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Group Decision-Making Problems Based on Mixed Aggregation Operations of Interval-Valued Fuzzy and Entropy Elements in Single- and Interval-Valued Fuzzy Environments

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Abstract: Fuzzy sets and interval-valued fuzzy sets are two kinds of fuzzy information expression forms in real uncertain and vague environments. Their mixed multivalued information expression and operational problems are very challenging and indispensable issues in group decision-making (GDM) problems. To solve single- and interval-valued fuzzy multivalued hybrid information expression, operations, and GDM issues, this study first presents the notion of a single- and interval-valued fuzzy multivalued set/element (SIVFMS/SIVFME) with identical and/or different fuzzy values. To effectively solve operational problems for various SIVFME lengths, SIVFMS/SIVFME is converted into the interval-valued fuzzy and entropy set/element (IVFES/IVFEE) based on the mean and information entropy of SIVFME. Then, the operational relationships of IVFEEs and the expected value function and sorting rules of IVFEEs are defined. Next, the IVFEE weighted averaging and geometric operators and their mixed-weighted-averaging operation are proposed. In terms of the mixed-weighted-averaging operation and expected value function of IVFEEs, a GDM method is developed to solve multicriteria GDM problems in the environment of SIVFMSs. Finally, the proposed GDM method was utilized for a supplier selection problem in a supply chain as an actual sample to show the rationality and efficiency of SIVFMSs. Through the comparative analysis of relative decision-making methods, we found the superiority of this study in that the developed GDM method not only compensates for the defects of existing GDM methods, but also makes the GDM process more reasonable and flexible.

Keywords: single- and interval-valued fuzzy multivalued set; interval-valued fuzzy and entropy set; interval-valued fuzzy and entropy element weighted averaging operator; interval-valued fuzzy and entropy element weighted geometric operator; mixed-weighted-averaging operation; group decision making

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1. Introduction

Fuzzy sets (FS) [1] and interval-valued fuzzy sets (IVFSs) [2] are two important tools of fuzzy information expressions in real uncertain and vague environments. A bag/fuzzy multiset [3,4] or an interval-valued fuzzy multiset (IVFM) [5] was proposed as the extension of FS or IVFS, where each element in a universe set can occur more times with different and/or identical fuzzy values or interval-valued fuzzy values. Therefore, they have been used in various areas [6–10]. In a hesitant situation, a hesitant fuzzy set (HFS) [11] can represent a set of a few of different fuzzy values of each element in the set. To express the hybrid information of HFS and IVFS, some researchers presented cubic HFSs and applied them to medical assessments of prostatic patients [12] and multicriteria decision-making problems [13]; then, other researchers introduced hesitant cubic fuzzy sets (HCFSs) and applied them to multicriteria (group) decision-making problems [14,15]. However, their hesitant information does not contain the same fuzzy values corresponding to the hesitant characteristics/concept [11], which is different from the fuzzy multiset concept.

Regarding the probability of an element belonging to a set, hesitant probabilistic fuzzy sets (HPFSs) [16,17] were introduced and applied to hesitant probabilistic fuzzy decision-making problems. However, an HPFS only contains the probabilistic values of a few of the same values, resulting in probabilistic distortion. Since the probabilistic method requires a lot of fuzzy data (more sample data) to maintain reasonable probabilistic values, the probabilistic values of small samples of data lead to irrationality/distortion. Therefore, it is difficult to apply the probabilistic method in actual group decision making (GDM) applications because the evaluation values of a lot of decision makers are required to ensure the rationality of the probabilistic values. Hence, it is obvious that the use of HPFSs may have some flaws from the perspective of probability.

Recently, Turkarslan et al. [18] introduced a consistency fuzzy set/element (CFS/CFE) based on the mean of a fuzzy sequence and the complement of the standard deviation of a fuzzy sequence in a fuzzy multiset to reasonably simplify the information expression and operation of different fuzzy sequence lengths, and then proposed a cosine similarity measure of CFSs for medical diagnosis in the case of fuzzy multisets. Furthermore, Du and Ye [19] presented cubic fuzzy multivalued sets (CFMSs) and converted them into cubic fuzzy consistency sets with the help of the mean of a fuzzy sequence and the complement of the standard deviation of a fuzzy sequence. Then, they developed a hybrid weighted arithmetic and geometric aggregation operator for GDM with CFMSs. In general, the concept of standard deviation is only applicable to the calculation of fuzzy sequences containing normal distributions, which exposes its limitations.

In real GDM problems, single- and interval-valued fuzzy hybrid multivalued information expression and operation problems are very challenging issues, due to the uncertainty and incompleteness of each decision-maker's judgement/cognition of the evaluated object. However, existing fuzzy multiset/HFS/HPFS/IVFM/CFMS cannot represent the single- and interval-valued fuzzy hybrid multivalued information with identical and/or different fuzzy values that are given by a group of decision makers in the GDM process. In the GDM problem, one of the experts/decision makers can assign his/her single-valued or interval-valued fuzzy evaluation value in terms of his cognition of the evaluated object in the assessment process. For example, five experts evaluate a car's "comfort" with a group of fuzzy values (0.5, 0.5, 0.6, [0.6, 0.7], [0.7, 0.8]). The fuzzy values 0.5, 0.5, and 0.6 are given by three of the five experts, and the interval-valued/uncertain fuzzy values [0.6, 0.7] and [0.7, 0.8] are given by the two of the five experts. In this issue, the existing fuzzy multiset/HFS/HPFS/IVFM/CFMS can only represent a fuzzy sequence or an interval-valued fuzzy sequence, but they cannot express such a group of single- and interval-valued fuzzy hybrid values (the hybrid set of two different fuzzy sequences) simultaneously. Meanwhile, there is no research on a single- and interval-valued fuzzy multivalued framework in the existing literature. Therefore, it is necessary to propose a new expression form to effectively express the single- and interval-valued fuzzy hybrid multivalued information to overcome the defect of existing various fuzzy expressions. Motivated by this new idea, this paper first puts forward the concept of a single- and interval-valued fuzzy multivalued set/element (SIVFMS/SIVFME). Then, a new information entropy measure of SIVFME is proposed to transform SIVFMS/SIVFME into an interval-valued fuzzy and entropy set/element (IVFES/IVFEE) based on the mean and information entropy of SIVFME, and then some operations of IVFEEs and the expected value function and sorting rules of IVFEEs are defined. Next, the IVFEE weighted averaging (IVFEEWA) and IVFEE weighted geometric (IVFEEWG) operators and their mixed-weighted-averaging operation are proposed to overcome the flaws of the IVFEEWA operator, which mainly attends to group arguments, and the IVFEEWG operator, which mainly attends to individual arguments [19], in the IVFEE aggregation process. According to the proposed mixed-weighted-averaging operation and the expected value function, a GDM method is developed to solve multicriteria GDM problems with SIVFMSs. Finally, the proposed GDM method is utilized for an actual supplier selection problem in a supply chain to show the rationality and effectiveness in the setting of SIVFMSs. The results indicate that the proposed GDM method makes the GDM process more reasonable and flexible.

This original study demonstrates the following main contributions and highlights:

(i). The proposed SIVFMS/SIVFME forms single- and interval-valued fuzzy multivalued framework to reasonably express the mixed information of the single-valued/certain fuzzy sequence and interval-valued/uncertain fuzzy sequence, which are given by different decision makers in the GDM process.

(ii). The IVFEE transformed based on the mean and information entropy of SIVFME can reasonably simplify the information expression and operation of different fuzzy sequence lengths in SIVFMEs; then, the proposed transformation method using the mean and information entropy of SIVFME can reveal the average level and consistency/consensus degree of the single- and interval-valued fuzzy sequence in SIVFME to keep much more useful information in the transformation process.

(iii). The mixed-weighted-averaging operation of the IVFEEWA and IVFEEWG operators can provide a useful modeling tool for their GDM method in the environment of SIVFMSs and overcome the flaw of having a single aggregation operator [19].

(iv). The developed GDM method can solve multicriteria GDM problems and make the decision results more flexible and more reasonable for SIVFMSs.

The remainder of this article is made up of the following structures. In Section 2, we present the concepts of SIVFMS, SIVFME, information entropy, and IVFEE. Then, we define the operational laws of IVFEEs, and the expected value function and sorting rules of IVFEEs. The IVFEEWA and IVFEEWG operators and their mixed-weighted-averaging operation are presented in Section 3. In Section 4, a GDM method is given by using the mixed-weighted-averaging operation and the expected value function. In Section 5, the proposed GDM method is applied to an actual supplier selection problem in a supply chain to show its rationality and effectiveness when dealing with SIVFMSs, and then the superiorities of the proposed method are indicated by comparative analysis. Section 6 depicts conclusions and future research.

2. SIVFMS and IVFES

Definition 1. *Let $U = \{u_1, u_2, \ldots, u_s\}$ be a finite universe set U. Then, a single- and interval-valued fuzzy multivalued set H in U is defined as follows:*

$$H = \{\langle u_k, F_H(u_k)\rangle | u_k \in U\} \tag{1}$$

where $F_H(u_k)$ for $u_k \in U$ ($k = 1, 2, \ldots, s$) is a single- and interval-valued fuzzy sequence of the element u_k in the set H, denoted as an increasing fuzzy sequence $F_H(u_k) = (\lambda_H^1(u_k), \lambda_H^2(u_k), \ldots, \lambda_H^{a_k}(u_k), [\lambda_H^{L1}(u_k), \lambda_H^{U1}(u_k)], [\lambda_H^{L2}(u_k), \lambda_H^{U2}(u_k)], \ldots, [\lambda_H^{Lb_k}(u_k), \lambda_H^{Ub_k}(u_k)])$ with identical and/or different fuzzy values, such that $0 \le \lambda_H^1(u_k) \le \lambda_H^2(u_k), \ldots, \le \lambda_H^{a_k}(u_k) \le 1$ with a_k single-valued fuzzy values and $[\lambda_H^{L1}(u_k), \lambda_H^{U1}(u_k)] \subseteq [\lambda_H^{L2}(u_k), \lambda_H^{U2}(u_k)] \subseteq, \ldots, \subseteq [\lambda_H^{Lb_k}(u_k), \lambda_H^{Ub_k}(u_k)] \subseteq [0,1]$ with b_k interval-valued fuzzy values.

Especially when all $b_k = 0$ or $a_k = 0$ for $k = 1, 2, \ldots, s$, SIVFMS degenerates to a fuzzy multiset or an IVFM.

For simplicity, the kth element $F_H(u_k)$ in H is denoted as the kth SIVFME: $F_{Hk} = (\lambda_{Hk}^1, \lambda_{Hk}^2, \ldots, \lambda_{Hk}^{a_k}, [\lambda_{Hk}^{L1}, \lambda_{Hk}^{U1}], [\lambda_{Hk}^{L2}, \lambda_{Hk}^{U2}], \ldots, [\lambda_{Hk}^{Lb_k}, \lambda_{Hk}^{Ub_k}])$.

To solve the difficult conversions between different single- and interval-valued fuzzy sequence lengths, it is necessary to convert SIVFMS into IVFES in terms of the mean and information entropy of SIVFME.

First, the concept of the Shannon/probability entropy [20] is introduced below.

Set $R = \{r_1, r_2, \ldots, r_s\}$ as a probability distribution on a set of random variables. Thus, the Shannon entropy of the probability distribution R is denoted as [20]

$$E(R) = -\sum_{i=1}^{s} r_i \ln(r_i) \tag{2}$$

where $r_i \in [0, 1]$ and $\sum_{i=1}^{s} r_i = 1$.

If all probability values of r_i (i = 1, 2, ..., s) in R are the same, the probability entropy can reach the maximum value of $E(R)$, which reflects the perfect consistency (the same probabilities) of all r_i. Generally, the larger the probability entropy measure value, the better the consistency level of all probability values.

According to the probability entropy notion, the interval-valued entropy concept of SIVFME (an information entropy measure of SIVFME) is proposed, and SIVFMS is converted into IVFES based on the mean and information entropy of SIVFME, which is given by the following definition.

Definition 2. *An IVFES Z of a SIVFMS H in a finite universe set $U = \{u_1, u_2, \ldots, u_s\}$ is defined as*

$$Z = \{(u_k, m_Z(u_k), e_Z(u_k)) | u_k \in U\},$$

where $m_Z(u_k) \subseteq [0, 1]$ and $e_Z(u_k) \subseteq [0, 1]$ (k = 1, 2, ..., s) are the interval-valued mean and interval-valued entropy of SIVFME, which are obtained by using the following formulae:

$$m_Z(u_k) = [m_Z^L(u_k), m_Z^U(u_k)] = \begin{bmatrix} \frac{1}{a_k+b_k}\left(\sum\limits_{i=1}^{a_k} \lambda_H^i(u_k) + \sum\limits_{i=1}^{b_k} \lambda_H^{Li}(u_k)\right), \\ \frac{1}{a_k+b_k}\left(\sum\limits_{i=1}^{a_k} \lambda_H^i(u_k) + \sum\limits_{i=1}^{b_k} \lambda_H^{Ui}(u_k)\right) \end{bmatrix}, m_Z(u_k) \subseteq [0, 1], \tag{3}$$

$$e_Z(u_k) = [e_Z^L(u_k), e_Z^U(u_k)]$$
$$= \begin{bmatrix} \min\left(\begin{array}{l} -\frac{1}{\ln(a_k+b_k)}\left(\begin{array}{l} \sum\limits_{i=1}^{a_k}\left(\frac{\lambda_H^i(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Li}(u_k)} \ln \frac{\lambda_H^i(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Li}(u_k)}\right)+ \\ \sum\limits_{i=1}^{b_k}\left(\frac{\lambda_H^{Li}(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Li}(u_k)} \ln \frac{\lambda_H^{Li}(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Li}(u_k)}\right) \end{array} \right), \\ -\frac{1}{\ln(a_k+b_k)}\left(\begin{array}{l} \sum\limits_{i=1}^{a_k}\left(\frac{\lambda_H^i(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Ui}(u_k)} \ln \frac{\lambda_H^i(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Ui}(u_k)}\right)+ \\ \sum\limits_{i=1}^{b_k}\left(\frac{\lambda_H^{Ui}(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Ui}(u_k)} \ln \frac{\lambda_H^{Ui}(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Ui}(u_k)}\right) \end{array} \right) \end{array} \right), \\ \max\left(\begin{array}{l} -\frac{1}{\ln(a_k+b_k)}\left(\begin{array}{l} \sum\limits_{i=1}^{a_k}\left(\frac{\lambda_H^i(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Li}(u_k)} \ln \frac{\lambda_H^i(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Li}(u_k)}\right)+ \\ \sum\limits_{i=1}^{b_k}\left(\frac{\lambda_H^{Li}(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Li}(u_k)} \ln \frac{\lambda_H^{Li}(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Li}(u_k)}\right) \end{array} \right), \\ -\frac{1}{\ln(a_k+b_k)}\left(\begin{array}{l} \sum\limits_{i=1}^{a_k}\left(\frac{\lambda_H^i(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Ui}(u_k)} \ln \frac{\lambda_H^i(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Ui}(u_k)}\right)+ \\ \sum\limits_{i=1}^{b_k}\left(\frac{\lambda_H^{Ui}(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Ui}(u_k)} \ln \frac{\lambda_H^{Ui}(u_k)}{\sum_{i=1}^{a_k}\lambda_H^i(u_k)+\sum_{i=1}^{b_k}\lambda_H^{Ui}(u_k)}\right) \end{array} \right) \end{array} \right) \end{bmatrix}, e_Z(u_k) \subseteq [0, 1] \tag{4}$$

It is obvious that the IVFES Z consists of interval-valued fuzzy average values and entropy values to reasonably solve the expression and operation problems of different sequence lengths in SIVFMEs.

Remark 1.

(1) *The entropy value indicates a degree of difference among various fuzzy values in the SIVFME $F_H(u_k)$. The larger the entropy value, the better the consistency of various fuzzy values in the SIVFME $F_H(u_k)$.*
(2) *All fuzzy values in $F_H(u_k)$ are identical when $e_Z(u_k) = \left[e_Z^L(u_k), e_Z^U(u_k)\right] = [1,1]$, which can indicate the complete consistency of the multiple fuzzy values.*
(3) *In GDM problems, the larger the average value and entropy value of the group evaluation, the better the group evaluation values and their consistency/consensus. When the entropy value of the group evaluation values is equal to one, this reflects complete consistency/consensus of the group evaluation values.*

Example 1. *Let us consider a GDM problem. When a group of four decision makers/experts is asked to assess product quality (u_1) and service quality (u_2) in U = {u_1, u_2} regarding a supplier A, they can give two groups of fuzzy assessment values, (u_1, 0.7, 0.8, [0.6, 0.8], [0.7, 0.9]) and (u_2, [0.6, 0.7], [0.6, 0.7], [0.6, 0.7], [0.6, 0.7], [0.6, 0.7]). Therefore, using Equations (2) and (3), their interval-valued fuzzy average values and entropy values are [0.7, 0.8] and [0.9963, 0.9972] for u_1 and [0.6, 0.7] and [1, 1] for u_2, respectively, which are expressed as the IVFES Z = {(u_1, [0.7, 0.8], [0.9963, 0.9972]), (u_2, [0.6, 0.7], [1, 1])} in the GDM example.*

In this example, it can be seen that the average values and entropy values can reflect the magnitude and consistency/consensus degree of the group evaluation values. The larger the entropy value, the better the consistency/consensus of the group evaluation values.

Then, the simplified expression form of a basic element $z(u_k) = \left(u_k, m_Z(u_k), e_{Zk}(u_k)\right)$ for $[m_Z^L(u_k), m_Z^U(u_k)] \subseteq [0,1]$ and $[e_Z^L(u_k), e_Z^U(u_k)] \subseteq [0,1]$ in the IVFES Z can be denoted as $z_k = (m_{Zk}, e_{Zk})$ for $\left[m_{Zk}^L, m_{Zk}^U\right] \subseteq [0,1]$ and $\left[e_{Zk}^L, e_{Zk}^U\right] \subseteq [0,1]$, which is named IVFEE.

Definition 3. *Set two IVFEEs as $z_1 = ([m_{Z1}^L, m_{Z1}^U], [e_{Z1}^L, e_{Z1}^U])$ and $z_2 = ([m_{Z2}^L, m_{Z2}^U], [e_{Z2}^L, e_{Z2}^U])$. Thus, their operational relationships are defined as follows:*

(1) $z_1 \supseteq z_2$ *if and if then* $m_{Z1}^L \geq m_{Z2}^L$, $m_{Z1}^U \geq m_{Z2}^U$, $e_{Z1}^L \geq e_{Z2}^L$, *and* $e_{Z1}^U \geq e_{Z2}^U$;
(2) $z_1 = z_2$ *if and if then* $z_1 \supseteq z_2$ *and* $z_2 \supseteq z_1$;
(3) $z_1 \cup z_2 = \left([m_{Z1}^L \vee m_{Z2}^L, m_{Z1}^U \vee m_{Z2}^U], [e_{Z1}^L \vee e_{Z2}^L, e_{Z1}^U \vee e_{Z2}^U]\right)$;
(4) $z_1 \cap z_2 = \left([m_{Z1}^L \wedge m_{Z2}^L, m_{Z1}^U \wedge m_{Z2}^U], [e_{Z1}^L \wedge e_{Z2}^L, e_{Z1}^U \wedge e_{Z2}^U]\right)$.

Definition 4. *Set two IVFEEs as $z_1 = ([m_{Z1}^L, m_{Z1}^U], [e_{Z1}^L, e_{Z1}^U])$ and $z_2 = ([m_{Z2}^L, m_{Z2}^U], [e_{Z2}^L, e_{Z2}^U])$. Thus, their operational laws are defined as follows:*

(1) $z_1 \oplus z_2 = \left(\begin{array}{c} [m_{Z1}^L + m_{Z2}^L - m_{Z1}^L m_{Z2}^L, m_{Z1}^U + m_{Z2}^U - m_{Z1}^U m_{Z2}^U], \\ [e_{Z1}^L + e_{Z2}^L - e_{Z1}^L e_{Z2}^L, e_{Z1}^U + e_{Z2}^U - e_{Z1}^U e_{Z2}^U] \end{array} \right)$;
(2) $z_1 \otimes z_2 = ([m_{Z1}^L m_{Z2}^L, m_{Z1}^U m_{Z2}^U], [e_{Z1}^L e_{Z2}^L, e_{Z1}^U e_{Z2}^U])$;
(3) $z_1^\lambda = ([(m_{Z1}^L)^\lambda, (m_{Z1}^U)^\lambda], [(e_{Z1}^L)^\lambda, (e_{Z1}^U)^\lambda])$ *for* $\lambda > 0$;
(4) $\lambda z_1 = ([1 - (1 - m_{Z1}^L)^\lambda, 1 - (1 - m_{Z1}^U)^\lambda], [1 - (1 - e_{Z1}^L)^\lambda, 1 - (1 - e_{Z1}^U)^\lambda])$ *for* $\lambda > 0$.

However, it is obvious that the above operational results are still IVFEEs.

To compare two IVFEEs $z_k = ([m_{Zk}^L, m_{Zk}^U], [e_{Zk}^L, e_{Zk}^U])$ for k = 1, 2, the expected value function is defined as

$$Q(z_k) = (m_{Zk}^L e_{Zk}^L + m_{Zk}^U e_{Zk}^U)/2 \text{ for } Q(z_k) \in [0,1] \tag{5}$$

Then, the sorting rules of the two IVFEEs are given as follows:

(1) If $Q(z_1) > Q(z_2)$, then $z_1 > z_2$;
(2) If $Q(z_1) = Q(z_2)$, then $z_1 \cong z_2$.

Example 2. *Assume that two IVFEEs are z_1 = ([0.7, 0.8], [0.8, 0.9]) and z_2 = ([0.6, 0.7], [0.7, 0.8]). Then, their sorting is yielded below:*

Using Equation (5), there are $Q(z_1)$ = (0.7 × 0.8 + 0.8 × 0.9)/2 = 0.56 and $Q(z_2)$ = (0.6 × 0.7 + 0.7 × 0.8)/2 = 0.54. Since $Q(z_1) > Q(z_2)$, their sorting is $z_1 > z_2$.

3. Two Weighted Aggregation Operators of IVFEEs and Their Mixed-Weighted-Averaging Operation

In this section, we propose the IVFEEWA and IVFEEWG operators according to the operational laws in Definition 4, and then define their mixed-weighted-averaging operation to make up for their flaws in aggregating IVFEEs; that is, the weighted averaging aggregation operator mainly tends to group arguments, and the weighted geometric aggregation operator tends to group personal arguments.

3.1. Weighted Averaging Aggregation Operator of IVFEEs

Based on the operational laws in Definition 4, the IVFEEWA operator is defined to aggregate IVFEE information.

Definition 5. *Let $z_k = ([m_{Zk}^L, m_{Zk}^U], [e_{Zk}^L, e_{Zk}^U])$ (k = 1, 2, …, s) be a group of IVFEEs and IVFEEWA: $\Omega^s \to \Omega$. Then, the IVFEEWA operator is defined as*

$$IVFEEWA(z_1, z_2, \ldots, z_s) = \mathop{\oplus}_{k=1}^{s} \lambda_k z_k \tag{6}$$

where λ_k is the weight of z_k with $0 \le \lambda_k \le 1$ and $\sum_{k=1}^{s} \lambda_k = 1$.

Theorem 1. *Let $z_k = ([m_{Zk}^L, m_{Zk}^U], [e_{Zk}^L, e_{Zk}^U])$ (k = 1, 2, …, s) be a group of IVFEEs with the weight vector = (λ_1, λ_2, …, λ_n) for $0 \le \lambda_k \le 1$ and $\sum_{k=1}^{s} \lambda_k = 1$. Then, the aggregated result of the IVFEEWA operator is still IVFEE, which is obtained by the equation:*

$$\begin{aligned} &IVFEEWA(z_1, z_2, \ldots, z_s) = \mathop{\oplus}_{k=1}^{s} \lambda_k z_k \\ &= \left(\left[1 - \prod_{k=1}^{s} (1 - m_{Zk}^L)^{\lambda_k}, 1 - \prod_{k=1}^{s} (1 - m_{Zk}^U)^{\lambda_k} \right], \left[1 - \prod_{k=1}^{s} (1 - e_k^L)^{\lambda_k}, 1 - \prod_{k=1}^{s} (1 - e_k^U)^{\lambda_k} \right] \right) \end{aligned} \tag{7}$$

Proof. Regarding mathematical induction, Equation (7) can be proved.

(1) When s = 2, by the operational laws in Definition 4, the aggregation result is yielded as follows:

$$\begin{aligned} &IVFEEWA(z_1, z_2) = \lambda_1 z_1 \oplus \lambda_1 z_2 \\ &= \left(\begin{array}{c} \left[\begin{array}{c} 1 - (1 - m_{Z1}^L)^{\lambda_1} + 1 - (1 - m_{Z2}^L)^{\lambda_2} - \left(1 - (1 - m_{Z1}^L)^{\lambda_1}\right)\left(1 - (1 - m_{Z2}^L)^{\lambda_2}\right), \\ 1 - (1 - m_{Z1}^U)^{\lambda_1} + 1 - (1 - m_{Z2}^U)^{\lambda_2} - \left(1 - (1 - m_{Z1}^U)^{\lambda_1}\right)\left(1 - (1 - m_{Z2}^U)^{\lambda_2}\right) \end{array} \right], \\ \left[\begin{array}{c} 1 - (1 - e_{Z1}^L)^{\lambda_1} + 1 - (1 - e_{Z2}^L)^{\lambda_2} - \left(1 - (1 - e_{Z1}^L)^{\lambda_1}\right)\left(1 - (1 - e_{Z2}^L)^{\lambda_2}\right), \\ 1 - (1 - e_{Z1}^U)^{\lambda_1} + 1 - (1 - e_{Z2}^U)^{\lambda_2} - \left(1 - (1 - e_{Z1}^U)^{\lambda_1}\right)\left(1 - (1 - e_{Z2}^U)^{\lambda_2}\right) \end{array} \right] \end{array} \right) \\ &= \left(\left[1 - \prod_{k=1}^{2} (1 - m_{Zk}^L)^{\lambda_k}, 1 - \prod_{k=1}^{2} (1 - m_{Zk}^U)^{\lambda_k} \right], \left[1 - \prod_{k=1}^{2} (1 - e_{Zk}^L)^{\lambda_k}, 1 - \prod_{k=1}^{2} (1 - e_{Zk}^U)^{\lambda_k} \right] \right). \end{aligned} \tag{8}$$

(2) When $s = n$, Equation (7) can keep the following result:

$$IVFEEWA(z_1, z_2, \ldots, z_n) = \mathop{\oplus}_{k=1}^{n} \lambda_k z_k = \left(\begin{array}{c} \left[1 - \prod_{k=1}^{n} (1 - m_{Zk}^L)^{\lambda_k}, 1 - \prod_{k=1}^{n} (1 - m_{Zk}^U)^{\lambda_k} \right], \\ \left[1 - \prod_{k=1}^{n} (1 - e_{Zk}^L)^{\lambda_k}, 1 - \prod_{k=1}^{n} (1 - e_{Zk}^U)^{\lambda_k} \right] \end{array} \right) \tag{9}$$

(3) When $s = n + 1$, by the operational laws in Definition 4 and Equations (8) and (9), the aggregated result is given as follows:

$$
\begin{aligned}
&IVFEEWA(z_1, z_2, \ldots, z_n, z_{n+1}) = \underset{k=1}{\overset{n}{\oplus}} \lambda_k z_k \oplus \lambda_{n+1} z_{n+1} \\
&= \left(\left[1 - \prod_{k=1}^{n} (1 - m_{Zk}^L)^{\lambda_k}, 1 - \prod_{k=1}^{n} (1 - m_{Zk}^U)^{\lambda_k} \right], \left[1 - \prod_{k=1}^{n} (1 - e_{Zk}^L)^{\lambda_k}, 1 - \prod_{k=1}^{n} (1 - e_{Zk}^U)^{\lambda_k} \right] \right) \oplus \lambda_{n+1} z_{n+1} \\
&= \begin{pmatrix} \left[1 - \prod_{k=1}^{n} (1 - m_{Zk}^L)^{\lambda_k} (1 - m_{Zn+1}^L)^{\lambda_{n+1}}, 1 - \prod_{k=1}^{n} (1 - m_{Zk}^U)^{\lambda_k} (1 - m_{Zn+1}^U)^{\lambda_{n+1}} \right], \\ \left[1 - \prod_{k=1}^{n} (1 - e_{Zk}^L)^{\lambda_k} (1 - e_{Zs+1}^L)^{\lambda_{n+1}}, 1 - \prod_{k=1}^{n} (1 - e_{Zk}^U)^{\lambda_k} (1 - e_{Zs+1}^U)^{\lambda_{n+1}} \right] \end{pmatrix} \\
&= \left(\left[1 - \prod_{k=1}^{n+1} (1 - m_{Zk}^L)^{\lambda_k}, 1 - \prod_{k=1}^{n+1} (1 - m_{Zk}^U)^{\lambda_k} \right], \left[1 - \prod_{k=1}^{n+1} (1 - e_{Zk}^L)^{\lambda_k}, 1 - \prod_{k=1}^{n+1} (1 - e_{Zk}^U)^{\lambda_k} \right] \right).
\end{aligned}
\tag{10}
$$

Obviously, Equation (7) exists for any s. □

Theorem 2. *The IVFEEWA operator implies these properties:*

(1) Idempotency: Set $z_k = ([m_{Zk}^L, m_{Zk}^U], [e_{Zk}^L, e_{Zk}^U])$ ($k = 1, 2, \ldots, s$) as a group of IVFEEs. There is $IVFEEWA(z_1, z_2, \ldots, z_s) = z$ if $z_k = z = ([m_Z^L, m_Z^U], [e_Z^L, e_Z^U])$ ($k = 1, 2, \ldots, s$).

(2) Boundedness: Set $z_k = ([m_{Zk}^L, m_{Zk}^U], [e_{Zk}^L, e_{Zk}^U])$ ($k = 1, 2, \ldots, s$) as a group of IVFEEs and let $z_{\min} = \left(\left[\min_k (m_{Zk}^L), \min_k (m_{Zk}^U) \right], \left[\min_k (e_{Zk}^L), \min_k (e_{Zk}^U) \right] \right)$ and $z_{\max} = \left(\left[\max_k (m_{Zk}^L), \max_k (m_{Zk}^U) \right], \left[\max_k (e_{Zk}^L), \max_k (e_{Zk}^U) \right] \right)$ be the minimum IVFEE and the maximum IVFEE, respectively. Then, $z_{\min} \leq IVFEEWA(z_1, z_2, \ldots, z_s) \leq z_{\max}$ exists.

(3) Monotonicity: Set $z_k = ([m_{Zk}^L, m_{Zk}^U], [e_{Zk}^L, e_{Zk}^U])$ and $z_k^* = \left([m_{Zk}^{L*}, m_{Zk}^{U*}], [e_{Zk}^{L*}, e_{Zk}^{U*}] \right)$ ($k = 1, 2, \ldots, s$) as two groups of IVFEEs. Then, there exists $IVFEEWA(z_1, z_2, \ldots, z_s) \leq IVFEEWA(z_1^*, z_2^*, \ldots, z_s^*)$ if $z_k \leq z_k^*$.

Proof. (1) For $z_k = z = ([m_Z^L, m_Z^U], [e_Z^L, e_Z^U])$ ($k = 1, 2, \ldots, s$), by Equation (7) the result is yielded below:

$$
\begin{aligned}
&IVFEEWA(z_1, z_2, \ldots, z_s) = \underset{k=1}{\overset{s}{\oplus}} \lambda_k z_k = \begin{pmatrix} \left[1 - \prod_{k=1}^{s} (1 - m_{Zk}^L)^{\lambda_k}, 1 - \prod_{k=1}^{s} (1 - m_{Zk}^U)^{\lambda_k} \right], \\ \left[1 - \prod_{k=1}^{s} (1 - e_{Zk}^L)^{\lambda_k}, 1 - \prod_{k=1}^{s} (1 - e_{Zk}^U)^{\lambda_k} \right] \end{pmatrix} \\
&= \left(\left[1 - (1 - m_Z^L)^{\sum_{k=1}^{s} \lambda_k}, 1 - (1 - m_Z^U)^{\sum_{k=1}^{s} \lambda_k} \right], \left[1 - (1 - e_Z^L)^{\sum_{k=1}^{s} \lambda_k}, 1 - (1 - e_Z^U)^{\sum_{k=1}^{s} \lambda_k} \right] \right) \\
&= \left([m_Z^L, m_Z^U], [e_Z^L, e_Z^U] \right) = z.
\end{aligned}
\tag{11}
$$

(2) There exists the inequality $z_{min} \leq z_k \leq z_{max}$ when z_{min} and z_{max} are the minimum and maximum IVFEEs. Thus, there also exists $\underset{k=1}{\overset{s}{\oplus}} \lambda_k z_{\min} \leq \underset{k=1}{\overset{s}{\oplus}} \lambda_k z_k \leq \underset{k=1}{\overset{s}{\oplus}} \lambda_k z_{\max}$. Then, the inequality $z_{min} \leq \underset{k=1}{\overset{s}{\oplus}} \lambda_k z_k \leq z_{max}$ can be kept regarding the above property (1); i.e., there is $z_{\min} \leq IVFEEWA(z_1, z_2, \ldots, z_s) \leq z_{\max}$.

(3) For $z_k \leq z_k^*$, there is the inequality $\underset{k=1}{\overset{s}{\oplus}} \lambda_k z_k \leq \underset{k=1}{\overset{s}{\oplus}} \lambda_k z_k^*$; i.e., $IVFEEWA(z_1, z_2, \ldots, z_s) \leq IVFEEWA(z_1^*, z_2^*, \ldots, z_s^*)$ exists.

Therefore, all the above properties are true. □

3.2. Weighted Geometric Aggregation Operator of IVFEEs

Definition 6. *Let $z_k = ([m^L_{Zk}, m^U_{Zk}], [e^L_{Zk}, e^U_{Zk}])$ (k = 1, 2, ..., s) be a group of IVFEEs and IVFEEWG: $\Omega^s \to \Omega$. Then, the IVFEEWG operator is defined as*

$$IVFEEWG(z_1, z_2, \ldots, z_s) = \overset{s}{\underset{k=1}{\otimes}} z_k^{\lambda_k} \tag{12}$$

where λ_k is the weight of z_k with $0 \le \lambda_k \le 1$ and $\sum_{k=1}^{s} \lambda_k = 1$.

Theorem 3. *Let $z_k = ([m^L_{Zk}, m^U_{Zk}], [e^L_{Zk}, e^U_{Zk}])$ (k = 1, 2, ..., s) be a group of IVFEEs along with the weight vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ for $0 \le \lambda_k \le 1$ and $\sum_{k=1}^{s} \lambda_k = 1$. Then, the aggregated result of the IVFEEWG operator is still IVFEE, which is yielded by the equation:*

$$IVFEEWG(z_1, z_2, \ldots, z_s) = \overset{s}{\underset{k=1}{\otimes}} z_k^{\lambda_k} = \left(\left[\prod_{k=1}^{s} \left(m^L_{Zk}\right)^{\lambda_k}, \prod_{k=1}^{s} \left(m^U_{Zk}\right)^{\lambda_k} \right], \left[\prod_{k=1}^{s} \left(e^L_{Zk}\right)^{\lambda_k}, \prod_{k=1}^{s} \left(e^U_{Zk}\right)^{\lambda_k} \right] \right) \tag{13}$$

Similarly to Theorem 1, Theorem 3 can easily be proved, which is omitted here.

Theorem 4. *The IVFEEWG operator implies these properties:*

(1) *Idempotency: Let $z_k = ([m^L_{Zk}, m^U_{Zk}], [e^L_{Zk}, e^U_{Zk}])$ (k = 1, 2, ..., s) be a group of IVFEEs. If $z_k = z = ([m^L_Z, m^U_Z], [e^L_Z, e^U_Z])$ (k= 1, 2, ..., s), then $IVFEEWG(z_1, z_2, \ldots, z_s) = z$.*
(2) *Boundedness: Let $z_k = ([m^L_{Zk}, m^U_{Zk}], [e^L_{Zk}, e^U_{Zk}])$ (k = 1, 2, ..., s) be a group of IVFEEs, and let $z_{\min} = \left(\left[\min_k \left(m^L_{Zk}\right), \min_k \left(m^U_{Zk}\right) \right], \left[\min_k \left(e^L_{Zk}\right), \min_k \left(e^U_{Zk}\right) \right] \right)$ and $z_{\max} = \left(\left[\max_k \left(m^L_{Zk}\right), \max_k \left(m^U_{Zk}\right) \right], \left[\max_k \left(e^L_{Zk}\right), \max_k \left(e^U_{Zk}\right) \right] \right)$ be the minimum and maximum IVFEEs. Then, $z_{\min} \le IVFEEWG(z_1, z_2, \ldots, z_s) \le z_{\max}$ exists.*
(3) *Monotonicity: Let $z_k = ([m^L_{Zk}, m^U_{Zk}], [e^L_{Zk}, e^U_{Zk}])$ and $z^*_k = \left([m^{L*}_{Zk}, m^{U*}_{Zk}], [e^{L*}_{Zk}, e^{U*}_{Zk}]\right)$ (k = 1, 2, ..., s) be two groups of IVFEEs. Then, there exists $IVFEEWG(z_1, z_2, \ldots, z_s) \le IVFEEWG(z^*_1, z^*_2, \ldots, z^*_s)$ for $z_k \le z^*_k$.*

Theorem 4 can be proved similarly to Theorem 2 (omitted).

3.3. Mixed-Weighted-Averaging Operation for the IVFEEWA and IVFEEWG Operators

Since the IVFEEWA operator and the IVFEEWG operator mainly tend to group arguments and individual arguments, respectively, here we propose a mixed-weighted-averaging operation for the IVFEEWA and IVFEEWG operators.

Definition 7. *Set $\eta \in [0, 1]$ as a weight parameter. Then, a mixed-weighted-averaging operation of the IVFEEWA and IVFEEWG operators with a weight parameter η is defined below:*

$$z(\eta) = \eta \times IVFEEWA(z_1, z_2, \ldots, z_s) \oplus (1 - \eta) \times IVFEEWG(z_1, z_2, \ldots, z_s) \tag{14}$$

Theorem 5. *Let $\eta \in [0, 1]$ be a weight parameter. Then, the operational result of Equation (14) with a weight parameter η is still IVFEE, which is obtained by the following equation:*

$$z(\eta) = \eta \times IVFEEWA(z_1, z_2, \ldots, z_s) \oplus (1-\eta) \times IVFEEWG(z_1, z_2, \ldots, z_s)$$
$$= \left(\begin{array}{c} \left[\begin{array}{c} 1 - \left(\prod_{k=1}^{s} (1 - m_{Zk}^L)^{\lambda_k} \right)^{\eta} \left(1 - \prod_{k=1}^{s} (m_{Zk}^L)^{\lambda_k} \right)^{(1-\eta)}, \\ 1 - \left(\prod_{k=1}^{s} (1 - m_{Zk}^U)^{\lambda_k} \right)^{\eta} \left(1 - \prod_{k=1}^{s} (m_{Zk}^U)^{\lambda_k} \right)^{(1-\eta)} \end{array} \right], \\ \left[\begin{array}{c} 1 - \left(\prod_{k=1}^{s} (1 - e_{Zk}^L)^{\lambda_k} \right)^{\eta} \left(1 - \prod_{k=1}^{s} (e_{Zk}^L)^{\lambda_k} \right)^{(1-\eta)}, \\ 1 - \left(\prod_{k=1}^{s} (1 - e_{Zk}^U)^{\lambda_k} \right)^{\eta} \left(1 - \prod_{k=1}^{s} (e_{Zk}^U)^{\lambda_k} \right)^{(1-\eta)} \end{array} \right] \end{array} \right) \tag{15}$$

Proof. Based on Equations (7), (13), and (14), along with the operational laws in Definition 4, the following result is obtained below:

$$z(\eta) = \eta \times IVFEEWA(z_1, z_2, \ldots, z_s) \oplus (1-\eta) \times IVFEEWG(z_1, z_2, \ldots, z_s)$$
$$= \left(\left[1 - \left(\prod_{k=1}^{s} (1 - m_{Zk}^L)^{\lambda_k} \right)^{\eta}, 1 - \left(\prod_{k=1}^{s} (1 - m_{Zk}^U)^{\lambda_k} \right)^{\eta} \right], \left[1 - \left(\prod_{k=1}^{s} (1 - e_{Zk}^L)^{\lambda_k} \right)^{\eta}, 1 - \left(\prod_{k=1}^{s} (1 - e_{Zk}^U)^{\lambda_k} \right)^{\eta} \right] \right) \oplus$$
$$\left(\left[1 - \left(1 - \prod_{k=1}^{s} (m_{Zk}^L)^{\lambda_k} \right)^{(1-\eta)}, 1 - \left(1 - \prod_{k=1}^{s} (m_{Zk}^U)^{\lambda_k} \right)^{(1-\eta)} \right], \left[1 - \left(1 - \prod_{k=1}^{s} (e_{Zk}^L)^{\lambda_k} \right)^{(1-\eta)}, 1 - \left(1 - \prod_{k=1}^{s} (e_{Zk}^U)^{\lambda_k} \right)^{(1-\eta)} \right] \right)$$
$$= \left(\begin{array}{c} \left[\begin{array}{c} 1 - \left(\prod_{k=1}^{s} (1 - m_{Zk}^L)^{\lambda_k} \right)^{\eta} + 1 - \left(1 - \prod_{k=1}^{s} (m_{Zk}^L)^{\lambda_k} \right)^{(1-\eta)} - \left(1 - \left(\prod_{k=1}^{s} (1 - m_{Zk}^L)^{\lambda_k} \right)^{\eta} \right) \left(1 - \left(1 - \prod_{k=1}^{s} (m_{Zk}^L)^{\lambda_k} \right)^{(1-\eta)} \right), \\ 1 - \left(\prod_{k=1}^{s} (1 - m_{Zk}^U)^{\lambda_k} \right)^{\eta} + 1 - \left(1 - \prod_{k=1}^{s} (m_{Zk}^U)^{\lambda_k} \right)^{(1-\eta)} - \left(1 - \left(\prod_{k=1}^{s} (1 - m_{Zk}^U)^{\lambda_k} \right)^{\eta} \right) \left(1 - \left(1 - \prod_{k=1}^{s} (m_{Zk}^U)^{\lambda_k} \right)^{(1-\eta)} \right) \end{array} \right], \\ \left[\begin{array}{c} 1 - \left(\prod_{k=1}^{s} (1 - e_{Zk}^L)^{\lambda_k} \right)^{\eta} + 1 - \left(1 - \prod_{k=1}^{s} (e_{Zk}^L)^{\lambda_k} \right)^{(1-\eta)} - \left(1 - \left(\prod_{k=1}^{s} (1 - e_{Zk}^L)^{\lambda_k} \right)^{\eta} \right) \left(1 - \left(1 - \prod_{k=1}^{s} (e_{Zk}^L)^{\lambda_k} \right)^{(1-\eta)} \right), \\ 1 - \left(\prod_{k=1}^{s} (1 - e_{Zk}^U)^{\lambda_k} \right)^{\eta} + 1 - \left(1 - \prod_{k=1}^{s} (e_{Zk}^U)^{\lambda_k} \right)^{(1-\eta)} - \left(1 - \left(\prod_{k=1}^{s} (1 - e_{Zk}^U)^{\lambda_k} \right)^{\eta} \right) \left(1 - \left(1 - \prod_{k=1}^{s} (e_{Zk}^U)^{\lambda_k} \right)^{(1-\eta)} \right) \end{array} \right] \end{array} \right) \tag{16}$$
$$= \left(\begin{array}{c} \left[1 - \left(\prod_{k=1}^{s} (1 - m_{Zk}^L)^{\lambda_k} \right)^{\eta} \left(1 - \prod_{k=1}^{s} (m_{Zk}^L)^{\lambda_k} \right)^{(1-\eta)}, 1 - \left(\prod_{k=1}^{s} (1 - m_{Zk}^U)^{\lambda_k} \right)^{\eta} \left(1 - \prod_{k=1}^{s} (m_{Zk}^U)^{\lambda_k} \right)^{(1-\eta)} \right], \\ \left[1 - \left(\prod_{k=1}^{s} (1 - e_{Zk}^L)^{\lambda_k} \right)^{\eta} \left(1 - \prod_{k=1}^{s} (e_{Zk}^L)^{\lambda_k} \right)^{(1-\eta)}, 1 - \left(\prod_{k=1}^{s} (1 - e_{Zk}^U)^{\lambda_k} \right)^{\eta} \left(1 - \prod_{k=1}^{s} (e_{Zk}^U)^{\lambda_k} \right)^{(1-\eta)} \right] \end{array} \right).$$

When η = 1, 0, $z(\eta)$ degenerates into the IVFEEWA operator of Equation (7) and the IVFEEWG operator of Equation (13), respectively. □

4. GDM Method Using the Mixed-Weighted-Averaging Operation and Expected Value Function

Here we propose a multicriteria GDM method using the mixed-weighted-averaging operation and expected value function for SIVFMSs.

A multicriteria GDM problem usually contains a set of alternatives $Y = \{Y_1, Y_2, \ldots, Y_m\}$, which is assessed by a set of criteria $U = \{u_1, u_2, \ldots, u_s\}$. To consider the importance of different criteria u_k (k = 1, 2, …, s) in U, decision makers specify a weigh vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ for the set of criteria. Regarding the uncertainty and certainty of decision makers' cognitions/judgments for the suitability assessment of alternatives over the criteria, the single- and interval-valued fuzzy values of the alternatives Y_j (j = 1, 2, …, m) over the criteria u_k (k = 1, 2, …, s) will be specified by various decision makers. Thus, the multicriteria GDM method is depicted by the following decision steps.

Step 1. A group of decision makers/experts is invited to give their single- and interval-valued fuzzy values of the alternatives Y_j (j = 1, 2, …, m) over the criteria u_k (k = 1, 2, …, s) and to set up the SIVFME decision matrix $D = (F_{Hjk})_{m \times s}$, where $F_{Hjk} = (\lambda_{Hjk}^1, \lambda_{Hjk}^2, \ldots, \lambda_{Hjk}^{a_{jk}}, [\lambda_{Hjk}^{L1}, \lambda_{Hjk}^{U1}], [\lambda_{Hjk}^{L2}, \lambda_{Hjk}^{U2}], \ldots, [\lambda_{Hjk}^{Lb_{jk}}, \lambda_{Hjk}^{Ub_{jk}}])$ composed of a_{jk} single-valued fuzzy values and b_{jk} interval-valued fuzzy values (j = 1, 2, …, m; k = 1, 2, …, s) are

SIVFMEs, such that $0 \leq \lambda^1_{Hjk} \leq \lambda^2_{Hjk}, \ldots, \leq \lambda^{a_{jk}}_{Hjk} \leq 1$ and $[\lambda^{L1}_{Hjk}, \lambda^{U1}_{Hjk}] \subseteq [\lambda^{L2}_{Hjk}, \lambda^{U2}_{Hjk}] \subseteq, \ldots, \subseteq [\lambda^{Lb_{jk}}_{Hjk}(u_k), \lambda^{Ub_{jk}}_{Hjk}] \subseteq [0,1]$ with identical and/or different fuzzy values.

Step 2. Using Equations (3) and (4) for the decision matrix $D = (F_{Hjk})_{m\times s}$, the interval-valued fuzzy average values m_{Zjk} and entropy values e_{Zjk} are obtained and IVFEEs are assembled by $z_{jk} = (m_{Zjk}, e_{Zjk})$ for $m_{Zjk} = \left[m^L_{Zjk}, m^U_{Zjk}\right] \subseteq [0,1]$ and $e_{Zjk} = \left[e^L_{Zjk}, e^U_{Zjk}\right] \subseteq [0,1]$ ($k = 1, 2, \ldots, s$; $j = 1, 2, \ldots, m$), which are constructed as the IVFEE decision matrix $M = (z_{jk})_{m\times s}$.

Step 3. Using Equation (15) with some values of η, the operational values of $z_j(\eta)$ for Y_j ($j = 1, 2, \ldots, m$) are obtained by the following equation:

$$
\begin{aligned}
z_j(\eta) &= \eta \times IVFEEWA(z_{j1}, z_{j2}, \ldots, z_{js}) \oplus (1-\eta) \times IVFEEWG(z_{j1}, z_{j2}, \ldots, z_{js}) \\
&= \left(\begin{array}{c}
\left[\begin{array}{c}
1 - \left(\prod\limits_{k=1}^{s} (1 - m^L_{Zjk})^{\lambda_k}\right)^{\eta} \left(1 - \prod\limits_{k=1}^{s} (m^L_{Zjk})^{\lambda_k}\right)^{(1-\eta)}, \\
1 - \left(\prod\limits_{k=1}^{s} (1 - m^U_{Zjk})^{\lambda_k}\right)^{\eta} \left(1 - \prod\limits_{k=1}^{s} (m^U_{Zjk})^{\lambda_k}\right)^{(1-\eta)}
\end{array} \right], \\
\left[\begin{array}{c}
1 - \left(\prod\limits_{k=1}^{s} (1 - e^L_{Zjk})^{\lambda_k}\right)^{\eta} \left(1 - \prod\limits_{k=1}^{s} (e^L_{Zjk})^{\lambda_k}\right)^{(1-\eta)}, \\
1 - \left(\prod\limits_{k=1}^{s} (1 - e^U_{Zjk})^{\lambda_k}\right)^{\eta} \left(1 - \prod\limits_{k=1}^{s} (e^U_{Zjk})^{\lambda_k}\right)^{(1-\eta)}
\end{array} \right]
\end{array} \right)
\end{aligned}
\quad (17)
$$

Step 4. The expected values of $Q(z_j(\eta))$ ($j = 1, 2, \ldots, m$) are given by Equation (5).

Step 5. Alternatives are sorted in descending order of the expected values, and the optimal one is selected depending on some specified value of η.

Step 6. End.

5. GDM Example of a Supplier Selection Problem and Comparative Analysis

5.1. Actual GDM Example

This section reports the application of the proposed GDM method to an actual example of a supplier selection problem in a supply chain to show the rationality and effectiveness of SIVFMSs.

Any enterprise tries to reduce the supply chain risks and uncertainty to improve customer service, inventory levels, and cycle times, which will increasing its competitiveness and profitability. Assume that a group of five suppliers is provided as a set of preliminary alternatives $Y = \{Y_1, Y_2, Y_3, Y_4, Y_5\}$. Then, a group of decision makers is invited to evaluate the five suppliers with three criteria: performance (e.g., quality, delivery, and price) (u_1), technology (e.g., design capability, manufacturing capability, and ability to deal with technology changes) (u_2), and organizational culture and strategy (e.g., external and internal integration of suppliers, feeling of trust, compatibility across levels, and functions of the supplier and buyer) (u_3). The weight vector of the three criteria is specified as $\boldsymbol{\lambda} = (0.3, 0.33, 0.37)$. Thus, the proposed GDM method can be applied to this GDM problem, which is depicted below.

Step 1. Suppose that three decision makers are invited to evaluate a set of five suppliers $Y = \{Y_1, Y_2, Y_3, Y_4, Y_5\}$ with a set of three criteria $U = \{u_1, u_2, u_3\}$. For instance, the three decision makers can declare the degree that an alternative Y_1 should satisfy a criterion u_1, and these values could be a group of three single- and interval-valued fuzzy values (0.7, 0.8, [0.7, 0.9]). In this manner, all their evaluation values of SIVFMEs are indicated in Table 1.

Table 1. Evaluation values of SIVFMEs provided by the three decision makers.

	u_1	u_2	u_3
Y_1	(0.7, 0.8, [0.7, 0.9])	(0.6, [0.6, 0.7], [0.7, 0.8])	(0.6, 0.7, [0.7, 0.8])
Y_2	(0.7, 0.8, [0.6, 0.7])	(0.6, 0.7, [0.7, 0.8])	(0.7, 0.7, [0.6, 0.9])
Y_3	(0.8, [0.8, 0.9], [0.8, 0.9])	(0.8, [0.7, 0.9], [0.8, 0.9])	(0.6, 0.7, [0.7, 0.9])
Y_4	(0.6, 0.6, [0.7, 0.8])	(0.6, 0.8, [0.7, 0.9])	(0.8, 0.8, [0.7, 0.9])
Y_5	(0.8, 0.9, [0.7, 0.8])	(0.8, 0.9, [0.7, 0.8])	(0.7, [0.6, 0.8], [0.7, 0.8])

Step 2. Using Equations (3) and (4) on Table 1, IVFEEs can be obtained based on the average values and entropy values of various SIVFMVEs, and the IVFEE decision matrix $M = (z_{jk})_{5\times3}$ is established as follows:

$$M = \begin{bmatrix} ([0.7333, 0.8000], [0.9952, 0.9981]) & ([0.6333, 0.7000], [0.9938, 0.9975]) & ([0.6667, 0.7000], [0.9938, 0.9977]) \\ ([0.7000, 0.7333], [0.9938, 0.9981]) & ([0.6667, 0.7000], [0.9938, 0.9977]) & ([0.6667, 0.7667], [0.9933, 0.9977]) \\ ([0.8000, 0.8667], [0.9986, 1.0000]) & ([0.7667, 0.8667], [0.9983, 0.9986]) & ([0.6667, 0.7333], [0.9870, 0.9977]) \\ ([0.6333, 0.6667], [0.9912, 0.9975]) & ([0.7000, 0.7667], [0.9876, 0.9938]) & ([0.7667, 0.8333], [0.9983, 0.9986]) \\ ([0.8000, 0.8333], [0.9952, 0.9986]) & ([0.8000, 0.8333], [0.9952, 0.9986]) & ([0.6667, 0.7667], [0.9977, 0.9983]) \end{bmatrix}$$

Step 3. Using Equation (17) with η = 0, 0.3, 0.5, 0.7, and 1, the operational values of $z_j(\eta)$ for Y_j (j = 1, 2, 3, 4, 5) and the decision results are indicated in Table 2.

Table 2. Decision results of the proposed GDM method with various weight values of η.

η	$z_1(\eta)$, $z_2(\eta)$, $z_3(\eta)$, $z_4(\eta)$, $z_5(\eta)$	$E(z_1(\eta))$, $E(z_2(\eta))$, $E(z_3(\eta))$, $E(z_4(\eta))$, $E(z_5(\eta))$	Sorting	Optimal One
0	([0.6745, 0.7286], [0.9942, 0.9978]), ([0.6765, 0.7341], [0.9936, 0.9978]), ([0.7374, 0.8147], [0.9942, 0.9987]), ([0.7025, 0.7582], [0.9926, 0.9967]), ([0.7478, 0.8080], [0.9961, 0.9984])	0.6988, 0.7024, 0.7734, 0.7265, 0.7758	$Y_5 > Y_3 > Y_4 > Y_2 > Y_1$	Y_5
0.3	([0.6756, 0.7303], [0.9942, 0.9978]), ([0.6767, 0.7347], [0.9936, 0.9978]), ([0.7399, 0.8187], [0.9951, 1.0000]), ([0.7047, 0.7620], [0.9933, 0.9969]), ([0.7510, 0.8090], [0.9962, 0.9984])	0.7002, 0.7027, 0.7775, 0.7298, 0.7780	$Y_5 > Y_3 > Y_4 > Y_2 > Y_1$	Y_5
0.5	([0.6764, 0.7315], [0.9942, 0.9978]), ([0.6768, 0.7351], [0.9936, 0.9978]), ([0.7416, 0.8213], [0.9956, 1.0000]), ([0.7061, 0.7645], [0.9937, 0.9970]), ([0.7532, 0.8096], [0.9963, 0.9985])	0.7012, 0.7030, 0.7798, 0.7320, 0.7794	$Y_3 > Y_5 > Y_4 > Y_2 > Y_1$	Y_3

Table 2. *Cont.*

η	$z_1(\eta)$, $z_2(\eta)$, $z_3(\eta)$, $z_4(\eta)$, $z_5(\eta)$	$E(z_1(\eta))$, $E(z_2(\eta))$, $E(z_3(\eta))$, $E(z_4(\eta))$, $E(z_5(\eta))$	Sorting	Optimal One
0.7	([0.6772, 0.7326], [0.9943, 0.9978]), ([0.6769, 0.7355], [0.9936, 0.9978]), ([0.7433, 0.8239], [0.9960, 1.0000]), ([0.7076, 0.7670], [0.9940, 0.9971]), ([0.7553, 0.8103], [0.9963, 0.9985])	0.7021, 0.7032, 0.7821, 0.7341, 0.7807	$Y_3 > Y_5 > Y_4 > Y_2 > Y_1$	Y_3
1	([0.6783, 0.7344], [0.9943, 0.9978]), ([0.6770, 0.7361], [0.9936, 0.9978]), ([0.7458, 0.8277], [0.9966, 1.0000]), ([0.7097, 0.7707], [0.9946, 0.9973]), ([0.7584, 0.8112], [0.9964, 0.9985])	0.7036, 0.7036, 0.7855, 0.7372, 0.7828	$Y_3 > Y_5 > Y_4 > Y_1 = Y_2$	Y_3

Step 4. By Equation (5), the expected values of $E(z_j(\eta))$ (j = 1, 2, 3, 4, 5) are given in Table 2.

Step 5. The sorting orders of the alternatives are $Y_5 > Y_3 > Y_4 > Y_2 > Y_1$, $Y_3 > Y_5 > Y_4 > Y_2 > Y_1$, and $Y_3 > Y_5 > Y_4 > Y_1 = Y_2$. The optimal one is Y_5 or Y_3, depending on some specified value of η.

Regarding the decision results in Table 2, there are different sorting orders for the IVFEEWA operator and the IVFEEWG operator when η = 0, 1 (two special cases), since the IVFEEWA operator tends to group arguments and the IVFEEWG operator tends to group personal arguments. The mixed-weighted-averaging operation of the IVFEEWA and IVFEEWG operators can compensate for the different tendencies of both when $\eta \neq 0, 1$.

5.2. Comparative Analysis

To verify the efficiency of the proposed GDM method, the proposed GDM method is compared with the existing consistency fuzzy decision-making method and various fuzzy decision-making methods.

First, the proposed GDM method is compared with the existing consistency fuzzy decision-making method [19]. For a convenient comparison with the existing consistency fuzzy decision-making method [19], assume that all interval-valued fuzzy values and entropy values in the IVFEE decision matrix M are fuzzy average values and consistency degrees as a special case of the actual example mentioned above. Thus, the IVFEE decision matrix M is reduced to the decision matrix of CFEs:

$$M' = \begin{bmatrix} (0.7667, 0.9881) & (0.6667, 0.9957) & (0.6834, 0.9958) \\ (0.7167, 0.9960) & (0.6834, 0.9958) & (0.7167, 0.9955) \\ (0.8334, 0.9993) & (0.8167, 0.9985) & (0.7000, 0.9924) \\ (0.6500, 0.9944) & (0.7334, 0.9907) & (0.8000, 0.9985) \\ (0.8167, 0.9969) & (0.8167, 0.9969) & (0.7167, 0.9980) \end{bmatrix}$$

Thus, the existing decision-making method [19] can be applied to the special case of the above actual example by the following CFE weighted averaging (CFEWA) and CFE weighted geometric (CFEWG) operators and score function [19]:

$$z'_j = CFEWA(z'_{j1}, z'_{j2}, \ldots, z'_{js}) = \mathop{\oplus}_{k=1}^{s} \lambda_k z'_{jk} = \left(1 - \prod_{k=1}^{s} (1 - m'_{Zjk})^{\lambda_k}, 1 - \prod_{k=1}^{s} (1 - e'_{Zjk})^{\lambda_k}\right) \quad (18)$$

$$z'_j = CFEWG(z'_{j1}, z'_{j2}, \ldots, z'_{js}) = \mathop{\otimes}_{k=1}^{s} (z'_{jk})^{\lambda_k} = \left(\prod_{k=1}^{s} (m'_{Zjk})^{\lambda_k}, \prod_{k=1}^{s} (e'_{Zjk})^{\lambda_k}\right) \quad (19)$$

$$F(z'_j) = (m'_{Zj} e'_{Zj} + (m'_{Zj} + e'_{Zj})/2)/2 \text{ for } F(z'_j) \in [0, 1] \quad (20)$$

Using Equations (18)–(20), the aggregated values of the CFEWA and CFEWG operators, the score values of $F(z'_j)$ for Y_i (i = 1, 2, 3, 4, 5), and the decision results were achieved. They are shown in Table 3.

Table 3. Decision results of the existing decision-making method in the case of CFEs [19].

Aggregation Operator	$z'_1, z'_2, z'_3, z'_4, z'_5$	$F(z'_1), F(z'_2), F(z'_3), F(z'_4), F(z'_5)$	Sorting	Optimal One
CFEWA	(0.7061, 0.9960), (0.7061, 0.9957), (0.7862, 0.9978), (0.7399, 0.9958), (0.7846, 0.9974)	0.7772, 0.7770, 0.8383, 0.8023, 0.8368	$Y_3 > Y_5 > Y_4 > Y_1 > Y_2$	Y_3
CFEWG	(0.7016, 0.9960), (0.7055, 0.9957), (0.7761, 0.9964), (0.7304, 0.9946), (0.7781, 0.9973)	0.7738, 0.7765, 0.8298, 0.7945, 0.8319	$Y_5 > Y_3 > Y_4 > Y_2 > Y_1$	Y_5

In the decision results in Table 3, there exists their sorting difference, since there are the different tendencies for the CFEWA and CFEWG operators. The optimal alternatives are Y_3 and Y_5 according to the existing decision-making method with CFE information. Although the optimal ones, Y_3 and Y_5, are the same according to the proposed GDM method and the existing decision-making method [19] in the example, the superiorities of the proposed GDM method over the existing decision-making method [19] are as follows:

(1) SIVFMSs can effectively express group evaluation values using identical and/or different single- and interval-valued fuzzy values, whereas CFMS introduced in [19] cannot.
(2) IVFEEs can reasonably reflect the mean and consistency/consensus degrees of the group evaluation values with the help of quantitative calculations corresponding to the mean and information entropy of a SIVFME in a SIVFMS. The transformation method introduced in [19] is only suitable for the normal distribution of fuzzy data, and there is no distribution limitation for the new transformation method proposed in this paper.
(3) The proposed GDM method not only demonstrated its decision flexibility, but also overcomes the flaws of the existing decision-making method using the single CFEWA operator or the CFEWG operator.

In comparison with the PFDM methods [16,17], the PFDM methods need a lot of fuzzy data to maintain the rationality (no distortion) of probabilistic fuzzy values from the probabilistic viewpoint; otherwise, the probabilistic fuzzy values are infeasible and irrational, since a lot of fuzzy data are created with difficultly by several decision makers and obviously unrealized in the GDM application. Hence, the PFDM methods cannot represent this decision example involving three decision makers and also cannot express single- and interval-valued fuzzy data. In the case of SIVFMSs, the proposed GDM method with the mean and information entropy only needs a few of decision makers to perform GDM problems with several single- and interval-valued fuzzy data, which are easily handled in actual applications. In this case, the proposed GDM method showed its rationality and efficiency and is superior to the existing PFDM methods regarding SIVFMSs.

Furthermore, with respect to the above GDM example in the SIVFMS setting, existing fuzzy multiset/IVFM/HFS/CHFS [9–15,18] cannot express SIVFMS, and then they also cannot be applied to this GDM problem with SIVFMS information.

However, our method not only solves the expression and operation problems of SIVFMEs, but also enhances the flexibility and rationality of GDM, which serve to highlight its advantages in the setting of SIVFMSs.

6. Conclusions

In this study, the presented SIVFMSs could effectively express single- and interval-valued fuzzy sequences in hybrid fuzzy multivalued situations to solve the difficult problems of various existing fuzzy expressions. The proposed information entropy of SIVFME provides a reasonable mathematical tool for converting SIVFMEs into IVFEEs when dealing with SIVFMSs. IVFEEs converted by the mean and information entropy of SIVFMEs in SIVFMS can reasonably reflect the average and consistency level of group evaluation values and effectively solve the operational problems of different fuzzy sequence lengths in SIVFMSs. In addition, the proposed mixed-weighted-averaging operation of the IVFEEWA and IVFEEWG operators can reasonably and flexibly aggregate IVFEE information with a changeable weight parameter and compensate for the flaws of the IVFEEWA and IVFEEWG operators. Next, the multicriteria GDM method developed based on the proposed mixed-weighted-averaging operation solved flexible decision-making problems involving SIVFMSs. Furthermore, the proposed GDM method was utilized for an actual example of a supplier selection problem to indicate its application. Through the comparative analysis with existing relative decision-making methods, the proposed GDM method demonstrated its rationality and effectiveness. However, this study not only effectively solved the expression and operation problems of the mixed information of single- and interval-valued fuzzy sequences with identical and/or different fuzzy values, but also strengthened the GDM rationality and flexibility with the help of the presented information entropy and the proposed mixed-weighted-averaging operation, which highlighted its merits when dealing with SIVFMSs.

This original study demonstrated new contributions in mixed fuzzy information expression, presented a transformation method based on the mean and information entropy of SIVFME, and presented mixed aggregation operations of IVFEEs and their GDM method in the environment of SIVFMSs. However, the new techniques proposed in this paper can only handle GDM problems with SIVFMSs, but cannot solve GDM problems with the fuzzy information of truth and falsity membership degrees. Regarding future research, this study will be further extended to image processing, pattern recognition, clustering analysis, and their applications in the setting of SIVFMSs. Then, the Aczel–Alsina operations and aggregation operators [21,22], and their applications, will be further developed in the intuitionistic and interval-valued intuitionistic fuzzy multivalued context.

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