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Article

Infinite Interval Problems for Fractional Evolution Equations

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Abstract: In this paper, we investigate infinite interval problems for the fractional evolution equations with Hilfer fractional derivative. By using the generalized Ascoli–Arzelà theorem and some new techniques, we prove the existence of mild solutions of Hilfer fractional evolution equations when the semigroup is compact as well as noncompact. In addition, an example is provided to illustrate the results.

Keywords: fractional evolution equations; Hilfer derivative; existence; infinite interval

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1. Introduction

Fractional differential equations have recently attracted a lot of attention due to their applications in science and engineering; in particular, they can describe much more nonlocal phenomena in physics, such as fluid mechanics, the diffusion phenomenon, and viscoelasticity. In lots of processes or phenomena with long-range temporal cumulative memory effects and/or long-range spatial interactions, theoretical and numerical results have also shown that fractional differential equations display more prominent advantages than integer order ones. In the past two decades, the theory of fractional differential equations has attracted the attention of researchers all over the world, as in the monographs [1–4] and the recent references.

Consider the Cauchy problem of fractional evolution equations on an infinite interval

$$\begin{cases} {}^{H}D_{0+}^{\mu,\lambda} y(t) = Ay(t) + g(t, y(t)), \quad t \in (0, \infty), \\ I_{0+}^{(1-\lambda)(1-\mu)} y(0) = y_0, \end{cases} \tag{1}$$

where ${}^{H}D_{0+}^{\mu,\lambda}$ is the Hilfer fractional derivative of order $0 < \lambda < 1$ and type $0 \le \mu \le 1$, $I_{0+}^{(1-\lambda)(1-\mu)}$ is Riemann–Liouville integral of order $(1-\lambda)(1-\mu)$, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (i.e., C_0 semigroup) $\{Q(t)\}_{t\ge 0}$ in Banach space X, $g : [0, \infty) \times X \to X$ is a function to be defined later.

The Hilfer fractional derivative is a natural generalization of Caputo derivative and Riemann–Liouville derivative [1]. It is obvious that fractional differential equations with Hilfer derivatives include fractional differential equations with a Riemann–Liouville derivative or Caputo derivative as special cases.

The well-posedness of fractional evolution equations is an important research topic of evolution equations, as many types of fractional partial differential equations, such as fractional diffusion equations, wave equations, Navier–Stokes equations, Rayleigh–Stokes equations, Fokker–Planck equations, Schrödinger equations, etc., can be abstracted as fractional evolution equations [5–7]. However, it seems that there are few works concerned with fractional evolution equations on an infinite interval. Most of these results involve the existence of solutions for fractional evolution equations on a finite interval $[0, T]$, where $T \in (0, \infty)$ (for example, see [8–11]). The Ascoli–Arzelà theorem and various fixed point theorems are widely used to study the existence of solutions. It is well known that the classical Ascoli–Arzelà theorem is powerful technique to give a necessary and sufficient condition for judging the relative compactness of a family of abstract continuous functions, while it is limited to finite closed interval.

In this paper, by using the generalized Ascoli–Arzelà theorem and some new techniques, we prove the existence of mild solutions for the infinite interval problem (1) when the semigroup is compact as well as noncompact. In particular, we do not need to assume that the $g(t, \cdot)$ satisfies the Lipschitz condition. The main methods of this paper are based on the generalization of Ascoli–Arzelà theorem on infinite intervals, Schauder's fixed point theorem, and Kuratowski's measure of noncompactness.

2. Preliminaries

We first introduce some notations and definitions about fractional calculus, Kuratowski's measure of noncompactness, and the definition of mild solutions. For more details, we refer to [1,2,12,13].

Assume that X is a Banach space with the norm $|\cdot|$. Let $\mathbb{R} = (-\infty, \infty)$ and J be an infinite interval of $\mathbb{R}$. By $C(J, X)$ we denote the space of all continuous functions from J to X with the norm $\|u\|_0 = \sup_{t \in J} |u(t)| < \infty$. We denote by $\mathcal{L}(X)$ the space of all bounded linear operators from X to X with the usual operator norm $\|\cdot\|_{\mathcal{L}(X)}$.

Definition 1 (see [2])**.** *The fractional integral of order λ for a function $u : [0, \infty) \to \mathbb{R}$ is defined as*

$$I_{0+}^{\lambda} u(t) = \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} u(s) ds, \quad \lambda > 0,\ t > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2 (Hilfer fractional derivative, see [1])**.** *Let $0 < \lambda < 1$ and $0 \le \mu \le 1$. The Hilfer fractional derivative of order λ and type μ for a function $u : [0, \infty) \to \mathbb{R}$ is defined as*

$${}^{H}D_{0+}^{\mu,\lambda} u(t) = I_{0+}^{\mu(1-\lambda)} \frac{d}{dt} I_{0+}^{(1-\lambda)(1-\mu)} u(t).$$

Remark 1. (i) *In particular, when $\mu = 0$, $0 < \lambda < 1$, then*

$${}^{H}D_{0+}^{0,\lambda} u(t) = \frac{d}{dt} I_{0+}^{1-\lambda} u(t) =: {}^{L}D_{0+}^{\lambda} u(t),$$

where ${}^{L}D_{0+}^{\lambda}$ is the Riemann–Liouville derivative.

(ii) *When $\mu = 1$, $0 < \lambda < 1$, we have*

$${}^{H}D_{0+}^{1,\lambda} u(t) = I_{0+}^{1-\lambda} \frac{d}{dt} u(t) =: {}^{C}D_{0+}^{\lambda} u(t),$$

where ${}^{C}D_{0+}^{\lambda}$ is Caputo derivative.

Let D be a nonempty subset of X. Kuratowski's measure of noncompactness β is said to be:

$$\beta(D) = \inf\left\{ d > 0 :\ D \subset \bigcup_{j=1}^{n} M_j \text{ and } \operatorname{diam}(M_j) \le d \right\},$$

where the diameter of M_j is given by $\operatorname{diam}(M_j) = \sup\{|x-y| :\ x, y \in M_j\}$, $j = 1, \ldots, n$.

Lemma 1 ([14]). *Let $\{u_n(t)\}_{n=1}^{\infty} : [0,\infty) \to X$ be a continuous function family. If there exists $\rho \in L^1[0,\infty)$ such that*

$$|u_n(t)| \leq \rho(t), \quad t \in [0,\infty),\ n = 1,2,\ldots.$$

Then $\beta(\{u_n(t)\}_{n=1}^{\infty})$ is integrable on $[0,\infty)$, and

$$\beta\Big(\Big\{\int_0^t u_n(s)ds :\ n = 1,2,\ldots\Big\}\Big) \leq 2\int_0^t \beta(\{u_n(s) :\ n = 1,2,\ldots\})ds.$$

Definition 3 ([15]). *The Wright function $W_\lambda(\theta)$ is defined by*

$$W_\lambda(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-\lambda n)},\ 0 < \lambda < 1,\ \theta \in \mathbb{C},$$

with the following property

$$\int_0^{\infty} \theta^{\delta} W_\lambda(\theta)d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\lambda\delta)},\ \textit{for } \delta \geq 0.$$

Lemma 2 ([8]). *The Cauchy problem (1) is equivalent to the integral equation*

$$\begin{aligned} y(t) =& \frac{y_0}{\Gamma(\mu(1-\lambda)+\lambda)} t^{(\mu-1)(1-\lambda)} \\ &+ \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1}[Ay(s) + G(s,y(s))]ds,\ t \in (0,\infty). \end{aligned} \tag{2}$$

Lemma 3 ([8]). *Assume that $y(t)$ satisfies integral Equation (2). Then*

$$y(t) = \mathcal{P}_{\mu,\lambda}(t)y_0 + \int_0^t \mathcal{Q}_\lambda(t-s)g(s,y(s))ds,\ t \in (0,\infty), \tag{3}$$

where

$$\mathcal{P}_{\mu,\lambda}(t) = I_{0+}^{\mu(1-\lambda)}\mathcal{Q}_\lambda(t),\ \ \mathcal{Q}_\lambda(t) = t^{\lambda-1}\mathcal{S}_\lambda(t),\ \ \textit{and}\ \ \mathcal{S}_\lambda(t) = \int_0^{\infty} \lambda\theta W_\lambda(\theta)Q(t^{\lambda}\theta)d\theta.$$

Due to Lemma 3, we give the following definition of the mild solution of (1).

Definition 4. *By the mild solution of the Cauchy problem (1), we mean that the function $y \in C((0,\infty),X)$ which satisfies*

$$y(t) = \mathcal{P}_{\mu,\lambda}(t)y_0 + \int_0^t \mathcal{Q}_\lambda(t-s)g(s,y(s))ds,\ t \in (0,\infty).$$

Suppose that A is the infinitesimal generator of a C_0 semigroup $\{Q(t)\}_{t\geq 0}$ of uniformly bounded linear operators on Banach space X. This means that there exists $L > 1$ such that $\sup_{t\in[0,\infty)} \|Q(t)\|_{\mathcal{L}(X)} \leq L$.

Lemma 4 ([4,8]). *If $\{Q(t)\}_{t>0}$ is a compact operator, then $\{\mathcal{P}_{\mu,\lambda}(t)\}_{t>0}$ and $\{\mathcal{S}_\lambda(t)\}_{t>0}$ are also compact operators.*

Lemma 5. *Assume that $\{Q(t)\}_{t>0}$ is a compact operator. Then $\{Q(t)\}_{t>0}$ is equicontinuous.*

Lemma 6 ([8]). *For any fixed $t > 0$, $\mathcal{S}_\lambda(t)$, $\mathcal{Q}_\lambda(t)$ and $\mathcal{P}_{\mu,\lambda}(t)$ are linear operators, i.e., for any $y \in X$*

$$|\mathcal{S}_\lambda(t)y| \leq \frac{L}{\Gamma(\lambda)}|y|,\ \ |\mathcal{Q}_\lambda(t)y| \leq \frac{L}{\Gamma(\lambda)}t^{\lambda-1}|y|,$$

and

$$|\mathcal{P}_{\mu,\lambda}(t)y| \leq \frac{L}{\Gamma(\mu(1-\lambda)+\lambda)} t^{-(1-\lambda)(1-\mu)}|y|.$$

Lemma 7 ([8])**.** *If $\{Q(t)\}_{t>0}$ is equicontinuous, then the operators $\{\mathcal{S}_\lambda(t)\}_{t>0}$, $\{\mathcal{Q}_\lambda(t)\}_{t>0}$ and $\{\mathcal{P}_{\mu,\lambda}(t)\}_{t>0}$ are strongly continuous, which means that, for $\forall\, y \in X$ and $t'' > t' > 0$, we have*

$$|\mathcal{S}_\lambda(t')y - \mathcal{S}_\lambda(t'')y| \to 0,\ |\mathcal{Q}_\lambda(t')y - \mathcal{Q}_\lambda(t'')y| \to 0,$$
$$|\mathcal{P}_{\mu,\lambda}(t')y - \mathcal{P}_{\mu,\lambda}(t'')y| \to 0,\ \textit{as } t'' \to t'.$$

Let

$$C_1([0,\infty),X) = \{u \in C([0,\infty),X) : \lim_{t\to\infty} \frac{|u(t)|}{1+t} = 0\}.$$

Then, $C_1([0,\infty),X)$ is a Banach space with the norm $\|u\| = \sup_{t\in[0,\infty)} |u(t)|/(1+t) < \infty$.

In the following, we state the generalized Ascoli–Arzelà theorem.

Lemma 8 ([16])**.** *The set $\Lambda \subset C_1([0,\infty),X)$ is relatively compact if and only if the following conditions hold:*

(a) *for any $h > 0$, the set $V = \{v : v(t) = x(t)/(1+t), x \in \Lambda\}$ is equicontinuous on $[0,h]$;*
(b) $\lim_{t\to\infty} |x(t)|/(1+t) = 0$ *uniformly for $x \in \Lambda$;*
(c) *for any $t \in [0,\infty)$, $V(t) = \{v(t) : v(t) = x(t)/(1+t), x \in \Lambda\}$ is relatively compact in X.*

3. Main Results

We introduce the following hypotheses:

(H0) $\{Q(t)\}_{t>0}$ is equicontinuous, i.e., $Q(t)$ is continuous in the uniform operator topology for $t > 0$.
(H1) $g(t,\cdot)$ is Lebesgue measurable with respect to t on $[0,\infty)$. $g(\cdot,y)$ is continuous with respect to y on X.
(H2) There exists a function $m : (0,\infty) \to (0,\infty)$ such that

$$I^\lambda_{0+}m(t) \in C((0,\infty),(0,\infty)), \quad |g(t,y)| \leq m(t), \quad \text{for all } y \in X,\ t \in (0,\infty),$$

and

$$\lim_{t\to 0+} t^{(1-\lambda)(1-\mu)} I^\lambda_{0+}m(t) = 0, \quad \lim_{t\to\infty} \frac{t^{(1-\lambda)(1-\mu)}}{1+t} I^\lambda_{0+}m(t) = 0.$$

Let

$$C_\lambda((0,\infty),X) = \Big\{y \in C((0,\infty),X) : \lim_{t\to 0+} t^{(1-\lambda)(1-\mu)}|y(t)| \text{ exists and is finite},$$
$$\lim_{t\to\infty} \frac{t^{(1-\lambda)(1-\mu)}|y(t)|}{1+t} = 0\Big\}.$$

Then $(C_\lambda((0,\infty),X), \|\cdot\|_\lambda)$ is a Banach space with the norm

$$\|y\|_\lambda = \sup_{t\in[0,\infty)} \frac{t^{(1-\lambda)(1-\mu)}|y(t)|}{1+t}.$$

For any $y \in C_\lambda((0,\infty),X)$, define an operator Ψ as follows

$$(\Psi y)(t) = (\Psi_1 y)(t) + (\Psi_2 y)(t),$$

where

$$(\Psi_1 y)(t) = \mathcal{P}_{\mu,\lambda}(t)y_0, \quad (\Psi_2 y)(t) = \int_0^t \mathcal{Q}_\lambda(t-s)g(s,y(s))ds, \text{ for } t \in (0,\infty).$$

For any $u \in C_1([0,\infty), X)$, set

$$y(t) = t^{-(1-\lambda)(1-\mu)}u(t), \quad \text{for } t \in (0,\infty).$$

Then, $y \in C_\lambda((0,\infty), X)$. Define an operator Φ as follows

$$(\Phi u)(t) = (\Phi_1 u)(t) + (\Phi_2 u)(t),$$

where

$$(\Phi_1 u)(t) = \begin{cases} t^{(1-\lambda)(1-\mu)}(\Psi_1 y)(t), & \text{for } t \in (0,\infty), \\ \dfrac{y_0}{\Gamma(\mu(1-\lambda)+\lambda)}, & \text{for } t = 0, \end{cases}$$
$$(\Phi_2 u)(t) = \begin{cases} t^{(1-\lambda)(1-\mu)}(\Psi_2 y)(t), & \text{for } t \in (0,\infty), \\ 0, & \text{for } t = 0. \end{cases}$$

Obviously, $y \in C_\lambda((0,\infty), X)$ is a mild solution of (1) if and only if the operator equation $y = \Psi y$ has a solution $y \in C_\lambda((0,\infty), X)$.

In view of (H2), we have

$$\lim_{t\to 0+} \frac{t^{(1-\lambda)(1-\mu)}}{1+t} I^\lambda_{0+} m(t) = 0, \quad \lim_{t\to\infty} \frac{t^{(1-\lambda)(1-\mu)}}{1+t} I^\lambda_{0+} m(t) = 0.$$

Thus, there exists a constant $r > 0$ such that

$$\sup_{t\in[0,\infty)} \left\{ \frac{L|y_0|}{\Gamma(\mu(1-\lambda)+\lambda)} + \frac{Lt^{(1-\lambda)(1-\mu)}}{1+t} I^\lambda_{0+} m(t) \right\} \le r,$$

i.e.,

$$\sup_{t\in[0,\infty)} \left\{ \frac{L|y_0|}{\Gamma(\mu(1-\lambda)+\lambda)} + \frac{L}{\Gamma(\lambda)} \frac{t^{(1-\lambda)(1-\mu)}}{1+t} \int_0^t (t-s)^{\lambda-1} m(s) ds \right\} \le r. \tag{4}$$

Let

$$\Omega_r = \{u \in C_1([0,\infty), X) : \|u\| \le r\}, \quad \widetilde{\Omega}_r = \{y \in C_\lambda((0,\infty), X) : \|y\|_\lambda \le r\}.$$

Clearly, Ω_r is a nonempty, convex, and closed subset of $C_1([0,\infty), X)$, and $\widetilde{\Omega}_r$ is a nonempty, convex, and closed subset of $C_\lambda((0,\infty), X)$.

Let

$$V := \{v : v(t) = (\Phi u)(t)/(1+t),\ u \in \Omega_r\}.$$

Lemma 9. *Assume that (H0), (H1) and (H2) hold. Then the set V is equicontinuous.*

Proof. Step I. We first prove that $\{v : v(t) = (\Phi_1 u)(t)/(1+t),\ u \in \Omega_r\}$ is equicontinuous.

As $\lim_{t\to 0+} \mathcal{S}_\lambda(t) y_0 = y_0/\Gamma(\lambda)$, we find

$$\begin{aligned}
\lim_{t\to 0+} t^{(1-\lambda)(1-\mu)} \mathcal{P}_{\mu,\lambda}(t) y_0 &= \lim_{t\to 0+} \frac{t^{(1-\lambda)(1-\mu)}}{\Gamma(\mu(1-\lambda))} \int_0^t (t-s)^{\mu(1-\lambda)-1} s^{\lambda-1} \mathcal{S}_\lambda(s) y_0 ds \\
&= \lim_{t\to 0+} \frac{1}{\Gamma(\mu(1-\lambda))} \int_0^1 (1-z)^{\mu(1-\lambda)-1} z^{\lambda-1} \mathcal{S}_\lambda(tz) y_0 dz \\
&= \frac{1}{\Gamma(\mu(1-\lambda))\Gamma(\lambda)} \int_0^1 (1-z)^{\mu(1-\lambda)-1} z^{\lambda-1} y_0 dz \\
&= \frac{y_0}{\Gamma(\mu(1-\lambda)+\lambda)}.
\end{aligned}$$

Hence, for $t_1 = 0$, $t_2 \in (0, \infty)$, we obtain

$$
\begin{aligned}
&\Big|\frac{(\Phi_1 u)(t_2)}{1+t_2} - (\Phi_1 u)(0)\Big| \\
\leq&\Big|\frac{1}{1+t_2} {t_2}^{(1-\lambda)(1-\mu)} \mathcal{P}_{\mu,\lambda}(t_2) y_0 - \frac{y_0}{\Gamma(\mu(1-\lambda)+\lambda)}\Big| \\
\to& \, 0, \quad \text{as } t_2 \to 0.
\end{aligned}
$$

For any $t_1, t_2 \in (0, \infty)$ and $t_1 < t_2$, we have

$$
\begin{aligned}
&\Big|\frac{(\Phi_1 u)(t_2)}{1+t_2} - \frac{(\Phi_1 u)(t_1)}{1+t_1}\Big| \\
=&\Big|\frac{{t_2}^{(1-\lambda)(1-\mu)} \mathcal{P}_{\mu,\lambda}(t_2) y_0}{1+t_2} - \frac{{t_1}^{(1-\lambda)(1-\mu)} \mathcal{P}_{\mu,\lambda}(t_1) y_0}{1+t_1}\Big| \\
\leq&\Big|\frac{{t_2}^{(1-\lambda)(1-\mu)} \mathcal{P}_{\mu,\lambda}(t_2) y_0}{1+t_2} - \frac{{t_2}^{(1-\lambda)(1-\mu)} \mathcal{P}_{\mu,\lambda}(t_2) y_0}{1+t_1}\Big| \\
&+ \Big|\frac{{t_2}^{(1-\lambda)(1-\mu)} \mathcal{P}_{\mu,\lambda}(t_2) y_0}{1+t_1} - \frac{{t_1}^{(1-\lambda)(1-\mu)} \mathcal{P}_{\mu,\lambda}(t_1) y_0}{1+t_1}\Big| \\
\leq&|{t_2}^{(1-\lambda)(1-\mu)} \mathcal{P}_{\mu,\lambda}(t_2) y_0| \frac{|t_2 - t_1|}{(1+t_2)(1+t_1)} \\
&+ |{t_2}^{(1-\lambda)(1-\mu)} \mathcal{P}_{\mu,\lambda}(t_2) y_0 - {t_1}^{(1-\lambda)(1-\mu)} \mathcal{P}_{\mu,\lambda}(t_1) y_0| \frac{1}{1+t_1} \\
\leq&|{t_2}^{(1-\lambda)(1-\mu)} \mathcal{P}_{\mu,\lambda}(t_2) y_0| \frac{|t_2 - t_1|}{(1+t_2)(1+t_1)} \\
&+ |{t_2}^{(1-\lambda)(1-\mu)}||\mathcal{P}_{\mu,\lambda}(t_2) y_0 - \mathcal{P}_{\mu,\lambda}(t_1) y_0| \frac{1}{1+t_1} \\
&+ |{t_2}^{(1-\lambda)(1-\mu)} - {t_1}^{(1-\lambda)(1-\mu)}||\mathcal{P}_{\mu,\lambda}(t_1) y_0| \frac{1}{1+t_1} \\
\to& \, 0, \quad \text{as } t_2 \to t_1.
\end{aligned}
$$

Hence, $\{v : v(t) = (\Phi_1 u)(t)/(1+t),\ u \in \Omega_r\}$ is equicontinuous.
Step II. We prove that $\{v : v(t) = (\Phi_2 u)(t)/(1+t),\ u \in \Omega_r\}$ is equicontinuous.
Let $y(t) = t^{-(1-\lambda)(1-\mu)} u(t)$, for any $u \in \Omega_r$, $t \in (0, \infty)$. Then $y \in \widetilde{\Omega}_r$.
For $\varepsilon > 0$, in view of (H2), there exists $T > 0$ such that

$$
\frac{L}{\Gamma(\lambda)} \frac{t^{(1-\lambda)(1-\mu)}}{1+t} \int_0^t (t-s)^{\lambda-1} m(s) ds < \frac{\varepsilon}{2}, \ \text{ for } t > T. \tag{5}
$$

For $t_1, t_2 > T$, in virtue of (H2) and (5), we find

$$
\begin{aligned}
\Big|\frac{(\Phi_2 u)(t_2)}{1+t_2} - \frac{(\Phi_2 u)(t_1)}{1+t_1}\Big| \leq&\Big|\frac{{t_2}^{(1-\lambda)(1-\mu)}}{1+t_2} \int_0^{t_2} \mathcal{Q}_\lambda(t_2 - s) g(s, y(s)) ds\Big| \\
&+ \Big|\frac{{t_1}^{(1-\lambda)(1-\mu)}}{1+t_1} \int_0^{t_1} \mathcal{Q}_\lambda(t_1 - s) g(s, y(s)) ds\Big| \\
\leq&\frac{L}{\Gamma(\lambda)} \frac{{t_2}^{(1-\lambda)(1-\mu)}}{1+t_2} \int_0^{t_2} (t_2 - s)^{\lambda-1} m(s) ds \\
&+ \frac{L}{\Gamma(\lambda)} \frac{{t_1}^{(1-\lambda)(1-\mu)}}{1+t_1} \int_0^{t_1} (t_1 - s)^{\lambda-1} m(s) ds \\
<&\varepsilon.
\end{aligned}
$$

When $t_1 = 0, 0 < t_2 < T$, we have

$$\begin{aligned}\Big|\frac{(\Phi_2 u)(t_2)}{1+t_2} - (\Phi_2 u)(0)\Big| =&\Big|\frac{{t_2}^{(1-\lambda)(1-\mu)}}{1+t_2}\int_0^{t_2} \mathcal{Q}_\lambda(t_2-s)g(s,y(s))ds\Big| \\ \leq&\frac{L}{\Gamma(\lambda)}\frac{{t_2}^{(1-\lambda)(1-\mu)}}{1+t_2}\int_0^{t_2}(t_2-s)^{\lambda-1}m(s)ds \\ \to& 0 \quad \text{as } t_2 \to 0.\end{aligned}$$

For $0 < t_1 < t_2 \leq T$, we find

$$\begin{aligned}&\Big|\frac{(\Phi_2 u)(t_2)}{1+t_2} - \frac{(\Phi_2 u)(t_1)}{1+t_1}\Big| \\ \leq&\Big|\frac{{t_1}^{(1-\lambda)(1-\mu)}}{1+t_1}\int_{t_1}^{t_2}(t_2-s)^{\lambda-1}\mathcal{S}_\lambda(t_2-s)g(s,y(s))ds\Big| \\ &+\Big|\frac{{t_1}^{(1-\lambda)(1-\mu)}}{1+t_1}\int_0^{t_1}\big((t_2-s)^{\lambda-1}-(t_1-s)^{\lambda-1}\big)\mathcal{S}_\lambda(t_2-s)g(s,y(s))ds\Big| \\ &+\Big|\frac{{t_1}^{(1-\lambda)(1-\mu)}}{1+t_1}\int_0^{t_1}(t_1-s)^{\lambda-1}\big(\mathcal{S}_\lambda(t_2-s)-\mathcal{S}_\lambda(t_1-s)\big)g(s,y(s))ds\Big| \\ &+\Big|\frac{{t_2}^{(1-\lambda)(1-\mu)}}{1+t_2}-\frac{{t_1}^{(1-\lambda)(1-\mu)}}{1+t_1}\Big|\Big|\int_0^{t_2}(t_2-s)^{\lambda-1}\mathcal{S}_\lambda(t_2-s)g(s,y(s))ds\Big| \\ \leq& I_1+I_2+I_3+I_4,\end{aligned}$$

where

$$\begin{aligned}I_1 =&\frac{L}{\Gamma(\lambda)}\frac{{t_1}^{(1-\lambda)(1-\mu)}}{1+t_1}\Big|\int_0^{t_2}(t_2-s)^{\lambda-1}m(s)ds-\int_0^{t_1}(t_1-s)^{\lambda-1}m(s)ds\Big|, \\ I_2 =&\frac{2L}{\Gamma(\lambda)}\frac{{t_1}^{(1-\lambda)(1-\mu)}}{1+t_1}\int_0^{t_1}\big((t_1-s)^{\lambda-1}-(t_2-s)^{\lambda-1}\big)m(s)ds, \\ I_3 =&\frac{{t_1}^{(1-\lambda)(1-\mu)}}{1+t_1}\Big|\int_0^{t_1}(t_1-s)^{\lambda-1}\big(\mathcal{S}_\lambda(t_2-s)-\mathcal{S}_\lambda(t_1-s)\big)g(s,y(s))ds\Big|, \\ I_4 =&\Big|\frac{{t_2}^{(1-\lambda)(1-\mu)}}{1+t_2}-\frac{{t_1}^{(1-\lambda)(1-\mu)}}{1+t_1}\Big|\frac{L}{\Gamma(\lambda)}\int_0^{t_2}(t_2-s)^{\lambda-1}m(s)ds.\end{aligned}$$

One can deduce that $\lim_{t_2\to t_1} I_1 = 0$, as $(I_{0+}^{\lambda}m)(t) \in C((0,\infty),(0,\infty))$. Noting that

$$((t_1-s)^{\lambda-1}-(t_2-s)^{\lambda-1})m(s) \leq (t_1-s)^{\lambda-1}m(s), \quad \text{for } s \in [0,t_1),$$

then by Lebesgue dominated convergence theorem, we find

$$\int_0^{t_1}\big((t_1-s)^{\lambda-1}-(t_2-s)^{\lambda-1}\big)m(s)ds \to 0, \quad \text{as } t_2 \to t_1,$$

so, $I_2 \to 0$ as $t_2 \to t_1$.

For $\varepsilon > 0$ be enough small, we have

$$\begin{aligned}
I_3 \leq & \frac{t_1{}^{(1-\lambda)(1-\mu)}}{1+t_1}\int_0^{t_1-\varepsilon}(t_1-s)^{\lambda-1}\big\|\mathcal{S}_\lambda(t_2-s)-\mathcal{S}_\lambda(t_1-s)\big\|_{\mathcal{L}(X)}|g(s,y(s))|ds \\
& + \frac{t_1{}^{(1-\lambda)(1-\mu)}}{1+t_1}\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\lambda-1}\big\|\mathcal{S}_\lambda(t_2-s)-\mathcal{S}_\lambda(t_1-s)\big\|_{\mathcal{L}(X)}|g(s,y(s))|ds \\
\leq & \frac{t_1{}^{(1-\lambda)(1-\mu)}}{1+t_1}\int_0^{t_1-\varepsilon}(t_1-s)^{\lambda-1}m(s)ds \sup_{s\in[0,t_1-\varepsilon]}\big\|\mathcal{S}_\lambda(t_2-s)-\mathcal{S}_\lambda(t_1-s)\big\|_{\mathcal{L}(X)} \\
& + \frac{2L}{\Gamma(\lambda)}\frac{t_1{}^{(1-\lambda)(1-\mu)}}{1+t_1}\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\lambda-1}m(s)ds \\
\leq & I_{31}+I_{32}+I_{33},
\end{aligned}$$

where

$$\begin{aligned}
I_{31} = & \frac{t_1{}^{(1-\lambda)(1-\mu)}}{1+t_1}\int_0^{t_1-\varepsilon}(t_1-s)^{\lambda-1}m(s)ds \sup_{s\in[0,t_1-\varepsilon]}\big\|\mathcal{S}_\lambda(t_2-s)-\mathcal{S}_\lambda(t_1-s)\big\|_{\mathcal{L}(X)}, \\
I_{32} = & \frac{2L}{\Gamma(\lambda)}\frac{t_1{}^{(1-\lambda)(1-\mu)}}{1+t_1}\Big|\int_0^{t_1}(t_1-s)^{\lambda-1}m(s)ds-\int_0^{t_1-\varepsilon}(t_1-\varepsilon-s)^{\lambda-1}m(s)ds\Big|, \\
I_{33} = & \frac{2L}{\Gamma(\lambda)}\frac{t_1{}^{(1-\lambda)(1-\mu)}}{1+t_1}\int_0^{t_1-\varepsilon}((t_1-\varepsilon-s)^{\lambda-1}-(t_1-s)^{\lambda-1})m(s)ds.
\end{aligned}$$

By (H0) and Lemma 7, it is easy to see that $I_{31}\to 0$ as $t_2\to t_1$. Similar to the proof that I_1, I_2 tend to zero, we obtain $I_{32}\to 0$ and $I_{33}\to 0$ as $\varepsilon\to 0$. Thus, I_3 tends to zero as $t_2\to t_1$. Clearly, $I_4\to 0$ as $t_2\to t_1$.

For $0<t_1<T<t_2$, if $t_2\to t_1$, then $t_2\to T$ and $t_1\to T$. Thus, for $u\in\Omega_r$

$$\begin{aligned}
& \Big|\frac{(\Phi_2 u)(t_2)}{1+t_2}-\frac{(\Phi_2 u)(t_1)}{1+t_1}\Big| \\
\leq & \Big|\frac{(\Phi_2 u)(t_2)}{1+t_2}-\frac{(\Phi_2 u)(T)}{1+T}\Big|+\Big|\frac{(\Phi_2 u)(T)}{1+T}-\frac{(\Phi_2 u)(t_1)}{1+t_1}\Big|\to 0, \quad \text{as } t_2\to t_1.
\end{aligned}$$

Consequently,

$$\Big|\frac{(\Phi_2 u)(t_2)}{1+t_2}-\frac{(\Phi_2 u)(t_1)}{1+t_1}\Big|\to 0, \quad \text{as } t_2\to t_1.$$

Therefore, $\{v : v(t)=(\Phi_2 u)(t)/(1+t),\ u\in\Omega_r\}$ is equicontinuous. Furthermore, V is equicontinuous. □

Lemma 10. *Assume that (H1) and (H2) hold. Then,* $\lim_{t\to\infty}|(\Phi u)(t)|/(1+t)=0$ *uniformly for* $u\in\Omega_r$.

Proof. In fact, for any $u\in\Omega_r$, by (H2) and Lemma 6, we find

$$\begin{aligned}
\frac{|(\Phi u)(t)|}{1+t} \leq & \Big|\frac{t^{(1-\lambda)(1-\mu)}}{1+t}\mathcal{P}_{\mu,\lambda}(t)y_0\Big|+\Big|\frac{t^{(1-\lambda)(1-\mu)}}{1+t}\int_0^t\mathcal{Q}_\lambda(t-s)g(s,y(s))ds\Big| \\
\leq & \frac{L|y_0|}{\Gamma(\mu(1-\lambda)+\lambda)(1+t)}+\frac{L}{\Gamma(\lambda)}\frac{t^{(1-\lambda)(1-\mu)}}{1+t}\int_0^t(t-s)^{\lambda-1}m(s)ds, \ \ t>0.
\end{aligned} \tag{6}$$

By (H2), we derive

$$\frac{|(\Phi u)(t)|}{1+t}\to 0, \quad \text{as } t\to\infty,$$

which implies that $\lim_{t\to\infty} |(\Phi u)(t)|/(1+t) = 0$ uniformly for $u \in \Omega_r$. This completes the proof. □

Lemma 11. *Assume that (H1) and (H2) hold. Then* $\Phi\Omega_r \subset \Omega_r$.

Proof. From Lemmas 9 and 10, we know that $\Phi\Omega_r \subset C_1([0,\infty), X)$. For $t > 0$ and any $u \in \Omega_r$, by (4) and (6), we have

$$\frac{|(\Phi u)(t)|}{1+t} \leq \frac{L|y_0|}{\Gamma(\mu(1-\lambda)+\lambda)} + \frac{L}{\Gamma(\lambda)} \frac{t^{(1-\lambda)(1-\mu)}}{1+t} \int_0^t (t-s)^{\lambda-1} m(s) ds \leq r.$$

For $t = 0$, we find

$$|(\Phi u)(0)| = \frac{y_0}{\Gamma(\mu(1-\lambda)+\lambda)} \leq \frac{L y_0}{\Gamma(\mu(1-\lambda)+\lambda)} \leq r.$$

Therefore, $\Phi\Omega_r \subset \Omega_r$. □

Lemma 12. *Suppose that (H1) and (H2) hold. Then* Φ *is continuous.*

Proof. Indeed, let $\{u_n\}_{n=1}^{\infty}$ be a sequence in Ω_r which is convergent to $u \in \Omega_r$. Consequently,

$$\lim_{n\to\infty} u_n(t) = u(t), \text{ and } \lim_{n\to\infty} t^{-(1-\lambda)(1-\mu)} u_n(t) = t^{-(1-\lambda)(1-\mu)} u(t), \text{ for } t \in (0,\infty).$$

Let $y(t) = t^{-(1-\lambda)(1-\mu)} u(t)$, $y_n(t) = t^{-(1-\lambda)(1-\mu)} u_n(t)$, $t \in (0,\infty)$. Then $y, y_n \in \widetilde{\Omega}_r$. In view of (H1), we have

$$\lim_{n\to\infty} g(t, y_n(t)) = \lim_{n\to\infty} g(t, t^{-(1-\lambda)(1-\mu)} u_n(t)) = g(t, t^{-(1-\lambda)(1-\mu)} u(t)) = g(t, y(t)).$$

On the one hand, using (H2), we get for each $t \in (0,\infty)$,

$$(t-s)^{\lambda-1} |g(s, y_n(s)) - g(s, y(s))| \leq 2(t-s)^{\lambda-1} m(s), \quad \text{a.e. in } [0,t).$$

On the other hand, the function $s \to 2(t-s)^{\lambda-1} m(s)$ is integrable for $s \in [0,t)$, $t \in [0,\infty)$. By Lebesgue dominated convergence theorem, we obtain

$$\int_0^t (t-s)^{\lambda-1} |g(s, y_n(s)) - g(s, y(s))| ds \to 0, \quad \text{as } n \to \infty.$$

Thus, for $t \in [0,\infty)$,

$$\begin{aligned}
&\left| \frac{(\Phi u_n)(t)}{1+t} - \frac{(\Phi u)(t)}{1+t} \right| \\
\leq & \frac{t^{(1-\lambda)(1-\mu)}}{1+t} \int_0^t |\mathcal{Q}_\lambda(t-s)(g(s, y_n(s)) - g(s, y(s)))| ds \\
\leq & \frac{L}{\Gamma(\lambda)} \frac{t^{(1-\lambda)(1-\mu)}}{1+t} \int_0^t (t-s)^{\lambda-1} |g(s, y_n(s)) - g(s, y(s))| ds \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore, $\|\Phi u_n - \Phi u\| \to 0$ as $n \to \infty$. Hence, Φ is continuous. The proof is completed. □

Theorem 1. *Assume that* $Q(t)(t > 0)$ *is compact. Furthermore suppose that (H1) and (H2) hold. Then the Cauchy problem (1) has at least one mild solution.*

Proof. Clearly, the problem (1) exists a mild solution $y \in \widetilde{\Omega}_r$ if and only if the operator Φ has a fixed point $u \in \Omega_r$, where $u(t) = t^{(1-\lambda)(1-\mu)} y(t)$. Hence, we only need to prove that the operator Φ has a fixed point in Ω_r. From Lemmas 11 and 12, we know that $\Phi\Omega_r \subset \Omega_r$ and Φ is continuous. In order to prove that Φ is a completely continuous operator, we need to prove that $\Phi\Omega_r$ is a relatively compact set. In view of Lemmas 9 and 10, the set $V = \{v : v(t) = (\Phi u)(t)/(1+t),\ u \in \Omega_r\}$ is equicontinuous on $[0,h]$ for any $h > 0$, and $\lim_{t\to\infty} |(\Phi u)(t)|/(1+t) = 0$ uniformly for $u \in \Omega_r$. According to Lemma 8, we only need to prove $V(t) = \{v(t) : v(t) = (\Phi u)(t)/(1+t),\ u \in \Omega_r\}$ is relatively compact in X for $t \in [0,\infty)$. Obviously, $V(0)$ is relatively compact in X. We only consider the case $t > 0$. For $\forall\ \varepsilon \in (0,t)$ and $\delta > 0$, define $\Phi_{\varepsilon,\delta}$ on Ω_r as follows:

$$
\begin{aligned}
&(\Phi_{\varepsilon,\delta}u)(t) := t^{(1-\lambda)(1-\mu)}(\Psi_{\varepsilon,\delta}y)(t)\\
=&t^{(1-\lambda)(1-\mu)}\left(\mathcal{P}_{\mu,\lambda}(t)y_0 + \int_0^{t-\varepsilon}\int_\delta^\infty \lambda\theta(t-s)^{\lambda-1}W_\lambda(\theta)Q((t-s)^\lambda\theta)g(s,y(s))d\theta ds\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{(\Phi_{\varepsilon,\delta}u)(t)}{1+t} = &\frac{t^{(1-\lambda)(1-\mu)}}{1+t}\Bigg(\mathcal{P}_{\mu,\lambda}(t)y_0\\
&+ Q(\varepsilon^\lambda\delta)\int_0^{t-\varepsilon}\int_\delta^\infty \lambda\theta(t-s)^{\lambda-1}W_\lambda(\theta)Q((t-s)^\lambda\theta-\varepsilon^\lambda\delta)g(s,y(s))d\theta ds\Bigg).
\end{aligned}
$$

By Lemma 4, we know that $\mathcal{P}_{\mu,\lambda}(t)$ is compact because $Q(t)$ is compact for $t > 0$. Further, $Q(\varepsilon^\lambda\delta)$ is compact, then the set $\{\frac{(\Phi_{\varepsilon,\delta}u)(t)}{1+t},\ u \in \Omega_r\}$ is relatively compact in X for any $\varepsilon \in (0,t)$ and for any $\delta > 0$. Moreover, for every $u \in \Omega_r$, we find

$$
\begin{aligned}
&\left|\frac{(\Phi u)(t)}{1+t} - \frac{(\Phi_{\varepsilon,\delta}u)(t)}{1+t}\right|\\
\le&\frac{t^{(1-\lambda)(1-\mu)}}{1+t}\left|\int_0^t\int_0^\delta \lambda\theta(t-s)^{\lambda-1}W_\lambda(\theta)Q((t-s)^\lambda\theta)g(s,y(s))d\theta ds\right|\\
&+\frac{t^{(1-\lambda)(1-\mu)}}{1+t}\left|\int_{t-\varepsilon}^t\int_\delta^\infty \lambda\theta(t-s)^{\lambda-1}W_\lambda(\theta)Q((t-s)^\lambda\theta)g(s,y(s))d\theta ds\right|\\
\le&\frac{\lambda L t^{(1-\lambda)(1-\mu)}}{1+t}\int_0^t(t-s)^{\lambda-1}m(s)ds\int_0^\delta\theta W_\lambda(\theta)d\theta\\
&+\frac{\lambda L t^{(1-\lambda)(1-\mu)}}{1+t}\int_{t-\varepsilon}^t(t-s)^{\lambda-1}m(s)ds\int_0^\infty\theta W_\lambda(\theta)d\theta\\
\to& 0, \quad \text{as } \varepsilon\to 0,\ \delta\to 0.
\end{aligned}
$$

Thus, $V(t)$ is also a relatively compact set in X for $t \in [0,\infty)$. Therefore, Schauder's fixed point theorem implies that Φ has at least a fixed point $u^* \in \Omega_r$. Let $y^*(t) = t^{-(1-\lambda)(1-\mu)}u^*(t)$. Thus,

$$
y^*(t) = \mathcal{P}_{\mu,\lambda}(t)y_0 + \int_0^t \mathcal{Q}_\lambda(t-s)g(s,y^*(s))ds,\ t \in (0,\infty),
$$

which implies that $y^* \in \widetilde{\Omega}_r$ is a mild solution of (1). The proof is completed. □

In the case that $Q(t)$ is noncompact for $t > 0$, we need the following hypothesis:

(H3) there exists a constant $K > 0$ such that for any bounded set $D \subseteq X$,

$$
\beta(g(t,D)) \le Kt^{(1-\lambda)(1-\mu)}\beta(D), \quad \text{for a.e. } t \in [0,\infty),
$$

where β is the Kuratowski's measure of noncompactness.

Theorem 2. *Assume that (H0), (H1), (H2) and (H3) hold. Then the Cauchy problem (1) has at least one mild solution.*

Proof. Let $u_0(t) = t^{(1-\lambda)(1-\mu)}\mathcal{P}_{\mu,\lambda}(t)y_0$ for all $t \in [0,\infty)$ and $u_{n+1} = \Phi u_n$, $n = 0,1,2,\cdots$. By Lemma 11, $\Phi u_n \in \Omega_r$, for $u_n \in \Omega_r$, $n = 0,1,2,\cdots$. Consider set $\mathcal{V} = \{v_n : v_n(t) = (\Phi u_n)(t)/(1+t),\ u_n \in \Omega_r\}_{n=0}^{\infty}$, and we will prove set $\mathcal{V}$ is relatively compact.

In view of Lemmas 9 and 10, the set $\mathcal{V}$ is equicontinuous and $\lim_{t\to\infty} |(\Phi u_n)(t)|/(1+t) = 0$ uniformly for $u_n \in \Omega_r$. According to Lemma 8, we only need to prove $\mathcal{V}(t) = \{v_n(t) : v_n(t) = (\Phi u_n)(t)/(1+t),\ u_n \in \Omega_r\}_{n=0}^{\infty}$ is relatively compact in X for $t \in [0,\infty)$.

Let $y_n(t) = t^{-(1-\lambda)(1-\mu)}u_n(t)$, $t \in (0,\infty)$, $n = 0,1,2,\cdots$. By the condition (H3) and Lemma 1, we have

$$\begin{aligned}
\beta(\mathcal{V}(t)) =&\beta\Big(\Big\{\frac{(\Phi u_n)(t)}{1+t}\Big\}_{n=0}^{\infty}\Big)\\
=&\beta\Big(\Big\{\frac{t^{(1-\lambda)(1-\mu)}}{1+t}\mathcal{P}_{\mu,\lambda}(t)y_0 + \frac{t^{(1-\lambda)(1-\mu)}}{1+t}\int_0^t \mathcal{Q}_\lambda(t-s)g(s,y_n(s))ds\Big\}_{n=0}^{\infty}\Big)\\
=&\beta\Big(\Big\{\frac{t^{(1-\lambda)(1-\mu)}}{1+t}\int_0^t \mathcal{Q}_\lambda(t-s)g(s,y_n(s))ds\Big\}_{n=0}^{\infty}\Big)\\
\le&\frac{2L}{\Gamma(\lambda)}\frac{t^{(1-\lambda)(1-\mu)}}{1+t}\int_0^t (t-s)^{\lambda-1}\beta\Big(g(s,\{s^{-(1-\lambda)(1-\mu)}u_n(s)\}_{n=0}^{\infty})\Big)ds\\
\le&\frac{2LK}{\Gamma(\lambda)}\frac{t^{(1-\lambda)(1-\mu)}}{1+t}\int_0^t (t-s)^{\lambda-1}s^{(1-\lambda)(1-\mu)}\beta\Big(\{s^{-(1-\lambda)(1-\mu)}u_n(s)\}_{n=0}^{\infty}\Big)ds\\
\le&\frac{2LK}{\Gamma(\lambda)}\frac{t^{(1-\lambda)(1-\mu)}}{1+t}\int_0^t (t-s)^{\lambda-1}(1+s)\beta\Big(\Big\{\frac{u_n(s)}{1+s}\Big\}_{n=0}^{\infty}\Big)ds.
\end{aligned}$$

On the other hand, by the properties of measure of noncompactness, for any $t \in [0,\infty)$ we have

$$\beta\Big(\Big\{\frac{u_n(t)}{1+t}\Big\}_{n=0}^{\infty}\Big) = \beta\Big(\Big\{\frac{u_0(t)}{1+t}\Big\} \cup \Big\{\frac{u_n(t)}{1+t}\Big\}_{n=1}^{\infty}\Big) = \beta\Big(\Big\{\frac{u_n(t)}{1+t}\Big\}_{n=1}^{\infty}\Big) = \beta(\mathcal{V}(t)).$$

Thus

$$\beta(\mathcal{V}(t)) \le \frac{2LKM^*}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}(1+s)\beta(\mathcal{V}(s))ds, \tag{7}$$

where $M^* = \max_{t\in[0,\infty)}\{\frac{t^{(1-\lambda)(1-\mu)}}{1+t}\}$. From (7), we know that

$$\beta(\mathcal{V}(t)) \le \frac{4LKM^*}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}\beta(\mathcal{V}(s))ds,$$

or

$$\beta(\mathcal{V}(t)) \le \frac{4LKM^*}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}s\beta(\mathcal{V}(s))ds,$$

holds. Therefore, by the inequality in ([17] p. 188), we obtain that $\beta(\mathcal{V}(t)) = 0$, then $\mathcal{V}(t)$ is relatively compact. Consequently, it follows from Lemma 8 that set $\mathcal{V}$ is relatively compact, i.e., there exists a convergent subsequence of $\{u_n\}_{n=0}^{\infty}$. With no confusion, let $\lim_{n\to\infty} u_n = u^*$, $u^* \in \Omega_r$.

Thus, by continuity of the operator Φ, we have

$$u^* = \lim_{n\to\infty} u_n = \lim_{n\to\infty} \Phi u_{n-1} = \Phi\Big(\lim_{n\to\infty} u_{n-1}\Big) = \Phi u^*.$$

Let $y^*(t) = t^{-(1-\lambda)(1-\mu)}u^*(t)$. Thus, $y^* \in \widetilde{\Omega}_r$ is a mild solution of (1). The proof is completed. □

By Theorems 1 and 2, we have the following corollaries.

Corollary 1. *Assume that $Q(t)$ is compact for $t > 0$ and (H1) holds. Furthermore suppose that* (H2)$'$ *there exists a function $m : (0,\infty) \to (0,\infty)$ and $\alpha \in (0,1), M > 0$ such that*

$$I_{0+}^{\lambda} m(t) \in C((0,\infty),(0,\infty)), \quad t^{(1-\lambda)(1-\mu)} I_{0+}^{\lambda} m(t) \le M t^{\alpha},$$

and

$$|g(t,y)| \le m(t), \quad for\ all\ \ y \in X, t \in (0,\infty).$$

Then the Cauchy problem (1) has at least one mild solution.

Corollary 2. *Assume that (H0), (H1), (H2)$'$ and (H3) hold. Then the Cauchy problem (1) has at least one mild solution.*

Example 1. *Let $X = L^2([0,\pi],\mathbb{R})$. Consider the following fractional partial differential equations on infinite interval*

$$\begin{cases} {}^{H}D_{0+}^{\mu,\lambda} y(t,z) = \partial_z^2 y(t,z) + t^{-\eta}, \quad z \in [0,\pi],\ t > 0, \\ y(t,0) = y(t,\pi) = 0, \quad t > 0 \\ I_{0+}^{(1-\mu)(1-\lambda)} y(0,z) = y_0(z),\ z \in [0,\pi]. \end{cases} \tag{8}$$

We define an operator A by $Av = v''$ with the domain

$$D(A) = \{v \in X : \ v, v'' \text{ are absolutely continuous},\ v'' \in X,\ v(0) = v(\pi) = 0\}.$$

Then A generates a compact, analytic, self-adjoint semigroup $\{T(t)\}_{t>0}$. Then problem (8) can be rewritten as follows

$$\begin{cases} {}^{H}D_{0+}^{\mu,\lambda} y(t) = Ay(t) + g_1(t,y(t)), \quad t > 0, \\ I_{0+}^{(1-\mu)(1-\lambda)} y(0) = y_0, \end{cases} \tag{9}$$

where $g_1(t,y) := t^{-\eta}$ for $\eta \in (\lambda, 1-\mu+\mu\lambda)$ satisfies (H1), and $|g_1(t,y(t))| \le t^{-\eta}$, $t \in (0,\infty)$. Let $m(t) = t^{-\eta}$, for $t > 0$. Then

$$I_{0+}^{\lambda} m(t) = \frac{\Gamma(1-\eta)}{\Gamma(1+\lambda-\eta)} t^{\lambda-\eta}, \quad t^{(1-\lambda)(1-\mu)} I_{0+}^{\lambda} m(t) = \frac{\Gamma(1-\eta)}{\Gamma(1+\lambda-\eta)} t^{\alpha},$$

where $\alpha = 1-\mu+\lambda\mu-\eta \in (0,1)$. This means that the condition (H2)$'$ is satisfied. By Corollary 1, the problem (8) has at least a mild solution.

4. Conclusions

In this paper, by using the generalized Ascoli–Arzelà theorem and some new techniques, we investigated the existence of mild solutions for Hilfer fractional evolution equations on infinite interval. We proved the existence theorems of mild solutions for both the cases in which the semigroup is compact and noncompact. In particular, we do not need to assume that the $g(t,\cdot)$ satisfies the Lipschitz condition. The methods in this paper can be applied to study infinite interval problems for non-autonomous evolution equations, fractional evolution equations with instantaneous/non-instantaneous impulses, fractional neutral functional evolution equations, and fractional stochastic evolution equations. We recommend readers to refer to relevant papers [10,18,19].

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