 1) \sum_{i=1}^{\lceil \sqrt[q]{n} \rceil - 1} i^{q} z^{i} $$

Finally, substituting Equation (2) in previous equation we get Equation (35). □

7.3. Corollaries of Section 6

Corollary 13. *For any $q, n \in \mathbb{N}$ and a finite sequence k_t, the following equation holds true:*

$$ \sum_{i=1}^{n} i^{q}(k_i - k_{i-1}) = (n+1)^{q} k_n - k_0 - \sum_{i=1}^{n} \sum_{t=0}^{q-1} \binom{q}{t} i^{t} k_i \tag{36} $$

Proof. Equation (36) can be derived by substituting $a = \frac{1}{q}, q \in \mathbb{N}$ and $b = 1$ in Equation (22). □

Corollary 14. *For any $q, n \in \mathbb{N}$ and a finite sequence k_t, the following equation holds true:*

$$ \sum_{i=1}^{n-1} i^{q}(k_{i+1} - k_i) = n^{q} k_n - \sum_{i=1}^{n} \sum_{t=0}^{q-1} (-1)^{q+t+1} \binom{q}{t} i^{t} k_i \tag{37} $$

Proof. Equation (37) can be derived by substituting $a = \frac{1}{q}, q \in \mathbb{N}$ and $b = 1$ in Equation (23). □

8. Relations of Fibonacci Numbers

Theorem 9 (Shah–Pingala Formula of order q)**.** *Let $n, q \in \mathbb{N}$ and let F_t be the t^{th} Fibonacci number of the sequence, then the following equation holds true:*

$$S(q,n) = \sum_{i=1}^{n} i^q F_i = (n-1)^q F_{n+1} + n^q F_n + (-1)^{q-1} - \sum_{t=0}^{q-1} \binom{q}{t} [(-1)^{q-k} - (-2)^{q-k}] S(k,n) \tag{38}$$

Proof. By substituting $k_i = F_{i+2}$ in Equation (36) or $k_i = F_{i+1}$ in Equation (37), we simply arrive at Equation (38). $\square$

Theorem 10 (Floor Fibonacci Series)**.** *Let $n, q \in \mathbb{N}$ and let F_t be the t^{th} Fibonacci number of the sequence then the following equation holds true:*

$$\sum_{i=1}^{n} F_{\lfloor \sqrt[q]{i} \rfloor + 2} = (n+1) F_{\lfloor \sqrt[q]{n} \rfloor + 2} - 1 - S(q, \lfloor \sqrt[q]{n} \rfloor) \tag{39}$$

Proof. Replace function f with the sequence F_t in Equation (11) with $y = 2, b = x = 1$, *and* $a = \frac{1}{q}, q \in \mathbb{N}$. $\square$

Theorem 11 (Ceiling Fibonacci Series)**.** *Let $n, q \in \mathbb{N}$ and let F_t be the t^{th} Fibonacci number of the sequence, then the following equation holds true:*

$$\sum_{i=1}^{n} F_{\lceil \sqrt[q]{i} \rceil + 1} = n F_{\lceil \sqrt[q]{n} \rceil + 1} - S(q, \lceil \sqrt[q]{n} \rceil - 1) \tag{40}$$

Proof. Replace function f with the sequence F_t in Equation (17) with $y = b = x = 1$, *and* $a = \frac{1}{q}, q \in \mathbb{N}$. $\square$

9. Results for Specific Values

9.1. Specific Values—Section 4

By taking $a = 1$ in Equations (28) and (29) or $q = 1$ in Equations (32) and (33), all of these are reduced to the Gauss formula (take Equation (28) for example):

$$^F S^{1,1}(n) = \sum_{i=1}^{n} \left\lfloor i^{\frac{1}{1}} \right\rfloor = (n+1) \left\lfloor n^{\frac{1}{1}} \right\rfloor - \sum_{i=1}^{\left\lfloor n^{\frac{1}{1}} \right\rfloor} \left\lceil i^1 \right\rceil \Rightarrow \sum_{i=1}^{n} i = (n+1)n - \sum_{i=1}^{n} i$$

$$\Rightarrow 2\sum_{i=1}^{n} i = (n+1)n \Rightarrow \sum_{i=1}^{n} i = \frac{(n+1)n}{2}$$

Similarly, if taking $a = \frac{1}{2}$ in Equation (28) or $q = 2$ in Equation (32), both of these reduce to:

$$^F S^{\frac{1}{2},1}(n) = \sum_{i=1}^{n} \left\lfloor \sqrt{i} \right\rfloor = \frac{6n\lfloor \sqrt{n} \rfloor - 2\lfloor \sqrt{n} \rfloor^3 - 3\lfloor \sqrt{n} \rfloor^2 + 5\lfloor \sqrt{n} \rfloor}{6} \tag{41}$$

Moreover, taking $a = \frac{1}{2}$ in Equation (29) or $q = 2$ in Equation (33), both of these reduce to:

$$\sum_{i=1}^{n} \left\lceil \sqrt{i} \right\rceil = \frac{6n\lceil \sqrt{n} \rceil - 2\lceil \sqrt{n} \rceil^3 + 3\lceil \sqrt{n} \rceil^2 - \lceil \sqrt{n} \rceil}{6} \tag{42}$$

Both equations are discussed by Knuth and Patashnik [17]. With Equations (32) and (33), one can go for $a = \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$.

9.2. Specific Values—Section 5

By substituting $a = b = x = 1$ in Equation (20) or Equation (21), we obtain Equation (3) of Section 2.3.

9.3. Specific Values—Section 6

9.3.1. Generalised Pascal Identities

Ceiling Pascal Identity

By substituting $k_t = 1$ $(\forall t)$ in Equation (22), we arrive at the "ceiling Pascal identity":

$$\sum_{i=1}^{n}\left[\left\lceil \frac{(i+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{i^{\frac{1}{a}}}{b} \right\rceil\right] = \left\lceil \frac{(n+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{1}{b} \right\rceil \tag{43}$$

Remark 3. *Pascal's identity (Equation (5)) is a special case of Equation (43) for $b = 1$ and $a = \frac{1}{q}, q \in \mathbb{N}$. The same can also be obtained by substituting $k_t = 1$ $(\forall t)$ in Equation (36).*

Floor Pascal Identity

By substituting $k_t = 1$ $(\forall t)$ in Equation (23), we arrive at "floor Pascal identity":

$$\sum_{i=1}^{n}\left[\left\lfloor \frac{i^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(i-1)^{\frac{1}{a}}}{b} \right\rfloor\right] = \left\lfloor \frac{n^{\frac{1}{a}}}{b} \right\rfloor \tag{44}$$

Remark 4. *An alternate form of Pascal's identity (Equation (6)) is a special case of Equation (44) for $b = 1$ and $a = \frac{1}{q}, q \in \mathbb{N}$, which can also be obtained by substituting $k_t = 1$ $(\forall t)$ in Equation (37).*

9.3.2. Sum of q^{th} Powers of First n Natural Numbers

By substituting $k_t = t$, Equation (36) reduces as:

$$\sum_{i=1}^{n} i^q = \frac{n(n+1)^q}{q+1} - \frac{1}{q+1}\sum_{i=1}^{n}\sum_{t=1}^{q-1}\binom{q}{t-1} i^t \tag{45}$$

Whereas by substituting $k_t = t$, Equation (37) reduces as:

$$\sum_{i=1}^{n} i^q = \frac{(n+1)n^q}{q+1} - \frac{1}{q+1}\sum_{i=1}^{n}\sum_{t=1}^{q-1}(-1)^{q+t}\binom{q}{t-1} i^t \tag{46}$$

Both of which relate to $F^q(n)$ in terms of lower powers $F^k(n), k = 1, 2, \ldots, q-1$.

9.4. Specific Values—Section 8

9.4.1. For $q = 0$

The Shah–Pingala formula for $q = 0$ reduces to the "Fibonacci Series" (Section 2.4).

9.4.2. For $q = 1$

Using Equation (38), the Shah–Pingala Formula of order 1 can be obtained as:

$$\sum_{i=1}^{n} iF_i = nF_{n+2} - F_{n+3} + 2 \tag{47}$$

9.4.3. For $q = 2$

Using Equations (38) and (47), the Shah–Pingala formula of order 2 can be obtained as:

$$\sum_{i=1}^{n} i^2F_i = n^2F_{n+2} - 2nF_{n+3} + F_{n+6} - 8 \tag{48}$$

Furthermore, equations for $q = 3, 4, \ldots$ can be derived using the same method.

10. Proofs Using Principle of Mathematical Induction

In this section, the proofs by principle of mathematical induction (for Corollaries 5 and 6) mentioned in Remark 2 are given.

Proofs

Corollary 5: let $P(n)$ be the following statement:

$$P(n): \sum_{i=1}^{n} \lfloor i^a \rfloor = (n+1)\lfloor n^a \rfloor - \sum_{t=1}^{\lfloor n^a \rfloor} \left\lceil t^{\frac{1}{a}} \right\rceil \tag{49}$$

Basic step of induction: $P(1)$ is clearly true:

$$P(1): 1 = (1+1)1 - 1 = 2 \times 1 - 1 = 1 \; \left(\because a \in \mathbb{R}^+ \Rightarrow \lfloor 1^a \rfloor = \left\lceil 1^{\frac{1}{a}} \right\rceil = 1 \right)$$

Inductive step: assume $P(k)$ is true for some $n = k$. One shall prove $P(k) \Rightarrow P(k+1)$

$$P(k): \sum_{i=1}^{k} \lfloor i^a \rfloor = (k+1)\lfloor k^a \rfloor - \sum_{t=1}^{\lfloor k^a \rfloor} \left\lceil t^{\frac{1}{a}} \right\rceil \tag{50}$$

It follows that

$$P(k+1): \sum_{i=1}^{k+1} \lfloor i^a \rfloor = (k+1)\lfloor k^a \rfloor - \sum_{t=1}^{\lfloor k^a \rfloor} \left\lceil t^{\frac{1}{a}} \right\rceil + \lfloor (k+1)^a \rfloor \tag{51}$$

Consider $m \in \mathbb{N}$ such that $m = \lfloor k^a \rfloor$,

$$m = \lfloor k^a \rfloor \Rightarrow m \le k^a < m+1 \Rightarrow m^{\frac{1}{a}} \le k < (m+1)^{\frac{1}{a}} (\because a \in \mathbb{R}^+)$$

$$\Rightarrow \left\lceil m^{\frac{1}{a}} \right\rceil \le k < \left\lceil (m+1)^{\frac{1}{a}} \right\rceil (\because k \in \mathbb{N})$$

$$\Rightarrow \left\lceil m^{\frac{1}{a}} \right\rceil \le k \le \left\lceil (m+1)^{\frac{1}{a}} \right\rceil - 1$$

This splits into two cases:
(I): $k = \left\lceil (m+1)^{\frac{1}{a}} \right\rceil - 1$ & (II): $\left\lceil m^{\frac{1}{a}} \right\rceil \le k < \left\lceil (m+1)^{\frac{1}{a}} \right\rceil - 1$
Case (I): $k = \left\lceil (m+1)^{\frac{1}{a}} \right\rceil - 1 \Rightarrow k+1 = \left\lceil (m+1)^{\frac{1}{a}} \right\rceil$
Here, $k \in \mathbb{N} \Rightarrow k+1 \in \mathbb{N} \Rightarrow \left\lceil (m+1)^{\frac{1}{a}} \right\rceil \in \mathbb{N} \Rightarrow \left\lceil (m+1)^{\frac{1}{a}} \right\rceil = (m+1)^{\frac{1}{a}}$
$\Rightarrow k+1 = (m+1)^{\frac{1}{a}} \Rightarrow (k+1)^a = m+1$
Again $m \in \mathbb{N} \Rightarrow m+1 \in \mathbb{N} \Rightarrow (k+1)^a \in \mathbb{N} \Rightarrow (k+1)^a = \lfloor (k+1)^a \rfloor$

$$\Rightarrow \lfloor (k+1)^a \rfloor = m+1 = \lfloor k^a \rfloor + 1 (\because m = \lfloor k^a \rfloor)$$

$\Rightarrow \lfloor (k+1)^a \rfloor = \lfloor k^a \rfloor + 1$, if $k = \left\lceil (m+1)^{\frac{1}{a}} \right\rceil - 1$ therefore, from Equation (51)

$$P(k+1): \sum_{i=1}^{k+1} \lfloor i^a \rfloor = (k+1)\lfloor k^a \rfloor - \sum_{t=1}^{\lfloor k^a \rfloor} \left\lceil t^{\frac{1}{a}} \right\rceil + \lfloor (k+1)^a \rfloor$$

$$\Rightarrow P(k+1): \sum_{i=1}^{k+1} \lfloor i^a \rfloor = (k+1)\left(\lfloor (k+1)^a \rfloor - 1\right) - \sum_{t=1}^{\lfloor (k+1)^a \rfloor - 1} \left\lceil t^{\frac{1}{a}} \right\rceil + \lfloor (k+1)^a \rfloor$$

$$\Rightarrow P(k+1) : \sum_{i=1}^{k+1} \lfloor i^a \rfloor = (k+1)\lfloor (k+1)^a \rfloor + \lfloor (k+1)^a \rfloor - (k+1) - \sum_{t=1}^{\lfloor (k+1)^a \rfloor - 1} \left\lceil t^{\frac{1}{a}} \right\rceil$$

Now, as $k+1 = \left\lceil (m+1)^{\frac{1}{a}} \right\rceil$ and $m+1 = \lfloor (k+1)^a \rfloor$, we obtain $k+1 = \left\lceil \lfloor (k+1)^a \rfloor^{\frac{1}{a}} \right\rceil$

$$\Rightarrow P(k+1) : \sum_{i=1}^{k+1} \lfloor i^a \rfloor = ((k+1)+1)\lfloor (k+1)^a \rfloor - \left\lceil \lfloor (k+1)^a \rfloor^{\frac{1}{a}} \right\rceil - \sum_{t=1}^{\lfloor (k+1)^a \rfloor - 1} \left\lceil t^{\frac{1}{a}} \right\rceil$$

$$\Rightarrow P(k+1) : \sum_{i=1}^{k+1} \lfloor i^a \rfloor = ((k+1)+1)\lfloor (k+1)^a \rfloor - \sum_{t=1}^{\lfloor (k+1)^a \rfloor} \left\lceil t^{\frac{1}{a}} \right\rceil$$

Therefore, $P(k) \Rightarrow P(k+1)$ for case (I).
Case (II) :

$$\begin{aligned} &\left\lceil m^{\frac{1}{a}} \right\rceil \leq k < \left\lceil (m+1)^{\frac{1}{a}} \right\rceil - 1 \Rightarrow \left\lceil m^{\frac{1}{a}} \right\rceil + 1 \leq k+1 < \left\lceil (m+1)^{\frac{1}{a}} \right\rceil \\ \Rightarrow &\left\lceil m^{\frac{1}{a}} \right\rceil < k+1 < \left\lceil (m+1)^{\frac{1}{a}} \right\rceil \Rightarrow m < (k+1)^a < m+1 \\ \Rightarrow &\lfloor (k+1)^a \rfloor = m \Rightarrow \lfloor (k+1)^a \rfloor = \lfloor k^a \rfloor = m \end{aligned}$$

Therefore, from Equation (51)

$$P(k+1) : \sum_{i=1}^{k+1} \lfloor i^a \rfloor = (k+1)\lfloor k^a \rfloor - \sum_{t=1}^{\lfloor k^a \rfloor} \left\lceil t^{\frac{1}{a}} \right\rceil + \lfloor (k+1)^a \rfloor$$

$$\Rightarrow P(k+1) : \sum_{i=1}^{k+1} \lfloor i^a \rfloor = (k+1)\lfloor (k+1)^a \rfloor - \sum_{t=1}^{\lfloor (k+1)^a \rfloor} \left\lceil t^{\frac{1}{a}} \right\rceil + \lfloor (k+1)^a \rfloor \ \left(\because \lfloor k^a \rfloor = \lfloor (k+1)^a \rfloor\right)$$

$$\Rightarrow P(k+1) : \sum_{i=1}^{k+1} \lfloor i^a \rfloor = ((k+1)+1)\lfloor (k+1)^a \rfloor - \sum_{t=1}^{\lfloor (k+1)^a \rfloor} \left\lceil t^{\frac{1}{a}} \right\rceil$$

Therefore, $P(k) \Rightarrow P(k+1)$ for case (II).
Hence,

$$P(k) \Rightarrow P(k+1) \forall\, k \ \ such\ that \left\lceil m^{\frac{1}{a}} \right\rceil \leq k \leq \left\lceil (m+1)^{\frac{1}{a}} \right\rceil - 1$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true.
Hence, by principle of mathematical induction, $P(n)$ is true $\forall n \in \mathbb{N}$.
Corollary (6): let $P(n)$ be the following statement:

$$P(n) : \sum_{i=1}^{n} \lceil i^a \rceil = n\lceil n^a \rceil - \sum_{t=1}^{\lceil n^a \rceil - 1} \left\lfloor t^{\frac{1}{a}} \right\rfloor \tag{52}$$

Basic step of induction : $P(1)$ is clearly true:

$$P(1) : 1 = 1(1) - 0 = 1 \left(\because a \in \mathbb{R}^+ \Rightarrow \left\lfloor 1^{\frac{1}{a}} \right\rfloor = \lceil 1^a \rceil = \left\lfloor \lceil 1^a \rceil^{\frac{1}{a}} \right\rfloor = 1\right)$$

Inductive step: assume $P(k)$ is true for some $n = k$. One shall prove $P(k) \Rightarrow P(k+1)$:

$$P(k) : \sum_{i=1}^{k} \lceil i^a \rceil = k\lceil k^a \rceil - \sum_{t=1}^{\lceil k^a \rceil - 1} \left\lfloor t^{\frac{1}{a}} \right\rfloor \tag{53}$$

It follows that:

$$P(k+1):\sum_{i=1}^{k+1}\lceil i^a\rceil = k\lceil k^a\rceil - \sum_{t=1}^{\lceil k^a\rceil-1}\left\lfloor t^{\frac{1}{a}}\right\rfloor + \lceil (k+1)^a\rceil \tag{54}$$

Consider $m \in \mathbb{N}$ such that $m = \lceil k^a\rceil$:

$$m = \lceil k^a\rceil \Rightarrow m-1 < k^a \le m \Rightarrow (m-1)^{\frac{1}{a}} < k \le m^{\frac{1}{a}} (\because a \in \mathbb{R}^+)$$

$$\Rightarrow \left\lfloor (m-1)^{\frac{1}{a}}\right\rfloor < k \le \left\lfloor m^{\frac{1}{a}}\right\rfloor (\because k \in \mathbb{N}) \Rightarrow \left\lfloor (m-1)^{\frac{1}{a}}\right\rfloor + 1 \le k \le \left\lfloor m^{\frac{1}{a}}\right\rfloor$$

This splits into two cases: (I): $k = \left\lfloor m^{\frac{1}{a}}\right\rfloor$ & (II): $\left\lfloor (m-1)^{\frac{1}{a}}\right\rfloor + 1 \le k < \left\lfloor m^{\frac{1}{a}}\right\rfloor$

Case (I) : $k = \left\lfloor m^{\frac{1}{a}}\right\rfloor \Rightarrow k = m^{\frac{1}{a}} \left(\because k \in \mathbb{N} \Rightarrow \left\lfloor m^{\frac{1}{a}}\right\rfloor \in \mathbb{N} \Rightarrow \left\lfloor m^{\frac{1}{a}}\right\rfloor = m^{\frac{1}{a}}\right)$

$$\Rightarrow k+1 = m^{\frac{1}{a}} + 1 \Rightarrow k+1 > m^{\frac{1}{a}} \Rightarrow (k+1)^a > m \Rightarrow \lceil (k+1)^a\rceil > \lceil m\rceil$$

$$\Rightarrow \lceil (k+1)^a\rceil > m (\because m \in \mathbb{N}) \Rightarrow \lceil (k+1)^a\rceil = m+1 \Rightarrow \lceil (k+1)^a\rceil = \lceil k^a\rceil + 1 (\because m = \lceil k^a\rceil)$$

$\Rightarrow \lceil (k+1)^a\rceil = \lceil k^a\rceil + 1$, if $k = \left\lfloor m^{\frac{1}{a}}\right\rfloor$. therefore, from Equation (54)

$$P(k+1):\sum_{i=1}^{k+1}\lceil i^a\rceil = k\lceil k^a\rceil - \sum_{t=1}^{\lceil k^a\rceil-1}\left\lfloor t^{\frac{1}{a}}\right\rfloor + \lceil (k+1)^a\rceil$$

$$\Rightarrow P(k+1):\sum_{i=1}^{k+1}\lceil i^a\rceil = k\left(\lceil (k+1)^a\rceil - 1\right) + \lceil (k+1)^a\rceil - \sum_{t=1}^{\lceil (k+1)^a\rceil-2}\left\lfloor t^{\frac{1}{a}}\right\rfloor$$

$$\Rightarrow P(k+1):\sum_{i=1}^{k+1}\lceil i^a\rceil = (k+1)\lceil (k+1)^a\rceil - k - \sum_{t=1}^{\lceil (k+1)^a\rceil-2}\left\lfloor t^{\frac{1}{a}}\right\rfloor$$

Furthermore, for case (I), we have $\lceil k^a\rceil = m$ and $\left\lfloor m^{\frac{1}{a}}\right\rfloor = k \Rightarrow \left\lfloor \lceil k^a\rceil^{\frac{1}{a}}\right\rfloor = k$

This yields $k = \left\lfloor \left(\lceil (k+1)^a\rceil - 1\right)^{\frac{1}{a}}\right\rfloor$

$$\Rightarrow P(k+1):\sum_{i=1}^{k+1}\lceil i^a\rceil = (k+1)\lceil (k+1)^a\rceil - \left\lfloor \left(\lceil (k+1)^a\rceil - 1\right)^{\frac{1}{a}}\right\rfloor - \sum_{t=1}^{\lceil (k+1)^a\rceil-2}\left\lfloor t^{\frac{1}{a}}\right\rfloor$$

$$\Rightarrow P(k+1):\sum_{i=1}^{k+1}\lceil i^a\rceil = (k+1)\lceil (k+1)^a\rceil - \sum_{t=1}^{\lceil (k+1)^a\rceil-1}\left\lfloor t^{\frac{1}{a}}\right\rfloor$$

Therefore, $P(k) \Rightarrow P(k+1)$ for case (I).

Case (II): $\left\lfloor (m-1)^{\frac{1}{a}}\right\rfloor + 1 \le k < \left\lfloor m^{\frac{1}{a}}\right\rfloor \Rightarrow \left\lfloor (m-1)^{\frac{1}{a}}\right\rfloor + 2 \le k+1 < \left\lfloor m^{\frac{1}{a}}\right\rfloor + 1$

$$\Rightarrow \left\lfloor (m-1)^{\frac{1}{a}}\right\rfloor < k+1 \le \left\lfloor m^{\frac{1}{a}}\right\rfloor \Rightarrow (m-1) < (k+1)^a \le m \Rightarrow \lceil (k+1)^a\rceil = m$$

$$\Rightarrow \lceil (k+1)^a\rceil = \lceil k^a\rceil = m$$

Therefore, from Equation (54)

$$P(k+1):\sum_{i=1}^{k+1}\lceil i^a\rceil = k\lceil k^a\rceil - \sum_{t=1}^{\lceil k^a\rceil -1}\left\lfloor t^{\frac{1}{a}}\right\rfloor + \lceil (k+1)^a\rceil$$

$$\Rightarrow P(k+1):\sum_{i=1}^{k+1}\lceil i^a\rceil = k\lceil (k+1)^a\rceil - \sum_{t=1}^{\lceil (k+1)^a\rceil -1}\left\lfloor t^{\frac{1}{a}}\right\rfloor + \lceil (k+1)^a\rceil$$

$$\Rightarrow P(k+1):\sum_{i=1}^{k+1}\lceil i^a\rceil = (k+1)\lceil (k+1)^a\rceil - \sum_{t=1}^{\lceil (k+1)^a\rceil -1}\left\lfloor t^{\frac{1}{a}}\right\rfloor$$

Therefore, $P(k) \Rightarrow P(k+1)$ for case (II).
Hence,

$$P(k) \Rightarrow P(k+1)\ \forall\ k\ \textit{ such that } \left\lfloor (m-1)^{\frac{1}{a}}\right\rfloor + 1 \le k \le \left\lfloor m^{\frac{1}{a}}\right\rfloor$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true.
Hence, by principle of mathematical induction, $P(n)$ is true $\forall n \in \mathbb{N}$.

11. Conclusions and Future Scope

We developed theorems of floor and ceiling functions and used them as the basis to provide more than 40 new results (as theorems or formulae or as their corollaries). Furthermore, these theorems and their corollaries create the possibility of finding hundreds of more results. In particular, implementing our theorems on the results of Schumacher's extended version of Faulhaber's formula [3] may have some more applications [4] in different fields. Considering the vast number of available finite summations, studying, analysing and providing results for all of them is not possible in the scope of one paper. Hence, we discussed just a few of the results which could be derived from the discussed lemmas, theorems and corollaries.

Therefore, we put forth the open problem for future studies to implement our results to different available finite summations (i.e., finite sums involving (1) exponential function, (2) logarithmic function, (3) trigonometric functions, (4) permutations and combinations, (5) extended Faulhaber's formula [3] or partial summations of any other available functions).

To inspire future studies, we list a few examples for reference:

(1) For the binomial coefficient $\binom{n}{k}$:

$$\sum_{i=0}^{n}\binom{n}{i} = \sum_{i=0}^{n} i\left(\frac{2i+1-n}{n+1}\right)\binom{n+1}{i+1} + 1 = 2^n$$

or (2) for the generalised Harmonic number $H_n^{(m)}$:

$$\sum_{i=1}^{n}\frac{1}{\lfloor \sqrt[q]{i}\rfloor^m} = \sum_{t=0}^{q-1}\binom{q}{t}H^{(m-t)}_{\lfloor \sqrt[q]{n}\rfloor} - \left(\lfloor \sqrt[q]{n}\rfloor^q - n\right)\frac{1}{\lfloor \sqrt[q]{n}\rfloor^m}$$

or (3) for logarithms:

$$\sum_{i=1}^{n} i\log\left[\frac{i+1}{i}\right] = \log\left[\frac{(n+1)^{(n+1)}}{(n+1)!}\right].$$

Author Contributions: Conceptualisation, D.S.; methodology, D.S.; validation, M.S., M.O.-L., E.L.-C. and R.S.; formal analysis, M.S., M.O.-L. and R.S.; writing—original draft preparation, D.S.; writing—review and editing, D.S., M.S. and E.L.-C.; supervision, M.S. and E.L.-C. All authors have read and agreed to the published version of the manuscript.

Funding: Author Leon-Castro acknowledges support from the Chilean Government through FONDECYT initiation grant No. 11190056.

Institutional Review Board Statement: Not Applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to acknowledge Nishant Doshi of Pandit Deendayal Energy University, Gandhinagar, Gujarat, India, for bringing us a very interesting exercise, from which we were able to work on the primitive forms of the results. The authors would equally like to thank Kenneth Beitler for contacting us and for making a very helpful suggestion for proving the results. We are especially grateful to have benefited from the guidance of Anand Sengupta and Atul Dixit of the Indian Institute of Technology, Gandhinagar, by the means of video conference and discussions. Furthermore, we are thankful to Meghna Parikh and Bhashin Thakore for their valuable contributions to the first draft.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Knuth, D. *The Art of Computer Programming*, 3rd ed.; Addison-Wesley: Boston, MA, USA, 2013; pp. 79–84.
2. Apostol, T. *Introduction to Analytic Number Theory*; Springer: New York, NY, USA, 1976.
3. Schumacher, R. An Extended Version of Faulhaber's Formula. *J. Integer Seq.* **2016**, *19*, 16.4.2.
4. Sahlmann, H.; Zilker, T. Quantum Surface Holonomies for Loop Quantum Gravity and Their Application to Black Hole Horizons. *Phys. Rev. D* **2020**, *102*, 026009. [CrossRef]
5. MacMillan, K.; Sondow, J. Proofs of Power Sum and Binomial Coefficient Congruences Via Pascal's Identity. *Am. Math. Mon.* **2011**, *118*, 549–551. [CrossRef]
6. McGown, K.; Parks, H. The generalization of Faulhaber's formula to sums of non-integral powers. *J. Math. Anal. Appl.* **2007**, *330*, 571–575. [CrossRef]
7. Merca, M. An Alternative to Faulhaber's Formula. *Am. Math. Mon.* **2015**, *122*, 599–601. [CrossRef]
8. Parks, H. Sums of non-integral powers. *J. Math. Anal. Appl.* **2004**, *297*, 343–349. [CrossRef]
9. Orosi, G. A Simple Derivation Of Faulhaber's Formula. *Appl. Math. E Notes* **2018**, *18*, 124–126.
10. Edwards, A. A Quick Route to Sums of Powers. *Am. Math. Mon.* **1986**, *93*, 451–455. [CrossRef]
11. Gregg, C. Relations between the sums of powers of the natural numbers. *Math. Gaz.* **1960**, *44*, 118–120. [CrossRef]
12. Chorlton, F. Finite Sums of Powers of the Natural Numbers. *Math. Gaz.* **1998**, *82*, 95–96. [CrossRef]
13. Beardon, A. Sums of Powers of Integers. *Am. Math. Mon.* **1996**, *103*, 201–213. [CrossRef]
14. Knuth, D. Johann Faulhaber Furthermore, Sums Of Powers. *Math. Comput.* **1993**, *61*, 277–294. [CrossRef]
15. Frontczak, R.; Srivastava, H.M.; Tomovski, Z. Some Families of Apéry-Like Fibonacci and Lucas Series. *Mathematics* **2021**, *9*, 1621. [CrossRef]
16. Conway, J.; Guy, R. *The Book of Numbers*; Copernicus: New York, NY, USA, 1998.
17. Graham, R.L.; Knuth, D.E.; Patashnik, O. *Concrete Mathematics: A Foundation for Computer Science*; Addison-Wesley: Boston, MA, USA, 1994.
18. The Computation of Polylogarithms. Technical Report 15–92. Available online: https://www.cs.kent.ac.uk/pubs/1992/110/ (accessed on 3 March 2022)
19. Hardy, G.H.; Wright, E.M. *An Introduction to the Theory of Numbers*, 5th ed.; Clarendon Press: Oxford, UK, 1980.