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On the Generalization of a Multiplicity Result

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Abstract: In this work, we shifted a recent multiplicity result by B. Ricceri from a Hilbert space to a Banach space setting by making use of a duality mapping relative to some increasing function. Using the min–max arguments, we provide conditions for an action functional to have at least two global minima.

Keywords: multiplicity; min–max; duality mapping

MSC: 49J27, 49J35

1. Introduction

Recently, in [1], the following result appeared:

Theorem 1. *Let X be a topological space, $(Y; \langle\cdot,\cdot\rangle)$ a real Hilbert space, $T \subset Y$ a convex set dense in Y, and $I : X \to \mathbb{R}$, $\varphi : X \to Y$ be two functions such that, for each $y \in T$, the function:*

$$x \to I(x) + \langle\varphi(x), y\rangle$$

is lower semicontinuous and inf-compact. Assume that there exists a point $x_0 \in X$ satisfying $\varphi(x_0) \neq 0$ and such that:

(a) x_0 *is a global minimum of both functions I and $\|\varphi(\cdot)\|^2$;*

(b) $\inf_{x\in X}\langle\varphi(x), \varphi(x_0)\rangle < \|\varphi(x_0)\|^2$.

Then, for each convex set $S \subseteq T$ dense in Y, there exists $\widetilde{y} \in S$ such that the function $x \to I(x) + \langle\varphi(x), \widetilde{y}\rangle$ has at least two global minima in X.

The proof relies on the Hilbert structure of the space Y, which leads to the application of a min–max theorem, which is due to B. Ricceri and was given in [2]; see Theorem 2 below. Since the application of Theorem 2 is not restricted to the setting of a Hilbert space, in this note, we ask the question whether in Theorem 1, one can replace the Hilbert space Y with some suitable Banach space. Moreover, the second question arises whether this generalization of Theorem 1 can be performed using the recent result [3] and its complement; see [4]. We answer positively these questions by utilizing the notion of the duality mapping relative to some increasing function. Our result seems to be of interest from a theoretical point of view since it shows how to replace with success the inner product with the duality mapping in certain cases. We show that results that require operations involving scalar products can be formulated in terms of duality mappings and monotonicity arguments about them.

This note is organized as follows. Firstly, we provide the necessary background information and next proceed to the main multiplicity result. Our main tools to prove it were provided by the recent results from [1,4]. Our approach to prove the main result must incorporate some optimization techniques related to constrained problems, which are not necessary in a Hilbert space setting, as well as arguments that are superfluous in that case, thereby showing that the idea of replacing the scalar product with the duality

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mapping is not merely the case of exchanging these two notions. We finish with some concluding remarks.

2. Preliminary Material

We provide a suitable background relying on the existing literature about the multiplicity theorem, the duality mapping, and the optimization tools coined so as to suit our setting. These notes came from various sources, which we cite when necessary, and are introduced in order to make our reasoning clear.

Following [2], we fix some notation and useful definitions. In $\overline{\mathbb{R}}$, the usual rules are applied. In this section, X is a topological space and Y is a convex set in a real Hausdorff topological vector space. Let $S \subset Y$ be a convex subset. By $\mathcal{A}_S$, we denote the family of functions $f : X \times Y \to \mathbb{R}$ such that for each fixed $y \in S$, the function $x \mapsto f(x,y)$ is lower semicontinuous and inf-compact on X. By $\mathcal{B}$, we mean the family of functions $f : X \times Y \to \mathbb{R}$ such that, either, for each $x \in X$, the function $y \mapsto f(x,y)$ is quasi-concave and continuous, or, for each $x \in X$, the function $y \mapsto f(x,y)$ is concave. We denote:

$$\alpha_f = \sup_{y\in Y} \inf_{x\in X} f(x,y),$$

$$\beta_f = \inf_{x\in X} \sup_{y\in Y} f(x,y).$$

We denote by C_f the family of all sets $S \subseteq Y$ such that:

$$\inf_{x\in X} \sup_{y\in Y} f(x,y) = \inf_{x\in X} \sup_{y\in S} f(x,y)$$

and by $\widetilde{C}_f$ the family of all sets $S \subseteq Y$ such that:

$$\sup_{y\in Y} f(x,y) = \sup_{y\in S} f(x,y) \text{ for all } x \in X.$$

If for each $x \in X$, there is a topology on Y for which S is dense and $y \mapsto f(x,y)$ is upper semicontinuous, then $S \subset \widetilde{C}_f$. We denote by τ_f the topology on Y generated by the family:

$$\{y \in Y : f(x,y) < r\}_{x\in X, r\in\mathbb{R}}.$$

As a consequence, τ_f is the weakest topology on Y for which $y \mapsto f(x,y)$ is upper semicontinuous for all $x \in X$. The theorem about the multiplicity due to B. Ricceri is as follows:

Theorem 2 (Theorem A)**.** *For every $g \in \mathcal{A}_Y \cap \mathcal{B}$, at least one of the following assertions holds:*

(a1) $\sup_Y \inf_X g = \inf_X \sup_Y g$;
(a2) *There exists $y^* \in Y$ such that the function $x \mapsto g(x,y^*)$ has at least two global minima.*

The above-mentioned theorem was used to prove Theorem 1 by showing that (a1) does not hold, so (a2) must hold.

Now from [3] and next from [4], a more complete version of Theorem 2 is given. We provide this result after [4]:

Theorem 3 (Theorem A′)**.** *Let $f : X \times Y \to \mathbb{R}$. Assume that there is a function $\xi : Y \to \mathbb{R}$ such that $f + \xi \in \mathcal{B}$ and:*

$$\alpha_{f+\xi} < \beta_{f+\xi}.$$

Then, for every convex set $S \subset C_{f+\psi}$, for every bounded function $\psi : X \to \mathbb{R}$, and for every $\lambda > 0$ such that $\lambda f + \psi \in \mathcal{A}_S$ and:

$$\lambda > \frac{\sup_{x\in X} \psi(x) - \inf_{x\in X} \psi(x)}{\beta_{f+\xi} - \alpha_{f+\xi}},$$

there exists $y^* \in Y$ such that the function $x \mapsto \lambda f(x, y^*) + \psi(x)$ has at least two global minima.

A normed linear space Y is called uniformly convex, if for each $\varepsilon \in (0, 2]$, there exists $\delta(\varepsilon) > 0$ such that if $\|x\| = 1, \|y\| = 1$ and if $\|x - y\| \geq \varepsilon$, then $\|x + y\| \leq 2(1 - \delta(\varepsilon))$. A uniformly convex space is necessarily reflexive due to the Pettis–Milman theorem.

In the proof of our main result, we need the notion of a duality mapping relative to a normalization function, which we recall from [5], in a special case when both Y and Y^* are uniformly convex. The notes in [5] were about a more general case, but we need only some specific settings. A continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a normalization function if it is strictly increasing, $\phi(0) = 0$ and $\phi(r) \to \infty$ with $r \to \infty$. A duality mapping on Y corresponding to a normalization function ϕ is an operator $A : Y \to Y^*$ such that:

$$\|A(u)\|_* = \phi(\|u\|), \langle A(u), u\rangle = \|A(u)\|_* \|u\| \text{ for all } u \in Y.$$

Then, $A : Y \to Y^*$ is a homeomorphism, d-monotone, and with the potential $\psi(u) = \int_0^{\|u\|} \phi(t)dt$.

We require the Karush–Kuhn–Tucker theorem, which provides necessary optimality conditions and which was taken after [6].

Theorem 4. *Let $f, g : Y \to \mathbb{R}$ be Fréchet differentiable. Assume that u_0 is a minimizer of a functional $f : Y \to \mathbb{R}$ satisfying the constraint:*

$$g(x) \leq 0.$$

Let the Slater constraint qualification hold, i.e., there is some x_0 that $g(x_0) < 0$. Then, there is a number $\mu \geq 0$ such that:

$$f'(u_0) + \mu g'(u_0) = 0.$$

3. A Multiplicity Result

Let $p \geq 2$ be fixed. In what follows, we assumed that Y and Y^* are uniformly convex, and we took a duality mapping $A : Y \to Y^*$ corresponding to a normalization function $t \mapsto t^{p-1}$. Note that when $p = 2$, we recover the original result by Ricceri, and we do not need to proceed in the way we do below.

Theorem 5. *Let X be a topological space, $T \subset Y$ be a convex set dense in Y, and $I, \psi : X \to \mathbb{R}$, $\varphi : X \to Y$ be three functions such that ψ is bounded and for each $y \in T$, the function:*

$$x \to I(x) + \psi(x) + \langle A(\varphi(x)), y\rangle$$

is lower semicontinuous and inf-compact. Moreover, assume that there exists a point $x_0 \in X$, with $\varphi(x_0) \neq 0$, such that:

(a) *x_0 is a global minimum of both I and $\|\varphi(\cdot)\|^{p-1}$;*
(b) *$\inf_{x \in X}\langle A(\varphi(x)), \varphi(x_0)\rangle < \|\varphi(x_0)\|^p$.*

Then, for each convex set $S \subseteq T$ that is dense in Y, there exists $\tilde{y} \in S$ such that the function $x \to I(x) + \psi(x) + \langle A(\varphi(x)), \tilde{y}\rangle$ has at least two global minima over X.

Proof. Using Assumption (b), we found $\tilde{x} \in X$ and $r > 0$ such that:

$$\begin{aligned} &\sup_{x \in X} \psi(x) - \inf_{x \in X} \psi(x) \\ &< I(x_0) - I(\tilde{x}) + \left(\|\varphi(x_0)\|^{p-1} - \frac{\langle A(\varphi(\tilde{x})), \varphi(x_0)\rangle}{\|\varphi(x_0)\|}\right) r \end{aligned} \tag{1}$$

Next, by Condition (a), we have:

$$I(x_0) + r\|\varphi(x_0)\| = \inf_{x \in X}(I(x) + r\|\varphi(x)\|).$$

Observe that since functional:

$$y \to \inf_{x \in X}(I(x) + \langle A(\varphi(x)), y \rangle)$$

is weakly upper semicontinuous, it has an argument of a maximum over a weakly compact set B_r (a closed ball centred at zero with radius r). Hence, there is some $\widetilde{y} \in B_r$ that:

$$\inf_{x \in X}(I(x) + \langle A(\varphi(x)), \widetilde{y} \rangle) = \sup_{y \in B_r} \inf_{x \in X}(I(x) + \langle A(\varphi(x)), y \rangle).$$

By the standard application of the Karush–Kuhn–Tucker Theorem, we know that functional $y \to \langle A(\varphi(x_0)), y \rangle$ attains its unique argument of a maximum at the boundary of the ball B_r at a point:

$$y_0 = \frac{r\varphi(x_0)}{\|\varphi(x_0)\|}. \tag{2}$$

Indeed, let us consider the following optimization problem:

$$-\langle A(\varphi(x_0)), y \rangle \to \inf$$

subject to the conditions:

$$\|y\|^p - r^p \leq 0.$$

This problem is obviously solvable by some y_0 since the action functional is sequentially weakly continuous and the constraint set is sequentially weakly compact. Since the Slater constraint qualification is satisfied (i.e., minimization takes place over a closed ball) and the functionals involved are continuously differentiable, we see that there is some $\lambda_0 \geq 0$ such that for all $h \in Y$, it holds that:

$$-\langle A(\varphi(x_0)), h \rangle + \lambda_0^{p-1} \langle A(y_0), h \rangle = 0, \lambda_0 \left(\|y_0\|^p - r^p\right) = 0$$

Let us consider two cases about λ_0. When $\lambda_0 = 0$, it follows that $\varphi(x_0) = 0$, which is impossible. Hence, $\lambda_0 > 0$; therefore, $\|y_0\|^p - r^p = 0$, and also by a direct calculation, we see that:

$$y_0 = A^{-1}\left(\frac{1}{\lambda_0^{p-1}} A(\varphi(x_0))\right) = \frac{1}{\lambda_0}\varphi(x_0)$$

We further see that $\lambda_0 = \frac{\|\varphi(x_0)\|}{r}$, and so, (2) holds. Moreover, the maximal value is calculated as follows:

$$\left\langle A(\varphi(x_0)), \frac{1}{\lambda_0}\varphi(x_0) \right\rangle = \frac{1}{\lambda_0}\langle A(\varphi(x_0)), \varphi(x_0) \rangle = \frac{1}{\lambda_0}\|\varphi(x_0)\|^p = r\|\varphi(x_0)\|^{p-1}.$$

With these calculations in mind, we consider two cases for $\widetilde{y}$, namely $\widetilde{y} = y_0$ and $\widetilde{y} \neq y_0$. Let us first take $\widetilde{y} \neq y_0$. Then, we have, due to the uniqueness of the argument of a maximum, that:

$$\inf_{x \in X}(I(x) + \langle A(\varphi(x)), \widetilde{y} \rangle) \leq I(x_0) + \langle A(\varphi(x_0)), \widetilde{y} \rangle < I(x_0) + r\|\varphi(x_0)\|^{p-1}.$$

When $\widetilde{y} = y_0$, we see using (1) that:

$$\inf\nolimits_{x \in X}(I(x) + \langle A(\varphi(x)), \widetilde{y} \rangle) \leq I(\widetilde{x}) + \langle A(\varphi(\widetilde{x})), \widetilde{y} \rangle = I(\widetilde{x}) + \langle A(\varphi(\widetilde{x})), \widetilde{y} \rangle =$$

$$I(\widetilde{x}) + \frac{r}{\|\varphi(x_0)\|}\langle A(\varphi(\widetilde{x})), \varphi(x_0) \rangle < I(x_0) + r\|\varphi(x_0)\|^{p-1}.$$

It now follows that:

$$\sup_{y \in B_r} \inf_{x \in X} (I(x) + \langle A(\varphi(x)), y \rangle) < \inf_{x \in X} \sup_{y \in B_r} (I(x) + \langle A(\varphi(x)), y \rangle)$$

For a set $S \subset T$, which is a convex set dense in Y, we obtain:

$$\sup_{y \in B_r \cap S} \langle A(\varphi(x)), y \rangle = \sup_{y \in B_r} \langle A(\varphi(x)), y \rangle.$$

Hence:

$$\sup\nolimits_{y \in B_r \cap S} \inf\nolimits_{x \in X} (I(x) + \langle A(\varphi(x)), y \rangle) < \inf\nolimits_{x \in X} \sup\nolimits_{y \in B_r} (I(x) + \langle A(\varphi(x)), y \rangle) =$$

$$\inf\nolimits_{x \in X} \sup\nolimits_{y \in B_r \cap S} (I(x) + \langle A(\varphi(x)), y \rangle)$$

Therefore, we have:

$$I(x_0) - I(\widetilde{x}) + \left(\|\varphi(x_0)\|^{p-1} - \frac{\langle A(\varphi(\widetilde{x}), \varphi(x_0) \rangle}{\|\varphi(x_0)\|} \right) r$$

$$\leq \inf_{x \in X} \sup_{y \in B_r \cap S} (I(x) + \langle A(\varphi(x)), y \rangle) - \sup_{y \in B_r \cap S} \inf_{x \in X} (I(x) + \langle A(\varphi(x)), y \rangle) .$$

Due to (1), we can choose $\lambda = 1$ in Theorem 3 in order to obtain the assertion. □

As a corollary, we obtain a direct counterpart of Theorem 1:

Theorem 6. *Let X be a topological space, $T \subset Y$ be a convex set dense in Y, and $I : X \to \mathbb{R}$, $\varphi : X \to Y$ be two functions such that, for each $y \in T$, the function:*

$$x \to I(x) + \langle A(\varphi(x)), y \rangle$$

is lower semicontinuous and inf-compact. Moreover, assume that there exists a point $x_0 \in X$, with $\varphi(x_0) \neq 0$, such that:

(a) x_0 is a global minimum of both functions I and $\|\varphi(\cdot)\|^{p-1}$;
(b) $\inf_{x \in X} \langle A(\varphi(x)), \varphi(x_0) \rangle < \|\varphi(x_0)\|^p$.

Then, for each convex set $S \subseteq T$ dense in Y, there exists $\widetilde{y} \in S$ such that the function $x \to I(x) + \langle A(\varphi(x)), \widetilde{y} \rangle$ has at least two global minima in X.

Proof. $\psi = 0$ in Theorem 5 in order to obtain the assertion. □

4. Conclusions

In this work, using the min–max arguments, we provided conditions under which an action functional acting on a uniformly convex Banach space with a uniformly convex dual has at least two global minima. We extended known results from a Hilbert space setting to the Banach space one. Thereby, we can ask questions about extensions of some already published works, such as for example [7] from the Hilbert space to the Banach one, as well as ask questions about providing applications for Theorem 1 as an example for non-local problems, such as those given in [8]. Theorem 1 was applied just once to a system of Dirichlet problems driven by the (negative) Laplacian in the same work, as well as to the case of systems involving the (negative) p-Laplacian in [9], and to the system of equations involving the (negative) Laplacian in [1].

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