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Hyers–Ulam–Rassias Stability of Hermite's Differential Equation

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Abstract: In this paper, we studied the Hyers–Ulam–Rassias stability of Hermite's differential equation, using Pachpatte's inequality. We compared our results with those obtained by Blaga et al. Our estimation for $|z(x) - y(x)|$, where z is an approximate solution and y is an exact solution of Hermite's equation, was better than that obtained by the authors previously mentioned, in some parts of the domain, especially in a neighborhood of the origin.

Keywords: Hermite's differential equation; Hyers–Ulam–Rassias stability; Pachpatte's inequality

MSC: 34D20; 34D10; 34K20

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1. Introduction

The importance of studying Hermite's equation results from the multiple applications that the equation has in theoretical physics, electrical engineering, economics, etc. This equation has the form:

$$y''(x) - 2xy'(x) + \lambda y(x) = 0, x \in \mathbb{R},\ y \in C^2(\mathbb{R}, \mathbb{R}),\ \lambda \in \mathbb{R}.$$

If $\lambda = 2n,\ n \in \mathbb{N}$, Hermite's equation becomes:

$$y''(x) - 2xy'(x) + 2ny(x) = 0, x \in \mathbb{R},\ y \in C^2(\mathbb{R}, \mathbb{R}),\ n \in \mathbb{N}, \tag{1}$$

and admits particular polynomial solutions H_n, called Hermite's polynomials (see [1]), given by:

$$H_n(x) = (-1)^n e^{x^2} \left(e^{-x^2}\right)^{(n)},\ x \in \mathbb{R}.$$

In what follows, we study the Hyers–Ulam–Rassias and generalized Hyers–Ulam–Rassias stability of Equation (1), using Pachpatte's inequality. The problem was also studied in [2] by Blaga et al., as an application of a result regarding the Ulam stability of a linear system of differential equations with nonconstant coefficients. We compared our results with those obtained in [2]. Pachpatte's next inequality is used in the paper.

Theorem 1 (Pachpatte [3], p. 39, Theorem 1.7.4.). *Let u, f, g be nonnegative continuous functions defined on $\mathbb{R}_+$, and h be a positive and nondecreasing continuous function defined on $\mathbb{R}_+$, for which the inequality:*

$$u(x) \leq h(x) + \int_0^x f(t)u(t)dt + \int_0^x f(t)\left[\int_0^t g(s)u(s)ds\right]dt,$$

holds for $x \in \mathbb{R}_+$. *Then:*

$$u(x) \le h(x)\left(1 + \int_0^x f(t)e^{\int_0^t (f(s)+g(s))ds} dt\right),$$

for $x \in \mathbb{R}_+$.

We recall that Ulam stability was proposed by Ulam in [4]. Hyers [5] established the first result in this direction. Obloza [6] established the first result regarding the Ulam stability of differential equations and also investigated the relation between Lyapunov and Ulam stability. After that, the domain has developed rapidly. The Hyers–Ulam stability of linear differential equations of first order was studied for example in [7–12], of higher order in [13], linear difference equations in [14], and others in [15,16]. More on this topic can be found in the books [17,18]. In [19], linear differential equations of second order of the form:

$$y'' + \alpha y' + \beta y = 0 \text{ and } y'' + \alpha y' + \beta y = f(x),$$

$y \in C^2[a,b], f \in C[a,b], -\infty < a < b < \infty$, were studied.

In [20], the equation:

$$y'' + p(x)y' + q(x)y + r(x) = 0,$$

$y \in C^2(a,b), f \in C[a,b], p, q, r \in C(a,b), -\infty < a < b < \infty$, was investigated.

Various other types of differential equations of second order have been further studied, for example in [21], the equation:

$$y''(x) + (1 + \psi(x))y(x) = 0,\ \psi \in C^1[0,\infty), y \in C^2[0,\infty),$$

in [22,23], the equation:

$$y''(x) + \beta(x)y(x) = 0,\ \beta \in C[a,b], y \in C^2[a,b],$$

in [22] with boundary conditions, in [23] with initial conditions, and in [24], the following equation:

$$y''(x) + \mu^2 y(x) = 0,\ y \in C^2[a,b], -\infty < a < b < \infty,$$

here using the Mahgoub transform.

In [25], the generalized Hyers–Ulam stability of the equation:

$$y'' + p(x)y' + q(x)y = f(x),\ y \in C^2(I),\ p, q, f \in C[I],\ I \subset \mathbb{R},$$

was studied, if there exists a solution $y_1 \in C^2(I)$ of the equation:

$$y'' + p(x)y' + q(x)y = 0.$$

As an application, some remarks regarding generalized the Hyers–Ulam–Rassias stability of Hermite's equation are given. In [26], the equation:

$$y''(x) + f(x, y(x)) = 0, \text{ if } |f(x, y(x))| \le A|y|^\alpha, \text{ and } f(x, 0) = 0,$$

was investigated, using the Gronwall lemma. In [27], the equations:

$$y''(x) + f(x, y(x)) = 0,$$

and:

$$y''(x) + f(x, y(x), y'(x)) = 0,$$

$y \in C^2(I), f \in C(I),\ I \subset \mathbb{R}$, were studied, using the Bihari lemma. The results obtained in [27] generalize those established in [26].

The outline of the paper is the following: In Section 2, we present the stability notions and prove several auxiliary results useful in the following sections (Remarks 1–4). The first main result (Theorem 2) is given in the next section and concerns the Hyers–Ulam–Rassias stability of Hermite's Equation (2). The second main result (Theorem 3), regarding the generalized Hyers–Ulam–Rassias stability of Hermite's Equation (2), is proven in Section 4. A graphical representation illustrating the difference between our result in Theorem 2 and that established by Blaga et al. in [2] is shown in Figure 1.

2. Preliminary Notions and Results

In what follows, we consider $n \in \mathbb{N}, I = [0, n+1], \alpha \in \mathbb{R}, \beta \in \mathbb{R}$, the equation:

$$y''(x) - 2xy'(x) + 2ny(x) = 0, x \in I,\ y \in C^2(I, \mathbb{R}), \tag{2}$$

and the conditions:

$$\begin{cases} y(0) = \alpha \\ y'(0) = \beta \end{cases}. \tag{3}$$

Let $\varepsilon > 0$. Let $\varphi \in C^2(I, \mathbb{R}_+)$. We also consider the inequalities:

$$\left|z''(x) - 2xz'(x) + 2nz(x)\right| \leq \varepsilon, x \in I,\ z \in C^2(I, \mathbb{R}), \tag{4}$$

and:

$$\left|z''(x) - 2xz'(x) + 2nz(x)\right| \leq \varepsilon\varphi(x), x \in I,\ z \in C^2(I, \mathbb{R}). \tag{5}$$

Definition 1. *Equation (2) is called Hyers–Ulam–Rassias stable if there is a real number $c > 0$ and a function $\psi : I \to [0, \infty)$ so that for any solution z of the inequality (4), satisfying (3), there is a solution y of Equation (2), satisfying (3), such that:*

$$|z(x) - y(x)| \leq c \cdot \varepsilon \cdot \psi(x),\ \forall x \in I.$$

Definition 2. *Equation (2) is called generalized Hyers–Ulam–Rassias stable if there is a real number $c > 0$ and a function $\psi : [a, b] \to [0, \infty)$ so that for any solution z of the inequality (5), satisfying (3), there is a solution y of Equation (2), satisfying (3), such that:*

$$|z(x) - y(x)| \leq c \cdot \varepsilon \cdot \psi(x),\ \forall x \in I.$$

In Definitions 1 and 2, z is called an approximate solution and y is called an exact solution of Hermite's Equation (2).

Remark 1. *A function $z \in C^2(I, \mathbb{R})$ is a solution of (4) if and only if there exists a function $g \in C^2(I, \mathbb{R})$ such that:*

1. $|g(x)| \leq \varepsilon, \forall x \in I;$
2. $z''(x) = 2xz'(x) - 2nz(x) + g(x), \forall x \in I.$

Remark 2. *If a function $z \in C^2(I, \mathbb{R})$ is a solution of (4), then z is a solution of the inequality:*

$$\left|z(x) - z(0) - z'(0)x - 2\int_0^x tz(t)dt + 2(n+1)\int_0^x \left[\int_0^t z(s)ds\right]dt\right| \leq \varepsilon\frac{x^2}{2}, \forall x \in I.$$

Indeed, integrating $z''(x) = 2xz'(x) - 2nz(x) + g(x)$, from zero to x, we have:

$$z'(x) - z'(0) = \int_0^x 2sz'(s)ds - 2n\int_0^x z(s)ds + \int_0^x g(s)ds;$$

hence:

$$z'(x) - z'(0) = 2xz(x) - \int_0^x 2z(s)ds - 2n\int_0^x z(s)ds + \int_0^x g(s)ds,$$

and:

$$z'(x) - z'(0) = 2xz(x) - 2(n+1)\int_0^x z(s)ds + \int_0^x g(s)ds.$$

Integrating again from zero to x, we have:

$$z(x) - z(0) - z'(0)x = 2\int_0^x tz(t)dt - 2(n+1)\int_0^x \left[\int_0^t z(s)ds\right]dt + \int_0^x \left[\int_0^t g(s)ds\right]dt.$$

Hence, we obtain:

$$\left|z(x) - z(0) - z'(0)x - 2\int_0^x tz(t)dt + 2(n+1)\int_0^x \left[\int_0^t z(s)ds\right]dt\right|$$
$$\leq \int_0^x \left[\int_0^t |g(s)|ds\right]dt \leq \varepsilon\frac{x^2}{2}.$$

Remark 3. *A function $z \in C^2(I,\mathbb{R})$ is a solution of (5) if and only if there exists a function $g \in C^2(I,\mathbb{R})$ such that:*

1. $|g(x)| \leq \varepsilon\varphi(x), \forall x \in I;$
2. $z''(x) = 2xz'(x) - 2nz(x) + g(x), \forall x \in I,$

where $\varphi \in C^2(I,\mathbb{R}_+)$.

Analogous to Remark 2, we can prove the following result.

Remark 4. *If a function $z \in C^2(I,\mathbb{R})$ is a solution of (5), then z is a solution of the inequality:*

$$\left|z(x) - z(0) - z'(0)x - 2\int_0^x tz(t)dt + 2(n+1)\int_0^x \left[\int_0^t z(s)ds\right]dt\right| \leq \varepsilon\int_0^x \left[\int_0^t \varphi(s)ds\right]dt,$$

$\forall x \in I.$

3. Hyers–Ulam–Rassias Stability of Hermite's Equation

We present below a result regarding the Hyers–Ulam–Rassias stability of Hermite's Equation (2), with the conditions (3). We also make a comparison between our results and those set out in [2].

Theorem 2. *Let $n \in \mathbb{N}, I = [0, n+1], \alpha \in \mathbb{R}, \beta \in \mathbb{R}$. Then:*

1. *Equation (2) has a unique solution with:*

$$\begin{cases} y(0) = \alpha \\ y'(0) = \beta \end{cases};$$

2. *Hermite's Equation (2), with the conditions (3), is Hyers–Ulam–Rassias stable.*

Proof. 1. It is a well-known result from differential equations of order two, since the coefficients of Equation (2), $p(x) = -2x, q(x) = 2n$, are continuous functions on I (see [28]);

2. Let z be a solution of (4) and y be the solution of (2) such that:

$$\begin{cases} z(0) = y(0) = \alpha \\ z'(0) = y'(0) = \beta \end{cases}.$$

From Remark 2, we have:

$$|z(x)-y(x)| \le \left|z(x)-z(0)-z'(0)x-2\int_0^x tz(t)dt+2(n+1)\int_0^x\left[\int_0^t z(s)ds\right]dt\right|$$
$$+2\int_0^x t|z(t)-y(t)|dt+2(n+1)\int_0^x\left[\int_0^t |z(t)-y(t)|ds\right]dt$$
$$\le \varepsilon\frac{x^2}{2}+2\int_0^x t|z(t)-y(t)|dt+2(n+1)\int_0^x\left[\int_0^t |z(t)-y(t)|ds\right]dt.$$

Since $t \le x \le n+1$, we have:

$$|z(x)-y(x)| \le \varepsilon\frac{x^2}{2}+2\int_0^x t|z(t)-y(t)|dt+2(n+1)\int_0^x\left[\int_0^t |z(t)-y(t)|ds\right]dt$$
$$\le \varepsilon\frac{x^2}{2}+\int_0^x 2(n+1)|z(t)-y(t)|dt+\int_0^x 2(n+1)\left[\int_0^t |z(t)-y(t)|ds\right]dt.$$

Using now Pachpatte's inequality from Theorem 1, we obtain:

$$|z(x)-y(x)| \le \varepsilon\frac{x^2}{2}\left(1+\int_0^x 2(n+1)e^{\int_0^t(2n+3)ds}dt\right)=\varepsilon\frac{x^2}{2}\left(1+\int_0^x 2(n+1)e^{(2n+3)t}dt\right)$$
$$=\varepsilon\frac{x^2}{2}\left(1+\frac{2n+2}{2n+3}\left(e^{(2n+3)x}-1\right)\right)$$
$$=\varepsilon\frac{x^2}{2}\left(\frac{1}{2n+3}+\frac{2n+2}{2n+3}e^{(2n+3)x}\right).$$

□

In the following, using the graphical representation from Figure 1, we make a comparison between the estimation $|z(x)-y(x)|$ obtained by Blaga et al. in [2] and that obtained by us in Theorem 2. For $n=0$ we have $I=[0,1]$, and we compare the functions $f_1(x)=\sqrt{\pi}e^{x^2+x}$, which appeared in [2], and $f_2(x)=x^2\left(\frac{1}{3}+\frac{2}{3}e^{3x}\right)$, which appears in Theorem 2. In Figure 1 we graphically represent both functions. It can be seen that in some parts of the domain, our estimation was better than the one obtained in [2], especially in a neighborhood of the origin.

For other values of n, we have similar results.

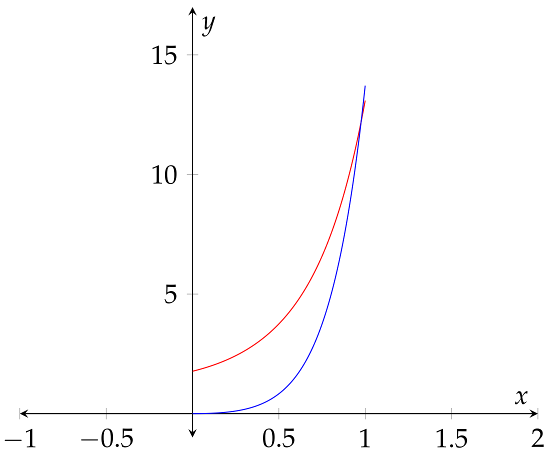

Figure 1. Representation of the curves $f_1(x)=\sqrt{\pi}e^{x^2+x}$ (red color), $f_2(x)=x^2(\frac{1}{3}+\frac{2}{3}e^{3x})$ (blue color), on $[0,1]$, together.

4. Generalized Hyers–Ulam–Rassias Stability of Hermite's Equation

We establish now a result regarding the generalized Hyers–Ulam–Rassias stability of Hermite's Equation (2), with the conditions (3).

Theorem 3. *Let $n \in \mathbb{N}, I = [0, n+1], \alpha \in \mathbb{R}, \beta \in \mathbb{R}$. If:*

(i) $\exists\, \lambda_\varphi > 0$ *such that* $\int_0^x \left[\int_0^t \varphi(s)ds\right]dt \le \lambda_\varphi \varphi(x), \forall x \in I$;

(ii) $\varphi \in C^2([0,\infty), \mathbb{R}_+)$ *is increasing;*

Then Hermite's Equation (2), with the conditions (3), is generalized Hyers–Ulam–Rassias stable.

Proof. Let z be a solution of (5) and y be the solution of (2) such that:

$$\begin{cases} z(0) = y(0) = \alpha \\ z'(0) = y'(0) = \beta \end{cases}.$$

From Remark 4, we have:

$$\begin{aligned} |z(x) - y(x)| &\le \left| z(x) - z(0) - z'(0)x - 2\int_0^x tz(t)dt + 2(n+1)\int_0^x \left[\int_0^t z(s)ds\right]dt \right| \\ &+ 2\int_0^x t|z(t) - y(t)|dt + 2(n+1)\int_0^x \left[\int_0^t |z(t) - y(t)|ds\right]dt \\ &\le \varepsilon \int_0^x \left[\int_0^t \varphi(s)ds\right]dt + 2\int_0^x t|z(t)-y(t)|dt + 2(n+1)\int_0^x \left[\int_0^t |z(t)-y(t)|ds\right]dt. \end{aligned}$$

Since $t \le x \le n+1$, we have:

$$\begin{aligned} |z(x) - y(x)| &\le \varepsilon\lambda_\varphi \varphi(x) + 2\int_0^x t|z(t)-y(t)|dt + 2(n+1)\int_0^x \left[\int_0^t |z(t)-y(t)|ds\right]dt \\ &\le \varepsilon\lambda_\varphi \varphi(x) + \int_0^x 2(n+1)|z(t)-y(t)|dt + \int_0^x 2(n+1)\left[\int_0^t |z(t)-y(t)|ds\right]dt. \end{aligned}$$

Using now Pachpatte's inequality from Theorem 1, we obtain:

$$\begin{aligned} |z(x) - y(x)| &\le \varepsilon\lambda_\varphi \varphi(x)\left(1 + \int_0^x 2(n+1)e^{\int_0^t (2n+3)ds}dt\right) = \varepsilon\frac{x^2}{2}\left(1 + \int_0^x 2(n+1)e^{(2n+3)t}dt\right) \\ &= \varepsilon\lambda_\varphi \varphi(x)\left(1 + \frac{2n+2}{2n+3}\left(e^{(2n+3)x} - 1\right)\right) \\ &= \varepsilon\lambda_\varphi \varphi(x)\left(\frac{1}{2n+3} + \frac{2n+2}{2n+3}e^{(2n+3)x}\right). \end{aligned}$$

□

5. Conclusions

In this paper, we studied Hyers–Ulam–Rassias and generalized Hyers–Ulam–Rassias stability of Hermite's equation, using a new approach, namely Pachpatte's inequality. The problem was also considered in [2] by Blaga et al., as a particular case of a general result regarding the Ulam stability of a linear system of differential equations with nonconstant coefficients. We remark, using the graphical representation from Figure 1, that our estimation for $|z(x) - y(x)|$, between an approximate solution and the exact solution of Equation (2), in some parts of the domain, was better than the one obtained in [2], especially in the neighborhood of the origin.

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