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Bootstrap Tests for the Location Parameter under the Skew-Normal Population with Unknown Scale Parameter and Skewness Parameter

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Abstract: In this paper, the inference on location parameter for the skew-normal population is considered when the scale parameter and skewness parameter are unknown. Firstly, the Bootstrap test statistics and Bootstrap confidence intervals for location parameter of single population are constructed based on the methods of moment estimation and maximum likelihood estimation, respectively. Secondly, the Behrens-Fisher type and interval estimation problems of two skew-normal populations are discussed. Thirdly, by the Monte Carlo simulation, the proposed Bootstrap approaches provide the satisfactory performances under the senses of Type I error probability and power in most cases regardless of the moment estimator or ML estimator. Further, the Bootstrap test based on the moment estimator is better than that based on the ML estimator in most situations. Finally, the above approaches are applied to the real data examples of leaf area index, carbon fibers' strength and red blood cell count in athletes to verify the reasonableness and effectiveness of the proposed approaches.

Keywords: skew-normal population; location parameter; moment estimation; maximum likelihood estimation; bootstrap

MSC: 62F25; 62F40

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1. Introduction

In many practical problems, the real-data distribution tends to be skew with unimodal and asymmetrical characteristics such as dental plaque index data [1], freeway speed data [2] and polarizer manufacturing process data [3]. For this reason, Azzalini [4,5] proposed the concept of the skew-normal distribution originally and gave its density function expression to characterize it. The random variable X follows a skew-normal distribution with location parameter $\xi \in R$, scale parameter $\eta^2 \in R^+$ and skewness parameter $\lambda \in R$, denoted by $X \sim SN(\xi, \eta^2, \lambda)$, if its density function is:

$$f(x;\xi,\eta^2,\lambda)=2\phi(x;\xi,\eta^2)\Phi[\lambda\eta^{-1}(x-\xi)], \quad (1)$$

where $\phi(x;\xi,\eta^2)$ is the normal probability density function with mean ξ and variance η^2, and $\Phi(\cdot)$ is the standard normal cumulative distribution function. When $\lambda = 0$, Equation (1) degenerates into the normal distribution with mean ξ and variance η^2.

In view of the wide applications of the skew-normal distribution, many scholars further explored its statistical properties. Some recent studies include: characterizations of distribution [6,7], characteristic functions [8], sampling distributions [9], distribution of quadratic forms [10–12], measures of skewness and divergence [13,14], asymptotic expansions for moments of the extremes [15], rates of convergence of the extremes [16], exact density of the sum of independent random variables [17], identifiability of finite mixtures of the skew-normal distributions [18], etc. On this basis, we can use the skew-normal distribution as the fitted distribution of real data and establish a statistical model to solve the practical problem. Some recent applications include: modelling of air pollution data [19], modelling of psychiatric measures [20], modelling of bounded health scores [21], modelling of insurance claims [22], asset pricing [23], individual loss reserving [24], robust portfolio estimation [25], growth estimates of cardinalfish [26], age-specific fertility rates [27], reliability studies [28], statistical process control [29], analysis of student satisfaction towards university courses [30], detecting differential expression to microRNA data [31], etc.

Due to the complex structure of the skew-normal distribution, the traditional parameter estimation method is difficult to be applied directly. To this end, Pewsey [32] studied the weaknesses of the direct parameterization in parameter estimation and proposed the centered parameterization method. Pewsey [33,34] applied this method to the wrapped skew-normal population and gave the methods of moment estimation and maximum likelihood (ML) estimation. Arellano-Valle and Azzalini [35] extended the centered parameterization method to the multivariate skew-normal distribution and studied its information matrix. Further, due to the wide application of location parameter in econometrics, medicinal chemistry and life testing, the research on the location parameter of skew-normal distribution has attracted many scholars' attention. For example, Wang et al. [36] discussed the interval estimation of location parameter when the coefficient of variation and skewness parameter are known. Thiuthad and Pal [37] considered the hypothesis testing problem of location parameter and constructed three testing statistics when the scale parameter and skewness parameter are known. Ma et al. [38] studied the interval estimation and hypothesis testing problems of location parameter with known scale parameter and skewness parameter. Based on the approximate likelihood equations, Gui and Guo [39] derived the explicit estimators of scale parameter and location parameter. But in practical applications, inferences on the location parameter with unknown scale parameter and skewness parameter is by no means an exception but a fact of life. For this, the statistical inference problems of location parameter for single and two skew-normal populations are researched when the scale parameter and skewness parameter are unknown.

This paper is organized as follows. In Section 2, for single skew-normal population, the centered parameterization and Bootstrap approaches are used for the hypothesis testing and interval estimation problems of location parameter with unknown scale parameter and skewness parameter. In Section 3, for two skew-normal populations, the Behrens-Fisher type and interval estimation problems of location parameters are discussed when the scale parameters and skewness parameters are unknown. In Section 4, the Monte Carlo simulation results of the above approaches are presented, which are compared from analytical perspective at the same time. In Section 5, the proposed approaches are applied to the real data examples of leaf area index (LAI), carbon fibers' strength and red blood cell (RBC) count in athletes. In Section 6, the summary of this paper is given.

2. Inference on the Location Parameter of Single Skew-Normal Population

In this section, the estimation problem of unknown parameters for single skew-normal population is considered firstly. Suppose that $X_1, \ldots, X_n$ are random samples from the skew-normal distribution $X \sim SN(\xi, \eta^2, \lambda)$ and all the samples are mutually independent. Let $(\bar{X}, S_2, S_3)$ denote the sample mean, the second and third central moments of the sample, respectively. Namely:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad S_2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \quad S_3 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^3. \tag{2}$$

Theorem 1. *Let* $\delta = \lambda/(1+\lambda^2)^{1/2}$. *If* $X \sim SN(\xi, \eta^2, \lambda)$, *then the moment estimators of* (ξ, η^2, λ) *are:*

$$\hat{\xi} = \bar{X} - cS_3^{1/3}, \hat{\eta}^2 = S_2 + c^2 S_3^{2/3}, \hat{\lambda} = \hat{\delta}/(1-\hat{\delta}^2)^{1/2}, \tag{3}$$

where $b = (2/\pi)^{1/2}, c = [2/(4-\pi)]^{1/3}, \hat{\delta} = cS_3^{1/3}/b(S_2 + c^2 S_3^{2/3})^{1/2}$.

Proof. Let $(\bar{x}, s_2, s_3)$ be the observed values of $(\bar{X}, S_2, S_3)$. $Y_1, \ldots, Y_n$ are the standardized samples where $Y_i = (X_i - \bar{x})/\sqrt{s_2}$ from $Y \sim SN(\xi_s, \eta_s^2, \lambda)$, $i = 1, \ldots, n$. Note that:

$$\xi_s = (\xi - \bar{x})/\sqrt{s_2}, \eta_s = \eta/\sqrt{s_2}. \tag{4}$$

The moment generating function (MGF) of Y is

$$M_Y(t) = E(\exp(tY)) = 2\exp\left(t\xi_s + \frac{t^2\eta_s^2}{2}\right)\Phi(t\eta_s\delta). \tag{5}$$

By Equation (5), we have:

$$\begin{array}{l} M_Y'(t)|_{t=0} = \xi_s + b\eta_s\delta = 0 \\ M_Y''(t)|_{t=0} = \xi_s^2 + 2b\xi_s\eta_s\delta + \eta_s^2 = 1 \\ M_Y'''(t)|_{t=0} = \xi_s^3 + 3b\xi_s^2\eta_s\delta + 3\xi_s\eta_s^2 + 3b\eta_s^3\delta - b\eta_s^3\delta^3 = s_2^{-3/2}s_3. \end{array} \tag{6}$$

According to Equations (4) and (6), the moment estimates of (ξ, η^2, λ) can be expressed as:

$$\hat{\xi}^* = \bar{x} - cs_3^{1/3}, \hat{\eta}^{*2} = s_2 + c^2 s_3^{2/3}, \hat{\lambda}^* = \frac{\hat{\delta}^*}{\sqrt{1 - \hat{\delta}^{*2}}},$$

where $\hat{\delta}^* = cs_3^{1/3}/b(s_2 + c^2 s_3^{2/3})^{1/2}$. By Equation (3), the moment estimators of (ξ, η^2, λ) are given, then the proof of Theorem 1 is completed. □

Further, the ML estimators of (ξ, η^2, λ) are considered. Pewsey [32] proved that the results of using numerical techniques to maximize the log-likelihood for direct parameters (ξ, η^2, λ), may be highly misleading as no unique solution exists in this case. For this, we derive the ML estimators of the unknown parameters based on the method of centered parametrization by References [4,32,34,35,40]. Firstly, we give the following definition.

Definition 1. *Suppose $X \sim SN(\xi, \eta^2, \lambda)$. Let $Z = \frac{X-\xi}{\eta} \sim SN(\lambda)$, then:*

$$X_C = \mu + \sigma\left(\frac{Z - E(Z)}{\sqrt{\mathrm{var}(Z)}}\right) \sim SN_C(\mu, \sigma^2, \gamma),$$

where $SN_C(\mu, \sigma^2, \gamma)$ denotes the skew-normal distribution with mean $\mu \in R$, variance $\sigma^2 \in R^+$ and skewness coefficient γ.

The centered parameterization removes the singularity of the expected Fisher information matrix at $\lambda = 0$. Furthermore, the components of centered parameters are less correlated than those of direct parameters. By Definition 1, the relationship between the direct parameters (ξ, η^2, λ) and centered ones (μ, σ^2, γ) is as follows (see [34]).

$$\xi = \mu - c\gamma^{1/3}\sigma\ , \eta^2 = \sigma^2(1 + c^2\gamma^{2/3}), \lambda = \frac{c\gamma^{1/3}}{\sqrt{b^2 + c^2(b^2 - 1)\gamma^{2/3}}}\ . \tag{7}$$

$X_{C1}, \ldots, X_{Cn}$ are assumed to be random samples from the skew-normal distribution $X_C \sim SN_C(\mu, \sigma^2, \gamma)$. The sample mean, the second and third central moments of the sample can be written respectively as:

$$\bar{X}_C = \frac{1}{n}\sum_{i=1}^{n} X_{Ci}, S_{C2} = \frac{1}{n}\sum_{i=1}^{n}(X_{Ci} - \bar{X}_C)^2, S_{C3} = \frac{1}{n}\sum_{i=1}^{n}(X_{Ci} - \bar{X}_C)^3.$$

Theorem 2. *Suppose $X \sim SN(\xi, \eta^2, \lambda)$. Let $Z = \frac{X-\xi}{\eta}$ and $X_C = \mu + \sigma\left(\frac{Z-E(Z)}{\sqrt{\mathrm{var}(Z)}}\right)$, then $X_C = X$.*

Proof. The first three derivatives of MGF $M_X(t)$ of X can be obtained as:

$$
\begin{aligned}
E(X) &= M_X^{'}(t)|_{t=0} = \xi + b\eta\delta,\\
E(X^2) &= M_X^{''}(t)|_{t=0} = \xi^2 + 2b\xi\eta\delta + \eta^2,\\
E(X^3) &= M_X^{'''}(t)|_{t=0} = \xi^3 + 3b\xi^2\eta\delta + 3\xi\eta^2 + 3b\eta^3\delta - b\eta^3\delta^3.
\end{aligned}
$$

By the above three equations, the skewness coefficient γ of X has the form of:

$$\gamma = \frac{E[X - E(X)]^3}{\left\{E[X - E(X)]^2\right\}^{3/2}} = \frac{b^3\delta^3}{c^3(1 - b^2\delta^2)^{3/2}}. \tag{8}$$

From Equations (7) and (8), we have:

$$\sigma = \frac{\eta}{\sqrt{1 + c^2\gamma^{2/3}}} = \eta\sqrt{1 - b^2\delta^2}, \mu = \xi + c\gamma^{1/3}\sigma = \xi + b\eta\delta. \tag{9}$$

Because $Z \sim SN(\lambda)$, we obtain that $E(Z) = b\delta$ and $Var(Z) = 1 - b^2\delta^2$. Then,

$$X_C = \mu + \sigma\left(\frac{Z - E(Z)}{\sqrt{\mathrm{var}(Z)}}\right) = \xi + b\eta\delta + \eta\sqrt{1 - b^2\delta^2}\left(\frac{\frac{X-\xi}{\eta} - b\delta}{\sqrt{1 - b^2\delta^2}}\right) = X.$$

Hence, the proof of Theorem 2 is completed. □

Remark 1. *By Theorem 1, if $|\lambda| \to \infty$, then $|\delta| \to 1$. Furthermore, we have $\gamma \in (-0.99527, 0.99527)$ by (8). More details see Pewsey [32].*

Besides, we consider the ML estimators of the centered parameters (μ, σ^2, γ). The observed values of $(\bar{X}_C, S_{C2}, S_{C3})$ are denoted by $(\bar{x}_C, s_{C2}, s_{C3})$. Similarly, let $X_{si} = (X_{Ci} - \bar{x}_C)/\sqrt{s_{C2}}$, $i = 1, \ldots, n$, where $X_{s1}, \ldots, X_{sn}$ are the standardized samples from $X_s \sim SN_C(\mu_s, \sigma_s^2, \gamma)$ with $\mu_s = (\mu - \bar{x}_C)/\sqrt{s_{C2}}$ and $\sigma_s = \sigma/\sqrt{s_{C2}}$. So the density function of X_s is:

$$
\begin{aligned}
f\left(x_s; \mu_s, \sigma_s^2, \gamma\right) = &\frac{2}{\sigma_s\sqrt{s_{C2}(1 + c^2\gamma^{2/3})}}\phi\left[\left(\frac{x_s - \mu_s}{\sigma_s} + c\gamma^{1/3}\right)\frac{1}{\sqrt{1 + c^2\gamma^{2/3}}}\right]\\
&\times \Phi\left\{\left(\frac{x_s - \mu_s}{\sigma_s} + c\gamma^{1/3}\right)\frac{c\gamma^{1/3}}{\sqrt{(1 + c^2\gamma^{2/3})\left[b^2 + c^2\gamma^{2/3}(b^2 - 1)\right]}}\right\}.
\end{aligned} \tag{10}
$$

By Equation (10), the logarithmic likelihood function (without constant terms) of $X_{s1}, \ldots, X_{sn}$ can be represented as:

$$
\begin{aligned}
l\left(x_{s1}, \cdots, x_{sn}; \mu_s, \sigma_s^2, \gamma\right) = &-n\log\sigma_s - \frac{n}{2}\log\left(1 + c^2\gamma^{2/3}\right)\\
&+ \sum_{i=1}^{n}\log\phi\left[\left(\frac{x_{si} - \mu_s}{\sigma_s} + c\gamma^{1/3}\right)\frac{1}{(1 + c^2\gamma^{2/3})^{1/2}}\right]\\
&+ \sum_{i=1}^{n}\log\Phi\left\{\frac{\frac{(x_{si} - \mu_s)c\gamma^{1/3}}{\sigma_s} + c^2\gamma^{2/3}}{(1 + c^2\gamma^{2/3})^{1/2}\left[b^2 + c^2\gamma^{2/3}(b^2 - 1)\right]^{1/2}}\right\}.
\end{aligned} \tag{11}
$$

In addition, the ML estimators of ξ and η^2 satisfy the constraint (see [4]):

$$\eta^2 = \sum_{i=1}^{n}(X_i - \xi)^2/n. \tag{12}$$

By Theorem 2, η^2 in Equation (12) can also be expressed as:

$$\eta^2 = \sum_{i=1}^{n}(X_{Ci} - \xi)^2/n. \tag{13}$$

From Equations (7) and (13), the ML estimators $(\tilde{\sigma}_s, \tilde{\mu}_s^2, \tilde{\gamma})$ of $(\sigma_s, \mu_s^2, \gamma)$ have the following relationship:

$$\sigma_s^2 = \left(\sqrt{1 + \mu_s^2 (1 + c^2 \gamma^{2/3})} - c\mu_s \gamma^{1/3} \right)^2. \tag{14}$$

Substituting Equation (14) into Equation (11), we have:

$$\begin{aligned} l\left(x_{s1}, \ldots, x_{sn}; \mu_s, \sigma_s^2, \gamma\right) = & -\frac{n}{2} \log\left(1 + c^2\gamma^{2/3}\right) - n \log\left\{ \left[1 + \mu_s^2 \left(1 + c^2\gamma^{2/3}\right)\right]^{1/2} - c\mu_s\gamma^{1/3} \right\} \\ & + \sum_{i=1}^{n} \log \phi \left\{ \frac{\frac{x_{si} - \mu_s}{\left[1 + \mu_s^2 (1 + c^2\gamma^{2/3})\right]^{1/2} - c\mu_s\gamma^{1/3}} + c\gamma^{1/3}}{\left(1 + c^2\gamma^{2/3}\right)^{1/2}} \right\} \\ & + \sum_{i=1}^{n} \log \Phi \left\{ \frac{\frac{(x_{si} - \mu_s) c\gamma^{1/3}}{\left[1 + \mu_s^2 (1 + c^2\gamma^{2/3})\right]^{1/2} - c\mu_s\gamma^{1/3}} + c^2\gamma^{2/3}}{\left(1 + c^2\gamma^{2/3}\right)^{1/2} \left[b^2 + c^2\gamma^{2/3}(b^2 - 1)\right]^{1/2}} \right\}. \end{aligned} \tag{15}$$

Therefore, we define $(\tilde{\mu}_s^*, \tilde{\sigma}_s^{*^2}, \tilde{\gamma}^*)$ as the ML estimates of $(\mu_s, \sigma_s^2, \gamma)$ with default starting values given by the moment estimates of $(\mu_s, \sigma_s^2, \gamma)$ from Equation (15). Namely:

$$\hat{\mu}_s^* = -cs_{C2}^{-1/2} s_{C3}^{1/3}, \hat{\sigma}_s^{*^2} = 1 + cs_{C2}^{-1/2} s_{C3}^{2/3}, \hat{\gamma}^* = \frac{b\hat{\delta}^{*^3}(2b^2 - 1)}{\left(1 - b^2 \hat{\delta}^{*^2}\right)^{3/2}}. \tag{16}$$

Further, the ML estimates of μ and σ^2 are obtained as follows:

$$\tilde{\mu}^* = \bar{x}_C + s_{C2}^{1/2} \tilde{\mu}_s^*, \tilde{\sigma}^{*^2} = s_{C2} \tilde{\sigma}_s^{*2}. \tag{17}$$

By Equation (7), the ML estimates of the direct parameters (ξ, η^2, λ) are:

$$\tilde{\xi}^* = \tilde{\mu}^* - c\tilde{\gamma}^{*^{1/3}} \tilde{\sigma}^*, \tilde{\eta}^{*^2} = \tilde{\sigma}^{*^2}(1 + c^2 \tilde{\gamma}^{*^{2/3}}), \tilde{\lambda}^* = \frac{c\tilde{\gamma}^{*^{1/3}}}{\sqrt{b^2 + c^2(b^2 - 1)\tilde{\gamma}^{*^{2/3}}}}.$$

Then the ML estimate of δ is $\tilde{\delta}^* = \tilde{\lambda}^* / (1 + \tilde{\lambda}^{*^2})^{1/2}$. Hence, we have the following result.

Theorem 3. *Suppose that $(\tilde{\mu}_s, \tilde{\sigma}_s^2, \tilde{\gamma})$ are the ML estimators corresponding to $(\tilde{\mu}_s^*, \tilde{\sigma}_s^{*^2}, \tilde{\gamma}^*)$ in Equation (16), then the ML estimators of direct parameters (ξ, η^2, λ) are:*

$$\tilde{\xi} = \tilde{\mu} - c\tilde{\gamma}^{1/3} \tilde{\sigma}, \tilde{\eta}^2 = \tilde{\sigma}^2(1 + c^2 \tilde{\gamma}^{2/3}), \tilde{\lambda} = \frac{c\tilde{\gamma}^{1/3}}{\sqrt{b^2 + c^2(b^2 - 1)\tilde{\gamma}^{2/3}}}, \tag{18}$$

where $\tilde{\mu}$ and $\tilde{\sigma}^2$ are the ML estimators corresponding to $\tilde{\mu}^*$ and $\tilde{\sigma}^{*^2}$ in Equation (17), respectively. Furthermore, the ML estimator of δ is $\tilde{\delta} = \tilde{\lambda} / (1 + \tilde{\lambda}^2)^{1/2}$.

It is well-known that when $\lambda \neq 0$, the location parameter of skew-normal population is a generalization of the mean of normal population. Therefore, it is especially important to study the statistical inference of location parameter of single skew-normal distribution. Then, the Bootstrap approach for the hypothesis testing problem of location parameter in the single skew-normal population $SN(\xi, \eta^2, \lambda)$ is proposed. Specifically, the hypothesis of interest is:

$$H_0 : \xi = \xi_0 \qquad \text{vs.} \qquad H_1 : \xi \neq \xi_0 \ , \tag{19}$$

where ξ_0 is a specified value. Based on the central limit theorem, under H_0 in (19) we have:

$$T = \frac{\bar{X} - \xi_0 - b\eta\delta}{\sqrt{\frac{1}{n}\eta^2(1 - b^2\delta^2)}}. \tag{20}$$

If η and δ are known, T can be the test statistic for hypothesis testing problem (19). Since η and δ are often unknown, the test statistics might be developed by replacing η and δ with their moment and ML estimators in Equation (20), respectively. Therefore, the test statistics have the form of:

$$T_1 = \frac{\bar{X} - \xi_0 - b\hat{\eta}\hat{\delta}}{\sqrt{\frac{1}{n}\hat{\eta}^2(1 - b^2\hat{\delta}^2)}}, \tag{21}$$

$$T_2 = \frac{\bar{X} - \xi_0 - b\tilde{\eta}\tilde{\delta}}{\sqrt{\frac{1}{n}\tilde{\eta}^2(1 - b^2\tilde{\delta}^2)}}. \tag{22}$$

As the exact distributions of T_1 and T_2 are often unknown, it is impossible to establish their exact test approaches, so we can construct an approximate test approach based on the central limit theorem. However, the Monte Carlo simulation results indicate that the Type I error probabilities of approximate approach exceed the nominal significance level in most cases. Namely, the above approach is liberal. This result may be attributed to its approximate distribution characteristic. In view of this, we propose the Bootstrap test statistic for hypothesis testing problem (19) in this paper.

Under H_0 in (19), we define $X_{BM1}, \ldots, X_{BMn}$ as the Bootstrap samples from $SN(\xi_0, \hat{\eta}^{*^2}, \hat{\lambda}^*)$, where $(\bar{X}_{BM}, S_{BM2}, S_{BM3})$ denote the sample mean, the second and third center moments of the sample and $(\bar{x}_{BM}, s_{BM2}, s_{BM3})$ are their observed values. By Theorem 1, the moment estimators of (ξ, η^2, δ) have the form of:

$$\hat{\xi}_{BM} = \bar{X}_{BM} - cS_{BM3}^{1/3}, \hat{\eta}_{BM}^2 = S_{BM2} + c^2 S_{BM3}^{2/3}, \hat{\delta}_{BM} = \frac{cS_{BM3}^{1/3}}{b\sqrt{S_{BM2} + c^2 S_{BM3}^{2/3}}}.$$

Let $(\hat{\xi}_{BM}^*, \hat{\eta}_{BM}^{*^2}, \hat{\delta}_{BM}^*)$ be the moment estimates corresponding to $(\hat{\xi}_{BM}, \hat{\eta}_{BM}^2, \hat{\delta}_{BM})$. Let $X_{BL1}, \cdots, X_{BLn}$ be the Bootstrap samples from $SN(\xi_0, \tilde{\eta}^{*^2}, \tilde{\lambda}^*)$ with the sample mean $\bar{X}_{BL}$. Then the ML estimators $(\tilde{\xi}_{BL}, \tilde{\eta}_{BL}^2, \tilde{\delta}_{BL})$ can be obtained by Theorem 3. Similar to T_1 and T_2, the Bootstrap test statistics can be expressed as:

$$T_{B1} = \frac{\bar{X}_{BM} - \xi_0 - b\hat{\eta}_{BM}\hat{\delta}_{BM}}{\sqrt{\frac{1}{n}\hat{\eta}_{BM}^2(1 - b^2\hat{\delta}_{BM}^2)}}, \tag{23}$$

$$T_{B2} = \frac{\bar{X}_{BL} - \xi_0 - b\tilde{\eta}_{BL}\tilde{\delta}_{BL}}{\sqrt{\frac{1}{n}\tilde{\eta}_{BL}^2(1 - b^2\tilde{\delta}_{BL}^2)}}. \tag{24}$$

Then the Bootstrap p-values for hypothesis testing problem (19) are defined as:

$$p_i = 2\min\{P(T_{Bi} > t_i), P(T_{Bi} < t_i)\}\ , i = 1, 2, \tag{25}$$

where t_1 and t_2 are the observed values of T_1 and T_2, respectively. The null hypothesis H_0 in (19) is rejected whenever the above p-values are less than the nominal significance level of α, which means that the difference between ξ and ξ_0 is significant.

Remark 2. *According to [41], similar to T_{B1} and T_{B2}, the Bootstrap pivot quantities of ξ can be constructed as T_{B1}^* and T_{B2}^* based on the moment estimator and ML estimator, respectively. Suppose that $T_{B1}^*(\beta)$ is the 100β empirical percentile of T_{B1}^*. Then the $100(1-\alpha)\%$ Bootstrap confidence interval for ξ is given by:*

$$\left[\bar{x} - b\hat{\eta}^*\hat{\delta}^* - T_{B1}^*(1-\alpha/2)\sqrt{\frac{1}{n}\hat{\eta}^{*^2}\left(1 - b^2\hat{\delta}^{*^2}\right)}, \bar{x} - b\hat{\eta}^*\hat{\delta}^* - T_{B1}^*(\alpha/2)\sqrt{\frac{1}{n}\hat{\eta}^{*^2}\left(1 - b^2\hat{\delta}^{*^2}\right)}\right]$$

Similarly, a $100(1-\alpha)\%$ Bootstrap confidence interval for ξ based on T_{B2}^* is also obtained.

3. Inference on the Location Parameters of Two Skew-Normal Populations

Let $X_{i1}, \ldots, X_{in_i}$ be random samples from $X_i \sim SN(\xi_i, \eta_i^2, \lambda_i)$ and all of them are mutually independent, $i = 1, 2$. The sample mean, the second and third central moments of the sample can be expressed respectively as:

$$\bar{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij}, S_{i2} = \frac{1}{n_i}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2, S_{i3} = \frac{1}{n_i}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^3, i = 1, 2.$$

Firstly, we consider the estimation problems of $(\xi_1, \eta_1^2, \lambda_1)$ and $(\xi_2, \eta_2^2, \lambda_2)$ of two skew-normal populations in this section. By Theorem 1, the moment estimators of $(\xi_1, \eta_1^2, \lambda_1)$ and $(\xi_2, \eta_2^2, \lambda_2)$ can be given by:

$$\hat{\xi}_i = \bar{X}_i - cS_{i3}^{1/3}, \hat{\eta}_i^2 = S_{i2} + c^2 S_{i3}^{2/3}, \hat{\lambda}_i = \frac{\hat{\delta}_i}{\sqrt{1 - \hat{\delta}_i^2}}, i = 1, 2, \tag{26}$$

where $\hat{\delta}_i = cS_{i3}^{1/3}/b\sqrt{S_{i2} + c^2 S_{i3}^{2/3}}$. By Theorem 3, the ML estimators of $(\xi_i, \eta_i^2, \lambda_i, \delta_i)$ can be written as $(\tilde{\xi}_i, \tilde{\eta}_i^2, \tilde{\lambda}_i, \tilde{\delta}_i)$, $i = 1, 2$. Thus, let $(\hat{\eta}_i^{*2}, \hat{\lambda}_i^*, \hat{\delta}_i^*)$ be the moment estimates corresponding to $(\hat{\eta}_i^2, \hat{\lambda}_i, \hat{\delta}_i)$ and $(\tilde{\eta}_i^{*2}, \tilde{\lambda}_i^*, \tilde{\delta}_i^*,)$ be the ML estimates corresponding to $(\tilde{\eta}_i^2, \tilde{\lambda}_i, \tilde{\delta}_i)$, $i = 1, 2$.

Next, the problem of interest here is to test:

$$H_0 : \xi_1 = \xi_2 = \xi_* \qquad \text{vs.} \qquad H_1 : \xi_1 \neq \xi_2\,, \tag{27}$$

where ξ_* is a specified value. By the central limit theorem, under H_0 in (27) we have:

$$T^* = \frac{(\bar{X}_1 - \bar{X}_2) - b(\eta_1\delta_1 - \eta_2\delta_2)}{\sqrt{\frac{1}{n_1}\eta_1^2(1 - b^2\delta_1^2) + \frac{1}{n_2}\eta_2^2(1 - b^2\delta_2^2)}}. \tag{28}$$

If η_i and δ_i are known, $i = 1, 2$, then T^* is a natural statistic for hypothesis testing problem (27). For $i = 1, 2$, since η_i and δ_i are often unknown in practical, the test statistics might be obtained by replacing η_i and δ_i by their moment estimators and ML estimators, respectively. They are given by:

$$T_3 = \frac{(\bar{X}_1 - \bar{X}_2) - b(\hat{\eta}_1\hat{\delta}_1 - \hat{\eta}_2\hat{\delta}_2)}{\sqrt{\frac{1}{n_1}\hat{\eta}_1^2(1 - b^2\hat{\delta}_1^2) + \frac{1}{n_2}\hat{\eta}_2^2(1 - b^2\hat{\delta}_2^2)}}, \tag{29}$$

$$T_4 = \frac{(\bar{X}_1 - \bar{X}_2) - b(\tilde{\eta}_1\tilde{\delta}_1 - \tilde{\eta}_2\tilde{\delta}_2)}{\sqrt{\frac{1}{n_1}\tilde{\eta}_1^2(1 - b^2\tilde{\delta}_1^2) + \frac{1}{n_2}\tilde{\eta}_2^2(1 - b^2\tilde{\delta}_2^2)}}. \tag{30}$$

The exact distributions of T_3 and T_4 are also unknown like T_1 and T_2. For this, the Bootstrap approach will be used to construct test statistics for hypothesis testing problem (27).

Under H_0 in (27), let $X_{BMi1}, \ldots, X_{BMin_i}$ denote the Bootstrap samples from $SN(\xi_*, \hat{\eta}_i^{*2}, \hat{\lambda}_i^*)$ with the sample mean $\bar{X}_{BMi}$, $i = 1, 2$. By Theorem 1, the moment estimators of $(\xi_i, \eta_i^2, \delta_i)$ are $(\hat{\xi}_{BMi}, \hat{\eta}_{BMi}^2, \hat{\delta}_{BMi})$, $i = 1, 2$. Likewise, let $X_{BLi1}, \ldots, X_{BLin_i}$ denote the Bootstrap samples from $SN(\xi_*, \tilde{\eta}_i^{*2}, \tilde{\lambda}_i^*)$ with the sample mean $\bar{X}_{BLi}$ and the ML estimators of $(\xi_i, \eta_i^2, \delta_i)$ be $(\tilde{\xi}_{BLi}, \tilde{\eta}_{BLi}^2, \tilde{\delta}_{BLi})$ by Theorem 3, $i = 1, 2$. Based on T_3 and T_4, the Bootstrap test statistics are defined as:

$$T_{B3} = \frac{(\bar{X}_{BM1} - \bar{X}_{BM2}) - b(\hat{\eta}_{BM1}\hat{\delta}_{BM1} - \hat{\eta}_{BM2}\hat{\delta}_{BM2})}{\sqrt{\frac{1}{n_1}\hat{\eta}_{BM1}^2(1 - b^2\hat{\delta}_{BM1}^2) + \frac{1}{n_2}\hat{\eta}_{BM2}^2(1 - b^2\hat{\delta}_{BM2}^2)}}, \tag{31}$$

$$T_{B4} = \frac{(\bar{X}_{BL1} - \bar{X}_{BL2}) - b(\tilde{\eta}_{BL1}\tilde{\delta}_{BL1} - \tilde{\eta}_{BL2}\tilde{\delta}_{BL2})}{\sqrt{\frac{1}{n_1}\tilde{\eta}_{BL1}^2(1 - b^2\tilde{\delta}_{BL1}^2) + \frac{1}{n_2}\tilde{\eta}_{BL2}^2(1 - b^2\tilde{\delta}_{BL2}^2)}}. \tag{32}$$

Then the Bootstrap p-values for hypothesis testing problem (27) are:

$$p_i = 2\min\{P(T_{Bi} > t_i), P(T_{Bi} < t_i)\}, i = 3, 4, \tag{33}$$

where t_3 and t_4 are the observed values of T_3 and T_4, respectively. The null hypothesis H_0 in (27) is rejected whenever the above p-values are less than the nominal significance level of α, which means that the difference between ξ_1 and ξ_2 is significant.

Remark 3. *Similar to Remark 2, the Bootstrap pivotal quantities of $\xi_1 - \xi_2$ are constructed as T^*_{B3} and T^*_{B4} based on the moment estimators and ML estimators, respectively. Let $T^*_{B3}(\beta)$ be the 100β empirical percentile of T^*_{B3}. The $100(1-\alpha)\%$ Bootstrap confidence interval for $\xi_1 - \xi_2$ is defined as:*

$$\Bigg[(\bar{x}_1 - \bar{x}_2) - b(\hat{\eta}^*_1\hat{\delta}^*_1 - \hat{\eta}^*_2\hat{\delta}^*_2) - T^*_{B3}(1-\alpha/2)\sqrt{\tfrac{1}{n_1}\hat{\eta}^{*^2}_1\left(1 - b^2\hat{\delta}^{*^2}_1\right) + \tfrac{1}{n_2}\hat{\eta}^{*^2}_2\left(1 - b^2\hat{\delta}^{*^2}_2\right)},$$
$$(\bar{x}_1 - \bar{x}_2) - b(\hat{\eta}^*_1\hat{\delta}^*_1 - \hat{\eta}^*_2\hat{\delta}^*_2) - T^*_{B3}(\alpha/2)\sqrt{\tfrac{1}{n_1}\hat{\eta}^{*^2}_1\left(1 - b^2\hat{\delta}^{*^2}_1\right) + \tfrac{1}{n_2}\hat{\eta}^{*^2}_2\left(1 - b^2\hat{\delta}^{*^2}_2\right)}\Bigg].$$

Similarly, a $100(1-\alpha)\%$ Bootstrap confidence interval for $\xi_1 - \xi_2$ based on T^*_{B4} is also obtained.

4. Simulation Results and Discussion

In this section, the Monte Carlo simulation is used to numerically investigate properties of the above hypothesis testing approaches from the aspects of the Type I error rates and powers. Type I error refers to the error of rejecting the actually established and correct hypothesis, which can measure whether the testing approach is liberal or conservative. For convenience, we only provide the steps of the Bootstrap approach based on the moment estimators for hypothesis testing problem (19).

Step 1: For a given $(n, \xi_0, \eta^2, \lambda)$, generate a group of random samples $x_1, \ldots, x_n$ from skew-normal distribution. And $(\bar{x}, s_2, s_3)$ are computed by Equation (2).

Step 2: By Theorem 1, the moment estimates of (ξ, η^2, δ) are computed and denoted by $(\xi^*, \hat{\eta}^{*^2}, \hat{\delta}^*)$. Then the observed value t_1 of T_1 is obtained by Equation (21).

Step 3: Under H_0 in (19), generate the Bootstrap samples $x_{_{BMi}} \sim SN(\xi_0, \hat{\eta}^{*^2}, \hat{\lambda}^*)$, $i = 1, \ldots, n$. And $(\bar{x}_{_{BM}}, s_{_{BM2}}, s_{_{BM3}})$ are computed.

Step 4: By Theorem 1, the moment estimates of (ξ, η^2, δ) from Bootstrap samples are computed and denoted by $(\hat{\xi}^*_{BM}, \hat{\eta}^{*^2}_{BM}, \hat{\delta}^*_{BM})$. Then T_{B1} is obtained by Equation (23).

Step 5: Repeat Steps 3–4 n_1 times and compute p_1 by Equation (25). If $p_1 \leq 0.05$, then $Q = 1$; otherwise, $Q = 0$.

Step 6: Repeat Steps 1–5 n_2 times and we get $Q_1, \ldots, Q_{n_2}$. Then the Type I error probability is $(1/n_2)\sum_{i=1}^{n_2} Q_i$.

Based on the above steps, the power of hypothesis testing problem (19) can be obtained similarly.

In this simulation, the parameters and sample sizes are set as follows. Firstly, let the nominal significance level be 5%, and the number of inner loops n_1 and number of outer loops n_2 both be 2500. Secondly, for hypothesis testing problem (19), we set $\xi_0 = 2$, $\eta^2 = 1$, $\lambda = (-8, -7.5, -7, -6.5, -6, -5.5, -5, -4.5, -4, 4, 4.5, 5, 5.5, 6, 6.5, 7, 7.5, 8)$, and $n =(40, 45, 50, 55, 60, 70, 100, 150, 200)$. Finally, for hypothesis testing problem (27), we suppose $\xi_* = 2$, $(\eta_1^2, \eta_2^2) = ((0.1, 0.3), (0.2, 0.5), (0.3, 0.7), (0.4, 0.9), (0.5, 1.0))$, $(\lambda_1, \lambda_2) = ((4, 5), (6, 8))$, and $(n_1, n_2) = ((40, 50), (50, 50), (50, 60), (60, 70), (70, 80), (80, 90), (90, 120), (120, 150), (150, 200))$.

For hypothesis testing problem (19), Tables A1–A3 in Appendix A present the simulated Type I error probabilities and powers of the proposed approaches. Since the simulated results are similar in the case of positive and negative skewness, only the positive situation is analyzed below. From Table A1 in Appendix A, the Type I error probabilities based on p_1 are close to those based on p_2 in most parameter settings. Specifically, regarding the small sample size and skewness parameter, these two approaches are slightly liberal; regarding the large sample size, both approaches control the Type I error probabilities well. Furthermore, with the increase of sample size, the actual levels of the above two approaches are close to the nominal significance level of 5%. From Tables A2 and A3 in Appendix A, it is clear that the powers of these two approaches based on p_1 and p_2 both increase with larger sample size, but the former approach always performs better than the latter.

For hypothesis testing problem (27), Tables A4–A6 in Appendix A give the simulated Type I error probabilities and powers of the proposed approaches. From Table A4 in Appendix A, the approach based on p_3 is slightly liberal when the sample size and skewness parameter are small, but it can effectively control the Type I error probabilities under other parameter settings. As the sample size increases, the actual level of this approach is close to the nominal significance level of 5%. The approach based on p_4 is conservative under most parameter settings. From Tables A5 and A6 in Appendix A, the powers of the approach based on p_3 are obviously better than those based on p_4 in most cases.

In a word, for hypothesis testing problems (19) and (27), the proposed Bootstrap approaches provide the satisfactory performances under the senses of Type I error probability and power in most cases regardless of the moment estimator or ML estimator. It is well-known that the ML estimator depends on the choice of initial value, which may influence its estimation accuracy. Hence, the Bootstrap test based on the moment estimator is better than that based on the ML estimator in most situations, which can provide a useful approach for the inference on location parameter in the real data examples.

Remark 4. *For hypothesis testing problem* (27)*, we only provide the simulation results in the case of positive skewness. When the skewness parameter is negative, the results are similar to those of positive skewness parameter, so we omit them.*

5. Illustrative Examples

In order to verify the rationality and validity of the proposed approaches, we apply them into the examples of LAI, carbon fibers' strength and RBC count in athletes in this section.

Example 1. *The above approaches are applied to the data of LAI of Robinia pseudoacacia Plantation in Huaiping Forest Farm, Yongshou County, Shanxi Province (see Ye et al. [42]). From Figures 1 and 2, the distribution of LAI does not follow the normal distribution but shows asymmetric right-biased distribution characteristics. To confirm the conclusion, we first test the normality of this data. It turns out that the p-values from R output of Shapiro–Wilk test, Anderson–Darling test and Lilliefor test are 0.0007, 0.0014 and 0.0458, respectively. Hence, the LAI is not normally distributed at the nominal significance level of 5%. Further, we should prove whether the distribution of LAI is skew-normal by the Chi-square goodness-of-fit test. By calculation, we have* $\chi^2 = 5.0929 < \chi^2_2(0.95) = 5.9915$*. Therefore, the LAI follows the skew-normal distribution* $SN(\xi, \eta^2, \lambda)$ *at the nominal significance level of 5%. Based on the method of moment estimation, the LAI is approximately distributed as SN* $(1.2585, 1.8332^2, 2.7966)$*, and its density curve is given in Figure 2.*

To illustrate the proposed approach for hypothesis testing problem (19), we suppose ξ_0 as the nearby value of moment estimate of ξ. Namely, consider the hypothesis testing problem:

$$H_0 : \xi = 2 \qquad \text{vs.} \qquad H_1 : \xi \neq 2.$$

Based on the moment and ML estimators, the p-values of Bootstrap test are 0.02584 and 0.00097, respectively. Hence, the null hypothesis H_0 is rejected at the nominal significance level of 5%, that is, the location parameter of LAI is not equal to 2 significantly.

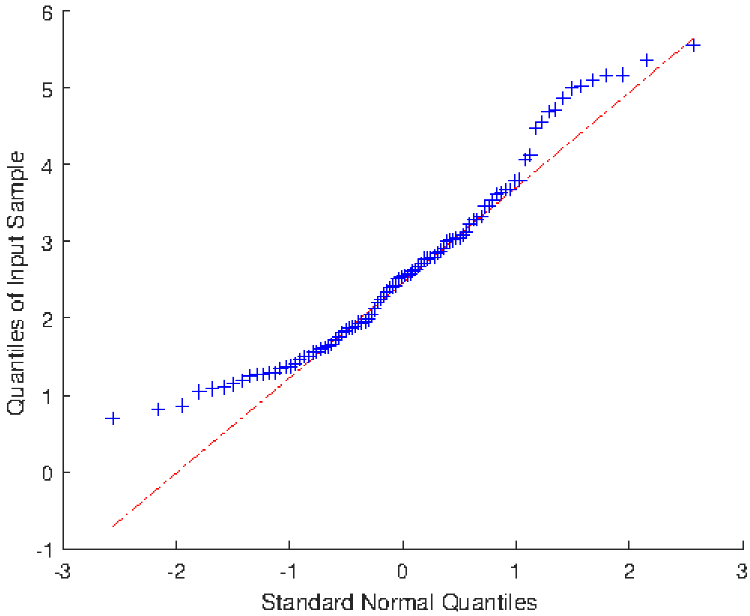

Figure 1. Q-Q plot of LAI data (Red line and blue line respectively denote the standard normal quantiles and sample quantiles).

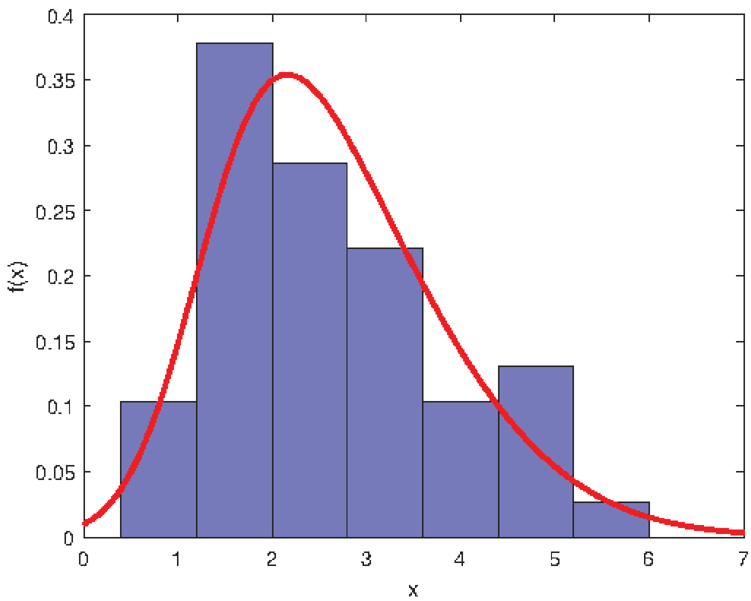

Figure 2. Frequency histogram of the LAI with superimposed skew-normal density curve.

Example 2. *Kundu and Gupta [43] presented a data set of the strength measured in GPA for single carbon fibers. The Shapiro-Wilk test, Anderson-Darling test and Lilliefor test are used to test the normality of the data. It turns out that the p-values of the data are 0.0108, 0.0109 and 0.0254 respectively. The P-P plot and the histogram of the data are given in Figures 3 and 4. Furthermore, the chi-square goodness-of-fit test is used to see whether the distribution of data is skew-normal, namely we set H_0: the carbon fibers' strength data is skew-normally distributed. We obtain that $\chi^2 = 1.1907 < \chi^2_3(0.95) = 7.8147$, then the null hypothesis H_0 is rejected at the nominal significance level of 5%. Similar to Example 1, the carbon fibers' strength data is considered to follow skew normal distribution SN (2.0917, 0.9230^2, 2.9668).*

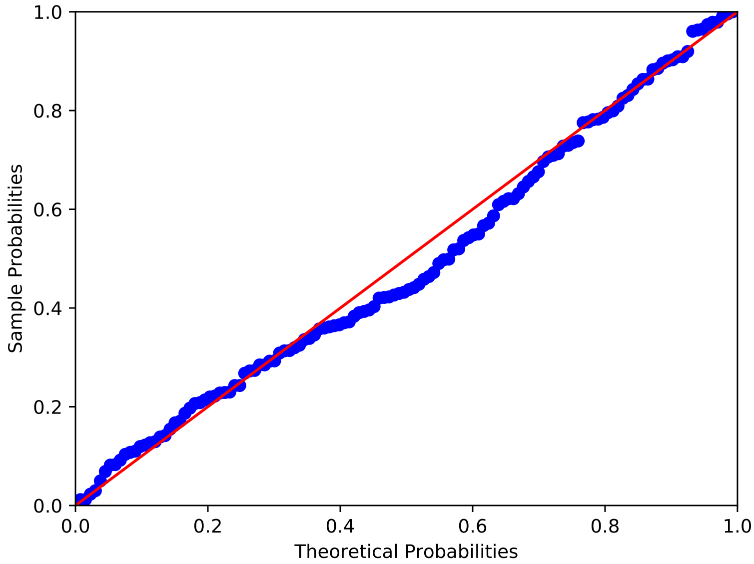

Figure 3. P-P plot of carbon fibers' strength data (Red line and blue line respectively denote the theoretical probabilities and sample probabilities).

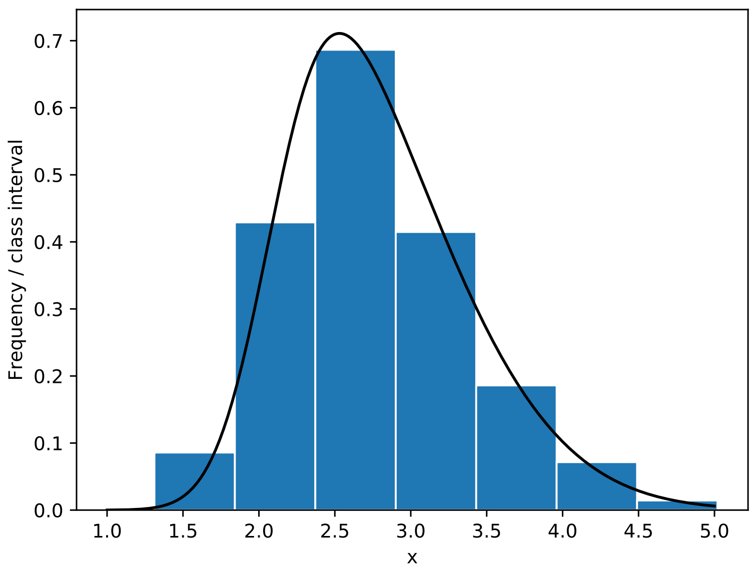

Figure 4. Frequency histogram of the carbon fibers' strength data with superimposed skew-normal density curve.

Consider the hypothesis testing problem:

$$H_0 : \xi = 2.1 \qquad \text{vs.} \qquad H_1 : \xi \neq 2.1 .$$

By the moment and ML estimators, the p-values of Bootstrap test are 0.9433 and 0.0814, respectively. Therefore, the null hypothesis is not rejected at the nominal significance level of 5%.

Example 3. *The data collected by the Australian Institute of Physical Education of RBC count in 102 male and 100 female athletes are analyzed in this example (see Cook and Weisberg [44]). Similar to Example 2, the Shapiro-Wilk test, Anderson-Darling test and Lilliefor test are used to test the normality of RBC count in male and female athletes. It shows that the p-values of RBC count in male athletes are 0.0000, 0.0019 and 0.0025 respectively, while those in female athletes are 0.0065, 0.0131 and 0.0181 respectively. The corresponding images are shown in Figures 5 and 6. Therefore, at the nominal significance level of 5%, the above tests all reject the null hypothesis that RBC counts in male and female* *athletes follow normal distributions. Furthermore, to verify the skew-normality of RBC count, we test the null hypothesis H_0: the RBC count is skew-normally distributed. It can be obtained by calculation that $\chi^2_m = 2.0527 < \chi^2_1(0.95) = 3.8415$ and $\chi^2_f = 0.3469 < \chi^2_2(0.95) = 5.9915$, which means the RBC count in male and female athletes follow the skew-normal distributions $SN(\xi_m, \eta^2_m, \lambda_m)$ and $SN(\xi_f, \eta^2_f, \lambda_f)$ respectively at the nominal significance level of 5%.*

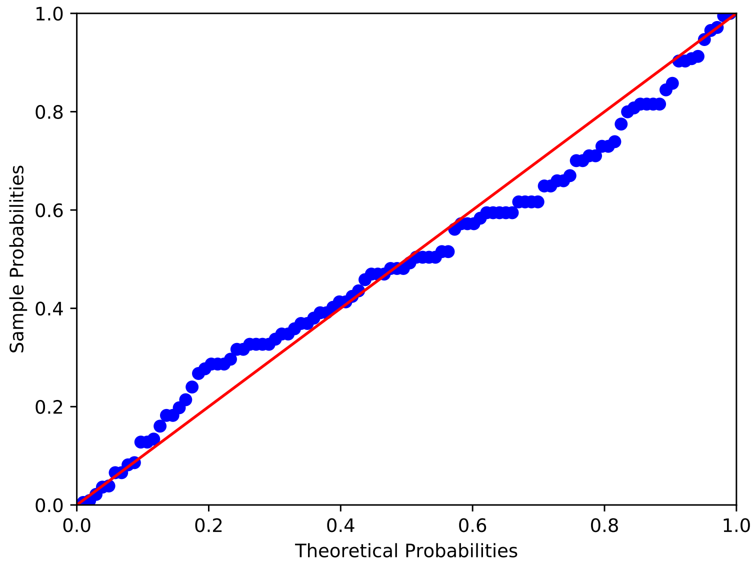

Figure 5. P-P plot of RBC count in male athletes (Red line and blue line respectively denote the theoretical probabilities and sample probabilities).

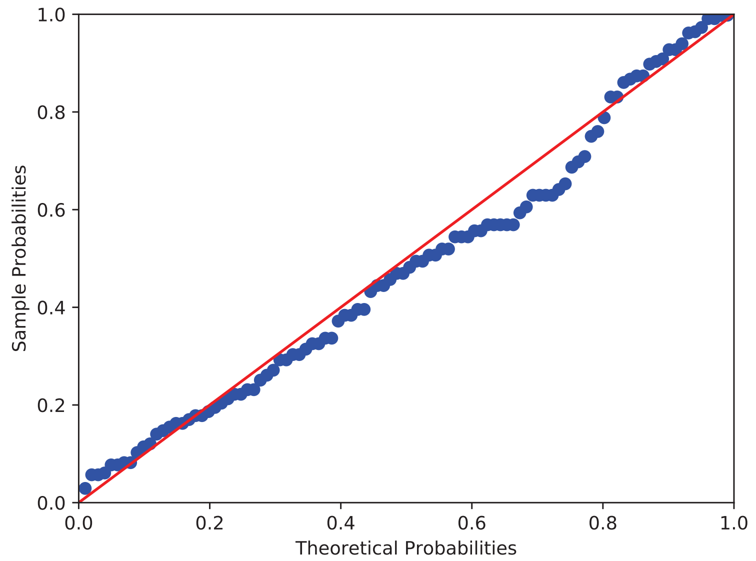

Figure 6. P-P plot of RBC count in female athletes (Red line and blue line respectively denote the theoretical probabilities and sample probabilities).

Consider the hypothesis testing problem:

$$H_0 : \xi_m = \xi_f \qquad \text{vs.} \qquad H_1 : \xi_m \neq \xi_f .$$

The p-values of Bootstrap test statistics based on the moment estimators and ML estimators are 0.00071 and 0.00153, respectively. Therefore, the null hypothesis H_0 is rejected at the nominal significance level of 5%, that is, the location parameters of RBC counts in male and female athletes have significant differences.

6. Conclusions

By using the centered parameterization and Bootstrap approaches, we study the hypothesis testing and interval estimation problems of location parameters for single and two skew-normal populations with unknown scale parameters and skewness parameters. Firstly, the Bootstrap test statistics and Bootstrap confidence intervals for location parameter of single population are constructed based on the moment estimators and ML estimators, respectively. Secondly, the Bootstrap approaches for Behrens-Fisher type and interval estimation problems are established for two skew-normal populations. Thirdly, the Monte Carlo simulation results show that the Bootstrap test based on the moment estimator is better than that based on the ML estimator in most parameter settings, whether in single population or two. Finally, the above approaches are applied to LAI, carbon fibers' strength and RBC count in athletes to verify the rationality and validity of the proposed approaches. In summary, the Bootstrap approach based on the moment estimator is preferentially suggested to be used for inference on the location parameter of the skew-normal population. In the future, we plan to consider the hypothesis testing and confidence interval problems for the location parameter vector in multivariate skew-normal population, and discuss the applications of Bootstrap approach in multivariate skew-normal data sets, expecting to provide a feasible solution for the multivariate skew data analysis problem.

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Appendix A

Table A1. Simulated Type I error probabilities of hypothesis testing problem (19).

λ	p	$n = 40$	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 70$	$n = 100$	$n = 150$
4	p_1	0.0672	0.0544	0.0540	0.0528	0.0512	0.0480	0.0452	0.0484
	p_2	0.0608	0.0556	0.0484	0.0492	0.0464	0.0456	0.0404	0.0460
4.5	p_1	0.0588	0.0456	0.0460	0.0476	0.0440	0.0416	0.0424	0.0440
	p_2	0.0520	0.0504	0.0448	0.0428	0.0416	0.0440	0.0380	0.0420
5	p_1	0.0512	0.0396	0.0404	0.0408	0.0404	0.0368	0.0388	0.0412
	p_2	0.0472	0.0476	0.0408	0.0376	0.0404	0.0400	0.0352	0.0408
5.5	p_1	0.0480	0.0376	0.0376	0.0356	0.0380	0.0336	0.0380	0.0372
	p_2	0.0444	0.0416	0.0372	0.0356	0.0388	0.0380	0.0328	0.0396
6	p_1	0.0444	0.0356	0.0360	0.0344	0.0360	0.0316	0.0356	0.0352
	p_2	0.0428	0.0384	0.0344	0.0352	0.0368	0.0364	0.0304	0.0384
6.5	p_1	0.0420	0.0332	0.0328	0.0332	0.0348	0.0312	0.0348	0.0332
	p_2	0.0408	0.0364	0.0344	0.0324	0.0360	0.0352	0.0300	0.0360
7	p_1	0.0396	0.0324	0.0316	0.0316	0.0336	0.0296	0.0340	0.0324
	p_2	0.0388	0.0360	0.0332	0.0312	0.0356	0.0340	0.0288	0.0360
7.5	p_1	0.0364	0.0300	0.0304	0.0304	0.0332	0.0296	0.0320	0.0312
	p_2	0.0376	0.0360	0.0328	0.0312	0.0348	0.0332	0.0280	0.0356
8	p_1	0.0360	0.0288	0.0304	0.0296	0.0324	0.0288	0.0308	0.0308
	p_2	0.0360	0.0344	0.0328	0.0304	0.0340	0.0332	0.0272	0.0356

Table A1. *Cont.*

λ	p	$n = 40$	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 70$	$n = 100$	$n = 150$
−4	p_1	0.0680	0.0604	0.0596	0.0532	0.0528	0.0484	0.0488	0.0524
	p_2	0.0576	0.0604	0.0520	0.0484	0.0488	0.0460	0.0380	0.0460
−4.5	p_1	0.0608	0.0524	0.0492	0.0456	0.0468	0.0448	0.0440	0.0464
	p_2	0.0512	0.0532	0.0444	0.0436	0.0424	0.0420	0.0356	0.0460
−5	p_1	0.0540	0.0480	0.0448	0.0408	0.0428	0.0396	0.0424	0.0420
	p_2	0.0452	0.0472	0.0412	0.0404	0.0400	0.0388	0.0340	0.0428
−5.5	p_1	0.0452	0.0452	0.0400	0.0376	0.0376	0.0372	0.0392	0.0392
	p_2	0.0400	0.0424	0.0388	0.0384	0.0396	0.0364	0.0312	0.0412
−6	p_1	0.0412	0.0408	0.0368	0.0348	0.0352	0.0336	0.0376	0.0364
	p_2	0.0368	0.0416	0.0364	0.0364	0.0376	0.0360	0.0296	0.0400
−6.5	p_1	0.0400	0.0376	0.0344	0.0320	0.0332	0.0328	0.0352	0.0340
	p_2	0.0344	0.0404	0.0356	0.0348	0.0364	0.0352	0.0284	0.0384
−7	p_1	0.0372	0.0372	0.0324	0.0312	0.0328	0.0300	0.0340	0.0316
	p_2	0.0332	0.0400	0.0356	0.0328	0.0348	0.0348	0.0276	0.0368
−7.5	p_1	0.0352	0.0364	0.0312	0.0308	0.0308	0.0292	0.0316	0.0308
	p_2	0.0320	0.0392	0.0344	0.0320	0.0344	0.0348	0.0272	0.0368
−8	p_1	0.0344	0.0348	0.0312	0.0300	0.0296	0.0292	0.0304	0.0296
	p_2	0.0316	0.0380	0.0336	0.0308	0.0340	0.0344	0.0264	0.0360

Table A2. Simulated powers of hypothesis testing problem (19) (positive skewness).

λ	n	p	ξ					
			1.9	**1.8**	**1.7**	**1.6**	**1.5**	**1.4**
4	40	p_1	0.1540	0.2936	0.4504	0.6184	0.7624	0.8556
		p_2	0.1288	0.2096	0.2684	0.3196	0.3700	0.4380
	50	p_1	0.1776	0.3388	0.5076	0.7028	0.8280	0.9016
		p_2	0.1368	0.2184	0.2860	0.3376	0.4060	0.4780
	60	p_1	0.1700	0.3604	0.5636	0.7444	0.8740	0.9408
		p_2	0.1372	0.2256	0.2832	0.3540	0.4288	0.5076
	70	p_1	0.1848	0.3884	0.5968	0.7892	0.9000	0.9620
		p_2	0.1348	0.2188	0.2872	0.3580	0.4416	0.5416
	100	p_1	0.2052	0.4424	0.7096	0.8872	0.9616	0.9904
		p_2	0.1376	0.2212	0.2980	0.3832	0.4892	0.5972
	150	p_1	0.2636	0.5704	0.8376	0.9608	0.9924	0.9992
		p_2	0.1532	0.2264	0.3012	0.4084	0.5324	0.6836
	200	p_1	0.2988	0.6620	0.9156	0.9876	0.9980	1.0000
		p_2	0.1472	0.2324	0.3348	0.4536	0.5960	0.7672
4.5	40	p_1	0.1476	0.2928	0.4580	0.6292	0.7836	0.8688
		p_2	0.1292	0.2188	0.2828	0.3404	0.3952	0.4632
	50	p_1	0.1708	0.3452	0.5204	0.7164	0.8420	0.9136
		p_2	0.1404	0.2396	0.3140	0.3632	0.4308	0.5092
	60	p_1	0.1708	0.3656	0.5780	0.7644	0.8888	0.9484
		p_2	0.1428	0.2380	0.3080	0.3836	0.4664	0.5456
	70	p_1	0.1868	0.4004	0.6156	0.8016	0.9128	0.9660
		p_2	0.1416	0.2452	0.3220	0.3956	0.4872	0.5816
	100	p_1	0.2100	0.4640	0.7256	0.9024	0.9696	0.9932
		p_2	0.1528	0.2496	0.3348	0.4284	0.5380	0.6452
	150	p_1	0.2720	0.5928	0.8528	0.9684	0.9940	0.9996
		p_2	0.1816	0.2644	0.3524	0.4660	0.5992	0.7288
	200	p_1	0.3088	0.6828	0.9260	0.9908	0.9992	1.0000
		p_2	0.1796	0.2804	0.3924	0.5108	0.6612	0.8100
5	40	p_1	0.1396	0.2944	0.4660	0.6404	0.7896	0.8768
		p_2	0.1284	0.2272	0.2992	0.3584	0.4180	0.4964
	50	p_1	0.1668	0.3472	0.5332	0.7268	0.8556	0.9260
		p_2	0.1400	0.2524	0.3300	0.3900	0.4568	0.5380
	60	p_1	0.1700	0.3736	0.5868	0.7744	0.8948	0.9532
		p_2	0.1468	0.2572	0.3348	0.4076	0.4908	0.5712
	70	p_1	0.1844	0.4064	0.6288	0.8176	0.9256	0.9732
		p_2	0.1476	0.2688	0.3504	0.4224	0.5156	0.6080
	100	p_1	0.2140	0.4792	0.7396	0.9108	0.9756	0.9936
		p_2	0.1620	0.2780	0.3724	0.4644	0.5720	0.6788
	150	p_1	0.2780	0.6088	0.8624	0.9712	0.9956	1.0000
		p_2	0.1992	0.2956	0.3916	0.5088	0.6416	0.7600
	200	p_1	0.3208	0.6968	0.9340	0.9912	0.9992	1.0000
		p_2	0.2104	0.3196	0.4316	0.5564	0.7104	0.8356

Table A2. *Cont.*

λ	n	p	ξ 1.9	1.8	1.7	1.6	1.5	1.4
5.5	40	p_1	0.1372	0.2980	0.4732	0.6532	0.7972	0.8852
		p_2	0.1272	0.2328	0.3156	0.3732	0.4348	0.5136
	50	p_1	0.1640	0.3540	0.5428	0.7436	0.8620	0.9348
		p_2	0.1424	0.2608	0.3420	0.4084	0.4804	0.5568
	60	p_1	0.1700	0.3800	0.5948	0.7848	0.9016	0.9604
		p_2	0.1496	0.2688	0.3556	0.4312	0.5092	0.5988
	70	p_1	0.1840	0.4136	0.6388	0.8252	0.9308	0.9752
		p_2	0.1504	0.2864	0.3696	0.4456	0.5332	0.6268
	100	p_1	0.2148	0.4904	0.7476	0.9192	0.9784	0.9952
		p_2	0.1672	0.3032	0.3984	0.4952	0.6008	0.7016
	150	p_1	0.2800	0.6220	0.8704	0.9744	0.9960	1.0000
		p_2	0.2128	0.3180	0.4228	0.5452	0.6792	0.7908
	200	p_1	0.3320	0.7148	0.9380	0.9924	0.9992	1.0000
		p_2	0.2336	0.3496	0.4652	0.5932	0.7440	0.8540
6	40	p_1	0.1356	0.3008	0.4772	0.6628	0.8040	0.8892
		p_2	0.1236	0.2368	0.3236	0.3868	0.4472	0.5264
	50	p_1	0.1620	0.3592	0.5496	0.7468	0.8688	0.9416
		p_2	0.1416	0.2704	0.3560	0.4236	0.4964	0.5768
	60	p_1	0.1700	0.3848	0.6040	0.7932	0.9072	0.9648
		p_2	0.1492	0.2800	0.3676	0.4428	0.5216	0.6156
	70	p_1	0.1840	0.4216	0.6488	0.8308	0.9364	0.9804
		p_2	0.1532	0.2980	0.3848	0.4660	0.5500	0.6440
	100	p_1	0.2140	0.4968	0.7572	0.9224	0.9824	0.9960
		p_2	0.1688	0.3232	0.4184	0.5188	0.6208	0.7212
	150	p_1	0.2816	0.6296	0.8784	0.9776	0.9964	1.0000
		p_2	0.2240	0.3416	0.4496	0.5748	0.7048	0.8116
	200	p_1	0.3416	0.7232	0.9420	0.9932	0.9996	1.0000
		p_2	0.2524	0.3788	0.4932	0.6296	0.7732	0.8676
6.5	40	p_1	0.1336	0.3040	0.4816	0.6684	0.8100	0.8940
		p_2	0.1208	0.2400	0.3328	0.4008	0.4616	0.5392
	50	p_1	0.1596	0.3628	0.5548	0.7528	0.8732	0.9448
		p_2	0.1436	0.2764	0.3616	0.4312	0.5064	0.5896
	60	p_1	0.1676	0.3900	0.6080	0.7992	0.9100	0.9680
		p_2	0.1516	0.2916	0.3836	0.4584	0.5384	0.6308
	70	p_1	0.1820	0.4264	0.6564	0.8348	0.9396	0.9828
		p_2	0.1528	0.3052	0.3988	0.4780	0.5656	0.6596
	100	p_1	0.2144	0.5048	0.7612	0.9256	0.9832	0.9960
		p_2	0.1704	0.3364	0.4328	0.5360	0.6420	0.7320
	150	p_1	0.2836	0.6368	0.8852	0.9784	0.9972	1.0000
		p_2	0.2320	0.3588	0.4676	0.5972	0.7248	0.8232
	200	p_1	0.3488	0.7296	0.9460	0.9944	0.9996	1.0000
		p_2	0.2628	0.4016	0.5180	0.6572	0.7896	0.8840
7	40	p_1	0.1324	0.3052	0.4856	0.6740	0.8140	0.8984
		p_2	0.1184	0.2440	0.3400	0.4088	0.4692	0.5472
	50	p_1	0.1588	0.3668	0.5608	0.7588	0.8756	0.9468
		p_2	0.1432	0.2804	0.3704	0.4404	0.5148	0.6016
	60	p_1	0.1676	0.3932	0.6144	0.8052	0.9116	0.9712
		p_2	0.1512	0.3028	0.3928	0.4688	0.5496	0.6396
	70	p_1	0.1800	0.4304	0.6616	0.8404	0.9420	0.9844
		p_2	0.1528	0.3160	0.4072	0.4876	0.5800	0.6736
	100	p_1	0.2168	0.5112	0.7652	0.9284	0.9836	0.9964
		p_2	0.1708	0.3476	0.4432	0.5448	0.6520	0.7468
	150	p_1	0.2856	0.6428	0.8868	0.9788	0.9976	1.0000
		p_2	0.2352	0.3740	0.4860	0.6156	0.7416	0.8360
	200	p_1	0.3512	0.7352	0.9496	0.9952	0.9996	1.0000
		p_2	0.2760	0.4168	0.5324	0.6724	0.8032	0.8956

Table A2. *Cont.*

λ	n	p	ξ					
			1.9	**1.8**	**1.7**	**1.6**	**1.5**	**1.4**
7.5	40	p_1	0.1304	0.3048	0.4876	0.6764	0.8176	0.9000
		p_2	0.1164	0.2484	0.3484	0.4148	0.4780	0.5560
	50	p_1	0.1584	0.3680	0.5668	0.7632	0.8776	0.9492
		p_2	0.1424	0.2856	0.3800	0.4500	0.5256	0.6136
	60	p_1	0.1660	0.3968	0.6188	0.8100	0.9136	0.9740
		p_2	0.1520	0.3084	0.3996	0.4756	0.5588	0.6500
	70	p_1	0.1796	0.4344	0.6664	0.8448	0.9436	0.9848
		p_2	0.1512	0.3212	0.4140	0.4960	0.5916	0.6844
	100	p_1	0.2160	0.5152	0.7716	0.9288	0.9852	0.9964
		p_2	0.1724	0.3572	0.4576	0.5556	0.6616	0.7572
	150	p_1	0.2864	0.6484	0.8900	0.9792	0.9976	1.0000
		p_2	0.2412	0.3856	0.5012	0.6308	0.7532	0.8456
	200	p_1	0.3536	0.7388	0.9516	0.9956	0.9996	1.0000
		p_2	0.2844	0.4288	0.5480	0.6864	0.8100	0.9016
8	40	p_1	0.1296	0.3064	0.4904	0.6808	0.8224	0.9024
		p_2	0.1148	0.2492	0.3500	0.4180	0.4848	0.5648
	50	p_1	0.1580	0.3692	0.5684	0.7680	0.8836	0.9500
		p_2	0.1424	0.2888	0.3848	0.4588	0.5340	0.6196
	60	p_1	0.1648	0.4008	0.6228	0.8128	0.9164	0.9756
		p_2	0.1512	0.3120	0.4052	0.4852	0.5660	0.6560
	70	p_1	0.1792	0.4392	0.6724	0.8472	0.9456	0.9856
		p_2	0.1520	0.3276	0.4216	0.5028	0.5996	0.6916
	100	p_1	0.2152	0.5188	0.7752	0.9316	0.9856	0.9968
		p_2	0.1728	0.3696	0.4672	0.5656	0.6680	0.7644
	150	p_1	0.2872	0.6520	0.8972	0.9808	0.9976	1.0000
		p_2	0.2444	0.3948	0.5108	0.6420	0.7592	0.8532
	200	p_1	0.3564	0.7440	0.9532	0.9964	0.9996	1.0000
		p_2	0.2896	0.4368	0.5616	0.6964	0.8192	0.9096

Table A3. Simulated powers of hypothesis testing problem (19) (negative skewness).

λ	n	p	ξ					
			2.1	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**
−4	40	p_1	0.1592	0.2980	0.4600	0.6344	0.7696	0.8628
		p_2	0.1292	0.2100	0.2652	0.3060	0.3660	0.4392
	50	p_1	0.1772	0.3344	0.5116	0.6992	0.8356	0.9100
		p_2	0.1420	0.2268	0.2796	0.3336	0.3976	0.4832
	60	p_1	0.1772	0.3588	0.5640	0.7472	0.8780	0.9404
		p_2	0.1408	0.2244	0.2916	0.3524	0.4296	0.5120
	70	p_1	0.1876	0.3884	0.6040	0.7920	0.9016	0.9620
		p_2	0.1296	0.2188	0.2916	0.3680	0.4420	0.5280
	100	p_1	0.2064	0.4460	0.6972	0.8764	0.9588	0.9908
		p_2	0.1412	0.2256	0.2976	0.3824	0.4800	0.5872
	150	p_1	0.2616	0.5700	0.8372	0.9648	0.9912	0.9992
		p_2	0.1560	0.2312	0.3052	0.4188	0.5416	0.6828
	200	p_1	0.2956	0.6496	0.9172	0.9888	0.9988	1.0000
		p_2	0.1488	0.2356	0.3396	0.4508	0.5872	0.7512
−4.5	40	p_1	0.1556	0.3000	0.4680	0.6464	0.7816	0.8756
		p_2	0.1264	0.2148	0.2820	0.3300	0.3960	0.4724
	50	p_1	0.1704	0.3368	0.5216	0.7148	0.8464	0.9248
		p_2	0.1404	0.2320	0.2968	0.3584	0.4340	0.5160
	60	p_1	0.1744	0.3704	0.5808	0.7664	0.8952	0.9512
		p_2	0.1440	0.2432	0.3144	0.3880	0.4712	0.5468
	70	p_1	0.1884	0.4028	0.6244	0.8092	0.9144	0.9696
		p_2	0.1364	0.2436	0.3248	0.4016	0.4792	0.5636
	100	p_1	0.2132	0.4628	0.7212	0.8932	0.9676	0.9932
		p_2	0.1520	0.2572	0.3352	0.4204	0.5248	0.6356
	150	p_1	0.2692	0.5876	0.8536	0.9716	0.9936	0.9996
		p_2	0.1832	0.2616	0.3580	0.4744	0.6000	0.7300
	200	p_1	0.3084	0.6708	0.9268	0.9912	0.9996	1.0000
		p_2	0.1820	0.2768	0.3916	0.5092	0.6524	0.8016

Table A3. *Cont.*

λ	n	p	ξ					
			2.1	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**
	40	p_1	0.1504	0.2992	0.4728	0.6560	0.7960	0.8852
		p_2	0.1260	0.2268	0.3008	0.3500	0.4192	0.4972
	50	p_1	0.1704	0.3444	0.5328	0.7280	0.8572	0.9336
		p_2	0.1404	0.2444	0.3124	0.3784	0.4544	0.5452
	60	p_1	0.1736	0.3804	0.5940	0.7792	0.9020	0.9564
		p_2	0.1468	0.2600	0.3400	0.4148	0.4948	0.5752
−5	70	p_1	0.1860	0.4136	0.6388	0.8188	0.9252	0.9752
		p_2	0.1424	0.2684	0.3476	0.4308	0.5104	0.5932
	100	p_1	0.2188	0.4764	0.7328	0.9020	0.9724	0.9944
		p_2	0.1616	0.2836	0.3680	0.4608	0.5656	0.6712
	150	p_1	0.2740	0.6004	0.8656	0.9732	0.9952	1.0000
		p_2	0.1996	0.2940	0.3940	0.5164	0.6452	0.7640
	200	p_1	0.3204	0.6924	0.9328	0.9932	0.9996	1.0000
		p_2	0.2048	0.3160	0.4240	0.5572	0.7000	0.8288
	40	p_1	0.1424	0.2972	0.4752	0.6588	0.8016	0.8896
		p_2	0.1220	0.2312	0.3096	0.3624	0.4352	0.5168
	50	p_1	0.1668	0.3480	0.5400	0.7834	0.8680	0.9400
		p_2	0.1436	0.2532	0.3288	0.3996	0.4732	0.5656
	60	p_1	0.1700	0.3844	0.6012	0.7892	0.9060	0.9612
		p_2	0.1476	0.2728	0.3584	0.4320	0.5152	0.5968
−5.5	70	p_1	0.1864	0.4188	0.6464	0.8260	0.9316	0.9796
		p_2	0.1460	0.2852	0.3660	0.4528	0.5288	0.6216
	100	p_1	0.2208	0.4888	0.7416	0.9100	0.9756	0.9956
		p_2	0.1676	0.3048	0.3940	0.4884	0.5924	0.6948
	150	p_1	0.2796	0.6120	0.8748	0.9764	0.9956	1.0000
		p_2	0.2148	0.3224	0.4244	0.5508	0.6772	0.7968
	200	p_1	0.3300	0.7032	0.9372	0.9940	0.9996	1.0000
		p_2	0.2308	0.3496	0.4568	0.5972	0.7336	0.8564
	40	p_1	0.1388	0.3000	0.4816	0.6652	0.8080	0.8976
		p_2	0.1196	0.2384	0.3228	0.3748	0.4472	0.5296
	50	p_1	0.1644	0.3544	0.5484	0.7492	0.8748	0.9440
		p_2	0.1424	0.2576	0.3412	0.4160	0.4900	0.5832
	60	p_1	0.1688	0.3892	0.6064	0.7956	0.9116	0.9660
		p_2	0.1508	0.2856	0.3716	0.4456	0.5296	0.6168
−6	70	p_1	0.1860	0.4220	0.6544	0.8328	0.9376	0.9808
		p_2	0.1468	0.3020	0.3812	0.4688	0.5488	0.6436
	100	p_1	0.2204	0.4964	0.7512	0.9140	0.9792	0.9960
		p_2	0.1700	0.3236	0.4100	0.5088	0.6120	0.7160
	150	p_1	0.2820	0.6252	0.8808	0.9784	0.9960	1.0000
		p_2	0.2220	0.3420	0.4544	0.5756	0.7008	0.8148
	200	p_1	0.3380	0.7160	0.9416	0.9940	0.9996	1.0000
		p_2	0.2508	0.3732	0.4820	0.6264	0.7632	0.8748
	40	p_1	0.1376	0.3024	0.4876	0.6708	0.8164	0.9032
		p_2	0.1188	0.2424	0.3296	0.3856	0.4560	0.5420
	50	p_1	0.1620	0.3580	0.5528	0.7544	0.8800	0.9484
		p_2	0.1432	0.2660	0.3504	0.4260	0.5044	0.5928
	60	p_1	0.1672	0.3916	0.6104	0.7996	0.9164	0.9684
		p_2	0.1524	0.2960	0.3832	0.4588	0.5432	0.6304
−6.5	70	p_1	0.1848	0.4284	0.6600	0.8376	0.9412	0.9824
		p_2	0.1492	0.3152	0.3928	0.4804	0.5620	0.6584
	100	p_1	0.2188	0.5056	0.7612	0.9192	0.9808	0.9960
		p_2	0.1704	0.3360	0.4284	0.5308	0.6268	0.7300
	150	p_1	0.2832	0.6360	0.8876	0.9800	0.9972	1.0000
		p_2	0.2312	0.3624	0.4720	0.5960	0.7172	0.8328
	200	p_1	0.3444	0.7216	0.9440	0.9948	0.9996	1.0000
		p_2	0.2660	0.3904	0.5040	0.6496	0.7820	0.8864

Table A3. *Cont.*

λ	n	p	ξ 2.1	2.2	2.3	2.4	2.5	2.6
	40	p_1	0.1348	0.3060	0.4896	0.6756	0.8220	0.9068
		p_2	0.1168	0.2436	0.3348	0.3964	0.4664	0.5520
	50	p_1	0.1608	0.3600	0.5584	0.7600	0.8840	0.9524
		p_2	0.1240	0.2460	0.3300	0.3940	0.4760	0.5680
	60	p_1	0.1664	0.3952	0.6132	0.8044	0.9208	0.9704
		p_2	0.1444	0.2728	0.3620	0.4384	0.5152	0.6056
-7	70	p_1	0.1832	0.4308	0.6640	0.8408	0.9444	0.9828
		p_2	0.1516	0.3244	0.4016	0.4912	0.5724	0.6728
	100	p_1	0.2160	0.5100	0.7688	0.9232	0.9816	0.9964
		p_2	0.1700	0.3472	0.4424	0.5416	0.6424	0.7448
	150	p_1	0.2836	0.6416	0.8908	0.9812	0.9976	1.0000
		p_2	0.2368	0.3788	0.4884	0.6140	0.7396	0.8404
	200	p_1	0.3496	0.7300	0.9472	0.9960	0.9996	1.0000
		p_2	0.2728	0.4068	0.5256	0.6696	0.7960	0.8952
	40	p_1	0.1308	0.3056	0.4924	0.6776	0.8260	0.9100
		p_2	0.1136	0.2460	0.3384	0.4044	0.4748	0.5576
	50	p_1	0.1604	0.3628	0.5624	0.7628	0.8860	0.9528
-7.5		p_2	0.1448	0.2812	0.3732	0.4448	0.5212	0.6160
	60	p_1	0.1644	0.3956	0.6180	0.8076	0.9212	0.9728
		p_2	0.1516	0.3112	0.3992	0.4740	0.5604	0.6492
	70	p_1	0.1816	0.4368	0.6700	0.8460	0.9464	0.9848
		p_2	0.1524	0.3284	0.4132	0.5008	0.5844	0.6816
	100	p_1	0.2152	0.5152	0.7740	0.9280	0.9824	0.9964
		p_2	0.1716	0.3552	0.4540	0.5500	0.6540	0.7568
	150	p_1	0.2852	0.6460	0.8932	0.9824	0.9976	1.0000
		p_2	0.2420	0.3880	0.5052	0.6256	0.7488	0.8488
	200	p_1	0.3528	0.7360	0.9504	0.9964	0.9996	1.0000
		p_2	0.2808	0.4200	0.5444	0.6860	0.8076	0.9000
	40	p_1	0.1300	0.3072	0.4952	0.6808	0.8288	0.9116
		p_2	0.1124	0.2476	0.3432	0.4108	0.4808	0.5624
	50	p_1	0.1592	0.3652	0.5656	0.7648	0.8908	0.9552
		p_2	0.1444	0.2856	0.3772	0.4508	0.5292	0.6252
	60	p_1	0.1648	0.3984	0.6236	0.8092	0.9224	0.9732
		p_2	0.1524	0.3176	0.4012	0.4784	0.5672	0.6592
-8	70	p_1	0.1808	0.4388	0.6748	0.8476	0.9472	0.9848
		p_2	0.1524	0.3320	0.4216	0.5080	0.5948	0.6884
	100	p_1	0.2148	0.5188	0.7784	0.9292	0.9824	0.9968
		p_2	0.1728	0.3604	0.4652	0.5588	0.6676	0.7652
	150	p_1	0.2856	0.6496	0.8968	0.9828	0.9984	1.0000
		p_2	0.2440	0.3988	0.5180	0.6384	0.7620	0.8576
	200	p_1	0.3580	0.7392	0.9508	0.9968	0.9996	1.0000
		p_2	0.2892	0.4296	0.5588	0.6984	0.8148	0.9072

Table A4. Simulated Type I error probabilities of hypothesis testing problem (27).

n_1	n_2	η_1^2	η_1^2	$\lambda_1 = 4, \lambda_2 = 5$ p_3	p_4	$\lambda_1 = 6, \lambda_2 = 8$ p_3	p_4
		0.1	0.3	0.0656	0.0676	0.0516	0.0428
		0.2	0.5	0.0688	0.0516	0.0508	0.0268
40	50	0.3	0.7	0.0684	0.0484	0.0512	0.0244
		0.4	0.9	0.0680	0.0476	0.0516	0.0244
		0.5	1.0	0.0652	0.0392	0.0472	0.0208
		0.1	0.3	0.0652	0.0760	0.0484	0.0528
		0.2	0.5	0.0628	0.0540	0.0504	0.0312
50	50	0.3	0.7	0.0612	0.0408	0.0504	0.0232
		0.4	0.9	0.0600	0.0348	0.0484	0.0188
		0.5	1.0	0.0540	0.0264	0.0440	0.0132
		0.1	0.3	0.0596	0.0576	0.0456	0.0520
		0.2	0.5	0.0580	0.0416	0.0460	0.0236
50	60	0.3	0.7	0.0564	0.0352	0.0448	0.0180
		0.4	0.9	0.0552	0.0304	0.0448	0.0168
		0.5	1.0	0.0500	0.0236	0.0376	0.0096

Table A4. *Cont.*

n_1	n_2	η_1^2	η_1^2	$\lambda_1 = 4, \lambda_2 = 5$		$\lambda_1 = 6, \lambda_2 = 8$	
				p_3	p_4	p_3	p_4
60	70	0.1	0.3	0.0592	0.0492	0.0464	0.0548
		0.2	0.5	0.0536	0.0356	0.0452	0.0300
		0.3	0.7	0.0508	0.0324	0.0448	0.0232
		0.4	0.9	0.0492	0.0300	0.0432	0.0184
		0.5	1.0	0.0444	0.0220	0.0372	0.0112
70	80	0.1	0.3	0.0576	0.0452	0.0488	0.0444
		0.2	0.5	0.0528	0.0324	0.0464	0.0332
		0.3	0.7	0.0512	0.0272	0.0448	0.0288
		0.4	0.9	0.0504	0.0224	0.0444	0.0260
		0.5	1.0	0.0404	0.0184	0.0396	0.0168
80	90	0.1	0.3	0.0512	0.0456	0.0440	0.0732
		0.2	0.5	0.0468	0.0332	0.0432	0.0532
		0.3	0.7	0.0456	0.0280	0.0416	0.0428
		0.4	0.9	0.0452	0.0284	0.0416	0.0372
		0.5	1.0	0.0404	0.0232	0.0380	0.0256
90	120	0.1	0.3	0.0584	0.0300	0.0476	0.0460
		0.2	0.5	0.0540	0.0364	0.0496	0.0428
		0.3	0.7	0.0508	0.0248	0.0480	0.0404
		0.4	0.9	0.0492	0.0272	0.0464	0.0320
		0.5	1.0	0.0436	0.0208	0.0420	0.0232
120	150	0.1	0.3	0.0576	0.0216	0.0460	0.0496
		0.2	0.5	0.0556	0.0220	0.0480	0.0496
		0.3	0.7	0.0560	0.0164	0.0472	0.0444
		0.4	0.9	0.0552	0.0220	0.0472	0.0408
		0.5	1.0	0.0504	0.0172	0.0472	0.0312
150	200	0.1	0.3	0.0548	0.0188	0.0448	0.0384
		0.2	0.5	0.0552	0.0192	0.0488	0.0340
		0.3	0.7	0.0552	0.0160	0.0476	0.0348
		0.4	0.9	0.0548	0.0168	0.0480	0.0368
		0.5	1.0	0.0512	0.0192	0.0484	0.0372

Table A5. Simulated powers of hypothesis testing problem (27) ($\lambda_1 = 4, \lambda_2 = 5$).

n_1	n_2	η_1^2	η_2^2	p	$\xi_1 - \xi_2$				
					0.1	**0.15**	**0.2**	**0.25**	**0.3**
40	50	0.1	0.3	p_3	0.5820	0.8376	0.9460	0.9820	0.9892
				p_4	0.5736	0.8284	0.9388	0.9736	0.9800
		0.2	0.5	p_3	0.3244	0.4960	0.6628	0.8068	0.8948
				p_4	0.2172	0.3604	0.5488	0.7092	0.8320
		0.3	0.7	p_3	0.2196	0.3356	0.4552	0.5672	0.6876
				p_4	0.1248	0.1972	0.2812	0.3952	0.5200
		0.4	0.9	p_3	0.1696	0.2504	0.3384	0.4272	0.5192
				p_4	0.0900	0.1328	0.1848	0.2424	0.3232
		0.5	1.0	p_3	0.1316	0.1912	0.2612	0.3456	0.4188
				p_4	0.0652	0.0848	0.1100	0.1488	0.1956
50	50	0.1	0.3	p_3	0.6068	0.8288	0.9440	0.9812	0.9888
				p_4	0.6408	0.8696	0.9492	0.9744	0.9840
		0.2	0.5	p_3	0.3308	0.5288	0.6900	0.8112	0.9004
				p_4	0.2740	0.4744	0.6692	0.8092	0.9040
		0.3	0.7	p_3	0.2264	0.3472	0.4912	0.6060	0.7228
				p_4	0.1492	0.2508	0.3900	0.5272	0.6608
		0.4	0.9	p_3	0.1780	0.2528	0.3532	0.4660	0.5568
				p_4	0.1108	0.1580	0.2372	0.3392	0.4480
		0.5	1.0	p_3	0.1428	0.2092	0.2764	0.3672	0.4616
				p_4	0.0636	0.0992	0.1364	0.1968	0.2672

Table A5. *Cont.*

n_1	n_2	η_1^2	η_2^2	p	$\xi_1 - \xi_2$				
					0.1	**0.15**	**0.2**	**0.25**	**0.3**
50	60	0.1	0.3	p_3	0.6376	0.8716	0.9632	0.9876	0.9944
				p_4	0.6388	0.8752	0.9624	0.9836	0.9888
		0.2	0.5	p_3	0.3440	0.5480	0.7220	0.8540	0.9276
				p_4	0.2736	0.4784	0.6812	0.8352	0.9184
		0.3	0.7	p_3	0.2252	0.3552	0.4988	0.6328	0.7476
				p_4	0.1300	0.2536	0.3900	0.5264	0.6448
		0.4	0.9	p_3	0.1748	0.2560	0.3624	0.4712	0.5800
				p_4	0.0944	0.1432	0.2456	0.3484	0.4496
		0.5	1.0	p_3	0.1416	0.2024	0.2800	0.3732	0.4696
				p_4	0.0524	0.0820	0.1216	0.1808	0.2640
60	70	0.1	0.3	p_3	0.6508	0.9000	0.9764	0.9940	0.9972
				p_4	0.6540	0.9172	0.9752	0.9896	0.9952
		0.2	0.5	p_3	0.3588	0.5568	0.7412	0.8804	0.9480
				p_4	0.2812	0.5208	0.7492	0.8936	0.9504
		0.3	0.7	p_3	0.2416	0.3720	0.5164	0.6464	0.7752
				p_4	0.1504	0.2748	0.4400	0.6124	0.7692
		0.4	0.9	p_3	0.1844	0.2748	0.3796	0.4932	0.5960
				p_4	0.0852	0.1576	0.2616	0.3920	0.5208
		0.5	1.0	p_3	0.1360	0.2156	0.3052	0.3996	0.4908
				p_4	0.0532	0.0840	0.1448	0.2292	0.3308
70	80	0.1	0.3	p_3	0.7184	0.9324	0.9896	0.9976	0.9992
				p_4	0.7136	0.9356	0.9864	0.9948	0.9972
		0.2	0.5	p_3	0.4024	0.6232	0.8060	0.9232	0.9680
				p_4	0.3084	0.5796	0.8160	0.9236	0.9712
		0.3	0.7	p_3	0.2664	0.4252	0.5740	0.7212	0.8368
				p_4	0.1684	0.3108	0.5108	0.6968	0.8304
		0.4	0.9	p_3	0.1932	0.3120	0.4356	0.5440	0.6712
				p_4	0.1092	0.1996	0.3216	0.4580	0.6220
		0.5	1.0	p_3	0.1596	0.2432	0.3460	0.4520	0.5552
				p_4	0.0700	0.1256	0.1980	0.3036	0.4316
80	90	0.1	0.3	p_3	0.7520	0.9484	0.9952	1.0000	1.0000
				p_4	0.7336	0.9496	0.9888	0.9980	1.0000
		0.2	0.5	p_3	0.4080	0.6536	0.8400	0.9392	0.9784
				p_4	0.3272	0.6064	0.8412	0.9416	0.9756
		0.3	0.7	p_3	0.2548	0.4280	0.6060	0.7592	0.8684
				p_4	0.1760	0.3352	0.5372	0.7252	0.8680
		0.4	0.9	p_3	0.1872	0.3004	0.4356	0.5808	0.6992
				p_4	0.1764	0.2136	0.3460	0.4980	0.6480
		0.5	1.0	p_3	0.1464	0.2412	0.3496	0.4740	0.5940
				p_4	0.0672	0.1232	0.2172	0.3336	0.4740
90	120	0.1	0.3	p_3	0.8372	0.9796	0.9992	1.0000	1.0000
				p_4	0.7480	0.9716	0.9988	0.9996	0.9996
		0.2	0.5	p_3	0.4680	0.7372	0.9164	0.9704	0.9948
				p_4	0.2884	0.6076	0.8640	0.9652	0.9952
		0.3	0.7	p_3	0.3152	0.4984	0.6868	0.8500	0.9320
				p_4	0.1416	0.3012	0.5368	0.7560	0.8960
		0.4	0.9	p_3	0.2396	0.3660	0.5124	0.6632	0.7924
				p_4	0.0956	0.1804	0.3140	0.4944	0.6756
		0.5	1.0	p_3	0.1876	0.3004	0.4164	0.5592	0.6856
				p_4	0.0660	0.1172	0.2052	0.3488	0.5124
120	150	0.1	0.3	p_3	0.9024	0.9972	0.9996	1.0000	1.0000
				p_4	0.7836	0.9960	0.9996	1.0000	1.0000
		0.2	0.5	p_3	0.5652	0.8260	0.9600	0.9948	0.9988
				p_4	0.3008	0.6672	0.9204	0.9904	0.9988
		0.3	0.7	p_3	0.3780	0.6032	0.7880	0.9180	0.9732
				p_4	0.1424	0.3564	0.6252	0.8408	0.9516
		0.4	0.9	p_3	0.2708	0.4440	0.6204	0.7596	0.8740
				p_4	0.0980	0.1996	0.3820	0.5836	0.7728
		0.5	1.0	p_3	0.2244	0.3668	0.5196	0.6680	0.7912
				p_4	0.0656	0.1368	0.2684	0.4480	0.6396

Table A6. Simulated powers of hypothesis testing problem (27) ($\lambda_1 = 6, \lambda_2 = 8$).

n_1	n_2	η_1^2	η_2^2	p	$\xi_1 - \xi_2$				
					0.1	**0.15**	**0.2**	**0.25**	**0.3**
40	50	0.1	0.3	p_3	0.6188	0.8600	0.9620	0.9896	0.9964
				p_4	0.7100	0.9144	0.9752	0.9884	0.9904
		0.2	0.5	p_3	0.3324	0.5268	0.7068	0.8412	0.9180
				p_4	0.2448	0.4528	0.6628	0.8176	0.9112
		0.3	0.7	p_3	0.2196	0.3468	0.4872	0.6092	0.7296
				p_4	0.1136	0.2100	0.3400	0.4896	0.6348
		0.4	0.9	p_3	0.1596	0.2528	0.3536	0.4624	0.5536
				p_4	0.0676	0.1208	0.1964	0.2856	0.3952
		0.5	1.0	p_3	0.1248	0.1932	0.2724	0.3664	0.4584
				p_4	0.0404	0.0592	0.0932	0.1380	0.2008
50	50	0.1	0.3	p_3	0.6444	0.8628	0.9608	0.9880	0.9940
				p_4	0.7884	0.9400	0.9808	0.9888	0.9920
		0.2	0.5	p_3	0.3508	0.5596	0.7280	0.8484	0.9268
				p_4	0.3448	0.5892	0.7932	0.9048	0.9584
		0.3	0.7	p_3	0.2348	0.3664	0.5244	0.6492	0.7556
				p_4	0.1620	0.3028	0.4652	0.6416	0.7752
		0.4	0.9	p_3	0.1776	0.2672	0.3780	0.5012	0.5980
				p_4	0.1008	0.1732	0.2756	0.4016	0.5256
		0.5	1.0	p_3	0.1428	0.2184	0.2984	0.4032	0.5044
				p_4	0.0472	0.0824	0.1360	0.2008	0.2908
50	60	0.1	0.3	p_3	0.6864	0.9016	0.9768	0.9940	0.9988
				p_4	0.8100	0.9608	0.9872	0.9936	0.9948
		0.2	0.5	p_3	0.3732	0.5852	0.7628	0.8864	0.9500
				p_4	0.3456	0.6116	0.8260	0.9324	0.9712
		0.3	0.7	p_3	0.2372	0.3876	0.5420	0.6864	0.7920
				p_4	0.1556	0.3040	0.4908	0.7104	0.8064
		0.4	0.9	p_3	0.1748	0.2712	0.3972	0.5136	0.6288
				p_4	0.0848	0.1748	0.2820	0.4216	0.5576
		0.5	1.0	p_3	0.1384	0.2164	0.3128	0.4144	0.5192
				p_4	0.0384	0.0756	0.1280	0.2028	0.2992
60	70	0.1	0.3	p_3	0.6956	0.9256	0.9880	0.9972	0.9992
				p_4	0.8476	0.9692	0.9928	0.9960	0.9984
		0.2	0.5	p_3	0.3976	0.6040	0.7916	0.9120	0.9652
				p_4	0.4684	0.7404	0.9084	0.9648	0.9824
		0.3	0.7	p_3	0.2560	0.4144	0.5640	0.7060	0.8260
				p_4	0.2464	0.4588	0.6588	0.8164	0.9180
		0.4	0.9	p_3	0.1868	0.2980	0.4220	0.5368	0.6472
				p_4	0.1444	0.2760	0.4452	0.6084	0.7352
		0.5	1.0	p_3	0.1460	0.2356	0.3376	0.4476	0.5512
				p_4	0.0680	0.1448	0.2520	0.3924	0.5268
70	80	0.1	0.3	p_3	0.7588	0.9536	0.9960	0.9996	1.0000
				p_4	0.8728	0.9800	0.9948	0.9984	0.9992
		0.2	0.5	p_3	0.4408	0.6696	0.8464	0.9452	0.9844
				p_4	0.5476	0.7940	0.9300	0.9784	0.9912
		0.3	0.7	p_3	0.2908	0.4616	0.6248	0.7720	0.8788
				p_4	0.3336	0.5676	0.7500	0.8776	0.9460
		0.4	0.9	p_3	0.2096	0.3424	0.4740	0.6020	0.7228
				p_4	0.2252	0.4000	0.5768	0.7200	0.8344
		0.5	1.0	p_3	0.1672	0.2756	0.3932	0.5044	0.6140
				p_4	0.1328	0.2616	0.4100	0.5716	0.7064
80	90	0.1	0.3	p_3	0.7948	0.9676	0.9976	1.0000	1.0000
				p_4	0.8956	0.9876	0.9996	1.0000	1.0000
		0.2	0.5	p_3	0.4520	0.7000	0.8764	0.9588	0.9896
				p_4	0.5752	0.8188	0.9456	0.9852	0.9964
		0.3	0.7	p_3	0.2804	0.4768	0.6552	0.8060	0.9012
				p_4	0.3608	0.5980	0.7816	0.8988	0.9596
		0.4	0.9	p_3	0.1992	0.3412	0.4888	0.6288	0.7528
				p_4	0.2500	0.4220	0.6048	0.7584	0.8608
		0.5	1.0	p_3	0.1604	0.2652	0.4056	0.5360	0.6552
				p_4	0.1696	0.3040	0.4660	0.6304	0.7684

Table A6. *Cont.*

n_1	n_2	η_1^2	η_2^2	p	$\xi_1 - \xi_2$				
					0.1	**0.15**	**0.2**	**0.25**	**0.3**
90	120	0.1	0.3	p_3	0.8720	0.9888	1.0000	1.0000	1.0000
				p_4	0.9152	0.9932	1.0000	1.0000	1.0000
		0.2	0.5	p_3	0.5252	0.7896	0.9372	0.9852	0.9992
				p_4	0.5560	0.8352	0.9652	0.9932	0.9992
		0.3	0.7	p_3	0.3468	0.5560	0.7432	0.8864	0.9500
				p_4	0.3412	0.5916	0.7936	0.9248	0.9764
		0.4	0.9	p_3	0.2616	0.3992	0.5748	0.7156	0.8440
				p_4	0.2404	0.4196	0.6148	0.7612	0.8808
		0.5	1.0	p_3	0.2068	0.3400	0.4756	0.6268	0.7424
				p_4	0.1680	0.3164	0.4988	0.6672	0.8004
120	150	0.1	0.3	p_3	0.9300	0.9976	1.0000	1.0000	1.0000
				p_4	0.9552	0.9984	1.0000	1.0000	1.0000
		0.2	0.5	p_3	0.6220	0.8628	0.9748	0.9972	0.9988
				p_4	0.5948	0.9008	0.9852	0.9992	0.9996
		0.3	0.7	p_3	0.4180	0.6512	0.8304	0.9444	0.9864
				p_4	0.3656	0.6460	0.8808	0.9676	0.9924
		0.4	0.9	p_3	0.3008	0.4964	0.6636	0.8060	0.9088
				p_4	0.2476	0.4532	0.6760	0.8544	0.9492
		0.5	1.0	p_3	0.2564	0.4144	0.5832	0.7260	0.8368
				p_4	0.2032	0.3680	0.5736	0.7588	0.8916

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