mathematics

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Interval Estimation of Generalized Inverted Exponential Distribution under Records Data: A Comparison Perspective

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Abstract: In this paper, the problem of interval estimation is considered for the parameters of the generalized inverted exponential distribution. Based on upper record values, different pivotal quantities are proposed and the associated exact and generalized confidence intervals are constructed for the unknown model parameters and reliability indices, respectively. For comparison purposes, conventional likelihood based approximate confidence intervals are also provided by using observed Fisher information matrix. Moreover, prediction intervals are also constructed for future records based on proposed pivotal quantities and likelihood procedures as well. Finally, numerical studies are carried out to investigate and compare the performances of the proposed methods and a real data analysis is presented for illustrative purposes.

Keywords: records; generalized inverted exponential distribution; confidence interval; pivotal quantities; Monte-Carlo simulation

MSC: 62F25; 62F40; 62N05

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Citation: Wang, L.; Lin, H.; Lio, Y.; Tripathi, Y.M. Interval Estimation of Generalized Inverted Exponential Distribution under Records Data: A Comparison Perspective. *Mathematics* **2022**, *10*, 1047. https://doi.org/10.3390/math10071047

Academic Editor: Christophe Chesneau

Received: 13 February 2022
Accepted: 23 March 2022
Published: 24 March 2022

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1. Introduction

Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of independent and identically distributed (i.i.d.) random variables. An observation X_j is called an upper record value if its value exceeds those of previous observations. Thus X_j is an upper record value if $X_j > X_i$ for each $j > i$. Similar definition deals with the lower record values. The concept of record values, initially introduced by [1], can be viewed as order statistics whose size is determined by the values and the order of occurrence of observations. In many situations such as sport matches, destructive stress testing, meteorology, hydrology, seismology, mining and aerology, we may use such data. Due to the commonality and the importance, many researchers have considered the use of record values in their studies. Interested readers may refer to the works of [2–6] and the references cited therein. For more details, one can see the monographs of [7,8].

For any lifetime distribution, the inferences of parameters and predictions of future observations are always of fundamental importance in statistical theory. Besides estimation, confidence interval is also of considerable interest and practical significance in statistical inference. Such problems have been widely discussed by many authors, for example, the works of [6,9–13] and the reference cited therein. However, likelihood based interval estimation is affected heavily by its sample size, and hence, the associated confidence sets may not have desirable properties when the sample size is not sufficiently large. Since data sets consisting of record values may often lack sufficiently large sample size in data analysis, it is of importance to pursue alternative inferential approach for statistical inference under small sample size when record values are utilized. Similar works for confidence interval estimation is also discussed by many authors. Kinaci et al. [14] studied the estimation problems for the generalized inverted exponential distribution based on progressively type-II censored order statistics and record values. Marques [15] has proposed the confidence interval expressions based on time-history records. Motivated by such reasons, this paper considers the study of inference for the interval estimation of a lifetime distribution based on data consisting of upper record values.

Let random variable, T, have the generalized inverted exponential distribution (GIED) with scale parameter $\lambda > 0$ and shape parameter $\theta > 0$, where the cumulative distribution function (CDF) and probability density function (PDF) are given by

$$F(t; \lambda, \theta) = 1 - [1 - e^{-\lambda/t}]^{\theta}, t > 0, \tag{1}$$

and

$$f(t; \lambda, \theta) = \frac{\theta\lambda}{t^2} e^{-\lambda/t}[1 - e^{-\lambda/t}]^{\theta-1}, t > 0. \tag{2}$$

The distribution of (1) was proposed by [16], wherein they carried out an extensive study on the properties of GIED. From various studies on GIED, (e.g., [16,17]), one can conclude GIED to be an useful lifetime model that could be treated as an potential alternative to the traditinal distributions such as gamma, Weibull, generalized exponential and inverted exponential distributions. Recently, the GIED has received considerable attention in literatures because of its practical significance in many situations like accelerated life testing, horse racing, supermarket queues, wind speed and so on. For more information, readers may refer to [2,18–20] and the references cited therein.

Due to the GIED's importance and practicability, for comparing with traditional asymptotic confidence intervals, our aim in this paper is to explore some alternative confidence intervals for the unknown parameters of the GIED based on upper record values. Another objective is to construct generalized prediction intervals for the future upper record values based on the past upper record values from the GIED because it is very important to correctly predict future record values given a sample of observed record values in many fields such as earthquakes, flood, and rainfall. As far as we know, no attempt has been made on different types of interval estimation and prediction for the parameters of the GIED based on upper record values. Moreover, for the purpose of comparison, likelihood based inferential procedures are also presented for model parameters and prediction of future records, respectively.

This rest of this paper is organized as follows. In Section 2, different kinds of confidence intervals using pivotal quantities are proposed for the unknown parameters and generalized confidence intervals (GCIs) for the future upper record values from the GIED based on the upper record values. In Section 3, simulation results and real-life examples are presented to assess the performance of the proposed methods. Finally, some discussions and remarks are provided in Section 4.

2. Confidence Interval Estimation

In this section, different types of interval estimators are proposed for the GIED parameters and the reliability indices through different approaches. The prediction intervals (PIs) of record values are also presented for a given significance level.

2.1. F Pivotal Based Interval Estimation for λ and θ

In order to obtain the CIs of GIED parameters, a useful result is given firstly as follows.

Theorem 1. *Let $R = (R_1, R_2, \ldots, R_n)$ be the first n upper records from the GIED of (1) with parameters λ and θ. Denote pivotal quantities*

$$P_1(\lambda; R) = (n-1)\left[\frac{\ln[1 - e^{-\lambda/R_n}]}{\ln[1 - e^{-\lambda/R_1}]} - 1\right]^{-1}$$

and

$$Q_1(\lambda, \theta; R) = -2\theta \ln[1 - e^{-\lambda/R_n}].$$

Then P_1 follows F distribution with 2 and $2(n-1)$ degrees of freedom, Q_1 is chi-square distributed with $2n$ degrees of freedom, and they are statistical independent.

Proof. See Appendix A. $\square$

Another useful lemma is also provided as follows.

Lemma 1. *Let a and b be arbitrary positive values satisfying $0 < a < b$ and $G(t) = \frac{\ln[1-e^{-t/b}]}{\ln[1-e^{-t/a}]}, t > 0$. Then*

$G(t)$ increases in t,
$\lim_{t\to 0} G(t) = 1$ *and* $\lim_{t\to\infty} G(t) = \infty$.

Proof. See Appendix B. $\square$

Corollary 1. *From Lemma 1, it is noted that pivotal quantity $P_1(\lambda; R)$ is a monotonically decreasing function with respect to λ and the associated range of $P_1(\lambda)$ is $(0, \infty)$.*

From Theorem 1, confidence interval based on pivotal quantity is constructed for parameter λ as follows.

Theorem 2. *Let $R = (R_1, R_2, \ldots, R_n)$ be the first n upper records from the GIED of (1) with parameters λ and θ. For arbitrary $0 < \gamma < 1$, a $100(1-\gamma)\%$ F pivotal quantity based exact confidence interval (FECI) for λ is given by*

$$\left(\rho_1\left(F^{\gamma/2}_{[2,2(n-1)]}; R\right), \rho_1\left(F^{1-\gamma/2}_{[2,2(n-1)]}; R\right)\right),$$

where $F^{\gamma}_{[k_1,k_2]}$ denotes the right-tail $100\gamma\%$ quantile of the F distribution with k_1 and k_2 degrees of freedom and $\rho_1(t)$ is the solution of λ for equation $P_1(\lambda; R) = t$.

Proof. See Appendix C. $\square$

Based on the proposed pivotal quantity $P_1(\lambda; R)$, the following hypotheses are provided as an application and complementary.

Remark 1. *Hypotheses are often used to compare whether the values of the parameter are same or not with past time. Here, we consider null hypothesis H_0 and alternative hypothesis H_1 as follows*

$$(a)\ H_0 :\ \lambda \le \lambda_0 \leftrightarrow H_1 :\ \lambda > \lambda_0,$$
$$(b)\ H_0 :\ \lambda \ge \lambda_0 \leftrightarrow H_1 :\ \lambda < \lambda_0,$$
$$(c)\ H_0 :\ \lambda = \lambda_0 \leftrightarrow H_1 :\ \lambda \ne \lambda_0.$$

From Theorem 1 and Lemma 1, it is observed that the pivotal quantity $P_1(\lambda)$ is F distributed with 2 and $2(n-1)$ degrees of freedom and decreases in λ. Therefore, for arbitrary $0 < \gamma < 1$, the decision rule to reject null hypothesis H_0 in $(a), (b), (c)$ can be respectively expressed as

$$(a)' \left\{P_1(\lambda_0) \ge F^{\gamma}_{[2,2(n-1)]}\right\}, \qquad (b)' \left\{P_1(\lambda_0) \le F^{\gamma}_{[2,2(n-1)]}\right\},$$
$$(c)' \left\{P_1(\lambda_0) \ge F^{\gamma/2}_{[2,2(n-1)]},\ or\ P_1(\lambda_0) \le F^{1-\gamma/2}_{[2,2(n-1)]}\right\}.$$

For parameter θ, a generalized confidence interval is constructed by using pivotal quantities $P_1(\lambda; R)$ and $Q_1(\lambda; R)$ for a given significance level. Using the same notation in

Theorem 2, denote $\rho_1(Y_1; R)$ as the unique solution of $P_1(\lambda) = Y_1$, where $Y_1 \sim F_{[2,2(n-1)]}$. Moreover, from Theorem 1, since pivotal $Q_1 = -2\theta \ln[1 - e^{-\lambda/R_n}]$ is distributed as chi-square distribution with $2n$ degrees of freedom, then one has

$$\theta = \frac{Q_1}{-2\ln[1 - e^{-\lambda/R_n}]}.$$

Following the substitution method of [21], a generalized pivotal quantities, namely, S_1, can be constructed by substituting $\rho_1(Y_1; R)$ for λ in expression of θ as follows

$$S_1 = \frac{Q_1}{-2\ln[1 - e^{-\rho_1(Y_1;r)/r_n}]} = \frac{\theta \ln[1 - e^{-\rho_1(Y_1;R)/R_n}]}{\ln[1 - e^{-\rho_1(Y_1;r)/r_n}]},$$

where $r = (r_1, r_2, \ldots, r_n)$ denotes the observation of records $R = (R_1, R_2, \ldots, R_n)$. It is noted that the distribution of S_1 is free from any unknown parameters from its first expression and S_1 reduces to θ when $R = r$. Therefore, we can conclude that S_1 is a generalized pivotal quantity for parameter θ. Furthermore, an extra Algorithm 1 is provided to derive the F pivotal quantity based generalized confidence interval (FGCI) of θ as follows.

Algorithm 1: FGCI for parameter θ.

Step 1 Generate a realization y_1 of Y_1 from F distribution with 2 and $2(n-1)$ degrees of freedom. Then an observation ρ_1 of $\rho_1(y_1; R)$ can be obtained from the equation $P_1(\lambda) = y_1$ for given records R.

Step 2 Generate random data for Q_1 from chi-square distribution with $2n$ degrees of freedom, and compute S_1.

Step 3 Repeat Step 1 and 2 M times, one can obtain M values of S_1.

Step 4 To construct the generalized confidence interval of θ, first arrange all estimates of S_1 in an ascending order as $S_1^{[1]}, S_1^{[2]}, \ldots, S_1^{[M]}$. For arbitrary $0 < \gamma < 1$, a $100(1-\gamma)$ confidence interval of θ can be obtained as

$$(S_1^{[j]}, S_1^{[j+M-[M\gamma+1]]}), j = 1, 2, \ldots, [M\gamma].$$

where $[t]$ denotes the greatest integer less than or equal to t. Therefore, the $100(1-\gamma)\%$ generalized confidence interval of θ can be constructed as the j^*th one satisfying

$$S_1^{[j^*+M-[M\gamma+1]]} - S_1^{[j^*]} = \min_{j=1}^{[M\gamma]}(S_1^{[j+M-[M\gamma+1]]} - S_1^{[j]}).$$

Remark 2. *It is worth mentioning that based on S_1, its CDF can be written as*

$$\begin{aligned} F_{S_1}(s) &= \int_0^\infty P(S_1 < s | Y_1 = y_1) f_{Y_1}(y_1) dy_1 \\ &= 1 - \int_0^\infty G_{\chi^2_{2n}}(-2s\ln[1 - e^{-\rho_1(y_1;r)/r_n}]) f_{[2,2(n-1)]}(y_1) dy_1, \end{aligned}$$

where notations $G_{\chi^2_k}(t)$ is the CDF of the chi-square distribution with k degrees of freedom and $f_{[k_1,k_2]}(t)$ is the PDF of F distribution with k_1 and k_2 degrees of freedom, respectively.

Using quantity $P_1(\lambda)$ and following similar procedure as previous, FGCI for parameter λ can also be constructed using following Algorithm 2.

Algorithm 2: FGCI for parameter λ.

Step 1 Generate a realization y_1 of Y_1 from F distribution with 2 and $2(n-1)$ degrees of freedom. Then an observation ρ_1 of $\rho_1(y_1;R)$ can be obtained from the equation $P_1(\lambda) = y_1$ for given records R.

Step 2 Repeat Step 1 M times, one can obtain M values of $\rho_1(y_1;R)$ as estimates for λ.

Step 3 Arrange all estimates of $\rho_1(y_1;R)$ in an ascending order as $\rho_1^{[1]}, \rho_1^{[2]}, \ldots, \rho_1^{[M]}$. For arbitrary $0 < \gamma < 1$, a $100(1-\gamma)$ confidence interval of θ can be obtained as

$$(\rho_1^{[j]}, \rho_1^{[j+M-[M\gamma+1]]}), j = 1, 2, \ldots, [M\gamma].$$

Then, the $100(1-\gamma)\%$ generalized confidence interval of λ can be provided as the j^*th one satisfying

$$\rho_1^{[j^*+M-[M\gamma+1]]} - \rho_1^{[j^*]} = \min_{j=1}^{[M\gamma]}(\rho_1^{[j+M-[M\gamma+1]]} - \rho_1^{[j]}).$$

2.2. Chi-Square Pivotal Based Interval Estimation for λ and θ

In this subsection, another set of alternative confidence intervals is proposed for GIED parameters λ and θ.

Theorem 3. *Let $R = (R_1, R_2, \ldots, R_n)$ be the first n upper records from the GIED of (1) with parameters λ and θ. Pivotal quantity*

$$P_2(\lambda;R) = -2\sum_{i=1}^{n-1} i \ln\left(\frac{\ln[1-e^{-\lambda/R_i}]}{\ln[1-e^{-\lambda/R_{i+1}}]}\right) = 2\sum_{i=1}^{n-1} \ln\left(\frac{\ln[1-e^{-\lambda/R_n}]}{\ln[1-e^{-\lambda/R_i}]}\right)$$

is chi-square distributed with $2(n-1)$ degrees of freedom. Moreover, P_2 is also independent with $Q_1(\lambda,\theta;R) = -2\theta\ln[1-e^{-\lambda/R_n}]$ defined in Theorem 1.

Proof. See Appendix D. □

Corollary 2. *From Lemma 1, it is also observed that pivotal quantity $P_2 = P_2(\lambda)$ increases in λ and the corresponding range of $P_2(\lambda)$ is $(0,\infty)$.*

Based on Theorem 3, another confidence interval is proposed for parameter λ as follows.

Theorem 4. *Let $R = (R_1, R_2, \ldots, R_n)$ be the first n upper records from the GIED of (1) with parameters λ and θ. For arbitrary $0 < \gamma < 1$, a $100(1-\gamma)\%$ chi-square pivotal quantity based exact confidence interval (CECI) for λ is given by*

$$\left(\rho_2(\chi^{1-\gamma/2}_{2(n-1)};R),\ \rho_2(\chi^{\gamma/2}_{2(n-1)};R)\right),$$

where χ_k^{γ} denotes the right-tail $100\gamma\%$ quantile of a chi-square distribution k degrees of freedom and $\rho_2(t;R)$ is the solution of λ for equation $P_2(\lambda) = t$.

Proof. See Appendix E. □

It should be mentioned that given $0 < p < 1$ the solution to $P_2(\lambda) = \chi^p_{2(n-1)}$ can be obtained by R function, uniroot, and labeled by $\rho_2(\chi^p_{2(n-1)};R)$. By using the distribution property of $P_2(\lambda)$, other hypotheses are presented as an application and complementary.

Remark 3. *Consider the same null hypothesis H_0 and alternative hypothesis H_1 as provided in Remark 1, since pivotal quantity $P_2(\lambda)$ is chi-square distributed and increases in λ with range $(0, \infty)$. Therefore, for arbitrary $0 < \gamma < 1$, the decision rule to reject null hypothesis H_0 in $(a), (b), (c)$ can be respectively written as*

$$(a)'' \left\{P_2(\lambda_0) \leq \chi^{\gamma}_{2(n-1)}\right\}, \qquad (b)'' \left\{P_2(\lambda_0) \geq \chi^{\gamma}_{2(n-1)}\right\},$$
$$(c)'' \left\{P_2(\lambda_0) \leq \chi^{\gamma/2}_{2(n-1)}, \text{ or } P_2(\lambda_0) \geq \chi^{1-\gamma/2}_{2(n-1)}\right\}.$$

It is noted that Remarks 1 and 3 provide two hypothesis-tests for λ. How to compare and pursue a better one under this case remains an open problem.

Following similar approach as Section 2.1, another generalized confidence interval could be proposed for parameter θ through pivotal quantities P_2 and Q_1.

Let $\rho_2(Y_2; R)$ be the unique solution of $P_2(\lambda; R) = Y_2$, where $Y_2 \sim \chi^2_{2(n-1)}$, then a generalized pivotal quantities, namely S_2, is constructed for λ as follows

$$S_2 = \frac{Q_1}{-2\ln[1 - e^{-\rho_2(Y_2;r)/r_n}]} = \frac{\theta \ln[1 - e^{-\rho_2(Y_2;R)/R_n}]}{\ln[1 - e^{-\rho_2(Y_2;r)/r_n}]}.$$

One can also observe that the distribution of S_2 is free of any unknown parameters from its first expression and S_2 reduces to θ when $R = r$. Furthermore, a realization of $\rho_2(Y_2; R)$ can be derived by generating random sample y_2 from chi-square distribution with $2(n-1)$ degrees of freedom and solving the equation $P_2(\lambda) = y_2$. Therefore, using similar Monte-Carlo procedure as Algorithm 1, the chi-square pivotal quantity based generalized confidence interval (CGCI) of θ could be obtained through replacing $\rho_1(y_1; R)$ by $\rho_1(y_1; R)$.The detailed procedure is shown in Algorithm 3.

Algorithm 3: CGCI for parameter θ.

Step 1 Generate a data y_2 of Y_2 from the chi-square distribution with $2(n-1)$ degrees of freedom and compute observation ρ_2 of $\rho_2(y_2; R)$ from $P_2(\lambda) = y_2$.
Step 2 Generate random sample for Q_1 from chi-square distribution with $2n$ degrees of freedom and compute S_2.
Step 3 Repeat Step 1 and 2 M times, then one can obtain M values of S_2.
Step 4 To construct the generalized confidence interval of θ, first arrange all estimates of S_1 in an ascend order as $S_2^{[1]}, S_2^{[2]}, \ldots, S_2^{[M]}$. For arbitrary $0 < \gamma < 1$, a $100(1-\gamma)$ confidence interval of θ can be obtained as

$$(S_2^{[j]}, S_2^{[j+M-[M\gamma+1]]}), j = 1, 2, \ldots, [M\gamma],$$

where $[t]$ denotes the greatest integer less than or equal to t. Therefore, the $100(1-\gamma)\%$ generalized confidence interval of θ can be constructed as the j^*th one that satisfies

$$S_2^{[j^*+M-[M\gamma+1]]} - S_2^{[j^*]} = \min_{j=1}^{[M\gamma]}(S_2^{[j+M-[M\gamma+1]]} - S_2^{[j]}).$$

Remark 4. *Based on the distribution of Y_2 and Q_1, the CDF of the generalized pivotal quantity S_2 is given by*

$$\begin{aligned} F_{S_2}(s) &= \int_0^\infty P(S_2 < s | Y_2 = y_2) f_{Y_2}(y_2) dy_2 \\ &= 1 - \int_0^\infty G_{\chi^2_{2n}}(-2s\ln[1 - e^{-\rho_2(y_2;r)/r_n}]) G_{\chi^2_{2(n-1)}}(y_2) dy_2. \end{aligned}$$

Furthermore, based on pivotal quantity $P_2(\lambda)$, a CGCI for parameter λ can be constructed based on following Algorithm 4.

Algorithm 4: CGCI for parameter λ.

Step 1 Generate a realization y_2 of Y_2 from the chi-square distribution with $2(n-1)$ degrees of freedom and obtain an observation ρ_2 of $\rho_2(y_2; R)$ from equation $P_2(\lambda) = y_2$.

Step 2 Repeat above 1 for M times to obtain M values of $\rho_2(y_2; R)$.

Step 3 Arrange all $\rho_2(y_2; R)$ in an ascending order as $\rho_2^{[1]}, \rho_2^{[2]}, \ldots, \rho_2^{[M]}$. For arbitrary $0 < \gamma < 1$, a $100(1-\gamma)$ confidence interval of θ can be obtained as $(h_2^{[j]}, h_2^{[j+M-[M\gamma+1]]})$, for $j = 1, 2, \ldots, [M\gamma]$. Then, a $100(1-\gamma)\%$ CGCI of λ is obtained as the j^*th one satisfying

$$\rho_2^{[j^*+M-[M\gamma+1]]} - \rho_2^{[j^*]} = \min_{j=1}^{[M\gamma]} (\rho_2^{[j+M-[M\gamma+1]]} - \rho_2^{[j]}).$$

Remark 5. *Based on the results obtained in Sections 2.1 and 2.2, generalized pivotal quantities for mean μ, $100\gamma\%$ quantile t_γ, reliability function $R(t_0)$ and failure rate function $r(t_0)$ of GIED are constructed. From (1), it is observed that μ, t_γ, $R(t_0)$ and $r(t_0)$ of GIED can be expressed as*

$$\mu = \int_0^\infty [1 - e^{-\lambda/t}]^\theta dt, \ t_\gamma = -\frac{\lambda}{\ln[1 - \gamma^{1/\theta}]},$$
$$R(t_0) = [1 - e^{-\lambda/t_0}]^\theta, \ r(t_0) = \frac{\lambda\theta}{t_0^2} \frac{e^{-\lambda/t_0}}{[1 - e^{-\lambda/t_0}]}.$$

Following similar approach as discussed earlier and using S_1, S_2 and Q_1, generalized pivotal quantities for $\mu, t_\gamma, R(t_0)$ and $r(t_0)$ can be constructed respectively as follows,

$$T_\mu = \int_0^\infty [1 - e^{-\rho_j(Y_j;r)/t}]^{S_j} dt, \quad T_{t_\gamma} = -\frac{\rho_j(Y_j;r)}{\ln[1 - \gamma^{1/S_j}]},$$
$$T_{R(t_0)} = [1 - e^{-\rho_j(Y_j;r)/t_0}]^{S_j} \quad \text{and} \quad T_{r(t_0)} = \frac{h_j(Y_j;r)S_j}{t_0^2} \frac{e^{-\rho_j(Y_j;r)/t_0}}{[1 - e^{-\rho_j(Y_j;r)/t_0}]},$$

for $j = 1$ and 2, respectively. Therefore, the Monte Carlo algorithm could be used to obtain the associated generalized confidence intervals for different reliability indices.

2.3. Approximate Confidence Interval Estimation

In this section, conventional approximate confidence intervals (ACIs) are presented based on the asymptotic theory of the maximum likelihood estimation, and the ACIs for the GIED parameters and reliability indices are constructed by using observed Fisher information matrix.

Let $R = (R_1, R_2, \ldots, R_n)$ be the upper records from the GIED (1), then the log-likelihood function of λ and θ, say $\ell(\lambda, \theta)$, can be expressed as

$$\ell(\lambda, \theta) = n \ln \theta + n \ln \lambda - \lambda \sum_{i=1}^n \frac{1}{r_i} + \theta \ln[1 - e^{-\lambda/r_n}] - \sum_{i=1}^n \ln[1 - e^{-\lambda/r_i}].$$

Therefore, the observed Fisher information matrix of (λ, θ) can be expressed as

$$I(\lambda, \theta) = \left[-\frac{\partial^2 \ell}{\partial\theta_i \partial\theta_j} \right]_{i,j=1,2} = \left[\begin{array}{cc} I_{11} & I_{12} \\ I_{12} & I_{22} \end{array} \right],$$

where the elements of Fisher information matrix is given by

$$I_{11} = -\frac{\partial^2 \ell(\lambda,\theta)}{\partial \lambda^2} = \frac{n}{\lambda^2} + \frac{\theta}{r_n^2}\frac{e^{-\lambda/r_n}}{[1-e^{-\lambda/r_n}]^2} - \sum_{i=1}^{n}\frac{1}{r_i^2}\frac{e^{-\lambda/r_i}}{[1-e^{-\lambda/r_i}]^2},$$
$$I_{21} = I_{12} = -\frac{\partial^2 \ell(\lambda,\theta)}{\partial \lambda \partial \theta} = -\frac{1}{r_n}\frac{e^{-\lambda/r_n}}{1-e^{-\lambda/r_n}} \text{ and } I_{22} = -\frac{\partial^2 \ell(\lambda,\theta)}{\partial \theta^2} = \frac{n}{\theta^2}.$$

For deriving the ACIs for GIED parameters λ and θ, the following asymptotic distribution is presented for MLEs of the parameters.

Theorem 5. *As $n \to \infty$, one has*

$$(\hat{\lambda} - \lambda, \hat{\theta} - \theta) \xrightarrow{d} N(0, I^{-1}(\lambda,\theta)),$$

where $\xrightarrow{d}$ means 'distributed as', $(\hat{\lambda},\hat{\theta})$ is the MLE of (λ,θ) being the solution of following equations

$$\frac{n}{\lambda} - \sum_{i=1}^{n}\frac{1}{r_i} + \frac{\theta}{r_n}\frac{e^{-\lambda/r_n}}{1-e^{-\lambda/r_n}} - \sum_{i=1}^{n}\frac{1}{r_i}\frac{e^{-\lambda/r_i}}{1-e^{-\lambda/r_i}} = 0 \text{ and}$$
$$\theta = -\frac{n}{\ln[1-e^{-\lambda/r_n}]}$$

and

$$I^{-1}(\lambda,\theta) = \frac{1}{I_{11}I_{22} - I_{12}^2}\begin{pmatrix} I_{22} & -I_{12} \\ -I_{12} & I_{11} \end{pmatrix}.$$

Proof. Using the asymptotic properties of MLEs for (λ,θ) under regularity conditions and multivariate central limit theorem, the result can be proved. □

Based on Theorem 5 and replacing (λ,θ) by its MLE $(\hat{\lambda},\hat{\theta})$, for arbitrary $0 < \gamma < 1$, the ACIs for λ and θ can be constructed respectively as

$$\left(\hat{\lambda} \mp z_{\gamma/2}\sqrt{Var(\hat{\lambda})}\right) \text{ and } \left(\hat{\theta} \mp z_{\gamma/2}\sqrt{Var(\hat{\theta})}\right),$$

where z_γ is the upper γ-th quantile of the standard normal distribution and

$$\begin{pmatrix} Var(\hat{\lambda}) & Cov(\hat{\lambda},\hat{\theta}) \\ Cov(\hat{\lambda},\hat{\theta}) & Var(\hat{\theta}) \end{pmatrix} = I^{-1}(\hat{\lambda},\hat{\theta}).$$

Moreover, in order to obtain ACIs for reliability indices $T_\mu, T_{t_\gamma}, T_{R(t_0)}, T_{r(t_0)}$, the following result is needed.

Theorem 6. *Let $K = K(\lambda,\theta)$ be the reliability index for GIED (1). When $n \to \infty$,*

$$(\hat{K} - K) \xrightarrow{d} N(0, V(\lambda,\theta)),$$

where $V(\lambda,\theta) = (\nabla K)^T I^{-1}(\lambda,\theta)(\nabla K)$ and

$$\nabla K = \left(\frac{\partial K}{\partial \lambda}, \frac{\partial K}{\partial \theta}\right)^T = \left(\frac{n}{\lambda} - \sum_{i=1}^{n}\frac{1}{r_i} + \frac{\frac{\theta}{r_n}e^{-\lambda/r_n}}{1-e^{-\lambda/r_n}} - \sum_{i=1}^{n}\frac{\frac{1}{r_i}e^{-\lambda/r_i}}{1-e^{-\lambda/r_i}}, \frac{n}{\theta} + \ln[1-e^{-\lambda/r_n}]\right).$$

Proof. See Appendix F. □

Let $K = K(\lambda,\theta) = T_\mu, T_{t_\gamma}, T_{R(t_0)}, T_{r(t_0)}$ and be the reliability indices of GIED model respectively. Based on Theorem 6 and replacing (λ,θ) by its MLE $(\hat{\lambda},\hat{\theta})$, for arbitrary $0<\gamma<1$, the ACI of K can be constructed as

$$\left(\hat{K} - z_{\gamma/2}\sqrt{\widehat{Var}(\hat{K})}, \hat{K} + z_{\gamma/2}\sqrt{\widehat{Var}(\hat{K})}\right),$$

where

$$\widehat{Var}(\hat{K}) = [\nabla\hat{K}]^T\widehat{Var}(\hat{\lambda},\hat{\theta})[\nabla\hat{K}], \quad \widehat{Var}(\hat{\lambda},\hat{\theta}) = I^{-1}(\hat{\lambda},\hat{\theta}),$$

and

$$\nabla\hat{K} = \left(\frac{\partial K}{\partial\lambda}, \frac{\partial K}{\partial\theta}\right)^T\Big|_{\lambda=\hat{\lambda},\theta=\hat{\theta}}.$$

Sometimes, the ACI obtained by previous procedure may have a negative lower bound. In order to overcome this drawback, the logarithmic transformation and delta methods can be used to obtain the asymptotic normality distribution of $\ln\hat{K}$ as $\frac{\ln\hat{K}-\ln K}{Var(\ln\hat{K})} \to N(0,1)$. Therefore, a $100(1-\gamma)\%$ ACI of K obtained in this manner can be constructed as

$$\left(\frac{\hat{K}}{\exp\left(z_{\gamma/2}\sqrt{\widehat{Var}(\ln\hat{K})}\right)}, \hat{K}\exp\left(z_{\gamma/2}\sqrt{\widehat{Var}(\ln\hat{K})}\right)\right),$$

where $\widehat{Var}(\ln\hat{K}) = \widehat{Var}(\hat{K})/\hat{K}$.

2.4. Prediction Interval of Future Records

Prediction problems have emerged as one of the basic methodologies in many practical studies such as reliability, industrial experiments, agricultural, clinical trials, mortality and survival analysis. This subsection discusses the prediction intervals of record values when first n upper records are available, and associated prediction intervals are obtained based on pivotal and likelihood procedures, respectively.

2.4.1. Pivotal Quantity Based Prediction

Using the same notations in the proof of Theorem 1, besides $Z_1, Z_2, \ldots, Z_n$, we further define

$$\begin{aligned}
Z_{n+1} &= -\theta(\ln[1-e^{-\lambda/R_{n+1}}] - \theta\ln[1-e^{-\lambda/R_n}]), \\
Z_{n+2} &= -\theta(\ln[1-e^{-\lambda/R_{n+2}}] - \theta\ln[1-e^{-\lambda/R_{n+1}}]), \\
&\cdots \\
Z_{n+k} &= -\theta(\ln[1-e^{-\lambda/R_{n+k}}] - \theta\ln[1-e^{-\lambda/R_{n+k-1}}]),
\end{aligned}$$

where $R_{n+1}, R_{n+2}, \ldots, R_{n+k}$ are future next k record values. Therefore, it is seen from Theorem 1 that $Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{n+k}$ are independent and identically distributed (i.i.d.) from the standard exponential distribution with mean 1. Along with notation $W_i = \sum_{j=1}^{i} Z_i = -\theta\ln[1-e^{-\lambda/R_i}], i = 1,2,\ldots,n+k$, one has that $2W_n = Q_1$ follows chi-square distribution with $2n$ degrees of freedom, $2\sum_{i=n+1}^{n+k} Z_i$ is independent of $2W_n$ and chi-square distribution with $2k$ degrees of freedom, and that $W_{n+k} = W_n + \sum_{i=n+1}^{n+k} Z_i$.

Furthermore, we further denote pivotal quantity

$$S = \frac{W_n}{W_{n+k}} = \left[1 + \frac{k}{n}\frac{2\sum_{i=n+1}^{n+k} Z_i/2k}{Q_1/2n}\right]^{-1} = \frac{\ln[1-e^{-\lambda/R_n}]}{\ln[1-e^{-\lambda/R_{n+k}}]}. \tag{3}$$

Using the independent and distribution properties of chi-square quantities Q_1 and $2\sum_{i=n+1}^{n+k} Z_i$, $T = \frac{2\sum_{i=n+1}^{n+k} Z_i/2k}{Q_1/2n}$ has F distribution with $2k$ and $2n$ degrees of freedom of which PDF is given by

$$f_T(t) = \frac{\Gamma(k+n)}{\Gamma(k)\Gamma(n)}\left(\frac{k}{n}\right)^k t^{k-1}\left(1+\frac{k}{n}t\right)^{-(k+n)}.$$

Therefore, one can conduct from theory of sampling distribution that the PDF of pivotal quantity S can be expressed as

$$f_S(s) = \frac{\Gamma(k+n)}{\Gamma(k)\Gamma(n)}s^{n-1}(1-s)^{k-1}, 0 < s < 1,$$

which implies that pivotal quantity S is Beta distributed with n and k degrees of freedom and that R_{n+k} from (3) can be expressed as

$$R_{n+k} = \frac{-\lambda}{\ln(1-[1-e^{-\lambda/R_n}]^{1/S})}.$$

Based on pivotal quantities P_1, P_2 and Q_1, for $j = 1, 2$ two kinds of generalized pivotal quantities can be constructed as

$$S_3 = \frac{-h_j(Y_j;r)}{\ln(1-[1-e^{-h_j(Y_j;r)/R_n}]^{1/S})}. \tag{4}$$

A Monte Carlo simulation Algorithm 5 is provided as follows to obtain the generalized prediction interval of R_{n+k}.

Algorithm 5: Generalized prediction interval for R_{n+k}.

Step 1 Generate a realization $h_j(y_j; R)$ of $h_j(Y_j; R)$ from equation $P_j(\lambda) = y_j$ with y_j being random sample of F distribution with 2 and $2(n-1)$ degrees of freedom or chi-square distribution with $2(n-1)$ degrees of freedom, respectively.

Step 2 Generate random data for S from beta distribution with n and k degrees of freedom and compute S_3 in (4).

Step 3 Repeat Step 1 and 2 for M times to obtain M values of S_3.

Step 4 To obtain the generalized prediction interval of R_{n+k}, arrange all estimates of S_3 in an ascending order as $S_3^{[1]}, S_3^{[2]}, \ldots, S_3^{[M]}$. For arbitrary $0 < \gamma < 1$, a $100(1-\gamma)$ prediction interval of R_{n+k} can be obtained as

$$(S_3^{[j]}, S_3^{[j+M-[M\gamma+1]]}), j = 1, 2, \ldots, [M\gamma].$$

Hence, the $100(1-\gamma)\%$ generalized prediction interval of R_{n+k} can be constructed as the j^*th one satisfying

$$S_3^{[j^*+M-[M\gamma+1]]} - S_3^{[j^*]} = \min_{j=1}^{[M\gamma]}(S_3^{[j+M-[M\gamma+1]]} - S_3^{[j]}).$$

2.4.2. Likelihood Based Plugging Prediction

Now, suppose that the first n record values from a population with CDF $F(\cdot)$ and PDF $f(\cdot)$, and the goal is to predict the kth upper record say $Y = R_k, k > n$. Due to the well-known Markovian property of record statistics, the conditional distribution of Y given

$r = (r_1, r_2, \ldots, r_n)$ is just the distribution of Y given $R_n = r_n$ that can be written as (see Arnold et al. [8])

$$f(y; r_n) = \frac{[H(y) - H(r_n)]^{k-n-1}}{\Gamma(k-n)} \frac{f(y)}{1 - F(r_n)}, r_n < y < \infty,$$

where $H(\cdot) = -\ln[1 - F(\cdot)]$.

For the GIED of (1), the prediction distribution can be expressed as

$$f(y, \lambda, \theta; r_n) = \frac{[\ln[1 - e^{-\lambda/r_n}]^\theta - \ln[1 - e^{-\lambda/y}]^\theta]^{k-n-1}}{\Gamma(k-n)} \frac{\frac{\theta\lambda}{y^2} e^{-\lambda/y} [1 - e^{-\lambda/y}]^{\theta-1}}{[1 - e^{-\lambda/r_n}]^\theta}, r_n < y. \quad (5)$$

Therefore, by substituting $\hat{\lambda}$ and $\hat{\theta}$ into (5), for $0 < \gamma < 1$, a simple $100(1-\gamma)\%$ likelihood based plugging prediction interval (LPI) for R_k can be obtained as (y_L, y_R), where y_L and y_R are the associated prediction bounds being the solutions of following equations

$$\int_{r_n}^{y_L} f(y, \hat{\lambda}, \hat{\theta}; r_n) = \frac{\gamma}{2} \quad \text{and} \quad \int_{y_R}^{\infty} f(y, \hat{\lambda}, \hat{\theta}; r_n) = \frac{\gamma}{2}. \quad (6)$$

3. Numerical Illustration

In this section, simulation studies and real-life examples are presented to assess the performance of the proposed methods.

3.1. Simulation Studies

Monte Carlo simulation study is performed for investigating the effectiveness of the proposed generalized interval estimates in terms of average width (AW) and coverage probability (CP) when upper records data is utilized.

The following Algorithm 6 is presented firstly to generate a group of record values as follows.

Algorithm 6: Generation procedure of record data

Step 1 Generate a group of random samples, namely $Z_1, Z_2, \ldots, Z_n$, from uniform distribution over [0, 1] interval.

Step 2 Make transformation $Y_i = -\ln(1 - Z_i)$, then $Y_i, i = 1, 2, \ldots, n$, are the i.i.d. samples from standard exponential distribution $Exp(1)$ with mean 1.

Step 3 Let $W_i = Y_1 + Y_2 + \ldots + Y_i$, for $1 \le i \le n$. Since the exponential distribution has the lack of memory property, then the sequence of $W_i, i = 1, 2, \ldots, n$, are the record values from standard exponential distribution.

Step 4 Denote $U_i = 1 - e^{-W_i}$, then the sequence $U_i, i = 1, 2, \ldots, n$, are the record values from uniform distribution over $[0, 1]$ interval.

Step 5 For arbitrary continuous CDF $F(\cdot)$, make transformation $R_i = F^{-1}(U_i)$, sequence $R_i, i = 1, 2, \ldots, n$ are the sequence of record values from population $F(\cdot)$, where $F^{-1}(\cdot)$ is the inverse function of $F(\cdot)$.

It should be mentioned that Step 5 will generate the record values, $R_i, i = 1, 2, \ldots, n$, from GIED with with scale parameter $\lambda > 0$ and shape parameter $\theta > 0$ by utilizing F of (1).

Using the above proposed record sampling algorithm, simulation studies are carried out to investigate the performance of different proposed confidence intervals for parameters λ, θ, R_{n+1} and reliability indices $\mu, t_\gamma, R(t_0), r(t_0)$, respectively. The simulation inputs include $n = 4, 5, 6, 7, 9, 10$, the number of upper records, $(\lambda, \theta) = (2, 1.1)$ and $(0.9, 0.5)$. For each combination inputs, the simulation were repeated 10,000 runs. In each simulation run, all confidence intervals with 90% confident level were obtained. Therefore, there are 10,000 confidence intervals for each type and parameter considered. The average width (AW) of the confidence interval for each type and parameter can be calculated. The coverage probability (CP) that is defined as the percentage of 10,000 confidence intervals containing the true input parameter can be evaluated, too. Parts of simulation results were presented in Tables 1 and 2, where the results of the FECI/CECI and FGCI/CGCI for parameter λ are shown in the first and second column of Table 1, respectively.

From the simulated results shown in Tables 1 and 2, it is observed that

1 The AWs of all interval estimates for parameters λ, θ, R_{n+1}, reliability indices $\mu, t_{0.1}, R(2)$, $r(2)$ decrease, and the associated CPs increase when the number, n, of records increases.
2 For fixed n, both FECIs and CECIs of λ have shorter interval lengths than corresponding FGCIs and CGCIs, respectively.
3 For all parameters and reliability indices, the FGCIs feature shortest interval lengths and ACIs have longest interval lengths among all interval estimates under same sample size; the prediction interval obtained under pivotal quantities perform better than the associated LPIs.
4 The CPs of all interval estimates are close to the nominal confidence level.

All confidence intervals were developed based on a high level of confidence, 90%. The simulated confidence interval with small AW and CP closed to 90% will be considered as a good confidence interval. To sum up, it is noted that the proposed pivotal quantities based confidence intervals perform better than conventional likelihood based confidence intervals in terms of AW and CP. Therefore, for parameter λ, the FECI may be an appropriate choice for interval estimation, whereas FGCIs for other parameters and reliability indices are recommended under same significance level which also features more concise expression than the associated chi-square pivotal quantity based intervals.

Table 1. AWs and CPs (within bracket) of interval estimates for GIED parameters.

(λ, θ)	n	F pivotal quantity based confidence intervals			
		λ (FECI)	λ (FGCI)	θ	R_{n+1}
(2, 1.1)	4	6.8527 [0.8791]	7.2971 [0.8821]	4.7835 [0.8741]	5.1010 [0.8921]
	6	5.7231 [0.8824]	6.3849 [0.8847]	3.4128 [0.8798]	3.8327 [0.8942]
	9	1.6693 [0.8938]	2.5714 [0.9011]	1.0519 [0.8893]	1.0061 [0.9016]
(0.9, 0.5)	5	4.5248 [0.8829]	5.6383 [0.8840]	5.3492 [0.8816]	3.8432 [0.8868]
	7	3.5915 [0.8904]	4.3739 [0.8916]	4.8221 [0.8873]	3.1459 [0.8897]
	10	1.6274 [0.9020]	2.4824 [0.8998]	3.5309 [0.8945]	1.4073 [0.8974]
(θ, λ)	n	Chi-squared pivotal quantity based confidence intervals			
		λ (CECI)	λ (CGCI)	θ	R_{n+1}
(2, 1.1)	4	7.3564 [0.8809]	8.4026 [0.8749]	5.2834 [0.8814]	6.2571 [0.8912]
	6	6.4287 [0.8825]	7.1215 [0.8832]	4.6290 [0.8839]	4.5684 [0.8973]
	9	2.8136 [0.9018]	3.3509 [0.8927]	2.1113 [0.8927]	2.2119 [0.9038]
(0.9, 0.5)	5	5.3552 [0.8838]	6.8742 [0.8862]	7.0229 [0.8852]	4.2203 [0.8872]
	7	4.7369 [0.8892]	5.3961 [0.8931]	6.4783 [0.8891]	3.8972 [0.8930]
	10	2.1250 [0.9017]	3.4926 [0.9022]	4.4812 [0.8980]	2.5614 [0.8992]
(θ, λ)	n	Approximate confidence interval and LPI			
		λ	θ	R_{n+1}	
(2, 1.1)	4	9.2738 [0.8753]	6.1283 [0.8837]	6.4769 [0.8929]	
	6	7.9254 [0.8840]	5.3679 [0.8872]	5.1253 [0.8968]	
	9	4.3210 [0.8996]	3.4142 [0.9005]	2.4561 [0.9049]	
(0.9, 0.5)	5	8.2441 [0.8906]	8.9238 [0.8891]	5.0937 [0.8901]	
	7	6.7532 [0.8945]	8.0763 [0.8934]	4.2311 [0.8931]	
	10	3.8728 [0.9028]	5.8345 [0.9019]	2.9148 [0.9006]	

Table 2. AWs and CPs (within bracket) of interval estimates for other indices from GIED.

(θ, λ)	n	F pivotal quantity based confidence intervals			
		μ	$t_{0.1}$	$R(2)$	$r(2)$
(2, 1.1)	4	4.7282 [0.8891]	7.8732 [0.8843]	0.3621 [0.8935]	5.7823 [0.8843]
	6	3.2971 [0.8914]	5.9012 [0.8862]	0.2935 [0.8954]	4.9637 [0.8879]
	9	1.1233 [0.8963]	3.2329 [0.8911]	0.1724 [0.8981]	3.2521 [0.8936]
(0.9, 0.5)	5	3.2986 [0.8903]	5.7925 [0.8855]	0.3151 [0.8920]	3.9319 [0.8865]
	7	2.4975 [0.8915]	4.8301 [0.8871]	0.2616 [0.8948]	3.1743 [0.8890]
	10	1.0821 [0.8982]	2.4219 [0.8913]	0.1543 [0.8996]	1.4176 [0.8948]
(θ, λ)	n	Chi-squared pivotal quantity based confidence intervals			
		μ	$t_{0.1}$	$R(2)$	$r(2)$
(2, 1.1)	4	5.5438 [0.8895]	11.8293 [0.8860]	0.4302 [0.8930]	6.4895 [0.8858]
	6	4.6210 [0.8926]	9.8278 [0.8875]	0.3525 [0.8947]	5.8750 [0.8892]
	9	2.9075 [0.8978]	6.6231 [0.8924]	0.2018 [0.8988]	3.6246 [0.8940]
(0.9, 0.5)	5	3.7128 [0.8911]	7.7823 [0.8862]	0.3529 [0.8929]	5.0121 [0.8863]
	7	2.9190 [0.8933]	6.6327 [0.8881]	0.2914 [0.8954]	3.9237 [0.8904]
	10	1.3013 [0.8978]	3.4996 [0.8933]	0.1698 [0.8989]	2.0632 [0.8952]
(θ, λ)	n	Approximate confidence interval			
		μ	$t_{0.1}$	$R(2)$	$r(2)$
(2, 1.1)	4	6.7291 [0.8886]	16.3527 [0.8848]	0.4827 [0.8944]	8.3823 [0.8857]
	6	5.5325 [0.8930]	13.6154 [0.8881]	0.3851 [0.8965]	6.7319 [0.8901]
	9	3.8092 [0.8998]	8.8231 [0.8932]	0.2530 [0.9006]	3.9072 [0.8964]
(0.9, 0.5)	5	4.5871 [0.8915]	9.2023 [0.8875]	0.3931 [0.8931]	7.2398 [0.8890]
	7	3.6992 [0.8930]	7.8921 [0.8893]	0.3656 [0.8960]	6.1243 [0.8917]
	10	2.1132 [0.9010]	5.8190 [0.8946]	0.2039 [0.9010]	3.1111 [0.8971]

3.2. Real-Life Examples

Example 1 (**Breakdown data of electrical insulating fluid**)**.** *The following real-life data set is taken from Lawless ([22] (p. 3)) and is presented under a logarithm transformation which represents the times to breakdown of electrical insulating fluid subjected to 30 kilovolts.*

$$2.836, 3.120, 3.045, 5.169, 4.934, 4.970, 3.018, 3.770, 5.272, 3.856, 2.046$$

Before progressing further, we first check whether the GIED of (1) can be used or not to analyze these breakdown data of the electrical insulating fluid. By computation, the Kolmogorov–Smirnov (K-S) distance and the corresponding p-value comes out to be 0.2091 and 0.6681 respectively, which suggests that the GIED of (1) fits the transformed breakdown data set well. Further, based on the breakdown data of electrical insulating fluid, a group of record data with sample size 4 can be generated as

$$2.836, \quad 3.120, \quad 5.169, \quad 5.272 \tag{7}$$

Therefore, under confidence level $1 - \alpha = 0.9$ and based on record data (7), the FECIs/FGCIs, CECIs/CGCIs and ACIs for model parameters are provided in Table 3, and associated interval estimates for reliability indices are presented in Table 4 respectively, where the interval lengths are also provided in squared brackets.

Table 3. Interval and prediction interval estimates of parameters using breakdown record data (7) of the electrical insulating fluid.

	FGCI	CGCI	ACI	FECI	CECI
λ	(0.1705, 1.7324)	(1.4462, 22.2995)	(1.5571, 24.5919)	(0.1630, 1.6924)	(1.4444, 21.0982)
	[1.5619]	[20.8533]	[23.0348]	[1.5294]	[19.6538]
θ	(2.8634, 47.4589)	(2.0978, 73.0060)	(0.3412, 156.7204)		
	[44.5955]	[70.9082]	[156.3792]		
R_5	(6.5381, 9.0154)	(5.2819, 8.2041)	(5.3469, 8.4665)		
	[2.4773]	[2.9222]	[3.1195]		

Table 4. Interval estimates of reliability indices using breakdown record data (7) of the electrical insulating fluid.

	μ	$t_{0.1}$	$R(4)$	$r(4)$
FGCI	(0.3994, 2.1898)	(0.3756, 4.1265)	(0.0001, 0.2033)	(0.2764, 1.9773)
	[1.7904]	[3.7509]	[0.2032]	[1.7009]
CGCI	(1.4450, 3.7198)	(2.4699, 6.5671)	(0.0284, 0.2479)	(0.3174, 2.5046)
	[2.2748]	[4.0972]	[0.2195]	[2.1872]
ACI	(2.0411, 4.3501)	(1.0291, 5.6470)	(0.0787, 0.3652)	(0.0074, 23.0982)
	[2.3090]	[4.6179]	[0.2865]	[23.0908]

From the results listed in Table 3, it is seen that the FECI of λ feature shorter interval length than that of the corresponding FGCI, and similar phenomenon also appears between the CECI and CGCI for λ. Meanwhile, one can observe that the F quantity pivotal based confidence interval is superior to the chi-square pivotal based confidence interval for parameter λ in terms of interval length. Moreover, all pivotal quantities based on exact and generalized confidence intervals have better performance than the associated likelihood based ACI of λ. For both parameters θ and prediction record R_5, we note that the proposed FGCIs have shorter interval lengths than those of the CGCIs, respectively. Similarly, for the confidence intervals as shown in Table 4, it is also observed that, for reliability indices $\mu, t_{0.1}, R(4)$ and $r(4)$, both FGCIs and CGCIs appears better performance than the ACIs, while the FGCI appears shortest interval lengths among three interval estimates.

Example 2 (**Lung cancer survival data**)**.** *The survival times in days of 16 lung cancer patients from Lawless ([22] (p. 319)) is presented as follows*

$$6.96,\ 9.30,\ 6.96,\ 7.24,\ 9.30,\ 4.90,\ 8.42,\ 6.05,$$
$$10.18,\ 6.82,\ 8.58,\ 7.77,\ 11.94,\ 11.25,\ 12.94,\ 12.94.$$

We first fit the GIED to the above data set. The K-S distance between the empirical distribution function and the fitted GIED is 0.1445 and the corresponding p value is 0.8650. Therefore, it is clear that the GIED distribution provides a very good fit to the above survival times of lung cancer patients.

Because upper recorded data set is a special ordered observations, we would like to use this type of ordered observations with small sample size to compare all the proposed estimation methods. From the 16 survival times of the lung cancer data, a set of record data with sample size 5 can be generated as

$$6.96,\quad 9.30,\quad 10.18,\quad 11.94,\quad 12.94. \tag{8}$$

Therefore, under same significance level $\gamma = 0.1$, different generalized and approximate intervals for model parameters and reliability indices as well as future one-step prediction interval are obtained as shown in Tables 5 and 6.

Table 5. Interval and prediction interval estimates using the lung cancer survival data (8).

	FGCI	CGCI	ACI	FECI	CECI
λ	(0.3847, 3.9446)	(0.4246, 20.7612)	(9.2335, 70.3682)	(0.3214, 3.7867)	(0.0002, 7.9787)
	[3.5599]	[20.3366]	[61.1347]	[3.4653]	[7.9785]
θ	(1.6131, 9.7976)	(5.2028, 94.3762)	(0.5363, 273.3536)		
	[8.1845]	[89.1734]	[272.8173]		
R_6	(14.8734, 18.5643)	(13.5022, 17.4767)	(12.0061, 16.2382)		
	[3.6909]	[3.9745]	[4.2321]		

Table 6. Interval and prediction interval estimates using the lung cancer survival data (8).

	μ	$t_{0.1}$	$R(4)$	$r(4)$
FGCI	(0.4890, 2.5751)	(0.5505, 4.7870)	(0.0001, 0.0402)	(0.1849, 1.0414)
	[2.0861]	[4.2365]	[0.0401]	[0.8565]
CGCI	(0.5286, 4.3892)	(5.8381, 10.5965)	(0.0249, 0.7472)	(0.1555, 1.1315)
	[3.8606]	[4.7584]	[0.7223]	[0.9760]
ACI	(5.4830, 10.3288)	(7.8962, 12.8412)	(0.2134, 1.0803)	(0.2830, 2.5722)
	[4.8458]	[4.9450]	[0.8669]	[2.2892]

Based on results in Tables 5 and 6, we observe that the exact confidence intervals perform better than the corresponding generalized interval estimates for the parameters λ, and the CGCIs have relatively longer interval lengths than FGCIs, but shorter interval lengths than ACIs for each parameters λ, θ, R_6 and reliability indices $\mu, t_{0.1}, R(4), r(4)$, respectively.

4. Discussions and Remarks

In this paper, inference of interval estimation is discussed for the unknown parameters as well as the reliability indices of generalized inverted exponential distribution. When upper record values are available, different pivotal quantities are proposed and associated exact and generalized confidence intervals are constructed for the model parameters. Moreover, generalized prediction intervals are also proposed for the future record values. Simulation studies and real-life examples show that the performance of the proposed exact and generalized confidence interval estimates provide better performance than conventional likelihood based approximate confidence intervals even in small sample sizes.

Based on the pivotal quantities proposed in this paper, there are several areas in which these methods have potential applications in statistical inference. For instance, an equal-tailed confidence region for parameters (λ, θ) could be constructed from Theorems 1 and 3 as

$$CR_1^{\gamma} = \Bigg\{ (\lambda, \theta) : \rho_1\left(\frac{n-1}{F_{[2,2(n-1)]}^{(1-\sqrt{1-\gamma})/2}} + 1\right) < \lambda < \rho_1\left(\frac{n-1}{F_{[2,2(n-1)]}^{1+\sqrt{1-\gamma}/2}} + 1\right),$$
$$\frac{\chi_{2n}^{1+\sqrt{1-\gamma}/2}}{-2\ln[1-e^{-\lambda/R_n}]} < \theta < \frac{\chi_{2n}^{1-\sqrt{1-\gamma}/2}}{-2\ln[1-e^{-\lambda/R_n}]} \Bigg\}$$

and

$$CR_2^{\gamma} = \left\{ (\lambda, \theta) : \rho_2\big(\chi_{2(n-1)}^{(1+\sqrt{1-\gamma})/2}\big) < \lambda < \rho_2\big(\chi_{2(n-1)}^{1-\sqrt{1-\gamma}/2}\big), \right.$$
$$\left. \frac{\chi_{2n}^{1+\sqrt{1-\gamma}/2}}{-2\ln[1-e^{-\lambda/R_n}]} < \theta < \frac{\chi_{2n}^{1-\sqrt{1-\gamma}/2}}{-2\ln[1-e^{-\lambda/R_n}]} \right\},$$

respectively, which further provide simultaneous confidence band for the GIED CDF $F(t; \lambda, \theta)$ at mission time t and is defined as

$$CDR_i^{\gamma} = \left\{ (t, F(t; \lambda, \theta)) : (\lambda, \theta) \in CR_i^{\gamma} \right\}, i = 1, 2.$$

In this cases, we need to minimize and maximize $F(t; \lambda, \theta)$ with $(\lambda, \theta) \in CR_i^{\gamma}, i = 1, 2$.

Moreover, pivotal inferential procedures proposed here can also be used for other problems. For example, the pivotal quantity approach can be employed to find one-sided confidence limits for reliability indices such as quantile and mean time to failure, among others. The regression model based on the generalized inverted exponential distribution is an interesting work that needs different approaches. It will be a future work.

Author Contributions: Conceptualization, L.W. and H.L.; Methodology, L.W., Y.M.T. and Y.L.; Software, L.W. and H.L.; Funding acquisition, L.W.; Writing—original draft preparation, L.W. and Y.L. All authors have read and agreed to the published version of the manuscript.

Funding: The work of Liang Wang was supported by the National Natural Science Foundation of China (No. 12061091), the Yunnan Fundamental Research Projects (No. 202101AT070103) and the Doctoral Research Foundation of Yunnan Normal University (No. 00800205020503129).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Proof of Theorem 1

Denote $X_i = -\theta \ln[1 - e^{-\lambda/R_i}], i = 1, 2, \ldots, n$; it can be observed that $X_1, X_2, \ldots, X_n$ are common records statistics from the standard exponential distribution with mean 1. According to [23], the exponential distribution has the lack of memory property. Consequently, the differences between successive records will be i.i.d. samples from the standard exponential distribution. Let $Z_1 = X_1$ and $Z_i = X_i - X_{i-1}, i = 1, 2, \ldots, n$; it is observed that $Z_i, i = 1, 2, \ldots, n$ are independent and identically distributed and follows standard exponential distribution with mean 1.

Furthermore, denoting

$$U_1 = 2Z_1 = -2\theta \ln[1 - e^{-\lambda/R_1}]$$

and

$$V_1 = 2\sum_{i=2}^{n} Z_i = 2\theta\{\ln[1 - e^{-\lambda/R_1}] - \ln[1 - e^{-\lambda/R_n}]\},$$

one can observe that quantities U_1 and V_1 follow chi-square distribution with 2 and $2(n-1)$ degrees of freedom respectively and are statistical independent.

Furthermore, let pivotal quantities be

$$P_1 = \frac{U_1/2}{V_1/2(n-1)} = (n-1)\left[\frac{\ln[1-e^{-\lambda/R_n}]}{\ln[1-e^{-\lambda/R_1}]} - 1\right]^{-1}$$

and

$$Q_1 = U_1 + V_1 = -2\theta \ln[1-e^{-\lambda/R_n}].$$

Then P_1 follows F distribution with 2 and $2(n-1)$ degrees of freedom, Q_1 has a chi-square distribution with $2n$ degrees of freedom and P_1 and Q_1 are statistical independent (see [24]).

Appendix B. Proof of Lemma 1

Taking the derivative for $G(t)$ with respect to t, one has that

$$\frac{dG(t)}{dt} = \frac{1}{t}\frac{\ln[1-e^{-t/b}]}{\ln[1-e^{-t/a}]}\left[\frac{t}{b}\frac{e^{-t/b}}{(1-e^{-t/b})\ln[1-e^{-t/b}]} - \frac{t}{a}\frac{e^{-t/a}}{(1-e^{-t/a})\ln[1-e^{-t/a}]}\right].$$

Therefore, showing that $G(t)$ increases in t is equivalent to proving that function $g(y) = \frac{y\ln y}{(1-y)\ln(1-y)}$ with $y = e^{-t} \in (0,1)$ decreases in t.

According Cauchy's mean-value theorem, for arbitrary $0 < y < 1$, there exist y_1 and y_2 with $0 < y_1 < y < y_2 < 1$ satisfying

$$\ln(1-y) - \ln(1-0) = -\frac{y}{1-y_1} \text{ and } \ln 1 - \ln y = \frac{1-y}{y_2},$$

which can be rewritten as

$$\ln(1-y) = -\frac{y}{1-y_1} \quad \text{and} \quad -\ln y = \frac{1-y}{y_2},\ 0 < y_1 < y < y_2 < 1.$$

Taking the derivative of $g(y)$ with respect to y, one has

$$\frac{dg(y)}{dy} = \frac{y\ln y + (\ln y + 1 - y)\ln(1-y)}{(1-y)^2\ln^2(1-y)} = \frac{y(1-y)(y_1-y_2)}{(1-y)^2\ln^2(1-y)y_2(1-y_1)} < 0.$$

Therefore, function $g(y)$ decreases in y and the proof of the first result is done.

Furthermore, by using the L'Hospital's rule, given $0 < a < b$

$$\lim_{t\to 0} G(t) = \lim_{t\to 0}\frac{a}{b}\cdot\frac{1-e^{-t/a}}{1-e^{-t/b}} = 1$$

and

$$\lim_{t\to\infty} G(t) = \lim_{t\to\infty}\frac{a}{b}\cdot\frac{e^{-t/b}}{e^{-t/a}} = \infty.$$

Therefore, the assertion is proved.

Appendix C. Proof of Theorem 2

From Theorem 1, $P_1(\lambda)$ is F distributed with 2 and $2(n-1)$ degrees of freedom. Hence, for arbitrary $0 < \gamma < 1$,

$$P\left(F^{1-\gamma/2}_{[2,2(n-1)]} < P_1(\lambda) < F^{\gamma/2}_{[2,2(n-1)]}\right) = 1-\gamma,$$

which is equivalent to

$$P\left(\frac{n-1}{F^{\gamma/2}_{[2,2(n-1)]}}+1<\frac{\ln[1-e^{-\lambda/R_n}]}{\ln[1-e^{-\lambda/R_1}]}<\frac{n-1}{F^{1-\gamma/2}_{[2,2(n-1)]}}+1\right)=1-\gamma,$$

where $P\left(P_1(\lambda)<F^{1-\gamma/2}_{[2,2(n-1)]}\right)=P\left(P_1(\lambda)>F^{\gamma/2}_{[2,2(n-1)]}\right)=\gamma/2.$

Based on Lemma 1 and Corollary 1, quantity $\frac{\ln[1-e^{-\lambda/R_{n:k}}]}{\ln[1-e^{-\lambda/R_{j:k}}]}$ increases in λ with range $(1,\infty)$, one further has

$$P\left(\rho_1\left(\frac{n-1}{F^{\gamma/2}_{[2,2(n-1)]}}+1\right)<\lambda<\rho_1\left(\frac{n-1}{F^{1-\gamma/2}_{[2,2(n-1)]}}+1\right)\right)=1-\gamma.$$

Therefore, the assertion is proved.

Appendix D. Proof of Theorem 3

Following the same notations as in Theorem 1, it is noted that

$$\begin{aligned}
Z_1 &= -\theta\ln[1-e^{-\lambda/R_1}],\\
Z_2 &= -\theta(\ln[1-e^{-\lambda/R_2}]-\ln[1-e^{-\lambda/R_1}]),\\
&\cdots\\
Z_n &= -\theta(\ln[1-e^{-\lambda/R_n}]-\ln[1-e^{-\lambda/R_{n-1}}])
\end{aligned}$$

are independent and identically distributed standard exponential distribution with mean 1. Let $W_i=\sum_{j=1}^{i}Z_i=-\theta\ln[1-e^{-\lambda/R_i}], i=1,2,\ldots,n$. According to Stephens [25] and Viveros and Balakrishnan [26],

$$\begin{aligned}
U_1 &= \left(\frac{W_1}{W_2}\right)^1=\left(\frac{\ln[1-e^{-\lambda/R_1}]}{\ln[1-e^{-\lambda/R_2}]}\right)^1,\\
U_2 &= \left(\frac{W_2}{W_3}\right)^2=\left(\frac{\ln[1-e^{-\lambda/R_2}]}{\ln[1-e^{-\lambda/R_3}]}\right)^2,\\
&\cdots\\
U_{n-1} &= \left(\frac{W_{n-1}}{W_n}\right)^{n-1}=\left(\frac{\ln[1-e^{-\lambda/R_{n-1}}]}{\ln[1-e^{-\lambda/R_n}]}\right)^{n-1},
\end{aligned}$$

are independent and identically distributed by standard uniform distribution $U(0,1)$ with sample size $n-1$. Moreover, $U_1,U_2,\ldots,U_{n-1}$ are also independent of $W_n=-2\theta\ln[1-e^{-\lambda/R_n}]$.

Furthermore, using the theory of sampling distribution, it is observed directly that

$$P_2=-2\sum_{i=1}^{n-1}\ln(U_i)=-2\sum_{i=1}^{n-1}i\ln\left(\frac{\ln[1-e^{-\lambda/R_i}]}{\ln[1-e^{-\lambda/R_{i+1}}]}\right)=2\sum_{i=1}^{n-1}\ln\left(\frac{\ln[1-e^{-\lambda/R_n}]}{\ln[1-e^{-\lambda/R_i}]}\right)$$

is chi-square distributed with $2(n-1)$ degrees of freedom, which is also independent with

$$Q_1=W_n=-2\theta\ln[1-e^{-\lambda/R_n}].$$

Therefore, the assertion is completed.

Appendix E. Proof of Theorem 4

Since $P_2 = P_2(\lambda)$ is chi-square distributed with $2(n-1)$ degrees of freedom, for arbitrary $0 < \gamma < 1$,

$$P\left(\chi^{1-\gamma/2}_{2(n-1)} < P_2(\lambda) < \chi^{\gamma/2}_{2(n-1)}\right) = 1-\gamma,$$

where $P\left(P_2(\lambda) > \chi^{\gamma/2}_{2(n-1)}\right) = P\left(P_2(\lambda) < \chi^{1-\gamma/2}_{2(n-1)}\right) = \gamma/2$. From Corollary 2, quantity $P_2(\lambda)$ increases in λ with range $(0,\infty)$, thus above expression is equivalent to

$$P\left(\rho_2(\chi^{1-\gamma/2}_{2(n-1)}) < \lambda < \rho_2(\chi^{\gamma/2}_{2(n-1)})\right) = 1-\gamma.$$

Therefore, the proof is done.

Appendix F. Proof of Theorem 6

Let $\alpha = (\lambda, \theta)$, $K(\hat{\alpha})$ can be expressed as

$$K(\hat{\alpha}) = K(\alpha) + [\nabla K(\alpha)]^T(\hat{\alpha}-\alpha) + \frac{1}{2}(\hat{\alpha}-\alpha)^T[\nabla^2 K(\alpha^*)](\hat{\alpha}-\alpha) \quad \text{(A1)}$$

by using Taylor series expansion and the differential mean value theorem, where $\nabla K(\alpha)$ and $\nabla^2 K(\alpha)$ denotes the matrices of the first and second derivatives for $K(\alpha)$ with respect to α, and α^* is some proper value between α and $\hat{\alpha}$. The expression (A1) implies $K(\hat{\alpha}) \to K(\alpha)$ when $n \to \infty$ by using $\hat{\alpha} \to \alpha$ from Theorem 5.

Based on delta method [27], the expression (A1) can be rewritten as

$$K(\hat{\alpha}) - K(\alpha) \approx [\nabla K(\alpha)]^T(\hat{\alpha}-\alpha),$$

Moreover, the variance of $K(\hat{\alpha})$ can be written as

$$\begin{aligned} Var[K(\hat{\alpha})] &\approx Var[[\nabla K(\alpha)]^T\hat{\alpha}] \\ &= [\nabla K(\alpha)]^T Var[\hat{\alpha}][\nabla K(\alpha)] \end{aligned}$$

Therefore, using the central limit theory and Theorem 5, one has

$$K(\hat{\alpha}) - K(\alpha) \to N(0, [\nabla K(\alpha)]^T Var[\hat{\alpha}][\nabla K(\alpha)]),$$

and the assertion is shown.

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