

Article

Monotone Iterative Technique for a New Class of Nonlinear Sequential Fractional Differential Equations with Nonlinear Boundary Conditions under the ψ -Caputo Operator

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Abstract: The main crux of this work is to study the existence of extremal solutions for a new class of nonlinear sequential fractional differential equations (NSFDEs) with nonlinear boundary conditions (NBCs) under the ψ -Caputo operator. The obtained outcomes of the proposed problem are derived by means of the monotone iterative technique (MIT) associated with the method of upper and lower solutions. Lastly, the desired findings are well illustrated by an example.

Keywords: sequential ψ -Caputo derivative; nonlinear boundary conditions; extremal solutions; monotone iterative technique; upper and lower solutions

MSC: 34A08; 26A33; 34A12



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1. Introduction

Currently, the study of initial or boundary value problems (BVPs) for fractional differential equations (FDEs) has received great recognition due to their important applications in various areas, such as mathematical, physical, and engineering models; see [1–3]. For more recent developments on this topic, one can see the monographs [4–7] and the references therein. As a consequence of the advancement made in the field of fractional calculus (FC), several new fractional operators have appeared ranging from Riemann–Liouville, Caputo, Hadamard and Hilfer to ψ -Caputo and ψ -Hilfer operators. For more clarifications and basic properties of these new fractional operators, the reader is referred to the following references [8–13]. The latest operators have the ability to recover the aforementioned operators. In this respect, real-world events are often nonlinear, and thus, they can be modeled by nonlinear FDEs. Recently, plenty of scholars studied the aforesaid field looking for some qualitative properties of their solutions. Generally speaking, getting the exact solution of FDEs involving nonlinearities is a tough task. Namely, in order to bypass the absence of exact solutions of nonlinear FDEs, many researchers have devoted themselves to developing various techniques to compute the approximate solutions to such problems of the considered FDEs. Among them, the monotone iterative technique (MIT) [14,15] linked with the method of upper and lower solutions is employed as a fundamental mechanism to prove the existence as well as the approximation of solutions to many applied problems of nonlinear differential equations and integral equations. In other words, the suggested approach has many interesting advantages. The main advantage of this tool is that it not only proves the existence of solutions but it can also provide calculable monotone sequences that converge to the extremal solutions. Recent results by means of the MIT are obtained in [16–22] and the references therein. To the best of our knowledge, NSFDEs involving the

ψ -fractional operator were not given enough consideration and were only studied by a few researchers [23], and it is the motivation of this paper. So, in this manuscript, we will explore the existence of extremal solutions for the following NSFDEs in the ψ -Caputo sense involving NBCs:

$$\begin{cases} ({}^c\mathbb{D}_{a+}^{\epsilon+1;\psi} + \omega {}^c\mathbb{D}_{a+}^{\epsilon;\psi})\mathbf{p}(\mathbf{r}) = \mathbb{M}(\mathbf{r}, \mathbf{p}(\mathbf{r})), & \mathbf{r} \in \Delta := [a, b], \\ \mathbb{W}(\mathbf{p}(a), \mathbf{p}(b)) = 0, & \mathbf{p}^{[1];\psi}(a) = \theta_1, \end{cases} \quad (1)$$

where ${}^c\mathbb{D}_{a+}^{\epsilon;\psi}$ is the ψ -Caputo fractional derivative of order $\epsilon \in (0, 1]$ (which will be specified in Definition 2), $\mathbb{M}: \Delta \times \mathbb{R} \rightarrow \mathbb{R}$, $\mathbb{W}: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are both continuous functions, ω is a positive real number, and $\theta_1 \in \mathbb{R}$.

This manuscript has the following structure: Section 2 offers some basic definitions and useful tools that are required in this paper. Section 3 is devoted to the principal findings concerning the existence of extremal solutions for the proposed model (1). An example is proposed in Section 4 to highlight the usefulness of our theoretical outcomes. At last, the manuscript ends with a brief conclusion and some suggestions for future work are also pointed out.

2. Preliminaries

Below, we provide some definitions and fundamental lemmas which will be employed and used as helping tools in our proofs later.

Let $\psi \in C^1(\Delta, \mathbb{R})$ be a given function such that $\psi'(\mathbf{r}) > 0$, for all $\mathbf{r} \in \Delta$.

Definition 1 ([4,8]). The R–L fractional integral of order $\epsilon > 0$ for an integrable function $\mathbf{p}: \Delta \rightarrow \mathbb{R}$ with respect to ψ is described by

$$\mathbb{I}_{a+}^{\epsilon;\psi} \mathbf{p}(\mathbf{r}) = \int_a^{\mathbf{r}} \frac{\psi'(\eta)(\psi(\mathbf{r}) - \psi(\eta))^{\epsilon-1}}{\Gamma(\epsilon)} \mathbf{p}(\eta) d\eta,$$

where $\Gamma(\epsilon) = \int_0^{+\infty} \mathbf{r}^{\epsilon-1} e^{-\mathbf{r}} d\mathbf{r}$, $\epsilon > 0$ is called the Gamma function.

Definition 2 ([8]). Let $\psi, \mathbf{p} \in C^n(\Delta, \mathbb{R})$. The Caputo fractional derivative of $\mathbf{p}$ of order $n - 1 < \epsilon < n$ with respect to ψ is defined by

$${}^c\mathbb{D}_{a+}^{\epsilon;\psi} \mathbf{p}(\mathbf{r}) = \mathbb{I}_{a+}^{n-\epsilon;\psi} \mathbf{p}^{[n];\psi}(\mathbf{r}),$$

where $n = [\epsilon] + 1$ for $\epsilon \notin \mathbb{N}$, $n = \epsilon$ for $\epsilon \in \mathbb{N}$, and $\mathbf{p}^{[n];\psi}(\mathbf{r}) = \left(\frac{d}{\psi'(\mathbf{r})} \right)^n \mathbf{p}(\mathbf{r})$.

Lemma 1 ([8]). Let $\epsilon, \mu > 0$, and $\mathbf{p} \in C^n(\Delta, \mathbb{R})$. Then, for each $\mathbf{r} \in \Delta$, the following statements are valid:

1. ${}^c\mathbb{D}_{a+}^{\epsilon;\psi} \mathbb{I}_{a+}^{\epsilon;\psi} \mathbf{p}(\mathbf{r}) = \mathbf{p}(\mathbf{r})$;
2. $\mathbb{I}_{a+}^{\epsilon;\psi} {}^c\mathbb{D}_{a+}^{\epsilon;\psi} \mathbf{p}(\mathbf{r}) = \mathbf{p}(\mathbf{r}) - \sum_{k=0}^{n-1} \frac{\mathbf{p}^{[k];\psi}(a)}{k!} [\psi(\mathbf{r}) - \psi(a)]^k$, $n - 1 < \epsilon \leq n$;
3. $\mathbb{I}_{a+}^{\epsilon;\psi} (\psi(\mathbf{r}) - \psi(a))^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu+\epsilon)} (\psi(\mathbf{r}) - \psi(a))^{\mu+\epsilon-1}$;
4. ${}^c\mathbb{D}_{a+}^{\epsilon;\psi} (\psi(\mathbf{r}) - \psi(a))^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu-\epsilon)} (\psi(\mathbf{r}) - \psi(a))^{\mu-\epsilon-1}$;
5. ${}^c\mathbb{D}_{a+}^{\epsilon;\psi} (\psi(\mathbf{r}) - \psi(a))^k = 0$, for all $k \in \{0, \dots, n-1\}$, $n \in \mathbb{N}$.

Definition 3 ([5]). For $u, v > 0$ and $\omega \in \mathbb{R}$, the Mittag–Leffler functions (MLFs) of one and two parameters are given by

$$\mathbb{E}_u(\omega) = \sum_{k=0}^{\infty} \frac{\omega^k}{\Gamma(uk+1)}, \quad \mathbb{E}_{u,v}(\omega) = \sum_{k=0}^{\infty} \frac{\omega^k}{\Gamma(uk+v)}. \quad (2)$$

Clearly, $\mathbb{E}_{u,1}(\omega) = \mathbb{E}_u(\omega)$.

Lemma 2 ([5,21]). Let $u \in (0,1)$, $v > u$ be arbitrary and $\omega \in \mathbb{R}$. The functions $\mathbb{E}_u, \mathbb{E}_{u,u}$ and $\mathbb{E}_{u,v}$ are nonnegative and have the properties listed below:

1. $\mathbb{E}_u(\omega) \leq 1, \mathbb{E}_{u,v}(\omega) \leq \frac{1}{\Gamma(v)},$ for any $\omega < 0$;
2. $\mathbb{E}_{u,v}(\omega) = \omega \mathbb{E}_{u,u+v}(\omega) + \frac{1}{\Gamma(v)}, \quad u, v > 0, \omega \in \mathbb{R}.$

Definition 4 ([11]). A function $u : [a, \infty) \rightarrow \mathbb{R}$ is said to be of ψ -exponential order if there exist nonnegative constants M, c, b such that

$$|u(r)| \leq M e^{c(\psi(r) - \psi(a))}, \quad r \geq b$$

Definition 5 ([11]). Let $p, \psi : [a, \infty) \rightarrow \mathbb{R}$ be real-valued functions such that ψ is continuous and $\psi'(r) > 0$ on $[a, \infty)$. The generalized Laplace transform of p is denoted by

$$\mathbb{L}_\psi \{p(r)\} = \int_a^\infty e^{-\lambda(\psi(r) - \psi(a))} p(r) \psi'(r) dr, \quad \text{for all } \lambda > 0. \quad (3)$$

This holds as long as the integral on the right-hand side exists.

Definition 6 ([11]). Let u and v be two functions which are piecewise continuous at each interval $[a, b]$ and of $\psi(r)$ -exponential order. We define the generalized convolution of u and v by

$$(u *_\psi v)(r) = \int_a^r \psi'(\eta) u(\eta) v(\psi^{-1}(\psi(r) + \psi(a) - \psi(\eta))) d\eta.$$

Lemma 3 ([11]). Let u and v be two functions which are piecewise continuous at each interval $[a, b]$ and of ψ -exponential order. Then,

$$\mathbb{L}_\psi \{u *_\psi v\} = \mathbb{L}_\psi \{u\} \mathbb{L}_\psi \{v\}.$$

In the following Lemma, we present the generalized Laplace transforms of some elementary functions

Lemma 4 ([11]). The following properties are satisfied:

1. $\mathbb{L}_\psi \{1\} = \frac{1}{\lambda}, \quad \lambda > 0;$
2. $\mathbb{L}_\psi \{(\psi(r) - \psi(a))^{\epsilon-1}\} = \frac{\Gamma(\epsilon)}{\lambda^\epsilon}, \quad \epsilon, \lambda > 0;$
3. $\mathbb{L}_\psi \{\mathbb{E}_u(\pm \omega(\psi(r) - \psi(a))^u)\} = \frac{\lambda^{u-1}}{\lambda^u \mp \omega}, \quad u > 0 \text{ and } \left| \frac{\omega}{\lambda^u} \right| < 1;$
4. $\mathbb{L}_\psi \{(\psi(r) - \psi(a))^{v-1} \mathbb{E}_{u,v}(\pm \omega(\psi(r) - \psi(a))^u)\} = \frac{\lambda^{u-v}}{\lambda^u \mp \omega}, \quad u > 0 \text{ and } \left| \frac{\omega}{\lambda^u} \right| < 1.$

In following theorems, we state the generalized Laplace transforms of the generalized fractional integrals and derivatives.

Lemma 5 ([11]). Let $\epsilon > 0$ and let p be a function of ψ -exponential order, piecewise continuous over each finite interval $[a, b]$. Then,

$$\mathbb{L}_\psi \{\mathbb{I}_{a+}^{\epsilon;\psi} p(r)\} = \frac{\mathbb{L}_\psi \{p(r)\}}{\lambda^\epsilon}, \quad \lambda > 0,$$

Lemma 6 ([11]). Let $\epsilon > 0$, $n = [\epsilon] + 1$, and p is a function such that $p, p^{[k];\psi}, k = 1, \dots, n-1$ are continuous on $[a, \infty)$ and of ψ -exponential order, while ${}^c\mathbb{D}_{a+}^{\epsilon;\psi} p(r)$ is piecewise continuous on $[a, \infty)$. Then, the following relation holds:

$$\mathbb{L}_\psi \{{}^c\mathbb{D}_{a+}^{\epsilon;\psi} p(r)\} = \lambda^\epsilon \mathbb{L}_\psi \{p(r)\} - \sum_{k=0}^{n-1} \lambda^{\epsilon-k-1} p^{[k];\psi}(a), \quad n-1 < \epsilon \leq n.$$

In particular, if $0 < \epsilon \leq 1$, then

$$\mathbb{L}_\psi \{ {}^c\mathbb{D}_{a^+}^{\epsilon;\psi} p(r) \} = \lambda^\epsilon \mathbb{L}_\psi \{ p(r) \} - \lambda^{\epsilon-1} p(a), \quad \lambda > 0,$$

and, if $1 < \epsilon \leq 2$, then

$$\mathbb{L}_\psi \{ {}^c\mathbb{D}_{a^+}^{\epsilon;\psi} p(r) \} = \lambda^\epsilon \mathbb{L}_\psi \{ p(r) \} - \lambda^{\epsilon-1} p(a) - \lambda^{\epsilon-2} p^{[1];\psi}(a), \quad \lambda > 0,$$

Lemma 7. For a given $f \in C(\Delta, \mathbb{R})$, $0 < \epsilon \leq 1$ and $\omega > 0$, the linear fractional initial value problem

$$\begin{cases} ({}^c\mathbb{D}_{a^+}^{\epsilon+1;\psi} + \omega {}^c\mathbb{D}_{a^+}^{\epsilon;\psi}) p(r) = f(r), & r \in \Delta := [a, b], \\ p(a) = \theta_0, & p^{[1];\psi}(a) = \theta_1, \end{cases} \quad (4)$$

has a unique solution given explicitly by

$$\begin{aligned} p(r) &= \theta_0 + \theta_1 (\psi(r) - \psi(a)) \mathbb{E}_{1,2}(-\omega(\psi(r) - \psi(a))) \\ &\quad + \int_a^r \psi'(\eta) (\psi(r) - \psi(\eta))^\epsilon \mathbb{E}_{1,\epsilon+1}(-\omega(\psi(r) - \psi(\eta))) f(\eta) d\eta. \end{aligned} \quad (5)$$

Proof. Performing the generalized Laplace transform to both sides of Equation (4) and then using Lemma 4, one obtains

$$\lambda^{\epsilon+1} \mathbb{L}_\psi \{ p(r) \} - \lambda^\epsilon p(a) - \lambda^{\epsilon-1} p^{[1];\psi}(a) + \omega \lambda^\epsilon \mathbb{L}_\psi \{ p(r) \} - \omega \lambda^{\epsilon-1} p(a) = \mathbb{L}_\psi \{ f(r) \}.$$

So,

$$\begin{aligned} \mathbb{L}_\psi \{ p(r) \} &= \omega \frac{\lambda^{-1}}{\lambda + \omega} \theta_0 + \frac{1}{\lambda + \omega} \theta_0 + \frac{\lambda^{-1}}{\lambda + \omega} \theta_1 + \frac{\lambda^{-\epsilon}}{\lambda + \omega} \mathbb{L}_\psi \{ f(r) \} \\ &= \omega \theta_0 \mathbb{L}_\psi \{ (\psi(r) - \psi(a)) \mathbb{E}_{1,2}(-\omega(\psi(r) - \psi(a))) \} \\ &\quad + \theta_0 \mathbb{L}_\psi \{ \mathbb{E}_1(-\omega(\psi(r) - \psi(a))) \} \\ &\quad + \theta_1 \mathbb{L}_\psi \{ (\psi(r) - \psi(a)) \mathbb{E}_{1,2}(-\omega(\psi(r) - \psi(a))) \} \\ &\quad + \mathbb{L}_\psi \{ (\psi(r) - \psi(a))^\epsilon \mathbb{E}_{1,\epsilon+1}(-\omega(\psi(r) - \psi(a))) \} \mathbb{L}_\psi \{ f(r) \}. \end{aligned}$$

Taking the inverse generalized Laplace transform on both sides of the last expression, we get

$$\begin{aligned} p(r) &= \theta_0 (\mathbb{E}_1(-\omega(\psi(r) - \psi(a))) + \omega(\psi(r) - \psi(a)) \mathbb{E}_{1,2}(-\omega(\psi(r) - \psi(a)))) \\ &\quad + \theta_1 (\psi(r) - \psi(a)) \mathbb{E}_{1,2}(-\omega(\psi(r) - \psi(a))) \\ &\quad + f(r) *_{\psi} (\psi(r) - \psi(a))^\epsilon \mathbb{E}_{1,\epsilon+1}(-\omega(\psi(r) - \psi(a))) \\ &= \theta_0 + \theta_1 (\psi(r) - \psi(a)) \mathbb{E}_{1,2}(-\omega(\psi(r) - \psi(a))) \\ &\quad + \int_a^r \psi'(\eta) (\psi(r) - \psi(\eta))^\epsilon \mathbb{E}_{1,\epsilon+1}(-\omega(\psi(r) - \psi(\eta))) f(\eta) d\eta. \end{aligned}$$

□

Lemma 8 (Comparison Result). Let $0 < \epsilon \leq 1$ and $\omega > 0$. If $\gamma \in C(\Delta, \mathbb{R})$ satisfying ${}^c\mathbb{D}_{a^+}^{\epsilon;\psi} \gamma(r), {}^c\mathbb{D}_{a^+}^{\epsilon+1;\psi} \gamma(r) \in C(\Delta, \mathbb{R})$ and

$$\begin{cases} ({}^c\mathbb{D}_{a^+}^{\epsilon+1;\psi} + \omega {}^c\mathbb{D}_{a^+}^{\epsilon;\psi}) \gamma(r) \geq 0, & r \in (a, b], \\ \gamma(a) \geq 0, & \gamma^{[1];\psi}(a) \geq 0, \end{cases}$$

then $\gamma(r) \geq 0$ for all $r \in \Delta$.

Proof. Let $f(r) = ({}^c\mathbb{D}_{a^+}^{\epsilon+1;\psi} + \omega {}^c\mathbb{D}_{a^+}^{\epsilon;\psi})\gamma(r) \geq 0$, $\gamma(a) = \theta_0 \geq 0$ and $\gamma^{[1];\psi}(a) = \theta_1 \geq 0$ in Lemma 7. Then, it follows by Equation (5) and Lemma 2 that the conclusion of Lemma 8 holds. $\square$

3. Main Results

Definition 7. A function $p \in C^1(\Delta, \mathbb{R})$ is called a lower solution of (1), if it satisfies

$$\begin{cases} ({}^c\mathbb{D}_{a^+}^{\epsilon+1;\psi} + \omega {}^c\mathbb{D}_{a^+}^{\epsilon;\psi})p(r) \leq \mathbb{M}(r, p(r)), & r \in \Delta, \\ \mathbb{W}(p(a), p(b)) \leq 0, & p^{[1];\psi}(a) \leq \theta_1. \end{cases} \quad (6)$$

An upper solution q of the problem (1) can be defined in a similar way by reversing the above inequality.

In order to obtain the existence of the extremal solutions for the initial value problem (1), we give the following assumptions

Hypothesis 1 (H1). There exist $p_0, \tilde{p}_0 \in C(\Delta, \mathbb{R})$ such that p_0 and $\tilde{p}_0$ are lower and upper solutions of problem (1), respectively, with $p_0(r) \leq \tilde{p}_0(r)$, $r \in \Delta$;

Hypothesis 2 (H2). $\mathbb{M} : \Delta \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and nondecreasing function with respect to the second variable;

Hypothesis 3 (H3). There exist constants $c > 0$ and $d \geq 0$, such that for $p_0(a) \leq u_1 \leq u_2 \leq \tilde{p}_0(a)$, $p_0(b) \leq v_1 \leq v_2 \leq \tilde{p}_0(b)$,

$$\mathbb{W}(u_2, v_2) - \mathbb{W}(u_1, v_1) \leq c(u_2 - u_1) - d(v_2 - v_1).$$

Theorem 1. Assume that (H_1) – (H_3) are satisfied. Then, there exist monotone iterative sequences $\{p_n\}$ and $\{\tilde{p}_n\}$, which converge uniformly on Δ to the extremal solutions of the problem (1) in the sector $[p_0, \tilde{p}_0]$, where

$$[p_0, \tilde{p}_0] = \{p \in C(\Delta, \mathbb{R}) : p_0(r) \leq p(r) \leq \tilde{p}_0(r), \quad r \in \Delta\}.$$

Proof. For any $p_0, \tilde{p}_0 \in C(\Delta, \mathbb{R})$, we define

$$\begin{cases} ({}^c\mathbb{D}_{a^+}^{\epsilon+1;\psi} + \omega {}^c\mathbb{D}_{a^+}^{\epsilon;\psi})p_{n+1}(r) = \mathbb{M}(r, p_n(r)), & r \in \Delta, \\ p_{n+1}(a) = p_n(a) - \frac{1}{c}\mathbb{W}(p_n(a), p_n(b)), & p_{n+1}^{[1];\psi}(a) = \theta_1, \end{cases} \quad (7)$$

and

$$\begin{cases} ({}^c\mathbb{D}_{a^+}^{\epsilon+1;\psi} + \omega {}^c\mathbb{D}_{a^+}^{\epsilon;\psi})\tilde{p}_{n+1}(r) = \mathbb{M}(r, \tilde{p}_n(r)), & r \in \Delta, \\ \tilde{p}_{n+1}(a) = \tilde{p}_n(a) - \frac{1}{c}\mathbb{W}(\tilde{p}_n(a), \tilde{p}_n(b)), & \tilde{p}_{n+1}^{[1];\psi}(a) = \theta_1. \end{cases} \quad (8)$$

By Lemma 7, we know that the linear problems (7) and (8) have unique solutions $p_n(r)$, $\tilde{p}_n(r)$, respectively, that are expressed as

$$\begin{aligned} p_{n+1}(r) = & p_n(a) - \frac{1}{c}\mathbb{W}(p_n(a), p_n(b)) + \theta_1(\psi(r) - \psi(a))\mathbb{E}_{1,2}(-\omega(\psi(r) - \psi(a))) \\ & + \int_a^r \psi'(\eta)(\psi(r) - \psi(\eta))^{\epsilon}\mathbb{E}_{1,\epsilon+1}(-\omega(\psi(r) - \psi(\eta)))\mathbb{M}(\eta, p_n(\eta))d\eta, \end{aligned} \quad (9)$$

and

$$\begin{aligned}\tilde{\mathfrak{p}}_{n+1}(r) &= \tilde{\mathfrak{p}}_n(a) - \frac{1}{c} \mathbb{W}(\tilde{\mathfrak{p}}_n(a), \tilde{\mathfrak{p}}_n(b)) + \theta_1(\psi(r) - \psi(a)) \mathbb{E}_{1,2}(-\omega(\psi(r) - \psi(a))) \\ &\quad + \int_a^r \psi'(\eta)(\psi(r) - \psi(\eta))^{\epsilon} \mathbb{E}_{1,\epsilon+1}(-\omega(\psi(r) - \psi(\eta))) \mathbb{M}(\eta, \tilde{\mathfrak{p}}_n(\eta)) d\eta.\end{aligned}$$

Firstly, let us prove that

$$\mathfrak{p}_0(r) \leq \mathfrak{p}_1(r) \leq \tilde{\mathfrak{p}}_1(r) \leq \tilde{\mathfrak{p}}_0(r), \quad r \in \Delta.$$

To this end, we set $\gamma(r) = \mathfrak{p}_1(r) - \mathfrak{p}_0(r)$. From (7) and Definition 7, we obtain

$$\begin{aligned}({}^c\mathbb{D}_{a^+}^{\epsilon+1;\psi} + \omega {}^c\mathbb{D}_{a^+}^{\epsilon;\psi})\gamma(r) &= ({}^c\mathbb{D}_{a^+}^{\epsilon+1;\psi} + \omega {}^c\mathbb{D}_{a^+}^{\epsilon;\psi})\mathfrak{p}_1(r) - ({}^c\mathbb{D}_{a^+}^{\epsilon+1;\psi} + \omega {}^c\mathbb{D}_{a^+}^{\epsilon;\psi})\mathfrak{p}_0(r) \\ &= \mathbb{M}(r, \mathfrak{p}_0(r)) - ({}^c\mathbb{D}_{a^+}^{\epsilon+1;\psi} + \omega {}^c\mathbb{D}_{a^+}^{\epsilon;\psi})\mathfrak{p}_0(r) \geq 0.\end{aligned}$$

Again, since

$$\begin{cases} \gamma(a) = -\frac{1}{c} \mathbb{W}(\mathfrak{p}_0(a), \mathfrak{p}_0(b)) \geq 0, \\ \gamma^{[1];\psi}(a) \geq 0. \end{cases}$$

Invoking Lemma 8, we get $\gamma(r) \geq 0$ for any $r \in \Delta$. Thus, $\mathfrak{p}_0(r) \leq \mathfrak{p}_1(r)$, $r \in \Delta$. By the same method, it can be shown that $\tilde{\mathfrak{p}}_1(r) \leq \tilde{\mathfrak{p}}_0(r)$, $r \in \Delta$.

Now, let $\gamma(r) = \tilde{\mathfrak{p}}_1(r) - \mathfrak{p}_1(r)$. Using (7) and (8) together with assumptions (H_1) – (H_2) , we get

$$({}^c\mathbb{D}_{a^+}^{\epsilon+1;\psi} + \omega {}^c\mathbb{D}_{a^+}^{\epsilon;\psi})\gamma(r) = \mathbb{M}(r, \tilde{\mathfrak{p}}_0(r)) - \mathbb{M}(r, \mathfrak{p}_0(r)) \geq 0.$$

On the other hand,

$$\begin{aligned}\gamma(a) &= (\tilde{\mathfrak{p}}_0(a) - \mathfrak{p}_0(a)) - \frac{1}{c} (\mathbb{W}(\tilde{\mathfrak{p}}_0(a), \tilde{\mathfrak{p}}_0(b)) - \mathbb{W}(\mathfrak{p}_0(a), \mathfrak{p}_0(b))) \\ &\geq \frac{d}{c} (\tilde{\mathfrak{p}}_0(b) - \mathfrak{p}_0(b)) \geq 0,\end{aligned}$$

and

$$\gamma^{[1];\psi}(a) = 0.$$

According to Lemma 8, we arrive at $\mathfrak{p}_1(r) \leq \tilde{\mathfrak{p}}_1(r)$, $r \in \Delta$.

Secondly, we need to show that $\mathfrak{p}_1$ and $\tilde{\mathfrak{p}}_1$ are the lower and upper solutions of problem (1), respectively. Taking into account that $\mathbb{M}$ is an increasing function with respect to the second variable, we get

$$({}^c\mathbb{D}_{a^+}^{\epsilon+1;\psi} + \omega {}^c\mathbb{D}_{a^+}^{\epsilon;\psi})\mathfrak{p}_1(r) = \mathbb{M}(r, \mathfrak{p}_0(r)) \leq \mathbb{M}(r, \mathfrak{p}_1(r)),$$

and

$$\begin{aligned}\mathbb{W}(\mathfrak{p}_1(a), \mathfrak{p}_1(b)) &\leq \mathbb{W}(\mathfrak{p}_0(a), \mathfrak{p}_0(b)) + c(\mathfrak{p}_1(a) - \mathfrak{p}_0(a)) - d(\mathfrak{p}_1(b) - \mathfrak{p}_0(b)) \\ &= -d(\mathfrak{p}_1(b) - \mathfrak{p}_0(b)) \\ &\leq 0.\end{aligned}$$

This means that $\mathfrak{p}_1$ is a lower solution of problem (1). Analogously, we can verify that $\tilde{\mathfrak{p}}_1$ is an upper solution of problem (1).

By the above arguments and mathematical induction, we can show that the sequences $\mathfrak{p}_n, \tilde{\mathfrak{p}}_n$, ($n \geq 1$) are lower and upper solutions of (1), respectively, and satisfy the following relation:

$$\mathfrak{p}_0(r) \leq \mathfrak{p}_1(r) \leq \cdots \leq \mathfrak{p}_n(r) \leq \cdots \leq \tilde{\mathfrak{p}}_n(r) \leq \cdots \leq \tilde{\mathfrak{p}}_1(r) \leq \tilde{\mathfrak{p}}_0(r), \quad r \in \Delta. \quad (10)$$

Thirdly, we show that the sequences $\{p_n\}$ and $\{\tilde{p}_n\}$ converge uniformly to their limit functions p^* and $\tilde{p}^*$, respectively. We show that the sequences $\{p_n\}$ and $\{\tilde{p}_n\}$ converge uniformly to their limit functions p^* and $\tilde{p}^*$, respectively.

First, we prove that $\{p_n\}$ is uniformly bounded. By considering supposition (H_2) , we may write

$$\mathbb{M}(r, p_0(r)) \leq \mathbb{F}(r, p_n(r)) \leq \mathbb{M}(r, \tilde{p}_0(r)), \quad r \in \Delta,$$

i.e.,

$$0 \leq \mathbb{M}(r, p_n(r)) - \mathbb{M}(r, p_0(r)) \leq \mathbb{M}(r, \tilde{p}_0(r)) - \mathbb{M}(r, p_0(r)).$$

Hence, we get

$$|\mathbb{M}(r, p_n(r)) - \mathbb{M}(r, p_0(r))| \leq |\mathbb{M}(r, \tilde{p}_0(r)) - \mathbb{M}(r, p_0(r))|.$$

Consequently, we arrive at

$$\begin{aligned} |\mathbb{M}(r, p_n(r))| &\leq |\mathbb{M}(r, p_n(r)) - \mathbb{M}(r, p_0(r))| + |\mathbb{M}(r, p_0(r))| \\ &\leq 2|\mathbb{M}(r, p_0(r))| + |\mathbb{M}(r, \tilde{p}_0(r))|. \end{aligned}$$

Since $p_0, \tilde{p}_0$ and $\mathbb{M}$ are continuous on Δ , we can find a constant $\mathbb{K}$ independent of n , such that

$$|\mathbb{M}(r, p_n(r))| \leq \mathbb{K}. \quad (11)$$

Furthermore, from (H_3) , we can obtain

$$p_0(a) - \frac{1}{c}\mathbb{W}(p_0(a), p_0(b)) \leq p_n(a) - \frac{1}{c}\mathbb{W}(p_n(a), p_n(b)) \leq \tilde{p}_0(a) - \frac{1}{c}\mathbb{W}(\tilde{p}_0(a), \tilde{p}_0(b)),$$

i.e.,

$$\begin{aligned} 0 &\leq p_n(a) - p_0(a) - \frac{1}{c}(\mathbb{W}(p_n(a), p_n(b)) - \mathbb{W}(p_0(a), p_0(b))) \\ &\leq \tilde{p}_0(a) - p_0(a) - \frac{1}{c}(\mathbb{W}(\tilde{p}_0(a), \tilde{p}_0(b)) - \mathbb{W}(p_0(a), p_0(b))). \end{aligned}$$

Hence, we get

$$\begin{aligned} &\left| p_n(a) - p_0(a) - \frac{1}{c}(\mathbb{W}(p_n(a), p_n(b)) - \mathbb{W}(p_0(a), p_0(b))) \right| \leq \\ &\left| \tilde{p}_0(a) - p_0(a) - \frac{1}{c}(\mathbb{W}(\tilde{p}_0(a), \tilde{p}_0(b)) - \mathbb{W}(p_0(a), p_0(b))) \right| \\ &\leq \left| p_0(a) - \frac{1}{c}\mathbb{W}(p_0(a), p_0(b)) \right| + \left| \tilde{p}_0(a) - \frac{1}{c}\mathbb{W}(\tilde{p}_0(a), \tilde{p}_0(b)) \right|. \end{aligned}$$

Thus,

$$\begin{aligned} \left| p_n(a) - \frac{1}{c}\mathbb{W}(p_n(a), p_n(b)) \right| &\leq \left| p_n(a) - p_0(a) - \frac{1}{c}(\mathbb{W}(p_n(a), p_n(b)) - \mathbb{W}(p_0(a), p_0(b))) \right| \\ &\quad + \left| p_0(a) - \frac{1}{c}\mathbb{W}(p_0(a), p_0(b)) \right| \\ &\leq 2 \left| p_0(a) - \frac{1}{c}\mathbb{W}(p_0(a), p_0(b)) \right| + \left| \tilde{p}_0(a) - \frac{1}{c}\mathbb{W}(\tilde{p}_0(a), \tilde{p}_0(b)) \right|. \end{aligned}$$

Since $p_0, \tilde{p}_0$ and $\mathbb{W}$ are continuous functions, we can find a constant $\mathbb{L}$ independent of n , such that

$$\left| p_n(a) - \frac{1}{c}\mathbb{W}(p_n(a), p_n(b)) \right| \leq \mathbb{L}. \quad (12)$$

Moreover, by (9), we have

$$|p_{n+1}(r)| = \left| p_n(a) - \frac{1}{c} \mathbb{W}(p_n(a), p_n(b)) \right| + |\theta_1|(\psi(r) - \psi(a)) \mathbb{E}_{1,2}(-\omega(\psi(r) - \psi(a))) \\ + \int_a^r \psi'(\eta)(\psi(r) - \psi(\eta))^\epsilon \mathbb{E}_{1,\epsilon+1}(-\omega(\psi(r) - \psi(\eta))) |\mathbb{M}(\eta, p_n(\eta))| d\eta, \quad r \in \Delta.$$

Using Lemma 2 along with (11) and (12), we get

$$|p_{n+1}(r)| = \mathbb{L} + |\theta_1|(\psi(r) - \psi(a)) + \frac{\mathbb{K}}{\Gamma(\epsilon+1)} \int_a^r \psi'(\eta)(\psi(r) - \psi(\eta))^\epsilon d\eta \\ \leq \mathbb{L} + |\theta_1|(\psi(b) - \psi(a)) + \frac{\mathbb{K}(\psi(b) - \psi(a))^{\epsilon+1}}{\Gamma(\epsilon+2)}.$$

Hence, $\{p_n\}$ are uniformly bounded in $C(\Delta, \mathbb{R})$. Similarly, we can prove that $\{\tilde{p}_n\}$ are uniformly bounded.

It remains to be shown that the sequences $\{p_n\}$ and $\{\tilde{p}_n\}$ are equicontinuous on Δ . To do this, choosing $r_1, r_2 \in \Delta$, with $r_1 \leq r_2$. By (11) and Lemma 2, we have

$$|p_n(r_2) - p_n(r_1)| \leq |\theta_1|(\psi(r_2) - \psi(a)) \mathbb{E}_{1,2}(-\omega(\psi(r_2) - \psi(a))) \\ - |\theta_1|(\psi(r_1) - \psi(a)) \mathbb{E}_{1,2}(-\omega(\psi(r_2) - \psi(a))) \\ + \int_a^{r_1} \frac{\psi'(\eta)[(\psi(r_1) - \psi(\eta))^{\epsilon-1} - (\psi(r_2) - \psi(\eta))^{\epsilon-1}]}{\Gamma(\epsilon)} |\mathbb{M}(\eta, p_n(\eta))| d\eta \\ + \int_{r_1}^{r_2} \frac{\psi'(\eta)(\psi(r_2) - \psi(\eta))^{\epsilon-1}}{\Gamma(\epsilon)} |\mathbb{M}(\eta, p_n(\eta))| d\eta \\ \leq \frac{\mathbb{K}}{\Gamma(\epsilon+1)} [(\psi(r_1) - \psi(a))^\epsilon + 2(\psi(r_2) - \psi(r_1))^\epsilon - (\psi(r_2) - \psi(a))^\epsilon] \\ + |\theta_1|(\psi(r_2) - \psi(a)) \mathbb{E}_{1,2}(-\omega(\psi(r_2) - \psi(a))) \\ - |\theta_1|(\psi(r_1) - \psi(a)) \mathbb{E}_{1,2}(-\omega(\psi(r_2) - \psi(a))) \\ \leq \frac{2\mathbb{K}}{\Gamma(\epsilon+1)} (\psi(r_2) - \psi(r_1))^\epsilon \\ + |\theta_1|(\psi(r_2) - \psi(a)) \mathbb{E}_{1,2}(-\omega(\psi(r_2) - \psi(a))) \\ - |\theta_1|(\psi(r_1) - \psi(a)) \mathbb{E}_{1,2}(-\omega(\psi(r_2) - \psi(a))).$$

By the continuity of the function $|\theta_1|(\psi(r) - \psi(a)) \mathbb{E}_{1,2}(-\omega(\psi(r_2) - \psi(a)))$ on Δ , the right-hand side of the previous inequality approaches to zero when $r_2 \rightarrow r_1$ independently of $\{p_n\}$. Hence, the family $\{p_n\}$ is equicontinuous on Δ . Likewise, we can demonstrate that $\{\tilde{p}_n\}$ is equicontinuous. Therefore, by Ascoli–Arzela’s Theorem, there exist subsequences $\{p_{n_k}\}$ and $\{\tilde{p}_{n_k}\}$ which converge uniformly to p^* and $\tilde{p}^*$, respectively, on Δ . This together with the monotonicity of sequences $\{p_n\}$ and $\{\tilde{p}_n\}$ implies

$$\lim_{n \rightarrow \infty} p_n(r) = p^*(r) \quad \text{and} \quad \lim_{n \rightarrow \infty} \tilde{p}_n(r) = \tilde{p}^*(r),$$

uniformly on $r \in \Delta$ and the limit functions $p^*, \tilde{p}^*$ satisfy problem (1).

Lastly, we prove the minimal and maximal property of p^* and $\tilde{p}^*$ on $[p_0, \tilde{p}_0]$. To do this, let $p \in [p_0, \tilde{p}_0]$ be any solution of (1). Suppose for some $n \in \mathbb{N}^*$ that

$$p_n(r) \leq p(r) \leq \tilde{p}_n(r), \quad r \in \Delta. \quad (13)$$

Setting $\gamma(r) = p(r) - p_{n+1}(r)$, it follows that

$$({}^c \mathbb{D}_{a^+}^{\epsilon+1; \psi} + \omega {}^c \mathbb{D}_{a^+}^{\epsilon; \psi}) \gamma(r) = \mathbb{M}(r, p(r)) - \mathbb{M}(r, p_n(r)) \geq 0.$$

Furthermore,

$$\begin{aligned} p_{n+1}(a) &= p_n(a) - \frac{1}{c} \mathbb{W}(p_n(a), p_n(b)) \\ &= p_n(a) + \frac{1}{c} \mathbb{W}(p(a), p(b)) - \frac{1}{c} \mathbb{W}(p_n(a), p_n(b)) \\ &\leq p(a) - \frac{d}{c} (p(b) - p_n(b)) \\ &\leq p(a). \end{aligned}$$

That is,

$$\gamma(a) \geq 0.$$

Furthermore, $\gamma^{[1];\psi}(a) = 0$. Thus, in light of Lemma 8, we have the inequality $\gamma(r) \geq 0$, $r \in \Delta$, and then, $p_{n+1}(r) \leq p(r)$, $r \in \Delta$. Analogously, it can be obtained that $p(r) \leq \tilde{p}_{n+1}(r)$, $r \in \Delta$. So, from mathematical induction, it follows that the relation (13) holds on Δ for all $n \in \mathbb{N}$. Taking the limit as $n \rightarrow \infty$ on both sides of (13), we get

$$p^*(r) \leq p(r) \leq \tilde{p}^*(r), \quad r \in \Delta.$$

This means that $p^*, \tilde{p}^*$ are the extremal solutions of (1) in $[p_0, \tilde{p}_0]$. Thus, the proof of Theorem 1 is finished. $\square$

4. An Example

To illustrate our abstract results (Theorem 1), let us consider problem (1) with specific data. More precisely, taking

$$\epsilon = \frac{1}{2}, \quad \omega = \frac{\sqrt{\pi}}{2}, \quad \psi(r) = r, \quad a = 0, \quad b = 1, \quad \mathbb{M}(r, x) = (1 - \sqrt{r})e^{x-1-r}, \quad \mathbb{W}(u, v) = u - 1, \quad \theta_1 = 0.$$

Then, our model (1) reduces to the following problem

$$\begin{cases} ({}^c \mathbb{D}_{0+}^{\frac{3}{2}} + \frac{\sqrt{\pi}}{2} {}^c \mathbb{D}_{0+}^{\frac{1}{2}}) p(r) = (1 - \sqrt{r})e^{p(r)-1-r}, & r \in J := [0, 1], \\ p(0) = 1, \quad p'(0) = 0. \end{cases} \quad (14)$$

Taking $p_0(r) = 1$ and $q_0(r) = 1 + r$, it is easy to see that p_0 and q_0 are lower and upper solutions of problem (14), respectively, and $p_0 \leq q_0$. So, condition (H_1) holds. In addition, it is obvious that the function $\mathbb{M}: \Delta \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and nondecreasing function with respect to the second variable. Hence, condition (H_2) is satisfied. Moreover, for $p_0(a) \leq u_1 \leq u_2 \leq q_0(a)$, $p_0(b) \leq v_1 \leq v_2 \leq q_0(b)$, we have

$$\mathbb{W}(u_2, v_2) - \mathbb{W}(u_1, v_1) \leq (u_2 - u_1).$$

Therefore, the hypothesis (H_2) of Theorem 1 is fulfilled with $c = 1$ and $d = 0$.

Thus, all assumptions of Theorem 1 are valid. As a result, the suggested problem (14) has extremal solutions on $[p_0, q_0]$.

5. Conclusions

The existence of extremal solutions for a new class of nonlinear sequential fractional differential equations (NSFDEs) with nonlinear boundary conditions (NBCs) containing the ψ -Caputo operator is the topic of our study. To arrive at the principal findings of this study, we used the interlinking between the monotone iterative technique (MIT) and the method of upper and lower solutions. We also tested the applicability and efficiency of the mentioned method by an example. For future research, we plan to look at the same outcomes for our present model (1) using other modern fractional operators. It would be

also intriguing to construct numerical approaches to approximate the solutions suggested by our Theorem 1.

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