mathematics

MDPI

Article

Applications of Certain *p*-Valently Analytic Functions

Georgia Irina Oros [1,†], Gheorghe Oros [1,†] and Shigeyoshi Owa [2,*,†]

[1] Department of Mathematics and Computer Science, Faculty of Informatics and Sciences, University of Oradea, 410087 Oradea, Romania; georgia_oros_ro@yahoo.co.uk (G.I.O.); gh_oros@yahoo.com (G.O.)
[2] 1 Decembrie 1918 University, 510009 Alba Iulia, Romania
* Correspondence: shige21@ican.zag.ne.jp
† These authors contributed equally to this work.

Abstract: In this paper, a new operator $D^s f$, with s a real number, is defined considering functions that belong to the known class of p-valent analytic functions in the open unit disk U. Applying this operator, a new subclass of p-valently analytic functions is introduced and some interesting subordination- and coefficient-related properties of the functions in this class are discussed. It is also shown that for certain values given to the parameters involved in the definition of the class, p-valently starlike and p-valently convex functions of certain orders can be obtained, respectively. Examples are also given as applications of the newly proven results.

Keywords: p-valently analytic function; p-valently starlike function; p-valently convex function; subordination; coefficient problem

MSC: 30C45

check for updates

Citation: Oros, G.I.; Oros, G.; Owa, S. Applications of Certain *p*-Valently Analytic Functions. *Mathematics* **2022**, *10*, 910. https://doi.org/10.3390/math10060910

Academic Editor: Carlo Bianca

Received: 30 January 2022
Accepted: 10 March 2022
Published: 12 March 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Let $A_p(n)$ denote the class of functions f of the form:

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots\}) \tag{1}$$

which are p-valently analytic in the open unit disk:

$$U = \{z \in \mathbb{C} : 0 \leq |z| < 1\}.$$

p-valently analytic functions have been investigated earlier regarding many aspects such as the starlikeness of order α [1], the introduction of new subclasses [2], subordination- and coefficient-related properties [3], and the use of operators in defining new subclasses [4].

p-valently analytic functions still inspire studies with interesting outcomes. Such results were recently published regarding the properties of p-valently analytic functions associated with cosine and exponential functions [5], concerning subordination and superordination results using operators on p-valently analytic functions [6,7], and the use of operators on p-valently analytic functions for introducing new classes [8].

The results of the study presented in this paper were also inspired and motivated by recently published papers, which are presented next. Generalised differential operators were used to introduce new classes of p-valent functions, which were investigated regarding many aspects including coefficient estimates and inclusion properties, in [9,10]. In [11], a subclass of multivalent functions was defined based on a newly introduced operator involving p-valent functions, and geometric properties were discussed for that subclass. The properties of p-valent analytic functions were obtained using the well-known Dziok–Srivastava operator in [12].

Having all those published studies as inspiration, in this paper, a new operator is next introduced, and using it, a new class of functions is defined and studied regarding several subordination and coefficient properties.

Definition 1. *For $f \in A_p(n)$, we define:*

$$D^0 f(z) = f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \tag{2}$$

$$D^1 f(z) = Df(z) = \frac{1}{p} z f'(z) = z^p + \sum_{k=p+n}^{\infty} \frac{k}{p} a_k z^k, \tag{3}$$

$$D^2 f(z) = D(Df(z)) = z^p + \sum_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^2 a_k z^k, \tag{4}$$

and:

$$D^j f(z) = D(D^{j-1} f(z)) = z^p + \sum_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^j a_k z^k \quad (j \in \mathbb{N}). \tag{5}$$

Furthermore, we introduce:

$$D^{-1} f(z) = p \int_0^z \frac{f(t)}{t} dt = z^p + \sum_{k=p+n}^{\infty} \frac{p}{k} a_k z^k, \tag{6}$$

$$D^{-2} f(z) = D^{-1}(D^{-1} f(z)) = z^p + \sum_{k=p+n}^{\infty} \left(\frac{p}{k}\right)^2 a_k z^k, \tag{7}$$

and:

$$D^{-j} f(z) = D^{-1}(D^{-j+1} f(z)) = z^p + \sum_{k=p+n}^{\infty} \left(\frac{p}{k}\right)^j a_k z^k \quad (j \in \mathbb{N}). \tag{8}$$

Definition 2. *With those definitions, we introduce:*

$$D^j f(z) = z^p + \sum_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^j a_k z^k \quad (z \in U) \tag{9}$$

for $j = \ldots, -1, 0, 1, 2, \ldots$.

Remark 1. *For a function f in the class $A_1(1)$ $(p = 1, n = 1)$, Flett [13] introduced:*

$$I^s f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1}{k}\right)^s a_k z^k \tag{10}$$

for some real number s. We know that:

$$I^s f(z) = \frac{1}{\Gamma(s)} \int_0^1 \left(\log \frac{1}{t}\right)^{s-1} f(tz) dt \quad (s > 0), \tag{11}$$

where $\Gamma(s)$ is the gamma function. Therefore, for $p = 1$ and $n = 1$ in (6) and $s = 1$ in (10), we have:

$$D^{-1} f(z) = I^1 f(z). \tag{12}$$

This show us that:

$$D^{-j} f(z) = I^j f(z) \quad (j \in \mathbb{N}) \tag{13}$$

for $p = 1$, $n = 1$, and $s = j$.

Considering $D^j f$ in (9), we introduce:

$$D^s f(z) = z^p + \sum_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^s a_k z^k \quad (f(z) \in A_p(n),\ z \in U) \tag{14}$$

for some real number $s \in \mathbb{R}$.

With the above operator $D^s f$, we define:

Definition 3. *If $f \in A_p(n)$ satisfies:*

$$\mathrm{Re}\left(\frac{D^s f(z)}{D^t f(z)}\right) > \alpha \quad (z \in U) \tag{15}$$

for $0 \leq \alpha < 1$, $s \in \mathbb{R}$, and $t \in \mathbb{R}$, then we say that f is in the class $A_p(n, s, t; \alpha)$.

Remark 2. *If $s = 1$ and $t = 0$ in (15), then f satisfies:*

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > p\alpha \quad (z \in U). \tag{16}$$

Therefore, $f \in A_p(n, 1, 0; \alpha)$ is p-valently starlike of order $p\alpha$ in U. If $s = 2$ and $t = 1$ in (15), then f satisfies:

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > p\alpha \quad (z \in U). \tag{17}$$

Thus, we know that $f \in A_p(n, 2, 1; \alpha)$ is p-valently convex of order $p\alpha$ in U (cf. Owa [14]).

Definition 4. *Let f and g be analytic in U. If there exists a function w that is analytic in U with $w(0) = 1$ and $|w(z)| < 1$ $(z \in U)$ and such that $f(z) = g(w(z))$ $(z \in U)$, then we say that f is subordinate to g in U.*

We denote this subordination by:

$$f(z) \prec g(z) \quad (z \in U). \tag{18}$$

Further, if g is univalent in U, then the subordination (18) is equivalent to $f(0) = g(0)$ and $f(U) \subseteq g(U)$ (cf. Pommerenke [15] and Miller and Mocanu [16]).

2. Subordination Properties

To discuss the subordination properties for our functions, we have to recall here the following results due to Miller and Mocanu [17].

Lemma 1 ([17]). *Let β and γ be complex numbers. Let h be convex (univalent) in U such that:*

$$\mathrm{Re}(\beta h(z) + \gamma) > 0 \quad (z \in U). \tag{19}$$

If a function p is analytic in U with:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \quad (p(0) = h(0)), \tag{20}$$

then:

$$p(z) \prec h(z) \quad (z \in U). \tag{21}$$

Furthermore, if the Briot–Bouquet differential equation:

$$g(z) + \frac{zg'(z)}{\beta g(z) + \gamma} = h(z) \quad (g(0) = h(0)) \tag{22}$$

has a univalent solution g, then:

$$p(z) \prec g(z) \prec h(z) \quad (z \in U) \tag{23}$$

and g is said to be the best dominant of (20).

Remark 3. *If the univalent function g in U has the property such that $p(z) \prec g(z)$ for all p satisfying (20), then g is called a dominant of (20). If $\widetilde{g}$ is a dominant of $g(z) \prec g(z)$ for all dominants g of (20), then $\widetilde{g}$ is said to be the best dominant of (20).*

The following lemma is due to Miller and Mocanu [18].

Lemma 2 ([18]). *If β is a real number such that $0 \leq \beta < 1$, then the differential equation:*

$$g(z) + \frac{zg'(z)}{g(z)} = \frac{1 + (1-2\beta)z}{1-z} \quad (z \in U) \tag{24}$$

has a univalent solution g given by:

$$g(z) = \begin{cases} \dfrac{(1-2\beta)z}{(1-z)(1-(1-z)^{1-2\beta})} & \left(\beta \neq \dfrac{1}{2}\right) \\ \dfrac{z}{(z-1)\log(1-z)} & \left(\beta = \dfrac{1}{2}\right). \end{cases} \tag{25}$$

Now, we derive:

Theorem 1. *Let β and γ be complex numbers and h be convex (univalent) in U such that:*

$$\mathrm{Re}(\beta h(z) + \gamma) > 0 \quad (z \in U). \tag{26}$$

If $f \in A_p(n)$ satisfies:

$$\frac{D^s f(z)}{D^t f(z)} + \left(\frac{D^{s+1} f(z)}{D^s f(z)} - \frac{D^{t+1} f(z)}{D^t f(z)}\right) \frac{p}{\beta + \gamma \dfrac{D^t f(z)}{D^s f(z)}} \prec h(z) \quad (z \in U) \tag{27}$$

for some real s and t, then:

$$\frac{D^s f(z)}{D^t f(z)} \prec h(z) \quad (z \in U). \tag{28}$$

Furthermore, the Briot–Bouquet differential equation:

$$g(z) + \frac{zg'(z)}{\beta g(z) + \gamma} = h(z) \tag{29}$$

has a univalent solution g, then:

$$\frac{D^s f(z)}{D^t f(z)} \prec g(z) \prec h(z) \quad (z \in U) \tag{30}$$

and g is the best dominant of (27).

Proof. Define a function p by:

$$p(z) = \frac{D^s f(z)}{D^t f(z)}.$$

Then, p is analytic in U and $p(0) = h(0) = 1$. Since:

$$\frac{zp'(z)}{p(z)} = p\left(\frac{D^{s+1}f(z)}{D^s f(z)} - \frac{D^{t+1}f(z)}{D^t f(z)}\right), \tag{31}$$

we see that:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} = \frac{D^s f(z)}{D^t f(z)} + \left(\frac{D^{s+1}f(z)}{D^s f(z)} - \frac{D^{t+1}f(z)}{D^t f(z)}\right)\frac{p}{\beta + \gamma\dfrac{D^t f(z)}{D^s f(z)}} \prec h(z) \quad (z \in U). \tag{32}$$

Thus, considering Lemma 1, we say that:

$$\frac{D^s f(z)}{D^t f(z)} \prec h(z) \quad (z \in U). \tag{33}$$

Furthermore, if g satisfies the conditions of the theorem, then g satisfies:

$$\frac{D^s f(z)}{D^t f(z)} \prec g(z) \prec h(z) \quad (z \in U) \tag{34}$$

and g is the best dominant of (27). □

Example 1. *We consider:*

$$f(z) = \left(\frac{z}{(1-z)^2}\right)^p = z^p + 2pz^{p+1} + p(2p+1)z^{p+2} + \dots \tag{35}$$

and:

$$h(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \dots . \tag{36}$$

Then $f \in A_p(1)$ and h is convex in U.
Take $s = 1$, $t = 0$, $\beta = 1$, and $\gamma = 0$ in Theorem 1. Then, the subordination (27) becomes:

$$1 + \frac{zf''(z)}{f'(z)} + \left(\frac{1-p}{p}\right)\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \quad (z \in U). \tag{37}$$

For the above subordination (37), the subordination (28) becomes:

$$\frac{zf'(z)}{pf(z)} \prec \frac{1+z}{1-z} \quad (z \in U). \tag{38}$$

This gives us that f is p-valently starlike in U. Further, considering a function g defined by:

$$g(z) + \frac{zg'(z)}{g(z)} = \frac{1+z}{1-z}, \tag{39}$$

we see that:

$$g(z) = \frac{1}{1-z} \quad (z \in U). \tag{40}$$

Thus, the subordination (30) becomes:

$$\frac{zf'(z)}{pf(z)} \prec \frac{1}{1-z} \prec \frac{1+z}{1-z} \quad (z \in U) \tag{41}$$

and g is the best dominant of (27).

Next, we derive the following:

Theorem 2. *If $f \in A_p(n)$ satisfies:*

$$\mathrm{Re}\left\{\frac{D^s f(z)}{D^t f(z)} + p\left(\frac{D^{s+1} f(z)}{D^s f(z)} - \frac{D^{t+1} f(z)}{D^t f(z)}\right)\right\} > \alpha \quad (z \in U) \tag{42}$$

for some real s and t with $0 \leq \alpha < 1$, then $f \in A_p(n, s, t; \delta(\alpha))$, where:

$$\delta(\alpha) = \begin{cases} \dfrac{2\alpha - 1}{2(1 - 2^{1-2\alpha})} & \left(\alpha \neq \dfrac{1}{2}\right) \\ \dfrac{1}{2\log 2} = 0.7213\ldots & \left(\alpha = \dfrac{1}{2}\right). \end{cases} \tag{43}$$

Proof. Define a function p by:

$$p(z) = \frac{D^s f(z)}{D^t f(z)} \tag{44}$$

and h_α by:

$$h_\alpha(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1). \tag{45}$$

It follows from (44) that:

$$\frac{zp'(z)}{p(z)} = p\left(\frac{D^{s+1} f(z)}{D^s f(z)} - \frac{D^{t+1} f(z)}{D^t f(z)}\right). \tag{46}$$

Applying Lemma 1 with $\beta = 1$ and $\gamma = 0$, if f satisfies:

$$\frac{D^s f(z)}{D^t f(z)} + p\left(\frac{D^{s+1} f(z)}{D^s f(z)} - \frac{D^{t+1} f(z)}{D^t f(z)}\right) \prec h_\alpha(z) \quad (z \in U), \tag{47}$$

then we have:

$$\frac{D^s f(z)}{D^t f(z)} \prec h_\alpha(z) \quad (z \in U). \tag{48}$$

Further, we consider a function g given by:

$$g(z) + \frac{zg'(z)}{g(z)} = h_\alpha(z) \quad (z \in U) \tag{49}$$

with $g(0) = h_\alpha(0) = 1$. It follows from (49) that:

$$g(z) = g_\alpha(z) = \begin{cases} \dfrac{(1 - 2\alpha)z}{(1 - z)(1 - (1 - z)^{1-2\alpha})} & \left(\alpha \neq \dfrac{1}{2}\right) \\ \dfrac{z}{(z - 1)\log(1 - z)} & \left(\alpha = \dfrac{1}{2}\right). \end{cases} \tag{50}$$

Furthermore, we note that g_α defined by (50) is univalent in U. Using Lemma 1, we say that:

$$\frac{D^s f(z)}{D^t f(z)} \prec g_\alpha(z) \prec h_\alpha(z) \quad (z \in U). \tag{51}$$

Since:

$$\min_{|z| \leq 1} \mathrm{Re}(g_\alpha(z)) = g_\alpha(-1) = \delta(\alpha), \tag{52}$$

we have that:

$$\mathrm{Re}\left(\frac{D^s f(z)}{D^t f(z)}\right) > \delta(\alpha) \quad (z \in U). \tag{53}$$

This means that $f \in A_p(n, s, t; \delta(\alpha))$. □

Considering $p = 1$ and $s = t + 1$ in Theorem 2, we say the following:

Corollary 1. *If* $f \in A_1(n, t+2, t+1; \alpha)$, *then* $f \in A_1(n, t+1, t; \delta(\alpha))$.

Example 2. *Let us consider* $p = 1$, $s = 1$, *and* $t = 0$ *in Theorem 2. Note that:*

$$\frac{D^s f(z)}{D^t f(z)} = \frac{D^{t+1} f(z)}{D^t f(z)} = \frac{z f'(z)}{f(z)} \tag{54}$$

and:

$$\frac{D^{s+1} f(z)}{D^s f(z)} = 1 + \frac{z f''(z)}{f'(z)}. \tag{55}$$

We consider f *satisfying:*

$$1 + \frac{z f''(z)}{f'(z)} = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (z \in U) \tag{56}$$

with $0 \leq \alpha < 1$. *Since:*

$$\mathrm{Re}\left(1 + \frac{z f''(z)}{f'(z)}\right) = \mathrm{Re}\left(\frac{1 + (1 - 2\alpha)z}{1 - z}\right) > \alpha \quad (z \in U), \tag{57}$$

f *satisfies the inequality (42). It follows from (56) that:*

$$f'(z) = \frac{1}{(1 - z)^{2(1-\alpha)}} \tag{58}$$

and:

$$f(z) = \frac{1}{1 - 2\alpha}\left(\frac{1}{(1 - z)^{1-2\alpha}} - 1\right). \tag{59}$$

This gives us that:

$$\frac{z f'(z)}{f(z)} = \frac{(1 - 2\alpha)z}{(1 - z)(1 - (1 - z)^{1-2\alpha})}. \tag{60}$$

Therefore, f *satisfies:*

$$\mathrm{Re}\left(\frac{D^s f(z)}{D^t f(z)}\right) = \mathrm{Re}\left(\frac{z f'(z)}{f(z)}\right) > \delta(\alpha) \quad (z \in U). \tag{61}$$

Thus, we see that $f \in A_1(1, 1, 0; \delta(\alpha))$.

To discuss our next results, we have to present the following lemma due to Miller and Mocanu ([16,19]) (also by Jack [20]).

Lemma 3. *Let the function* $w(z)$ *given by:*

$$w(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots \quad (n \in \mathbb{N}) \tag{62}$$

be analytic in U *with* $w(0) = 0$. *If* $|w(z)|$ *attains its maximum value on the circle* $|z| = r < 1$ *at a point* $z_0 \in U$, *then there exists a real number* $k \geq n$ *such that:*

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \tag{63}$$

and:

$$\mathrm{Re}\left(1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right) \geq k. \tag{64}$$

Applying this lemma, we derive the following:

Theorem 3. *If $f \in A_p(n)$ satisfies:*

$$\mathrm{Re}\left(\frac{D^{s+1}f(z)}{D^s f(z)} - \frac{D^{t+1}f(z)}{D^t f(z)}\right) > -\frac{n\alpha}{2p(1-\alpha)} \quad (z \in U) \tag{65}$$

for some real s and t and for $0 \leq \alpha < 1$, then $f \in A_p(n, s, t; \alpha)$. If $f \in A_p(n)$ satisfies:

$$\left|\frac{D^{s+1}f(z)}{D^s f(z)} - \frac{D^{t+1}f(z)}{D^t f(z)}\right| < \frac{n(1-\alpha)}{2p\alpha} \quad (z \in U) \tag{66}$$

for some real s and t and for $\frac{1}{2} \leq \alpha < 1$, then $f \in A_p(n, s, t; \alpha)$.

Proof. Let us define the function w by:

$$\frac{D^s f(z)}{D^t f(z)} = \frac{1 + (1-2\alpha)w(z)}{1 - w(z)} \quad (w(z) \neq 1). \tag{67}$$

Then, w is analytic in U, $w(0) = 0$, and:

$$w(z) = b_n z^n + b_{n+1} z^{n+1} + \dots . \tag{68}$$

It follows from (67) that:

$$p\left(\frac{D^{s+1}f(z)}{D^s f(z)} - \frac{D^{t+1}f(z)}{D^t f(z)}\right) = \frac{(1-2\alpha)zw'(z)}{1 + (1-2\alpha)w(z)} + \frac{zw'(z)}{1 - w(z)}. \tag{69}$$

We suppose that there exists a point $z_0 \in U$ such that:

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq 1). \tag{70}$$

Then, applying Lemma 3, we write that:

$$z_0 w'(z_0) = k w(z_0) \quad (k \geq n). \tag{71}$$

This implies that:

$$\begin{aligned} p\mathrm{Re}\left(\frac{D^{s+1}f(z_0)}{D^s f(z_0)} - \frac{D^{t+1}f(z_0)}{D^t f(z_0)}\right) &= \mathrm{Re}\left\{\frac{(1-2\alpha)kw(z_0)}{1 + (1-2\alpha)w(z_0)} + \frac{kw(z_0)}{1 - w(z_0)}\right\} \\ &= k\mathrm{Re}\left\{\frac{(1-2\alpha)e^{i\theta}}{1 + (1-2\alpha)e^{i\theta}} + \frac{e^{i\theta}}{1 - e^{i\theta}}\right\} \quad (w(z_0) = e^{i\theta}) \\ &= k\left\{\frac{(1-2\alpha)((1-2\alpha) + \cos\theta)}{1 + (1-2\alpha)^2 + 2(1-2\alpha)\cos\theta} - \frac{1}{2}\right\} \\ &\leq k\left\{\frac{1-2\alpha}{2(1-\alpha)} - \frac{1}{2}\right\} \\ &= -\frac{\alpha k}{2(1-\alpha)} \leq -\frac{n\alpha}{2(1-\alpha)}. \end{aligned} \tag{72}$$

This contradicts the condition (65) of the theorem. This means that there is no z_0 $(0 < |z_0| < 1)$ such that $|w(z_0)| = 1$. Since $|w(z)| < 1$ for all $z \in U$, we have that:

$$\mathrm{Re}\left(\frac{D^s f(z)}{D^t f(z)}\right) > \alpha \quad (z \in U), \tag{73}$$

that is $f \in A_p(n, s, t; \alpha)$.

Furthermore, we have that:

$$\begin{aligned}\left|\frac{D^{s+1}f(z_0)}{D^s f(z_0)} - \frac{D^{t+1}f(z_0)}{D^t f(z_0)}\right| &= \frac{k}{p}\left|\frac{(1-2\alpha)w(z_0)}{1+(1-2\alpha)w(z_0)} + \frac{w(z_0)}{1-w(z_0)}\right| \\ &= \frac{k}{p}\left|\frac{1-2\alpha}{1+(1-2\alpha)e^{i\theta}} + \frac{1}{1-e^{i\theta}}\right| \\ &= \frac{k}{p}\,\frac{2(1-\alpha)}{|(1+(1-2\alpha)e^{i\theta})(1-e^{i\theta})|} \\ &\geq \frac{k(1-\alpha)}{2p\alpha} \geq \frac{n(1-\alpha)}{2p\alpha} \quad \left(\frac{1}{2} \leq \alpha < 1\right).\end{aligned} \tag{74}$$

Since the above inequality contradicts (66), we say that $|w(z)| < 1$ for all $z \in U$. Thus, we say that $f \in A_p(n, s, t; \alpha)$. □

Taking $s = t + 1$ in Theorem 3, we have the following:

Corollary 2. *If $f \in A_p(n)$ satisfies:*

$$\mathrm{Re}\left(1 + \frac{z(D^t f(z))''}{(D^t f(z))'} - \frac{z(D^t f(z))'}{D^t f(z)}\right) > -\frac{n\alpha}{2(1-\alpha)} \quad (z \in U) \tag{75}$$

for some real t and $0 \leq \alpha < 1$, then $f(z) \in A_p(n, t+1, t; p\alpha)$. This means that f is p-valently starlike of order $p\alpha$ in U. If $f \in A_p(n)$ satisfies:

$$\left|1 + \frac{z(D^t f(z))''}{(D^t f(z))'} - \frac{z(D^t f(z))'}{D^t f(z)}\right| < \frac{n(1-\alpha)}{2\alpha} \quad (z \in U) \tag{76}$$

for some real t and $\frac{1}{2} \leq \alpha < 1$, then $f \in A_p(n, t+1, t; p\alpha)$.

Further, we derive the following:

Theorem 4. *If $f \in A_p(n)$ satisfies:*

$$\left|\frac{D^{s+2}f(z)}{D^{s+1}f(z)} - 1\right| < \frac{(2p+3n)\alpha - 2p\alpha^2 - n}{2p\alpha} \tag{77}$$

for some real s and α $\left(\frac{1}{3} \leq \alpha \leq \frac{1}{2}\right)$, or:

$$\left|\frac{D^{s+2}f(z)}{D^{s+1}f(z)} - 1\right| < \frac{(2p-n)\alpha - 2p\alpha^2 + n}{2p\alpha} \tag{78}$$

for some real s and α $\left(\frac{1}{2} \leq \alpha < 1\right)$, then $D^s f$ is p-valently starlike of order $p\alpha$ in U with $\frac{1}{3} \leq \alpha < 1$.

Proof. We consider a function w defined by:

$$\frac{D^{s+1}f(z)}{D^s f(z)} = \frac{1+(1-2\alpha)w(z)}{1-w(z)} \quad (w(z) \neq 1). \tag{79}$$

Since:

$$p\left\{\frac{D^{s+2}f(z)}{D^{s+1}f(z)} - \frac{D^{s+1}f(z)}{D^s f(z)}\right\} = \frac{(1-2\alpha)zw'(z)}{1+(1-2\alpha)w(z)} + \frac{zw'(z)}{1-w(z)}, \tag{80}$$

we have:

$$\frac{D^{s+2}f(z)}{D^{s+1}f(z)}-1=\left(\frac{w(z)}{1-w(z)}\right)\left\{2(1-\alpha)+\frac{1}{p}\left(\frac{zw'(z)}{w(z)}\right)\left(1+\frac{(1-2\alpha)(1-w(z))}{1+(1-2\alpha)w(z)}\right)\right\}. \quad (81)$$

Suppose that there exists a point $z_0 \in U$ such that:

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1 \quad (w(z_0)\neq 1). \quad (82)$$

Then, we can write that $w(z_0)=e^{i\theta}$ and:

$$z_0w'(z_0)=kw(z_0) \quad (k\geq n) \quad (83)$$

by Lemma 3. This implies that:

$$\begin{aligned}\left|\frac{D^{s+2}f(z_0)}{D^{s+1}f(z_0)}-1\right| &= \left|\left(\frac{e^{i\theta}}{1-e^{i\theta}}\right)\left(2(1-\alpha)+\frac{k}{p}\left(1+\frac{(1-2\alpha)(1-e^{i\theta})}{1+(1-2\alpha)e^{i\theta}}\right)\right)\right| \\ &\geq \frac{2p(1-\alpha)+k}{2p}-\frac{k|1-2\alpha|}{2p\alpha}.\end{aligned} \quad (84)$$

It follows from (84) that:

$$\begin{aligned}\left|\frac{D^{s+2}f(z_0)}{D^{s+1}f(z_0)}-1\right| &\geq \frac{2p\alpha(1-\alpha)+(3\alpha-1)k}{2p\alpha} \\ &\geq \frac{(2p+3n)\alpha-2p\alpha^2-n}{2p\alpha}\end{aligned} \quad (85)$$

for $\frac{1}{3}\leq\alpha\leq\frac{1}{2}$, and that:

$$\begin{aligned}\left|\frac{D^{s+2}f(z_0)}{D^{s+1}f(z_0)}-1\right| &\geq \frac{2p\alpha(1-\alpha)+(1-\alpha)k}{2p\alpha} \\ &\geq \frac{(2p-n)\alpha-2p\alpha^2+n}{2p\alpha}\end{aligned} \quad (86)$$

for $\frac{1}{2}\leq\alpha<1$. Thus, we say that there is no $z_0\in U$ such that $|w(z_0)|=1$ with the conditions of the theorem. Since $|w(z)|<1$ $(z\in U)$, we conclude that:

$$\mathrm{Re}\left(\frac{D^{s+1}f(z)}{D^{s}f(z)}\right)>\alpha \quad (z\in U) \quad (87)$$

with $\frac{1}{3}\leq\alpha<1$. This means that $f\in A_p(n,s+1,s;\alpha)$ with $\frac{1}{3}\leq\alpha<1$. Noting that:

$$D^{s+1}f(z)=\frac{1}{p}z(D^sf(z))', \quad (88)$$

we say that f is p-valently starlike of order $p\alpha$ in U with $\frac{1}{3}\leq\alpha<1$. □

Remark 4. *For Theorem 4, we leave the condition for* $\alpha\left(0\leq\alpha<\frac{1}{3}\right)$ *that* f *is* p*-valently starlike of order* α *in* U.

3. Coefficient Problems

We discuss some coefficient problems for $f \in A_p(n, s, t; \alpha)$. We need the following lemma by Hayman [21].

Lemma 4 ([21]). *Let a function:*

$$h(z) = \sum_{k=1}^{\infty} h_k z^k \tag{89}$$

be analytic in U and a function:

$$g(z) = \sum_{k=1}^{\infty} g_k z^k \tag{90}$$

be analytic and convex in U. If:

$$h(z) \prec g(z) \quad (z \in U), \tag{91}$$

then:

$$|h_k| \leq |g_1| \quad (k \geq 2). \tag{92}$$

The above lemma leads us to the following result:

Theorem 5. *If $f(z) \in A_p(n, s, t; \alpha)$, then:*

$$|a_k| \leq \beta_k \prod_{j=p+n}^{k-1} \left(1 + \left(\frac{j}{p}\right)^t \beta_j\right) \quad (k = p+n+1, p+n+2, \ldots) \tag{93}$$

and:

$$|a_{p+n}| \leq \beta_{p+n}, \tag{94}$$

where:

$$\beta_j = \frac{2(1-\alpha)}{\left|\left(\frac{j}{p}\right)^s - \left(\frac{j}{p}\right)^t\right|}. \tag{95}$$

Proof. We introduce a function p by:

$$p(z) = \frac{D^s f(z)}{D^t f(z)} = \frac{z^p + \sum_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^s a_k z^k}{z^p + \sum_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^t a_k z^k} \tag{96}$$

and a function q by:

$$q(z) = \frac{p(z) - 1}{1 - \alpha} = q_1 z + q_2 z^2 + \ldots . \tag{97}$$

Since:

$$\mathrm{Re} p(z) > \alpha \quad (z \in U), \tag{98}$$

we see that:

$$\mathrm{Re} q(z) > -1 \quad (z \in U). \tag{99}$$

This gives us that:

$$q(z) \prec \frac{2z}{1-z} = 2(z + z^2 + z^3 + \ldots). \tag{100}$$

Applying Lemma 4, we have:

$$|q_k| \leq 2 \quad (k = 1, 2, 3, \ldots). \tag{101}$$

Since:

$$\begin{aligned} p(z) &= 1 + (1-\alpha)q(z) \\ &= 1 + (1-\alpha)(q_1 z + q_2 z + \ldots), \end{aligned} \tag{102}$$

we obtain:

$$\begin{aligned} \left(\left(\frac{k}{p}\right)^3 - \left(\frac{k}{p}\right)^t\right) a_k = (1-\alpha)\Bigg\{ & q_{k-p} + \left(\frac{p+n}{p}\right)^t q_{k-p-1} a_{p+n} \\ & + \left(\frac{k+n+1}{p}\right)^t q_{k-p-2} a_{p+n+1} + \ldots + \left(\frac{k-1}{p}\right)^t q_1 a_{k-1} \Bigg\} \end{aligned} \tag{103}$$

and that:

$$\begin{aligned} \left|\left(\frac{k}{p}\right)^s - \left(\frac{k}{p}\right)^t\right| |a_k| \leq 2(1-\alpha)\Bigg\{ & 1 + \left(\frac{p+n}{p}\right)^t |a_{p+n}| \\ & + \left(\frac{p+n+1}{p}\right)^t |a_{p+n+1}| + \ldots + \left(\frac{k-1}{p}\right)^t |a_{k-1}| \Bigg\}. \end{aligned} \tag{104}$$

If $k = p + n$, then we have:

$$|a_{p+n}| \leq \frac{2(1-\alpha)}{\left|\left(\frac{k}{p}\right)^s - \left(\frac{k}{p}\right)^t\right|} \equiv \beta_{p+n}. \tag{105}$$

If $k = p + n + 1$, then we have:

$$|a_{p+n+1}| \leq \beta_{p+n+1}\left(1 + \left(\frac{p+n}{p}\right)^t \beta_{p+n}\right). \tag{106}$$

If $k = p + n + 2$, then we have:

$$\begin{aligned} |a_{p+n+2}| &\leq \beta_{p+n+2}\left\{1 + \left(\frac{p+n}{p}\right)^t \beta_{p+n} + \left(\frac{p+n+1}{p}\right)^t \beta_{p+n+1}\left(1 + \left(\frac{p+n}{p}\right)^t \beta_{p+n}\right)\right\} \\ &= \beta_{p+n+2}\left(1 + \left(\frac{p+n}{p}\right)^t \beta_{p+n}\right)\left(1 + \left(\frac{p+n+1}{p}\right)^t \beta_{p+n+1}\right). \end{aligned} \tag{107}$$

Similarly, we obtain that:

$$|a_k| \leq \beta_k \prod_{j=p+n}^{k-1}\left(1 + \left(\frac{j}{p}\right)^t \beta_j\right) \tag{108}$$

for $k = p + n + 1, p + n + 2, \ldots$. This completes the proof of the theorem. □

Letting $s = t + 1$ in Theorem 5, we have:

Corollary 3. *If $f \in A_p(n, t+1, t; \alpha)$, then:*

$$|a_k| \leq \frac{2p^{t+1}(1-\alpha)}{k^t(k-p)} \prod_{j=p+n}^{k-1}\left(1 + \frac{2p(1-\alpha)}{j-p}\right) \quad (k = p+n+1, p+n+2, \ldots) \tag{109}$$

and:

$$|a_{p+n}| \leq \frac{2p^{t+1}(1-\alpha)}{n(p+n)^t}. \tag{110}$$

Remark 5. *If we take $t = 0$ and $p = n = 1$ in Corollary 3, we say that if f is starlike of order α in U, then:*

$$|a_k| \leq \frac{\prod_{j=2}^{k}(j-2\alpha)}{(k-1)!} \quad (k = 2, 3, 4, \ldots). \tag{111}$$

If we make $t = p = n = 1$ in Corollary 3, we say that if f is convex of order α in U, then, according to [22] we write:

$$|a_k| \leq \frac{\prod_{j=2}^{k}(j-2\alpha)}{k!} \quad (k = 2, 3, 4, \ldots). \tag{112}$$

Next, we prove the following:

Theorem 6. *If $f \in A_p(n)$ satisfies:*

$$\sum_{k=p+n}^{\infty} \left\{ \left| \left(\frac{k}{p}\right)^s - \left(\frac{k}{p}\right)^t \right| + (1-\alpha)\left(\frac{k}{p}\right)^t \right\} |a_k| \leq 1-\alpha \tag{113}$$

for some real s and t with $0 \leq \alpha < 1$, then $f \in A_p(n, s, t; \alpha)$.

Proof. Consider that:

$$\left| \frac{D^s f(z)}{D^t f(z)} - 1 \right| = \left| \frac{\prod_{k=p+n}^{\infty} \left(\left(\frac{k}{p}\right)^s - \left(\frac{k}{p}\right)^t \right) a_k z^{k-p}}{1 + \prod_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^t a_k z^{k-p}} \right| \tag{114}$$

$$< \frac{\prod_{k=p+n}^{\infty} \left| \left(\frac{k}{p}\right)^s - \left(\frac{k}{p}\right)^t \right| |a_k|}{1 - \prod_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^t |a_k|} \leq 1-\alpha \quad (z \in U). \tag{115}$$

Then, we say that $f \in A_p(n, s, t; \alpha)$. Therefore, if f satisfies the inequality (113), then $f \in A_p(n, s, t; \alpha)$. □

Letting $s = t + 1$ in the above theorem, we have the following:

Corollary 4. *If $f \in A_p(n)$ satisfies:*

$$\sum_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^t \left(\frac{k}{p} - \alpha\right) |a_k| \leq 1-\alpha \tag{116}$$

for some real t with $0 \leq \alpha < 1$, then $f \in A_p(n, t+1, t; \alpha)$.

Remark 6. *Let us consider $t = 0$ in Corollary 4. Then, we see that if $f \in A_p(n)$ satisfies:*

$$\sum_{k=p+n}^{\infty} (k - p\alpha)|a_k| \leq p(1-\alpha) \tag{117}$$

for $0 \leq \alpha < 1$, *then* $f \in A_p(n, 1, 0; \alpha)$. *This implies that:*

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > p\alpha \quad (z \in U), \tag{118}$$

that is that f *is p-valently starlike of order* $p\alpha$ *in* U. *We consider:*

$$\begin{aligned}\frac{1}{2}\int_0^z \frac{t^{p+n-1-p\alpha}(z-t)^2}{1-t}dt &= \frac{1}{2}\left\{z^2\int_0^z \frac{t^{p+n-1-p\alpha}}{1-t}dt - 2z\int_0^z \frac{t^{p+n-p\alpha}}{1-t}dt + \int_0^z \frac{t^{p+n+1-p\alpha}}{1-t}dt\right\} \\ &= \sum_{k=p+n}^{\infty} \frac{z^{k+2-p\alpha}}{(k-p\alpha)(k+1-p\alpha)(k+2-p\alpha)}.\end{aligned} \tag{119}$$

With the above integration, we define:

$$\begin{aligned}f(z) &= z^p + \frac{p(1-\alpha)(p+n+1-p\alpha)z^{p\alpha-2}}{2}\int_0^z \frac{t^{p+n-1-p\alpha}(z-t)^2}{1-t}dt \\ &= z^p + \sum_{k=p+n}^{\infty} a_k z^k,\end{aligned} \tag{120}$$

where:

$$a_k = \frac{p(1-\alpha)(p+n+1-p\alpha)}{(k-p\alpha)(k+1-p\alpha)(k+2-p\alpha)} \quad (k = p+n, p+n+1, \ldots). \tag{121}$$

For this function f, *we obtain that:*

$$\begin{aligned}\sum_{k=p+n}^{\infty}(k-p\alpha)|a_k| &= p(1-\alpha)(p+n+1-p\alpha)\sum_{k=p+n}^{\infty}\left(\frac{1}{k+1-p\alpha} - \frac{1}{k+2-p\alpha}\right) \\ &= p(1-\alpha).\end{aligned} \tag{122}$$

Therefore, the function f *given by (120) belongs to the class* $A_p(n, 1, 0; \alpha)$.

Remark 7. *If we consider a function* f *given by:*

$$\begin{aligned}f(z) &= z + \frac{(1-\alpha)^2 z^{\alpha}}{2}\int_0^z \frac{t^{-\alpha}(z-t)^2}{1-t}dt \\ &= z + \sum_{k=3}^{\infty}\frac{(1-\alpha)^2}{(k-\alpha)(k-1-\alpha)(k-2-\alpha)}z^k \\ &= z + \sum_{k=3}^{\infty} a_k z^k,\end{aligned} \tag{123}$$

where:

$$a_k = \frac{(1-\alpha)^2}{(k-\alpha)(k-1-\alpha)(k-2-\alpha)} \quad (k = 3, 4, 5, \ldots), \tag{124}$$

then we have:

$$\begin{aligned}\sum_{k=2}^{\infty}(k-\alpha)|a_k| &= (1-\alpha)^2\sum_{k=3}^{\infty}\frac{1}{(k-1-\alpha)(k-2-\alpha)} \\ &= (1-\alpha)^2\sum_{k=3}^{\infty}\left(\frac{1}{k-2-\alpha} - \frac{1}{k-1-\alpha}\right) \\ &= 1-\alpha.\end{aligned} \tag{125}$$

This implies that f is starlike of order α in U. Further, if we consider:

$$\begin{aligned} f(z) &= \frac{1}{2}z + \frac{3}{4}z^2 - \frac{(1-z)^2}{2}\log(1-z) \\ &= z + \sum_{k=3}^{\infty} \frac{1}{k(k-1)(k-2)} z^k \\ &= z + \sum_{k=3}^{\infty} a_k z^k, \end{aligned} \tag{126}$$

where:

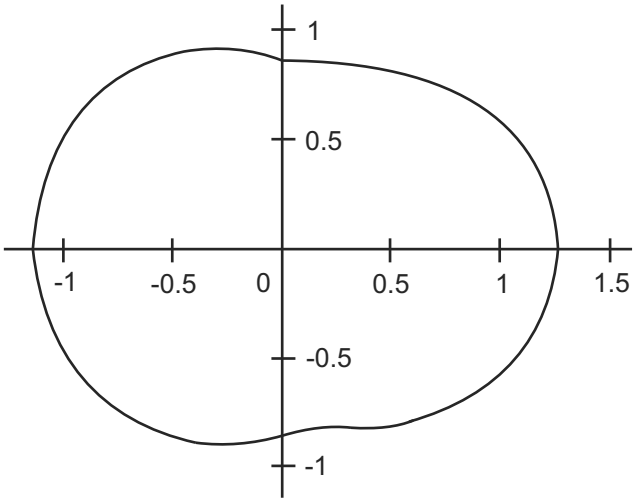
$$a_k = \frac{1}{k(k-1)(k-2)} \quad (k = 3, 4, 5, \ldots), \tag{127}$$

then we see:

$$\begin{aligned} \sum_{k=2}^{\infty} k|a_k| &= \sum_{k=3}^{\infty} \frac{1}{(k-1)(k-2)} \\ &= \sum_{k=3}^{\infty} \left(\frac{1}{k-2} - \frac{1}{k-1} \right) = 1. \end{aligned} \tag{128}$$

This implies that f given by (126) is starlike in U.

Finally, we know that the function f given by (126) maps U onto the following starlike domain.



4. Conclusions

Following the pattern seen in recently published papers cited in the Introduction, a new operator was introduced in Definition 2, and using it, a new class of p-valently analytic functions $A_{p(}n, s, t; \alpha)$ was given in Definition 3. Subordination results involving functions from class $A_{p(}n, s, t; \alpha)$ were contained in the theorems and corollaries in Section 2, and also, two examples were constructed based on the proven results. In Section 3, coefficient estimates were found for functions in the class $A_{p(}n, s, t; \alpha)$.

As future directions, the most appealing appears to be adding quantum calculus to the study of the class $A_{p(}n, s, t; \alpha)$ as was performed in [23] for a class of p-valent analytic functions introduced using the q-difference operator and Janowski functions. The Fekete–Szegö inequality was obtained for that class, and coefficient estimates, convexity, and starlikeness were investigated there; hence, similar studies could be further developed for class $A_{p(}n, s, t; \alpha)$. In [24], a q-difference operator was also used for investigating subclasses

of multivalent analytic functions, and this paper can also be used as inspiration for future studies related to class $A_{p(}n, s, t; \alpha)$.

In [25,26], q-type operators were investigated related to multivalent functions, which suggests the same future approach on the operator introduced in this paper.

As seen in paper [11], p-valent functions are associated with special functions such as hypergeometric functions. In [27], the Mittag–Leffler function was used for introducing classes of p-valent functions. Those studies suggest further investigations on p-valent functions involving other special functions.

Applications of the p-valent functions in nonlinear differential equations can be further investigated since those functions can be used for the initial solution of the variational iteration method.

Author Contributions: Conceptualisation, S.O.; methodology, G.I.O., G.O., and S.O; software, G.I.O.; validation, G.O.I, G.O., and S.O.; formal analysis, G.I.O., G.O., and S.O.; investigation, G.I.O., G.O., and S.O.; resources, G.I.O. and S.O.; data curation, G.I.O. and S.O.; writing—original draft preparation, S.O.; writing—review and editing, G.I.O., G.O., and S.O.; visualisation, G.I.O.; supervision, S.O.; project administration, G.I.O.; funding acquisition, G.I.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank the Reviewers for their valuable suggestions, which improved the paper.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Aouf, M.K. On a class of p-valent starlike functions of order α. *Int. J. Math. Math. Sci.* **1987**, *10*, 181790. [CrossRef]
2. Cho, N.E. On certain classes of p-valent analytic functions. *Internat. Int. J. Math. Math. Sci.* **1993**, *16*, 319–328. [CrossRef]
3. Oros, G.I.; Oros, G.; Owa, S. Differential subordinations on p-valent functions of missing coefficients. *J. Appl. Math.* **2009**, *22*, 1021–1030.
4. Ghanim, F.; Darus, M. Some results of p-valent meromorphic functions defined by a linear operator. *Far East J. Math. Sci. (FJMS)* **2010**, *44*, 155–165.
5. Khan, Q.; Dziok, J.; Raza, M.; Arif, M. Sufficient conditions for p-valent functions. *Math. Slovaca* **2021**, *71*, 1089–1102. [CrossRef]
6. Cho, N.E.; Aouf, M.K.; Srivastava, R. The Principle of Differential Subordination and Its Application to Analytic and p-Valent Functions Defined by a Generalized Fractional Differintegral Operator. *Symmetry* **2019**, *11*, 1083. [CrossRef]
7. Atshan, W.G.; Hadi, R.A. Some Differential Subordination and Superordination Results of p-valent Functions Defined by Differential Operator. *J. Phys. Conf. Ser.* **2020**, *1664*, 012043. [CrossRef]
8. Owa, S.; Güney, H.Ö. New Applications of the Bernardi Integral Operator. *Mathematics* **2020**, *8*, 1180. [CrossRef]
9. Yousef, A.T.; Salleh, Z.; Al-Hawary, T. On a class of p-valent functions involving generalized differential operator. *Afr. Mat.* **2021**, *32*, 275–287. [CrossRef]
10. Yousef, A.T.; Salleh, Z.; Al-Hawary, T. Some properties on a class of p-valent functions involving generalized differential operator. *Aust. J. Math. Anal. Appl.* **2021**, *18*, 6.
11. Al-Janaby, H.F.; Ghanim, F. A subclass of Noor-type harmonic p-valent functions based on hypergeometric functions. *Kragujev. J. Math.* **2021**, *45*, 499–519. [CrossRef]
12. Ali, E.E.; Aouf, M.K.; El-Ashwah, R.M. Some properties of p-valent analytic functions defined by Dziok–Srivastava operator. *Asian-Eur. J. Math.* **2021**, *14*, 2150084. [CrossRef]
13. Flett, T.M. Lipschitz spaces of functions on the circle and the disc. *J. Math. Anal. Appl.* **1972**, *39*, 125–158. [CrossRef]
14. Owa, S. On certain classes of p-valent functions with negative coefficients. *Simon Stevin* **1985**, *25*, 385–402.
15. Pommerenke, C. *Univalent Functions, Vanderhoeck and Ruprecht*; CiNii: Japan, 1975.
16. Miller, S.S.; Mocanu, P.T. *Differential Subordinations, Theory and Applications*; Marcel Dekker Inc.: New York, NY, USA, 2000.
17. Miller, S.S.; Mocanu, P.T. Briot–Bouquet differential equations and differential subordinations. *Complex Var.* **1997**, *33*, 217–237. [CrossRef]
18. Miller, S.S.; Mocanu, P.T. Univalent solutions of Briot–Bouquet differential equations. *J. Differ. Equations* **1985**, *56*, 297–309. [CrossRef]

19. Miller, S.S.; Mocanu, P.T. Second order differential equations in the complex plane. *J. Math. Anal. Appl.* **1978**, *65*, 289–305. [CrossRef]
20. Jack, I.S. Functions starlike and convex of order a. *J. London Math. Soc.* **1971**, *3*, 469–474. [CrossRef]
21. Hayman, W.K. *Multivalent Functions*; Cambridge University Press: Cambridge, UK, 1994.
22. Robertson, M.S. On the theory of univalent functions. *Ann. Math.* **1936**, *37*, 374–408. [CrossRef]
23. Wang, B.; Srivastava, R.; Liu, J.-L. A Certain Subclass of Multivalent Analytic Functions Defined by the q-Difference Operator Related to the Janowski Functions. *Mathematics* **2021**, *9*, 1706. [CrossRef]
24. Ali, E.E.; Bulboacă, T. Subclasses of Multivalent Analytic Functions Associated with a q-Difference Operator. *Mathematics* **2020**, *8*, 2184. [CrossRef]
25. Khan, B.; Srivastava, H.M.; Arjika, S.; Khan, S.; Khan, N.; Ahmad, Q.Z. A certain q-Ruscheweyh type derivative operator and its applications involving multivalent functions. *Adv. Differ. Equ.* **2021**, *279*, 1–14. [CrossRef]
26. Aldawish, I.; Aouf, M.K.; Frasin, B.A.; Al-Hawary, T. New subclass of analytic functions defined by q-analogue of p-valent Noor integral operator. *AIMS Math.* **2021**, *6*, 10466–10484. [CrossRef]
27. Aouf, M.K.; Mostafa, A.O.; Madian, S.M. Certain Classes of Meromorphic p-Valent Functions Associated with Mittag-Leffler Function. *Proc. Pakistan Acad. Sci.—A* **2021**, *58*, 97–106. [CrossRef]