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Global Existence for an Implicit Hybrid Differential Equation of Arbitrary Orders with a Delay

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Abstract: In this paper, we present a qualitative study of an implicit fractional differential equation involving Riemann–Liouville fractional derivative with delay and its corresponding integral equation. Under some sufficient conditions, we establish the global and local existence results for that problem by applying some fixed point theorems. In addition, we have investigated the continuous and integrable solutions for that problem. Moreover, we discuss the continuous dependence of the solution on the delay function and on some data. Finally, further results and particular cases are presented.

Keywords: fractional differential equations; hybrid differential equations; Rothe fixed-point theorem; nonlinear alternative of Leray–Schauder type; Kolmogorov compactness criterion; continuous dependence

MSC: 34A08; 34A38; 34A12

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1. Preliminaries and Introduction

The theory of fractional differential equations has gained a lot of circulation lately. It is of great importance because of its widespread applications in the fields of science and geometry as a mathematical model (see [1–3]).

Recently, a new class of mathematical modelings based on hybrid fractional differential equations with hybrid or non-hybrid boundary value conditions have been investigated in many papers and monographs using different techniques [2–16]. Fractional hybrid differential equations can be used in modeling and describing some non-homogeneous physical phenomena. The importance of investigations into these problems lies in the fact that they include many dynamic systems as special cases [12–14].

Implicit differential and integral equations have gained great attention, for example, Sun et al. [17] have studied a fractional hybrid boundary value problem under mixed Lipschitz and Carathéodory conditions. Benchohra et al. [5] have studied the existence of integrable solutions of an implicit differential equation with infinite delay involving Caputo fractional derivatives. Srivastava et al. [18] have studied the existence of monotonic integrable a.e. solution of nonlinear hybrid implicit functional differential inclusions of arbitrary fractional orders by using the measure of noncompactness technique. El-Sayed et al. [15] have discussed the existence of a solution and continuous dependence of the solution on some data for an implicit hybrid delay functional integral equation (see [4,6,7,18–24]).

Motivated by these results, here, we shall investigate hybrid differential equations of arbitrary order

$$
\begin{cases} D^{\alpha}\left(\frac{x(t)-h_1(t,x(t))}{h_2(t,x(t))}\right) \;=\; f\left(t, \int_0^{\varphi(t)} \; g\left(s, D^{\gamma}(\frac{x(s)-h_1(s,x(s))}{h_2(s,x(s))})\right)ds\right),\ \gamma<\alpha,\ \ t\ \in\ (0,T], \\ \frac{x(t)-h_1(t,x(t))}{h_2(t,x(t))}\Big|_{t=0} \;=\; 0, \end{cases} \tag{1}
$$

and prove the existence of L_1—solutions of this problem where D^{α} refers to the fractional derivative of Riemann–Liouville of order $\alpha \in (0,1)$.

The Riemann–Liouville differential operator is very important in the modeling of many physical phenomena. In addition, we shall study the continuous dependence of the solution on the delay function ϕ. Furthermore, a case when $\alpha = \gamma$ will be studied.

2. Main Results

Let $I = [0, T]$ and the class $E = C(I)$ with supremum norm $||z|| = \sup_{t\in I} |z(t)|$, for any $z \in E$.

Consider the following assumptions

(i) $f, g\ :\ I \times \mathbb{R} \to \mathbb{R}$ satisfy Carathéodory conditions and there exist two bounded measurable functions m_i and $b_i \geq 0,\ i = 1, 2$. Moreover

$$
|\ f(t,x)\ |\ \leq\ m_1(t)\ +\ b_1\ |\ x\ |,\quad |\ g(t,x)\ |\ \leq\ m_2(t)\ +\ b_2\ |\ x\ |,\quad t \in I,\ \ x \in \mathbb{R}.
$$

Let

$$
M = \max\{\sup_{t\in I} m_1(t), \sup_{t\in I} m_2(t)\}.
$$

(ii) $h_i : I \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ satisfy Carathéodory conditions and there exist $m_j \in L_1(I)$ and $b_j \geq 0$, $j = 3, 4$ such that

$$
|\ h_i(t,x)\ |\ \leq\ m_j(t)\ +\ b_j\ |\ x\ |,\quad t \in I,\ \ x \in \mathbb{R}.
$$

(iii) $\varphi : I \to I$ is continuous and monotonic nondecreasing.

(iv) $\frac{b_1\ b_2\ T^{1+\alpha-\gamma}}{\Gamma(1+\alpha-\gamma)}\ <\ 1.$

Taking

$$
y(t) \;=\; \frac{x(t) - h_1(t, x(t))}{h_2(t, x(t))},
$$

then

$$
x(t) \;=\; h_1(t, x(t)) + y(t)\ h_2(t, x(t)),\quad t\ \in\ I, \tag{2}
$$

and

$$
\begin{cases} D^{\alpha}\ y(t) \;=\; f\left(t, \int_0^{\varphi(t)} g(s, D^{\gamma}y(s))\ ds\right),\quad t\ \in\ (0,T], \\ y(0) \;=\; 0. \end{cases} \tag{3}
$$

from (3), we get

$$
y(t) \;=\; I^{\alpha}\ f\left(t, \int_0^{\varphi(t)} g(s, D^{\gamma}y(s))\ ds\right). \tag{4}
$$

Operating by $I^{1-\gamma}$ on both sides of the last equation, then

$$
I^{1-\gamma}\ y(t) \;=\; I^{1+\alpha-\gamma}\ f\left(t, \int_0^{\varphi(t)} g(s, D^{\gamma}y(s))\ ds\right).
$$

Differentiating both sides, we get

$$
\frac{d}{dt} I^{1-\gamma}\ y(t) \;=\; I^{\alpha-\gamma}\ f\left(t, \int_0^{\varphi(t)} g(s, D^{\gamma}y(s))\ ds\right).
$$

Let $z(t) = D^{\gamma} y(t)$, then

$$z(t) \;=\; I^{\alpha-\gamma} f\left(t, \int_0^{\varphi(t)} g(s, z(s))\, ds\right), \quad t \in I. \tag{5}$$

Now, we shall prove the existence of a continuous solution of the integral Equation (5) by applying a nonlinear alternative of Leray–Schauder type [9].

Theorem 1. *Let assumptions (i), (iii) and (iv) hold, then Equation (5) has at least a solution $z \in C(I)$.*

Proof. Define the operator F_1 on Ω by

$$F_1 z(t) \;=\; I^{\alpha-\gamma} f\left(t, \int_0^{\varphi(t)} g(s, z(s))\, ds\right), \quad t \in I,$$

where $\Omega = \{z \in C(I) : ||z|| \le m\}$.

Let

$$A \;=\; \frac{b_1\, b_2\, T^{1+\alpha-\gamma}}{\Gamma(1+\alpha-\gamma)},$$

and

$$B \;=\; \frac{T^{\alpha-\gamma} M}{\Gamma(1+\alpha-\gamma)}\Big(1 + b_1\, T\Big).$$

Then, according to condition (iv), we deduce that $A < 1$. It is also clear that $B > 0$. Take $m = B(1-A)^{-1}$ and suppose that $z \in \partial\Omega, \lambda > 1$ such that $F_1 z = \lambda z$, then

$$\begin{aligned}
\lambda\, m \;&=\; \lambda\, ||z|| \;=\; ||F_1 z|| \;=\; \sup_{t\in I} |(F_1 z)(t)| \\
&\le\; \sup_{t\in I} \left| I^{\alpha-\gamma} f\left(t, \int_0^{\varphi(t)} g(s, z(s))\, ds\right) \right| \\
&\le\; \sup_{t\in I} \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \left| f\left(s, \int_0^{\varphi(s)} g(u, z(u))\, du\right) \right| ds \\
&\le\; \sup_{t\in I} \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \left(m_1(s) + b_1 \int_0^{\varphi(s)} |g(u, z(u))|\, du \right) ds \\
&\le\; \sup_{t\in I} \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \left(m_1(s) + b_1 \int_0^{\varphi(s)} \Big[m_2(u) + b_2 |z(u)|\Big] du \right) ds \\
&\le\; \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \left(M + b_1 \int_0^{T} \Big[M + b_2 |z(u)|\Big] du \right) ds \\
&\le\; \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \left(M + b_1\, T \Big[M + b_2 ||z||\Big] \right) ds \\
&\le\; \frac{T^{\alpha-\gamma}}{\Gamma(1+\alpha-\gamma)} \Big(M + b_1\, T (M + b_2 ||z||) \Big) \\
&\le\; \frac{T^{\alpha-\gamma} M}{\Gamma(1+\alpha-\gamma)} \Big(1 + b_1\, T\Big) + \frac{b_1 b_2 T^{1+\alpha-\gamma}}{\Gamma(1+\alpha-\gamma)} ||z|| \\
&=\; B \,+\, A\, ||z|| \;=\; B \,+\, A\, m.
\end{aligned} \tag{6}$$

Therefore,

$$\lambda \;\le\; \frac{B}{m} \,+\, A \;=\; \frac{B}{B(1-A)^{-1}} \,+\, A \;=\; (1 \,-\, A) \,+\, A \;=\; 1,$$

this contradicts $\lambda > 1$. Therefore, if $F_1 : \overline{\Omega} \to E$ is a completely continuous operator, then it has a fixed point $z \in \overline{\Omega}$.

Now, we shall prove that F_1 is a completely continuous operator. For any $z \in \bar{\Omega}$, let $t_1, t_2 \in I$, $t_1 < t_2$, then we have

$$
\begin{aligned}
|F_1 z(t_2) - F_1 z(t_1)| &= \left| I^{\alpha-\gamma} f\left(t_2, \int_0^{\varphi(t_2)} g(s, z(s))\, ds\right) - I^{\alpha-\gamma} f\left(t_1, \int_0^{\varphi(t_1)} g(s, z(s))\, ds\right) \right| \\
&= \left| \int_0^{t_2} \frac{(t_2 - s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f\left(s, \int_0^{\varphi(s)} g(u, z(u))\, du\right) ds \right. \\
&\qquad \left. - \int_0^{t_1} \frac{(t_1 - s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f\left(s, \int_0^{\varphi(s)} g(u, z(u))\, du\right) ds \right| \\
&\leq \int_0^{t_1} \frac{(t_1 - s)^{\alpha-\gamma-1} - (t_2 - s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \left| f\left(s, \int_0^{\varphi(s)} g(u, z(u))\, du\right) \right| ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \left| f\left(s, \int_0^{\varphi(s)} g(u, z(u))\, du\right) \right| ds \\
&\leq \int_0^{t_1} \frac{(t_1 - s)^{\alpha-\gamma-1} - (t_2 - s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \left(m_1(s) + b_1 \int_0^{\varphi(s)} |g(u, z(u))| du \right) ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \left(m_1(s) + b_1 \int_0^{\varphi(s)} |g(u, z(u))|\, du \right) ds \\
&\leq \int_0^{t_1} \frac{(t_1 - s)^{\alpha-\gamma-1} - (t_2 - s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \left(m_1(s) \right. \\
&\qquad \left. + b_1 \int_0^{\varphi(s)} \Big(m_2(u) + b_2 |z(u)| \Big) du \right) ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \left(m_1(s) + b_1 \int_0^{\varphi(s)} \Big(m_2(u) + b_2 |z(u)| \Big) du \right) ds \\
&\leq \int_0^{t_1} \frac{(t_1 - s)^{\alpha-\gamma-1} - (t_2 - s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \left(M + b_1 \int_0^{T} \Big(M + b_2 |z(u)| \Big) du \right) ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \left(M + b_1 \int_0^{T} \Big(M + b_2 |z(u)| \Big) du \right) ds \\
&\leq \int_0^{t_1} \frac{(t_1 - s)^{\alpha-\gamma-1} - (t_2 - s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \Big(M + b_1\, T\, M + b_1 b_2 T ||z|| \Big) ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \Big(M + b_1\, T\, M + b_1 b_2 T ||z|| \Big) ds \\
&\leq \frac{2(t_2 - t_1)^{\alpha-\gamma} + t_2^{\alpha-\gamma} - t_1^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \Big(M + b_1\, T\, M + b_1 b_2 T m \Big).
\end{aligned}
$$

The above inequality shows that

$$
|\ F_1 z(t_2)\ -\ F_1 z(t_1)\ | \ \to\ 0 \ \text{ as } \ t_2\ \to\ t_1, \tag{7}
$$

then $F_1 z$ is uniformly continuous in I, and hence $F_1 : \overline{\Omega} \to E$ is well-defined. We deduce from (6) and (7) that the family $F_1 z$ is uniformly bounded and equicontinuous, thus the Arzela–Ascoli Theorem [8] guarantees that $F_1 : \overline{\Omega} \to E$ is compact operator, which completes the proof. □

Consequently, we can deduce an existence result for Equation (4).
Since $||z|| \leq m$ and $z(t) = D^{\gamma}y(t) \Rightarrow y(t) = I^{\gamma}z(t)$, we have $||y|| \leq \frac{mT^{\gamma}}{\Gamma(\gamma+1)}$.

Corollary 1. *Suppose that assumptions of Theorem 1 hold, then there exists a solution* $y \in C(I)$ *for Equation (4) which satisfies* $||y|| \leq k$, *where* $k = \frac{mT^{\gamma}}{\Gamma(\gamma+1)}$.

Proof. From $z(t) = D^{\gamma}y(t)$, we get $y(t) = I^{\gamma}z(t)$, and

$$
\begin{aligned}
|y(t)| &\leq \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}|z(s)|ds \\
&\leq m\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}ds \\
&\leq \frac{mT^{\gamma}}{\Gamma(\gamma+1)}.
\end{aligned}
$$

□

Now, we shall investigate the existence of integrable solution x for the quadratic integral Equation (2).

Let $B_r = \left\{x \in L_1(I) : ||x||_{L_1} \leq r,\ r > 0\right\}$.

Theorem 2. *Let the assumptions of Corollary 2 be satisfied in addition to assumption* (ii). *If* $b_3 + k\, b_4 < 1$, *then Equation (2) has a solution* $x \in L_1(I)$.

Proof. Let x be an arbitrary element in B_r. The operator F_2 is given by

$$F_2\, x(t) = h_1(t, x(t)) + y(t)\, h_2(t, x(t)). \tag{8}$$

Then from the assumptions (ii), we have

$$
\begin{aligned}
||F_2\, x||_{L_1} &= \int_0^T |\, F_2\, x(t)\,|\, dt \\
&= \int_0^T |\, h_1(t, x(t)) + y(t)\, h_2(t, x(t))\,|\, dt \\
&\leq \int_0^T \Big(m_3(t) + b_3|x(t)|\Big)\, dt + \int_0^T y(t)\Big(m_4(t) + b_4|x(t)|\Big)\, dt \\
&\leq ||m_3||_{L_1} + b_3\, ||x||_{L_1} + k(||m_4||_{L_1} + b_4\, ||x||_{L_1}).
\end{aligned}
$$

The last estimate shows that the operator F_2 maps $L_1(I)$ into $L_1(I)$. Next, for $x \in \partial B_r$, so, $||x||_{L_1} = r$, then

$$||F_2\, x||_{L_1} \leq ||m_3||_{L_1} + b_3\, r + k(||m_4||_{L_1} + b_4\, r).$$

Then $F_2(\partial B_r) \subset \bar{B}_r$ (closure of B_r) if

$$||F_2 x||_{L_1} \leq ||m_3||_{L_1} + b_3\, r + k(||m_4||_{L_1} + b_4\, r) \leq r.$$

Therefore

$$r \leq \frac{||m_3||_{L_1} + k||m_4||_{L_1}}{1 - (b_3 + k\, b_4)}.$$

Using inequality $b_3 + k\, b_4 < 1$, then we deduce that $r > 0$.
From assumption (ii) we have that $F_2 : B_r \to L_1(I)$ is continuous.

In what follows, we show that F_2 is compact, and to reach this purpose we will apply Kolmogorov compactness criterion [10]. So, let $\mathbf{\Omega}$ be a bounded subset of B_r. Then, $F_2(\mathbf{\Omega})$ is bounded in $L_1(I)$, i.e., condition (i) of Kolmogorov compactness criterion [10] is verified. Now, we prove that $(F_2x)_h \to F_2x \in L_1(I)$ as $h \to 0$, uniformly with respect to $F_2x \in F_2\,\mathbf{\Omega}$. Then

$$\begin{aligned}
||(F_2x)_h \,-\, F_2x||_{L_1} &= \int_0^T |(F_2x)_h(t) \,-\, (F_2x)(t)|\,dt \\
&= \int_0^T |\frac{1}{h}\int_t^{t+h} (F_2x)(s)\,ds \,-\, (F_2x)(t)|\,dt \\
&\leq \int_0^T \left(\frac{1}{h}\int_t^{t+h} |(F_2x)(s) \,-(F_2x)(t)|\,ds \right) dt \\
&\leq \int_0^T \frac{1}{h}\int_t^{t+h} \Big| \Big(h_1(s,x(s)) + y(s)\,h_2(s,x(s)) \Big) \\
&\qquad\qquad - \Big(h_1(t,x(t)) + y(t)\,h_2(t,x(t)) \Big) \Big|\,ds\,dt.
\end{aligned}$$

Since $h_i \in L_1(I), i = 1,2$, we have (see [25])

$$\frac{1}{h}\int_t^{t+h} \Big| \Big(h_1(s,x(s)) + y(s)\,h_2(s,x(s)) \Big) \,-\, \Big(h_1(t,x(t)) + y(t)\,h_2(t,x(t)) \Big) \Big|\,ds \,\to\, 0,$$

for a.e. $t \in (0,T]$. Therefore, $F_2(\mathbf{\Omega})$ is relatively compact, i.e., F_2 is a compact operator.

Hence, applying Rothe fixed-point Theorem [9] implies that F_2 has a fixed point. This completes the proof. □

Next, in order to have a global solution for the quadratic integral equation of fractional order, we have the following result.

Theorem 3. *Let the assumptions of Theorem 2 be satisfied in addition to the following assumption:*

(v) *Assume that every solution $x \in L_1(I)$ of the equation*

$$x(t) = \eta\Big(h_1(t,x(t)) + y(t)\,h_2(t,x(t)) \Big) \;\; a.e. \; on \;\; (0,T], \; \eta \in (0,1)$$

satisfies $||x||_{L_1} \neq r$ (r is fixed and arbitrary).

Then, Equation (2) has a solution $x \in L_1(I)$.

Proof. Let x be an arbitrary element in the open set $\mathbb{B}_r = \{x : ||x||_{L_1} < r, r > 0\}$. Then, from the assumption (ii), we have

$$||F_2x||_{L_1} \;\leq\; ||m_3||_{L_1} \;+\; b_3\,||x||_{L_1} + k(||m_4||_{L_1} \;+\; b_4\,||x||_{L_1}).$$

The above inequality means that the operator F_2 maps $\mathbb{B}_r$ into $L_1(I)$. Moreover, as a consequence of Theorem 2. we get that F_2 maps $\mathbb{B}_r$ continuously into $L_1(I)$ and F_2 is compact.

Then, in the view of assumption (v), F_2 has a fixed point. This completes the proof. □

3. Continuous Dependence of the Solution

In order to study the continuous dependence of the solution on some data, we assume the following assumptions:

(ii^*) $\Big|h_i(t,u_1) - h_i(t,u_2)\Big| \leq l_i\,|u_1 - u_2|,\ i=1,2\ \forall\, u_1,\, u_2 \in \mathbb{R}$ and $t \in I$.

(vi) $\Big|f(t,u_1) - f(t,u_2)\Big| \leq b_1'\,|u_1 - u_2|,\ \forall\, u_1, u_2 \in \mathbb{R}$ and $t \in I$.

(vii) $\Big|g(t,u_1) - g(t,u_2)\Big| \leq b_2'\,|u_1 - u_2|,\ \forall\, u_1, u_2 \in \mathbb{R}$ and $t \in I$.

Theorem 4. *Let the assumptions of Theorem 1 be satisfied with replacing condition (i) by (vi) and (vii). If* $\frac{b_1' b_2' T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} < 1$*, then the functional integral Equation (5) has a unique solution.*

Proof. Let $z_1,\ z_2$ be solutions of Equation (5), then

$$\begin{aligned}
|z_1(t) - z_2(t)| &= \left| I^{\alpha-\gamma} f\left(t, \int_0^{\varphi(t)} g(s,z_1(s))ds\right) - I^{\alpha-\gamma} f\left(t, \int_0^{\varphi(t)} g(s,z_2(s))ds\right)\right| \\
&\leq I^{\alpha-\gamma}\left| f\left(t, \int_0^{\varphi(t)} g(s,z_1(s))ds\right) - f\left(t, \int_0^{\varphi(t)} g(s,z_2(s))ds\right)\right| \\
&\leq b_1'\ I^{\alpha-\gamma}\left| \int_0^{\varphi(t)} g(s,z_1(s))ds - \int_0^{\varphi(t)} g(s,z_2(s))ds\right| \\
&\leq b_1'\ I^{\alpha-\gamma} \int_0^t \Big| g(s,z_1(s)) - g(s,z_2(s))\Big|\, ds \\
&\leq b_1'\ b_2'\ I^{\alpha-\gamma} \int_0^t |z_1(s) - z_2(s)|\, ds \\
&\leq b_1'\ b_2' \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} |z_1(s) - z_2(s)|\, ds \\
&\leq b_1'\ b_2'\ ||z_1 - z_2|| \frac{T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \\
&\Rightarrow \left(1 - b_1' b_2' \frac{T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\right) ||z_1 - z_2|| \leq 0.
\end{aligned}$$

Since $\frac{b_1' b_2' T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} < 1$, we have $z_1 = z_2$. Hence the solution of the problem (5) is unique. Similarly, we can prove a uniqueness result for Equation (4). Hence for (2), we have the following Theorem □

Theorem 5. *Let the assumptions of Theorems 2 and 4 be satisfied with replacing condition (ii) by (ii^*) equipped with $\big(l_1 + k\, l_2\big) < 1$. Then, the solution $x \in L_1(I)$ of the functional Equation (2) is unique.*

Proof. Firstly, Theorem 2 proved that the functional integral Equation (2) has at least one solution.

Now, let $x_1, x_2 \in L_1(I)$ be two solutions of (2). Then, for $t \in I$, we have

$$\begin{aligned}
|x_1(t) - x_2(t)| &= |h_1(t,x_1(t)) + y(t)h_2(t,x_1(t)) - h_1(t,x_2(t)) - y(t)h_2(t,x_2(t))| \\
&\leq \big|h_1(t,x_1(t)) - h_1(t,x_2(t))\big| + |y(t)||h_2(t,x_1(t)) - h_2(t,x_2(t))| \\
&\leq l_1|x_1(t) - x_2(t)| + |y(t)|\, l_2|x_1(t) - x_2(t)|.
\end{aligned}$$

Then, for $t \in I$, and $|y(t)| < k$, we get

$$\begin{aligned}\|x_1 - x_2\| &\leq l_1 \int_0^T |x_1(t) - x_2(t)|\, dt + k\, l_2 \int_0^T |x_1(t) - x_2(t)|\ dt \\ &\leq l_1 \|x_1 - x_2\| + k\, l_2 \|x_1 - x_2\| \ \leq (l_1 + k\, l_2) \|x_1 - x_2\|.\end{aligned}$$

Hence

$$\big[1 - (l_1 + k\, l_2)\big] \|x_1 - x_2\| \leq 0,$$

and then the solution of (2) is unique. □

Now, we are in position to state an existence result for the uniqueness of the solution for the hybrid implicit functional differential Equation (3).

Theorem 6. *Let the assumptions of Theorems 3 and 4 be satisfied. Then the solution $x \in L_1(I)$ of the implicit hybrid delay functional differential Equation (3) is unique.*

Theorem 7. *Suppose that assumptions (iii)–(iv) of Theorem 1 are satisfied in addition to (vi) and (vii). If $\frac{b_1' b_2' T^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+1)} < 1$, then the solution z of Equation (5) depends continuously on the delay function φ.*

Proof. Let $\forall\, \varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, we shall show that

$$|\varphi(t) - \varphi^*(t)| \leq \delta \ \Rightarrow \ \|z - z^*\| \leq \epsilon,$$

where

$$z^*(t) = I^{\alpha-\gamma} f\left(t, \int_0^{\varphi^*(t)} g(s, z^*(s)) ds\right).$$

Now

$$\begin{aligned}|z(t) - z^*(t)| \ &= \ \left| I^{\alpha-\gamma} f\left(t, \int_0^{\varphi(t)} g(s, z(s)) ds\right) - I^{\alpha-\gamma} f\left(t, \int_0^{\varphi^*(t)} g(s, z^*(s)) ds\right) \right| \\ &\leq \ I^{\alpha-\gamma} \left| f\left(t, \int_0^{\varphi(t)} g(s, z(s)) ds\right) - f\left(t, \int_0^{\varphi^*(t)} g(s, z^*(s)) ds\right) \right| \\ &= \ I^{\alpha-\gamma} \left| f\left(t, \int_0^{\varphi(t)} g(s, z(s)) ds\right) - f\left(t, \int_0^{\varphi^*(t)} g(s, z^*(s)) ds\right) \right. \\ &- \ \left. f\left(t, \int_0^{\varphi(t)} g(s, z^*(s)) ds\right) + f\left(t, \int_0^{\varphi(t)} g(s, z^*(s)) ds\right) \right| \\ &\leq \ I^{\alpha-\gamma} \left[\, \left| f\left(t, \int_0^{\varphi(t)} g(s, z(s)) ds\right) - f\left(t, \int_0^{\varphi(t)} g(s, z^*(s)) ds\right) \right| \right.\end{aligned}$$

$$
\begin{aligned}
&+ \quad \left| f\left(t, \int_0^{\varphi(t)} g(s, z^*(s))ds\right) - f\left(t, \int_0^{\varphi^*(t)} g(s, z^*(s))ds\right) \right| \Bigg] \\
&\leq \quad b_1' \, I^{\alpha-\gamma} \Bigg[\bigg| \int_0^{\varphi(t)} g(s, z(s))ds - \int_0^{\varphi(t)} g(s, z^*(s))ds \bigg| \\
&+ \quad \bigg| \int_0^{\varphi(t)} g(s, z^*(s))ds - \int_0^{\varphi^*(t)} g(s, z^*(s))ds \bigg| \Bigg] \\
&\leq \quad b_1' \, I^{\alpha-\gamma} \Bigg[\int_0^{\varphi(t)} |g(s, z(s)) - g(s, z^*(s))| \, ds + \int_{\varphi^*(t)}^{\varphi(t)} |g(s, z^*(s))| ds \Bigg] \\
&\leq \quad b_1' \, I^{\alpha-\gamma} \Bigg[\int_0^{\varphi(t)} |g(s, z(s)) - g(s, z^*(s))| \, ds + \varepsilon_1 \Bigg] \\
&\leq \quad b_1' \, b_2' \, I^{\alpha-\gamma} \Bigg[\int_0^T |z(s) - z^*(s)| \, ds + \varepsilon_1 \Bigg] \\
&\leq \quad b_1' \, b_2' \, I^{\alpha-\gamma} \Bigg[T \, ||z - z^*|| + \varepsilon_1 \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
||z - z^*|| \quad &\leq \quad \frac{b_1' b_2' \, T^{\alpha-\gamma+1}}{\Gamma(\alpha - \gamma + 1)} \, ||z - z^*|| + \frac{b_1' \, b_2' \, T^{\alpha-\gamma} \varepsilon_1}{\Gamma(\alpha - \gamma + 1)}, \\
||z - z^*|| \quad &\leq \quad \frac{b_1' \, b_2' \, T^{\alpha-\gamma} \varepsilon_1}{\Gamma(\alpha - \gamma + 1)} \left(1 - \frac{b_1' b_2' \, T^{\alpha-\gamma+1}}{\Gamma(\alpha - \gamma + 1)} \right)^{-1}
\end{aligned}
$$

Since $\frac{b_1' b_2' T^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+1)} < 1$, we obtain $||z - z^*|| \leq \varepsilon$. □

Corollary 2. *Since z depends continuously on the delay function φ, then y depends continuously on the delay function φ.*

Theorem 8. *Suppose that the conditions of Theorem 5 are satisfied, then the solution x of Equation (2) depends continuously on φ.*

Proof. Let $\forall \, \varepsilon' > 0$, there exists $\delta'(\varepsilon') > 0$, such that $|y - y^*| < \delta'$. Now

$$
\begin{aligned}
|x - x^*| \quad &= \quad \Big| h_1(t, x(t)) + y(t) \, h_2(t, x(t)) - h_1(t, x^*(t)) - y^*(t) \, h_2(t, x^*(t)) \Big| \\
&= \quad \Big| h_1(t, x(t)) + y(t) \, h_2(t, x(t)) - h_1(t, x^*(t)) - y^*(t) \, h_2(t, x^*(t)) \\
&\qquad\qquad + y^*(t) \, h_2(t, x(t)) - y^*(t) \, h_2(t, x(t)) \Big| \\
&\leq \quad |h_1(t, x(t)) - h_1(t, x^*(t))| + |h_2(t, x(t))| \, |y(t) - y^*(t)| \\
&\qquad\qquad + |y^*(t)| \, |h_2(t, x(t)) - h_2(t, x^*(t))| \\
&\leq \quad l_1 \, |x - x^*| + (h_2(t, 0) + l_2 |x|) \, |y(t) - y^*(t)| + k \, l_2 \, |x(t) - x^*(t)|,
\end{aligned}
$$

then

$$||x-x^*||_{L_1} \leq l_1\,||x-x^*||_{L_1} + (||h_2(t,0)||_{L_1} + l_2||x||_{L_1})\,||y-y^*|| + k\,l_2\,||x-x^*||_{L_1},$$

$$\begin{aligned} ||x-x^*||_{L_1} &\leq \frac{(||h_2(t,0)||_{L_1} + l_2||x||_{L_1})}{(1-l_1-k\,l_2)}\,||y-y^*|| \\ &\leq \frac{(||h_2(t,0)||_{L_1} + l_2||x||_{L_1})}{(1-l_1-k\,l_2)}\,\delta' = \epsilon'. \end{aligned}$$

From Corollary 2, we get the result. □

Remark 1. *By direct calculations as above we can prove that the solution* $z \in C(I)$ *of Equation (5) depends continuously on the function* f_2 *and thus* $x \in L_1(I)$ *of the Equation (3) depends continuously on the function* g.

4. Some Remarks and Particular Cases

Remark 2. *As a particular case of our results when* $\gamma = 1-\alpha$, *we can deduce the existence of at least one solution for the problem of conjugate orders*

$$\begin{cases} D^\alpha\left(\frac{x(t)-h_1(t,x(t))}{h_2(t,x(t))}\right) = f\left(t, \int_0^{\varphi(t)} g\left(s, D^{1-\alpha}\left(\frac{x(s)-h_1(s,x(s))}{h_2(s,x(s))}\right)\right)ds\right),\ \alpha \in (\frac{1}{2},1),\ \ t \in (0,T], \\ \left.\frac{x(t)-h_1(t,x(t))}{h_2(t,x(t))}\right|_{t=0} = 0. \end{cases}$$

Remark 3. *As a particular case of our results when* $f(t,x(t)) = a(t) + x(t)$, *we can deduce the existence of at least one solution for the following problem*

$$\begin{cases} D^\alpha\left(\frac{x(t)-h_1(t,x(t))}{h_2(t,x(t))}\right) = a(t) + \int_0^{\varphi(t)} g\left(s, D^{\gamma}\left(\frac{x(s)-h_1(s,x(s))}{h_2(s,x(s))}\right)\right)ds,\ \gamma < \alpha,\ \ t \in (0,T], \\ \left.\frac{x(t)-h_1(t,x(t))}{h_2(t,x(t))}\right|_{t=0} = 0. \end{cases}$$

where $a \in L_1(I)$.

Existence Results of the Problem (3) when $\alpha = \gamma$

In this section, we consider the hybrid differential equation

$$\begin{cases} D^\alpha\left(\frac{x(t)-h_1(t,x(t))}{h_2(t,x(t))}\right) = f\left(t, \int_0^{\varphi(t)} g\left(s, D^{\alpha}\left(\frac{x(s)-h_1(s,x(s))}{h_2(s,x(s))}\right)\right)ds\right),\ \ t \in (0,T], \\ \left.\frac{x(t)-h_1(t,x(t))}{h_2(t,x(t))}\right|_{t=0} = 0. \end{cases} \tag{9}$$

By putting $y(t) = \frac{x(t)-h_1(t,x(t))}{h_2(t,x(t))}$, then problem (9) has the form

$$\begin{cases} D^\alpha\, y(t) = f\left(t, \int_0^{\varphi(t)} g(s, D^\alpha y(s))\, ds\right),\ \ t \in (0,T], \\ y(0) = 0. \end{cases} \tag{10}$$

Let $w(t) = D^\alpha y(t)$, then

$$w(t) = f\left(t, \int_0^{\varphi(t)} g(s, w(s))\, ds\right),\ \ t \in I. \tag{11}$$

Now, consider this assumption:

(i^*) $f,\ g : I \times \mathbb{R} \to \mathbb{R}$ are continuous functions and satisfy conditions (vi) and (vii).
to prove the existence of a continuous solution of the integral Equation (11).

Theorem 9. *Let the assumptions (i^*) be satisfied. If $b_1' b_2' T < 1$, then the functional Equation (11) has a unique solution $w \in C(I)$.*

Proof. Define the operator $\mathcal{F}_1$ on Ω by

$$\mathcal{F}_1 w(t) \ = \ f\left(t, \int_0^{\varphi(t)} g(s, w(s))\, ds\right), \quad t \ \in \ I,$$

where $\Omega = \{w \in C(I) : ||w|| \le m^*\}$, $m^* = \frac{T \sup_{t\in I} |g(s,0)| + \sup_{t\in I} |f(t,0)|}{1 - b_1' b_2' T}$.

$$\begin{aligned}
\lambda\, ||w|| \ = \ ||\mathcal{F}_1 w|| \quad &= \quad \sup_{t\in I} |(F_1 w)(t)| \\
&\le \ \sup_{t\in I} \left| f\left(t, \int_0^{\varphi(t)} g(s, w(s))\, ds\right) \right| \\
&\le \ \sup_{t\in I} \left| f\left(t, \int_0^{\varphi(t)} g(s, w(s))\, ds\right) - f(t, 0) \right| + \sup_{t\in I} |f(t,0)| \\
&\le \ b_1' \sup_{t\in I} \left| \int_0^{\varphi(t)} g(s, w(s))\, ds \right| + \sup_{t\in I} |f(t,0)| \\
&\le \ b_1' \sup_{t\in I} \int_0^{\varphi(t)} |g(s, w(s)) - g(s, 0)|\, ds \ + b_1'\, T \sup_{t\in I} |g(s,0)| + \sup_{t\in I} |f(t,0)| \\
&\le \ b_1'\, b_2'\, T\, ||w|| \ + b_1'\, T \sup_{t\in I} |g(s,0)| + \sup_{t\in I} |f(t,0)| \\
&\le \ b_1'\, b_2'\, T\, m^* \ + b_1'\, T \sup_{t\in I} |g(s,0)| + \sup_{t\in I} |f(t,0)|.
\end{aligned} \tag{12}$$

In view of assumption (i^*), we show that $\mathcal{F}_1$ is a continuous operator.
For any $w, w' \in \Omega$, then we have

$$\begin{aligned}
|\mathcal{F}_1 w(t) - \mathcal{F}_1 w'(t)| \quad &= \quad \left| f\left(t, \int_0^{\varphi(t)} g(s, w(s))\, ds\right) - f\left(t, \int_0^{\varphi(t)} g(s, w'(s))\, ds\right) \right| \\
&\le \ b_1' \left| \int_0^{\varphi(t)} g(s, w(s))\, ds - \int_0^{\varphi(t)} g(s, w'(s))\, ds \right| \\
&\le \ b_1' \int_0^{\varphi(t)} |g(s, w(s)) - g(s, w'(s))|\, ds \\
&\le \ b_1' \int_0^{T} |g(s, w(s)) - g(s, w'(s))|\, ds \\
&\le \ b_1'\, b_2' \int_0^{T} |w(s) - w'(s)|\, ds \\
&\le \ b_1'\, b_2'\, T \sup_{t\in I} |w(s) - w'(s)| \\
&\le \ b_1'\, b_2'\, T\, ||w - w'||.
\end{aligned}$$

The above inequality shows that

$$||\mathcal{F}_1 z - \mathcal{F}_1 u|| \ \le \ ||w - w'||, \ \forall\, w,\ w' \in \Omega,$$

then $\mathcal{F}_1 : \Omega \to \Omega$ is a contraction, and hence $\mathcal{F}_1$ has a unique fixed point in Ω, which completes the proof. $\square$

5. Conclusions

Here, we have studied some qualitative results for a hybrid implicit differential equation of arbitrary order (3) involving a Riemann–Liouville fractional derivative (in case $\gamma < \alpha$) with a nonlocal initial condition. The Rothe fixed-point Theorem, Nonlinear alternative of Leray–Schauder type and Kolmogorov compactness criterion have been used with the aim of proving the main results. Next, we proved the existence of the global solution of that problem. Furthermore, we have established the continuous dependence of our solution on the delay function and on other functions. Finally, we considered the problem (3) when $\alpha = \gamma$, which cannot be a special case of the problem (3) because of the insufficiently of the assumption (i). So, f, g have been assumed to satisfy Lipchitz conditions. Thus, the solvability of (3) has been discussed for all $\gamma \leq \alpha$.

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